ON THE HOMOLOGY OF PARTIAL GROUP ACTIONS

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ABSTRACT. We study the partial group (co)homology of partial group actions using simplicial methods. We introduce the concept of universal globalization of a partial group action on a K-module and prove that, given a partial representation of G on M, the partial group homology $H_\bullet^{par}(G, M)$ is naturally isomorphic to the usual group homology $H_\bullet(G, KG \otimes_{G_{par}} M)$, where $KG \otimes_{G_{par}} M$ is the universal globalization of the partial group action associated to M. We dualize this result into a cohomological spectral sequence converging to $H^\bullet_{par}(G, M)$.

CONTENTS

INTRODUCTION

The general definition of a partial group action of a group on a set was introduced in [11], but partial group actions have been arising in mathematics over time. Naturally appearing with the question: what happens with a family of symmetries when we look its behavior locally? For example, if we have an action of a group G on a space X, when we focus on the symmetry structure of a subspace Y of X, generally we do not obtain a group action; instead, we obtain what we call a partial group action of G on Y. Besides the important applications of partial group actions in the context of C^*-algebras, the concept

of a partial group action has stimulated rich investigations in pure algebraic subjects (see for example the surveys [5], [3]). With the vast development of partial group action theory, a natural question arises: How to define a proper (co)homology theory for partial group actions? A first approach was introduced in [7], a cohomology theory based on partial group actions on semigroups with a strong relation to the partial Schur multiplier (see [10]). A second approach was studied in [1], based on partial representations of groups. This time, this cohomology appears as a component of a cohomological spectral sequence that computes the Hochschild cohomology of the partial crossed product $\mathcal{A} \rtimes G$ by a unital partial action of a group G on a unital algebra $\mathcal{A}$.

The primary objective of this work is to delve into the study of partial group (co)homology based on partial group representations. With this goal in mind, we adopt an approach to this (co)homology theory employing simplicial methods. This framework enables us to grasp the fundamental nature of the (co)homology theories for partial group actions. Consequently, it provides insights into employing other techniques to address the natural question: what is the relationship between the (co)homology of a partial group action and that of its globalization?

We begin by examining partial group actions on sets and subsequently extend our analysis to encompass partial representations. The principal results of this work are as follows: Firstly, we establish the existence of a universal global action for a partial group action of a group G on a K-module M. This universal global action serves as a proper *globalization* for partial group actions arising from partial representations. Secondly, we prove that the various (co)homology theories (explored in works such as [1], [7], [2] and [8]) naturally emerge as the simplicial (co)homology of the nerve of the group G, with coefficients in a suitable coefficient system. Lastly, our work culminates in the proof that the partial group homology of a partial representation $\pi : G \to \mathrm{End}_K(M)$ is naturally isomorphic to the usual (global) group homology of its universal globalization $KG \otimes_{G_{par}} M$. We extend this result to the cohomological framework by demonstrating the existence of a cohomological spectral sequence that converges to the partial group cohomology $H^{\bullet}_{par}(G, M)$. In several natural cases where this sequence collapses, it allows for a natural isomorphism between the partial group cohomology and the global group cohomology.

The first section is dedicated to recalling the basic theory and definitions necessary for the development of this work. Important concepts for this work are defined here, such as the definition of partial group actions on modules and right partial group actions. For the basic theory of partial group actions, we follow Exel's book [12]. For homological algebra and simplicial methods, we refere to the books [14], [17], [18], and [19].

The second section is dedicated to the construction of the *partial* tensor product of two partial group actions on modules. This partial tensor product will play a central role in the results obtained in this work. We prove that this partial tensor product satisfies the usual properties of tensor products on modules, such as certain universal property and *associativity*. A direct consequence of the existence of the partial tensor product is the existence of a universal global action for a partial group action on a K-module. This universal global action may not be a globalization in the usual sense; contrary to set-theoretical partial group actions, partial group actions on modules may not be globalizable, and if the globalization exists, it may not be unique. We give necessary and sufficient conditions for a partial group action on modules to be globalizable. Consequently, partial group actions arising from partial representations are always globalizable. We obtain, in this way, a functor from partial representations to global representations that sends a partial representation to its universal globalization. We call this functor the globalization functor.

The third section is the main part of this work. Given a set-theoretical partial group action θ of a group G on a set X, we begin by studying the simplicial set associated with θ. We show that this simplicial set is isomorphic to the simplicial set obtained from the groupoid associated with θ. As a consequence, we establish that the (co)homology of the associated simplicial set of θ is isomorphic to the (co)homology of the simplicial set of the globalization of θ. This sheds light on what may occur in the (co)homology for partial representations and their universal globalization. We note that the globalization problem for partial group actions based on partial representations was previously studied in [8] for a unital partial group action on a unital K-algebra. We conclude the first subsection by showing that for any group G, the cohomology theory based on unital partial group actions on semigroups and the cohomology theory based on partial group representations naturally arise as the simplicial cohomology of the nerve of G with an appropriate coefficient systems. We consider the simplicial module obtained from a partial group action θ on a module, and we demonstrate that if θ arises from a partial representation $\pi : G \to \mathrm{End}_K(M)$, then this complex is isomorphic to the canonical complex that computes the partial group homology of G with coefficients in M. Finally, we establish that the globalization functor from partial representations to global representations is exact. Consequently, we obtain a natural isomorphism between $H_\bullet^{par}(G, M)$ and $H_\bullet(G, KG \otimes_{G_{par}} M)$, where $KG \otimes_{G_{par}} M$ is the universal globalization of M. We derive a dual version for cohomology expressed in a cohomological spectral sequence, and we establish conditions for such spectral sequence to collapse.

1. Preliminaries

Let K be a commutative unital ring. Throughout this work, we fix K as our ground ring. Thus, we refer to K-modules simply as modules, and we denote the tensor product of two K-modules $X \otimes_K Y$ simply as $X \otimes Y$. If $\mathcal{A}$ is a K-algebra and X is an $\mathcal{A}$-bimodule, the $\mathcal{A}$-tensor product of n copies of X will be denoted by $X^{\otimes_{\mathcal{A}} n}$. When $\mathcal{A} = K$, it will be simply denoted as $X^{\otimes n}$. The capital letter G will be used to denote a group, and the unit element of an algebraic structure will be denoted simply by 1 if there is no ambiguity. We use the notation $\mathbb{N}$ to denote the set of natural numbers, $\{0, 1, 2, \ldots\}$. This work is dedicated to partial actions and partial representations of groups; hence, we will use the terms "*partial action*" and "*partial representation*" to refer to this specific type of partial action and partial representation, respectively.

1.1. **Partial actions and partial representations.** Here we introduce the basic theory of partial group actions and partial representations of groups used in this work, following the book [11].

Definition 1.1. A **partial representation** of G on the K-module M is a map $\pi : G \to \mathrm{End}_K(M)$ such that, for any $s, t \in G$, we have:
  (a) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$,
  (b) $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$,
  (c) $\pi(1_G) = 1$,
where $1 = id_M$.

Definition 1.2. Let $\pi : G \to \mathrm{End}_K(M)$ and $\pi' : G \to \mathrm{End}_K(W)$ be two partial representations of G. A **morphism of partial representations** is a morphism of K-modules $f : M \to W$, such that $f \circ \pi(g) = \pi'(g) \circ f \ \forall g \in G$.

The category of partial representations of G, denoted **ParRep**$_G$ is the category whose objects are pairs (M, π), where M is a K-module and $\pi : G \to \mathrm{End}_K(M)$ is a partial representation of G on M,

and whose morphisms are morphisms of partial representations. Denote by $\mathcal{S}(G)$ the Exel's semigroup of G, i.e., it is the inverse monoid defined by the generators $[t]$, $t \in G$, and relations:

(i) $[1_G] = 1$;
(ii) $[s^{-1}][s][t] = [s^{-1}][st]$;
(iii) $[s][t][t^{-1}] = [st][t^{-1}]$;

for any $t, s \in G$. We denote the inverse (in the inverse semigroup sense) of an element $w \in \mathcal{S}(G)$ as w^*. The set of idempotent elements of $\mathcal{S}(G)$ will be denoted by $E(\mathcal{S}(G))$. For more details, refer to [12].

Definition 1.3. We define the **partial group algebra** $K_{par}G$ as the semigroup K-algebra generated by $\mathcal{S}(G)$. We set $\mathcal{B}$ as the semigroup K-algebra generated by $E(\mathcal{S}(G))$.

Here, we recall some well-known computation rules for $K_{par}G$ that will be heavily used throughout this work. We will omit the proofs since they can be easily found in the literature (for example, refer to [12]).

Proposition 1.4. *Define* $e_g := [g][g^{-1}]$, *then*

(i) $[g]e_h = e_{gh}[g]$ *for all* $g, h \in G$,
(ii) $e_g e_h = e_h e_g$ *for all* $g, h \in G$
(iii) $e_g e_g = e_g$ *for all* $g \in G$
(iv) *the set* $\{e_g : g \in G\}$ *generates the semigroup* $E(\mathcal{S}(G))$

Recall that the subalgebra $\mathcal{B} \subseteq K_{par}G$ is a left $K_{par}G$-module with the action

$$[g] \triangleright u = [g]u[g^{-1}]$$

and a right $K_{par}G$-module with the action

$$u \triangleleft [g] = [g^{-1}]u[g].$$

One verifies that

(i) $w \triangleright u = wu$ for all $w, u \in \mathcal{B}$,
(ii) $u \triangleleft w = wu$ for all $w, u \in \mathcal{B}$,
(iii) $[g] \triangleright 1 = e_g$ for all $g \in G$,
(iv) $1 \triangleleft [g] = e_{g^{-1}}$ for all $g \in G$.

Hence, we obtain a well-defined morphism of right $K_{par}G$-modules

(1.1)
$$\varepsilon : K_{par}G \to \mathcal{B}$$
$$z \to 1 \triangleleft z.$$

Lemma 1.5. *Let* ε *be the map* (1.1), *then*

(i) $\varepsilon([g_1][g_2]\ldots[g_n]) = [g_n^{-1}]\ldots[g_2^{-1}][g_1^{-1}][g_1][g_2]\ldots[g_n] = e_{g_n^{-1}}e_{g_n^{-1}g_{n-1}^{-1}}\ldots e_{g_n^{-1}g_{n-1}^{-1}\ldots g_2^{-1}g_1^{-1}}$ *for all* $g_1, \ldots, g_n \in G$;
(ii) $\varepsilon(e_g) = e_g$ *for all* $g \in G$, *and consequently* $\varepsilon(w) = w$ *for all* $w \in \mathcal{B}$;
(iii) $[h^{-1}]\varepsilon(z) = \varepsilon(z[h])[h^{-1}]$ *for all* $h \in G$.

Proof. Items (i) and (ii) are direct consequences of Proposition 1.4. For (iii) observe that

$$[h^{-1}]\varepsilon([g_1][g_2]\ldots[g_n]) = [h^{-1}]e_{g_n^{-1}}e_{g_n^{-1}g_{n-1}^{-1}}\ldots e_{g_n^{-1}g_{n-1}^{-1}\ldots g_2^{-1}g_1^{-1}}$$

$$(\text{By Proposition 1.4}) = e_{h^{-1}g_n^{-1}}e_{h^{-1}g_n^{-1}g_{n-1}^{-1}}\ldots e_{h^{-1}g_n^{-1}g_{n-1}^{-1}\ldots g_2^{-1}g_1^{-1}}e_{h^{-1}}[h^{-1}]$$

$$= e_{h^{-1}}e_{h^{-1}g_n^{-1}}e_{h^{-1}g_n^{-1}g_{n-1}^{-1}}\ldots e_{h^{-1}g_n^{-1}g_{n-1}^{-1}\ldots g_2^{-1}g_1^{-1}}[h^{-1}]$$

$$= \varepsilon([g_1]\ldots[g_n][h])[h^{-1}].$$

$\square$

Proposition 1.6 (Proposition 2.5 [11]). *Let G be a group, then any element of $z \in \mathcal{S}(G)$ admits a decomposition*

$$z = e_{g_1}e_{g_2}\ldots e_{g_n}[h],$$

where $n \geq 0$ and $g_1,\ldots,g_n \in G$. In addition, one can assume that

(i) $g_i \neq g_j$, for $i \neq j$,
(ii) $g_i \neq s$ for any i.

Furthermore, this decomposition is unique up to the order of the $\{g_i\}$.

Proposition 1.7 (Proposition 10.5 [12]). *The map*

$$g \in G \mapsto [g] \in K_{par}G$$

*is a partial representation, which we will call the **universal partial representation** of G. In addition, for any partial representation π of G in a unital K-algebra A there exists a unique algebra homomorphism $\phi : K_{par}G \to A$, such that $\pi(g) = \phi([g])$, for any $g \in G$.*

A direct consequence of Proposition 1.7 is the following well-known theorem:

Theorem 1.8. *The categories $\textbf{\textit{ParRep}}_G$ and $K_{par}G\textbf{-Mod}$ are isomorphic.*

The key constructions in Theorem 1.8 are as follows: if $\pi : G \to \text{End}_K(M)$ is a partial group representation, then M becomes a $K_{par}G$-module with the action defined as $[g] \cdot m := \pi_g(m)$. Conversely, if M is a $K_{par}G$-module, we obtain a partial representation $\pi : G \to \text{End}_K(M)$ such that $\pi_g(m) := [g] \cdot m$. Keeping in mind Theorem 1.8, we use the terms *partial group representation* and $K_{par}G$-*module* interchangeably.

Definition 1.9. A (left) partial group action $\theta = \big(G, X, \{X_g\}_{g\in G}, \{\theta_g\}_{g\in G}\big)$ of a group G on a set X consists of a family indexed by G of subsets $X_g \subseteq X$ and a family of bijections $\theta_g : X_{g^{-1}} \to X_g$ for each $g \in G$, satisfying the following conditions:

(i) $X_1 = X$ and $\theta_1 = 1_X$,
(ii) $\theta_g(X_{g^{-1}} \cap X_{g^{-1}h}) \subseteq X_g \cap X_h$,
(iii) $\theta_g\theta_h(x) = \theta_{gh}(x)$, for each $x \in X_{h^{-1}} \cap X_{h^{-1}g^{-1}}$.

Let X be a set, and let A, B, D, C be subsets of X. Suppose $\psi : A \to B$ and $\phi : D \to C$ are bijective functions. The partial composition of $\phi \circ \psi$ is defined as the function whose domain is the larger set where $\phi(\psi(x))$ makes sense, i.e., $\text{dom}(\phi \circ \psi) = \psi^{-1}(A \cap C)$. It is worth mentioning that conditions (ii) and (iii) are equivalent to $\theta_g\theta_h$ being a restriction of θ_{gh} for all $g, h \in G$.

Definition 1.10. Let $\alpha = (G, X, \{X_g\}, \{\alpha_g\})$ and $\beta = (S, Y, \{Y_h\}, \{\beta_h\})$ be partial actions, a morphism of partial actions ϕ is a pair (ϕ_0, ϕ_1) where $\phi_0 : X \to Y$ is a map of sets and $\phi_1 : G \to S$ is a morphism of groups such that

(i) $\phi_0(X_g) \subseteq Y_{\phi_1(g)}$,
(ii) $\phi_0(\alpha_g(x)) = \beta_{\phi_1(g)}(\phi_0(x))$, for all $x \in X_{g^{-1}}$.

In Definition 1.10, if we have $G = S$ and $\phi_0 = 1_G$, then we obtain the classical definition of a G-equivariant map for partial group actions of a group G.

Definition 1.11. Let α be a partial group action of G on X and β a partial group action of G on Y. A G-**equivariant map** $\phi : X \to Y$ is a function such that:

a) $\phi(X_g) \subseteq Y_g$ for all $g \in G$,
b) $\phi(\alpha_g(x)) = \beta_g(\phi(x))$ for all $g \in G$ and $x \in X_{g^{-1}}$.

The category of partial group actions of a fixed group G on sets with G-equivariant maps will be denoted by G_{par}-**Set**.

Definition 1.12. Let G be a group, $\mathcal{A}$ a K-algebra, and M be a left (right) $\mathcal{A}$-module. A (left) partial group action of G on M is a set-theoretical partial action $\theta = (G, M, \{M_g\}, \{\theta_g\})$ of G on M such that each domain M_g is a left (right) $\mathcal{A}$-submodule of M and $\theta_g : M_{g^{-1}} \to M_g$ is an isomorphism of left (right) $\mathcal{A}$-modules.

We can extend the concept of partial group action morphism and G-equivariant map to partial group actions on $\mathcal{A}$-modules simply by requiring to the corresponding maps to be $\mathcal{A}$-linear maps. The category whose objects are partial group actions on modules and morphisms of partial group actions on left (right) $\mathcal{A}$-modules will be denoted by **PA-$_{\mathcal{A}}$Mod** (**PA-Mod$_{\mathcal{A}}$**), and for a fixed group G the respective category whose objects are partial group actions of G on left (right) $\mathcal{A}$-modules and the morphisms are partial G-equivariant maps of modules will be denoted by G_{par}-$_{\mathcal{A}}$**Mod** (G_{par}-**Mod$_{\mathcal{A}}$**). By abuse of notation, if $\mathcal{A} = K$, we denote the category of partial group actions on K-modules simply by **PA-Mod** and the category of partial group action of a group G on K-modules by G_{par}-**Mod**. Furthermore, if θ is a (left) partial group action of G on the module M, we will refer to M as a (left) G_{par}-module, always keeping in mind the partial action θ.

Example 1.13. Let α be a group action of G on a K-module M. Then it is clear that α is a partial group action. We will refer to this kind of partial action as a **global action** to emphasize its global nature. Furthermore, there exist and obvious embedding functor G-**Mod** $\to G_{par}$-**Mod**.

Example 1.14. Let $\pi : G \to \mathrm{End}_K(M)$ be a partial representation. Then, for all $g \in G$ we define $M_g := e_g \cdot M$ and $\theta_g := \pi_g|_{M_{g^{-1}}}$. Then, $\theta := (G, M, \{M_g\}, \{\theta_g\})$ is a partial group action of K-modules of G on M. Indeed, it is clear that $M_1 = M$ and $\theta_1 = id_M$. Since $[g][g^{-1}]e_g \cdot m = e_g \cdot m$ for all $g \in G$ and $m \in M$ we conclude that $\theta_g : M_{g^{-1}} \to M_g$ is a well-defined K-linear isomorphism. Let $m \in M_{g^{-1}} \cap M_{g^{-1}h}$, then:

$$\theta_g(m) = [g] \cdot m = [g]e_{g^{-1}}e_{g^{-1}h} \cdot m = e_g e_h [g] \cdot m \in M_g \cap M_h.$$

Let $m \in M_{h^{-1}} \cap M_{h^{-1}g^{-1}}$, then

$$\theta_g \theta_h(m) = [g][h]e_{h^{-1}}e_{h^{-1}g^{-1}} \cdot m = [gh]e_{h^{-1}}e_{h^{-1}g^{-1}} \cdot m = \theta_{gh}(m).$$

Thus, θ is a partial group action we call this partial group action as the **induced partial action** of G on the $K_{par}G$-module M. Let $\phi : M \to N$ be a morphism of $K_{par}G$-modules. Then ϕ is a G-equivariant map with respect to the partial group actions determined by M and N. Indeed, observe that

$$\phi(M_g) = \phi(e_g \cdot M) = e_g \cdot \phi(M) \subseteq e_g \cdot N = N_g,$$

and

$$\phi(\alpha_g(e_{g^{-1}} \cdot m)) = \phi([g]e_{g^{-1}} \cdot m) = [g] \cdot \phi(e_{g^{-1}} \cdot m) = \beta_g(\phi(e_{g^{-1}} \cdot m).$$

Hence, we obtain a well-defined functor $\Lambda : \mathbf{ParRep}_G \to G_{par}\text{-}\mathbf{Mod}$.

Remark 1.15. Observe that in Example 1.14 we have that $M_g = [g] \cdot M$. Indeed, it is clear that $e_g \cdot M \subseteq [g] \cdot M$ for all $g \in G$, then $[g] \cdot M = [g]e_{g^{-1}} \cdot M \subseteq [g][g^{-1}] \cdot M = e_g \cdot M$ for all $g \in G$.

Example 1.16. Let $X = K_{par}G$, then we have a partial group action of G on the right $K_{par}G$-module X defined as follows:

   (i) $X_g := e_{g^{-1}}K_{par}G$
   (ii) $\theta_g(e_{g^{-1}}z) := [g]e_{g^{-1}}z = [g]z$ for all $z \in K_{par}G$.

It is clear that each X_g is a right $K_{par}G$-module with the inherited structure of $K_{par}G$ and that θ_g is an isomorphism of right $K_{par}G$-modules.

Proposition 1.17. *Let* $\theta = (G, X, \{X_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ *be a set-theoretical partial group action of G on X. Then, θ determines a partial group action* $\hat{\theta} := (G, KX, \{KX_g\}_{g \in G}, \{\hat{\theta}_g\}_{g \in G})$ *of G on the free K-module KX, where $\hat{\theta}_g : KX_{g^{-1}} \to KX_g$ is the K-linear isomorphism determined by θ_g.*

Proof. Let $z \in KX_{g^{-1}} \cap KX_h$, then $z = \sum_i \lambda_i x_i$, such that $x_i \in X_{g^{-1}} \cap X_h$ for all i. Therefore,

$$\hat{\theta}_g(z) = \sum_i \lambda_i \theta_g(x_i) \in KX_g \cap KX_{gh}.$$

Let $z \in X_{g^{-1}} \cap X_{g^{-1}h^{-1}}$, then $z = \sum_i \lambda_i x_i$, such that $x_i \in X_{g^{-1}} \cap X_{g^{-1}h^{-1}}$ for all i. Thus,

$$\hat{\theta}_h\hat{\theta}_g(z) = \sum_i \lambda_i \theta_h\theta_g(x_i) = \sum_i \lambda_i \theta_{hg}(x_i) = \hat{\theta}_{hg}(z).$$

$\square$

Let $\alpha = (G, X, \{X_g\}, \{\alpha_g\})$ and $\beta = (H, Y, \{Y_g\}, \{\beta_g\})$ be two partial group actions and $\phi = (f, \varphi) : \alpha \to \beta$ a morphism of partial actions. Then, ϕ induces a morphism of partial actions $(\hat{f}, \varphi) : \hat{\alpha} \to \hat{\beta}$ where $\hat{\alpha}$ and $\hat{\beta}$ are the partial actions on modules determined by α and β respectively as in Proposition 1.17 and $\hat{f} : KX \to KY$ is the K-linear extension of f. Therefore, we obtain a functor $\mathbf{PA} \to \mathbf{PA}\text{-}\mathbf{Mod}$.

Proposition 1.18. *Let* $\theta = (G, X, \{X_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ *be a set-theoretical partial group action of G on X, and K be a commutative unital ring. Then, θ determines a partial representation $\pi^\theta := \pi$ of G on KX, such that for all $x \in X$ we set*

$$\pi_g(x) = \begin{cases} \theta_g(x) & \text{if } x \in X_{g^{-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Clearly, $\pi_1 = id_{KX}$. Let $g, h \in G$ and $x \in X$, then
   (i) if $x \notin X_{h^{-1}}$, then $\pi_{g^{-1}}\pi_g\pi_h(x) = 0$, and $\pi_{g^{-1}}\pi_{gh}(x) = 0$ since:

(a) if $x \notin X_{h^{-1}g^{-1}}$, then
$$\pi_{g^{-1}}\pi_{gh}(x) = 0,$$
(b) if $x \in X_{h^{-1}g^{-1}}$, then
$$\pi_{g^{-1}}\pi_{gh}(x) = \pi_{g^{-1}}(\theta_{gh}(x)),$$
(1) if $\theta_{gh}(x) \notin X_g$, then
$$\pi_{g^{-1}}\pi_{gh}(x) = \pi_{g^{-1}}(\theta_{gh}(x)) = 0,$$
(2) if $\theta_{gh}(x) \in X_g$, then
$$\theta_{g^{-1}}\theta_{gh}(x) = \theta_h(x),$$

therefore $x \in X_{h^{-1}}$, which contradicts the hypotheses of (i). Thus, $\theta_{gh}(x) \notin X_g$.

(ii) If $x \in X_{h^{-1}}$, then $\pi_{g^{-1}}\pi_g\pi_h(x) = \pi_{g^{-1}}\pi_g(\theta_h(x))$,

(a) if $\theta_h(x) \in X_{g^{-1}}$, then
$$\pi_{g^{-1}}\pi_g\pi_h(x) = \pi_{g^{-1}}\theta_g(\theta_h(x)) = \pi_{g^{-1}}\theta_{gh}(x) = \pi_{g^{-1}}\pi_{gh}(x),$$
(b) if $\theta_h(x) \notin X_{g^{-1}}$, then $\pi_{g^{-1}}\pi_g\pi_h(x) = 0$, and
(1) if $x \notin X_{h^{-1}g^{-1}}$, then $\pi_{g^{-1}}\pi_{gh}(x) = 0$,
(2) if $x \in X_{h^{-1}g^{-1}}$, then $\theta_{gh}(x) \notin X_g$ and consequently $\pi_{g^{-1}}\pi_{gh}(x) = \pi_{g^{-1}}(\theta_{gh}(x)) = 0$. Indeed, note that if $\theta_{gh}(x) \in X_g$, then $\theta_h(x) = \theta_{g^{-1}}\theta_{gh}(x) \in X_{g^{-1}}$, which contradicts (b) of (ii), thus $\theta_{gh}(x) \notin X_g$.

Henceforth, $\pi_{g^{-1}}\pi_g\pi_h = \pi_{g^{-1}}\pi_{gh}$ for all $g, h \in G$, analogously one proves that $\pi_h\pi_g\pi_{g^{-1}} = \pi_{hg}\pi_{g^{-1}}$ for all $g, h \in G$. $\qquad\square$

Remark 1.19. Let θ be a partial group action of a group G on a set X, then by Proposition 1.18 we have a partial group representation $\pi^\theta : G \to \operatorname{End}_K(KX)$. Then, by Theorem 1.8 KX is a $K_{par}G$-module with the left action determined by

$$(1.2) \qquad\qquad [g] \cdot x = \begin{cases} \theta_g(x) & \text{if } x \in X_{g^{-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.18 allows us to study partial group actions via the partial group representation that they determine. However, maps of partial group actions do not necessarily determine morphisms of partial representations.

Example 1.20. Let $X = \{x, y\}$ be the set with two points, and let $G = \{1, g : g^2 = 1\}$ be the cyclic group of order 2. Define the partial action $\alpha := (G, X, \{D_g\}, \{\alpha_g\})$ such that $D_g := \{x\}$ and $\alpha_g(x) = x$, consider β as the trivial group action of G on X. It is clear that the identity map id $: X \mapsto X$ is a G-equivariant map from (α, X) to (β, X). But the K-linear map id $: KX \to KX$ is not a morphism of $K_{par}G$-modules since

$$\text{id}([g] \cdot_\alpha y) = \text{id}(0) = 0 \text{ and } [g] \cdot_\beta \text{id}(y) = [g] \cdot_\beta y = y,$$

where $\cdot_\alpha$ and $\cdot_\beta$ denotes the left actions of $K_{par}G$ on KX induced by α and β respectively.

Remark 1.21. Let θ be a set-theoretical partial group action, then by Proposition 1.17 we have a partial group action $\hat{\theta}$ of G on KX, and by Proposition 1.18 we obtain a partial group representation π^θ. Thus, it is immediately that the partial action determined by π^θ is $\hat{\theta}$.

Analogously to the definition of left partial group action, we can define a right partial group action. In particular, we will use right partial group actions on $\mathcal{A}$-modules, where $\mathcal{A}$ is an algebra.

Definition 1.22. A right partial group action β of G on a left (right) $\mathcal{A}$-module X is a left partial group action of G^{op} on X, explicitly a right partial group action of G on X is a tuple $\beta := \big(G, X, \{X_g\}, \{\beta_g\}\big)$, where $\beta_g : X_{g^{-1}} \to X_g$ is an isomorphism of left (right) $\mathcal{A}$-modules for all $g \in G$, such that

   (i) $X_1 = X$ and $\beta_1 = 1_X$;
   (ii) $(X_h \cap X_{g^{-1}})\beta_g \subseteq X_{hg} \cap X_g$;
   (iii) $(x)\beta_g\beta_h = (x)\beta_{gh}$ for all $x \in X_{h^{-1}g^{-1}} \cap X_{g^{-1}}$,

where $(x)\beta_g := \beta_g(x)$.

The category of right partial group actions on modules of a group G will be denoted by **Mod-**G_{par}. Thus, by definition we have a natural isomorphism between the categories G_{par}^{op}-**Mod** and **Mod-**G_{par}. Analogously, to the left case, if β is a right partial group action on the module M we say that M is a right G_{par}-module.

   Notation. Given a group G and a set/module X we use the notation $\alpha : G \curvearrowright X$ to denote the partial group action of G on X, such that $\alpha := (G, X, \{X_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$. Analogously, we use the notation $\beta : X \curvearrowleft G$ to denote the right partial group action of G on X, such that $\beta := (G, X, \{X_g\}_{g \in G}, \{\beta_g\}_{g \in G})$.

Example 1.23. Analogously to Example 1.14, if M is a right $K_{par}G$-module, it determines a right partial group action $\theta := (G, M, \{M_g\}, \{\theta_g\})$ of G on M such that:

   (i) $M_g := M \cdot e_{g^{-1}}$;
   (ii) $\theta_g : M_{g^{-1}} \to M_g$ is such that $(m)\theta_g := m \cdot [g]$, for all $g \in G$ and $m \in M_{g^{-1}}$.

Thus, we obtain a functor **Mod-**$K_{par}G \to$ **Mod-**G_{par}.

Example 1.24. Let $X = K_{par}G$, then we have a right partial action of left $K_{par}G$-modules defined as follows:

   (i) $X_g := K_{par}Ge_{g^{-1}}$
   (ii) $(ze_g)\theta_g := ze_g[g] = z[g]$ for all $z \in K_{par}G$.

It is clear that each X_g is a left $K_{par}G$-module with the inherited structure of $K_{par}G$ and that θ_g is an isomorphism of left $K_{par}G$-modules.

Partial actions induced by partial representations will be one of the principal objects of study in this work. Thus, here we show some properties of this type of partial group action.

Lemma 1.25. *Let M and N be two left $K_{par}G$-modules, and let $f : M \to N$ be a morphism between them. Let $\alpha : G \curvearrowright M$ and $\beta : G \curvearrowright N$ be the respective induced partial action. Then,*

   (i) $[g] \cdot m = \alpha_g(e_{g^{-1}} \cdot m)$ for all $g \in G$ and $m \in M$;
   (ii) $\operatorname{dom} \alpha_{g_1}\alpha_{g_2} \ldots \alpha_{g_n} = \varepsilon([g_1][g_2] \ldots [g]) \cdot M$ for all $g_1, \ldots, g_n \in G$;
   (iii) Let $y \in N_g \cap \operatorname{im} f$, then there exist $x \in M_g$ such that $f(x) = y$.

Proof. By the definition of α we know that $e_{g^{-1}} \cdot m \in M_{g^{-1}}$ for all $g \in G$ and $m \in M$. Then, $\alpha_g(e_{g^{-1}} \cdot m) = [g]e_{g^{-1}} \cdot m = [g] \cdot m$. This proves (i).

By the axioms of partial action we know that $\operatorname{dom} \alpha_g\alpha_h = M_{h^{-1}} \cap M_{h^{-1}g^{-1}} = e_{h^{-1}}e_{h^{-1}g^{-1}} \cdot M$, and $e_{h^{-1}}e_{h^{-1}g^{-1}} = [h^{-1}][g^{-1}][g][h] = \varepsilon([g][h])$. Thus, by induction suppose that $\operatorname{dom} \alpha_{g_1}\alpha_{g_2} \ldots \alpha_{g_n} =$

$\varepsilon([g_1][g_2]\ldots[g_n]) \cdot M$, then

$$\begin{aligned}
\operatorname{dom} \alpha_{g_1}\ldots\alpha_{g_n}\alpha_{g_{n+1}} &= \alpha_{g_{n+1}^{-1}}\left(\operatorname{dom}\alpha_{g_1}\ldots\alpha_{g_n} \cap \operatorname{im}\alpha_{g_{n+1}}\right) \\
&= \alpha_{g_{n+1}^{-1}}\left(\varepsilon([g_1][g_2]\ldots[g_n]) \cdot M \cap e_{g_{n+1}} \cdot M\right) \\
&= \alpha_{g_{n+1}^{-1}}\left(e_{g_{n+1}}\varepsilon([g_1][g_2]\ldots[g_n]) \cdot M\right) \\
&= [g_{n+1}^{-1}]e_{g_{n+1}}\varepsilon([g_1][g_2]\ldots[g_n]) \cdot M \\
\text{(by Lemma 1.5)} &= \varepsilon([g_1][g_2]\ldots[g_n][g_{n+1}])[g_{n+1}^{-1}] \cdot M \\
\text{(by Remark 1.15)} &= \varepsilon([g_1][g_2]\ldots[g_n][g_{n+1}])e_{g_{n+1}^{-1}} \cdot M \\
&= \varepsilon([g_1][g_2]\ldots[g_n][g_{n+1}]) \cdot M,
\end{aligned}$$

what proves (ii). For (iii), observe that if $y \in N_g \cap \operatorname{im}, f$, then $e_g \cdot y = y$, and there exists $m \in M$ such that $f(m) = y$. Hence, $y = e_g \cdot y = e_g \cdot f(m) = f(e_g \cdot m)$. Therefore, setting $x = e_g \cdot m \in M_g$ yields the desired element. $\square$

<u>Form now on</u>, when dealing with a partial representation on M (or equivalently a $K_{par}G$-module), we will always take into account the partial group action on M inherited from the partial representation. This is crucial to consider, as we will carry out constructions using partial representations and immediately regard them as partial group actions without explicitly mentioning the functor Λ.

1.2. **Homological algebra.** We use Weibel's book [19] and Rotman's book [18] as our primary references for homological algebra theory. The homology theory for a G-module M is based on the space of coinvariants $M_G := M/DM$, where DM is the submodule of M generated by $\{g \cdot m - m : m \in M,\ g \in G\}$. In this case, one can verify that $M_G \cong k \otimes_{KG} M$. Hence, the functor $(-)_G : G\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$ is a right exact functor, and we can define the homology of G with coefficients in a G-module M as the right derived functor of the functor of coinvariants. Thus, we define

$$H_\bullet(G, M) := \operatorname{Tor}_\bullet^{KG}(k, M).$$

The standard resolution $C''_\bullet \to K$ of right KG-modules is given by $C''_n = KG^{\otimes n+1}$, with augmentation map $g \in KG \mapsto 1 \in K$, and with differential determined by the face maps

$$d_i(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes g_{n+1}) := \begin{cases} g_2 \otimes \ldots \otimes g_n \otimes g_{n+1} & \text{if } i = 0; \\ g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_{n+1} & \text{if } 0 < i \leq n. \end{cases}$$

Thus, if M is a KG-module, the canonical complex that computes the group homology of G with coefficients in M is $C''_\bullet \otimes_{KG} M \cong (C_n(G, M), d)$, such that $C_n(G, M) := KG^{\otimes n} \otimes M$ and d is determined by the face maps

$$d_i(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m) := \begin{cases} g_2 \otimes \ldots \otimes g_n \otimes m & \text{if } i = 0; \\ g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes m & \text{if } 0 < i < n; \\ g_1 \otimes \ldots \otimes g_{n-1} \otimes g_n \cdot m & \text{if } i = n. \end{cases}$$

In the category $G_{par}\text{-}\mathbf{Mod}$, if $\alpha : G \curvearrowright M$ is a partial group action on the K-module M, we can define the space of coinvariants of M as $M_G := M/DM$ where DM is the K-submodule of M generated by $\{\alpha_g(m) - m : g \in G,\ m \in M_{g^{-1}}\}$. This is clearly a generalization of the global case. We can consider

the trivial group action on M_G, then the canonical map $\pi : M \to M_G$ is a G-equivariant map. Indeed, for all $g \in G$ and $m \in M_g$ we have

$$\pi_G(\alpha_g(m)) = \overline{\alpha_g(m)} = \overline{m} = g \cdot \overline{m} = g \cdot \pi_G(m).$$

Furthermore, the space M_G together with the G-equivariant map π_G are universal in the following sense: For all any trivial G-module W and G-equivariant map $\psi : M \to W$ there exist a unique G-equivariant map $\tilde{\psi} : M_G \to W$ such that the following diagram commutes

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & W \\
{\scriptstyle \pi_G}\downarrow & \nearrow & \\
M_G & {\scriptstyle \tilde{\psi}} &
\end{array}
$$

what is exactly the same property that satisfies the space of coinvariants of a global group action.

For any $K_{par}G$-module M we can define the space of coinvariants as $M_G := M/DM$, where DM is the K-submodule of M generated by $\{[g] \cdot m - e_{g^{-1}} \cdot m : g \in G,\ m \in M\}$, note that this coinvariant space coincides with the coinvariant space of the partial group action determined by the partial representation on M.

Proposition 1.26. *Let M be a left $K_{par}G$-module. Then, $M_G \cong \mathcal{B} \otimes_{K_{par}G} M$ as K-modules.*

Proof. Define the K-linear map $f_0 : M \to \mathcal{B} \otimes_{K_{par}G} M$ such that $f_0(m) := 1 \otimes_{K_{par}G} m$, then

$$
\begin{aligned}
f_0([g] \cdot m) &= 1 \otimes_{K_{par}G} [g] \cdot m = 1 \triangleleft [g] \otimes_{K_{par}G} m = e_{g^{-1}} \otimes_{K_{par}G} m \\
&= 1 \triangleleft e_{g^{-1}} \otimes_{K_{par}G} m = 1 \otimes_{K_{par}G} e_{g^{-1}} \cdot m \\
&= f_0(e_{g^{-1}} \cdot m).
\end{aligned}
$$

Thus, we obtain a K-linear map $f : M_G \to \mathcal{B} \otimes_{K_{par}G} M$ such that $f(m) = 1 \otimes_{K_{par}G} m$. Conversely, the map $f' : \mathcal{B} \otimes_{K_{par}G} M \to M_G$ given by $f'(w \otimes_{K_{par}G} m) := \overline{w \cdot m}$ is a well-defined K-linear map. Indeed, note that

$$
\begin{aligned}
f'(w \triangleleft [g] \otimes_{K_{par}G} m) &= \overline{[g^{-1}]w[g] \cdot m} = \overline{e_g w[g] \cdot m} \\
&= \overline{w e_g[g] \cdot m} = \overline{w[g] \cdot m} \\
&= f'(w \otimes_{K_{par}G} [g] \cdot m).
\end{aligned}
$$

Finally, note that f and f' are mutually inverses

$$ff'(w \otimes_{K_{par}G} m) = f(\overline{w \cdot m}) = 1 \otimes_{K_{par}G} w \cdot m = w \otimes_{K_{par}G} m,$$

and

$$f'f(\overline{m}) = f'(1 \otimes_{K_{par}G} m) = \overline{m}.$$

$\square$

Hence, based on the previous discussion, we have the following definition:

Definition 1.27 ([1], [2])**.** Let G be a group, we define the partial group homology of G with coefficients in a left $K_{par}G$-module M as

$$(1.3) \qquad\qquad H_\bullet^{par}(G, M) := \mathrm{Tor}_\bullet^{K_{par}}(\mathcal{B}, M).$$

Dually, we define the partial group cohomology of G with coefficients in a right $K_{par}G$-module M as

$$(1.4) \qquad H^\bullet_{par}(G, M) := \mathrm{Ext}^\bullet_{K_{par}}(\mathcal{B}, M).$$

To compute the partial (co)homology we can use a projective resolution of right $K_{par}G$-modules of $\mathcal{B}$. We can utilize the projective resolution obtained in [6, Section 6] for the particular case $\mathcal{H}_{par} = K_{par}G$. Hence, we obtain a projective resolution $(C'_\bullet, d) \xrightarrow{\varepsilon} \mathcal{B}$ of right $K_{par}G$-modules, such that $C'_n := (K_{par}G)^{\otimes_\mathcal{B} n+1}$, the face maps are

$$(1.5) \qquad d_0(z_0 \otimes_\mathcal{B} z_1 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n) = z_0^* z_0 z_1 \otimes_\mathcal{B} z_2 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n,$$

$$(1.6) \qquad d_i(z_i \otimes_\mathcal{B} z_1 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n) = z_0 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_{i-1} z_i \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n, \text{ for } 0 < i \leq n,$$

where $z_i \in \mathcal{S}(G)$; the differential is given by

$$d = \sum_{i=0}^n (-1)^i d_i : C'_n \to C'_{n-1}.$$

We can obtain clearer description of the face maps on the basic n-chains $z_0 \otimes_\mathcal{B} z_1 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n$, where $z_i \in \mathcal{S}(G)$. By Proposition 1.4 and Proposition 1.6, there exist $u \in E(\mathcal{S}(G))$ and $g_0, g_1, \ldots, g_n \in G$ such that

$$z_0 \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} z_n = [g_0] \otimes_\mathcal{B} [g_1] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n]u.$$

Hence, the Equations (1.5) and (1.6) takes the form

$$(1.7) \qquad d_0([g_0] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n]u) = e_{g_0^{-1}}[g_1] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n]u$$
$$= [g_1] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n] e_{g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1} g_0^{-1}} u,$$

$$(1.8) \qquad d_i([g_0] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n]u) = [g_0] \otimes_\mathcal{B} [g_1] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_{i-1}][g_i] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n]u$$
$$= [g_0] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_{i-1}g_i] \otimes_\mathcal{B} \ldots \otimes_\mathcal{B} [g_n] e_{g_i^{-1} g_{i-1}^{-1} \cdots g_1^{-1} g_0^{-1}} u,$$

for $0 < i \leq n$.

1.3. **Simplicial objects.** For this section we follow [14], [17], and [19]. For all $n \in \mathbb{N}$ we define $\mathbf{n}$ as the ordered set of $n+1$ points $\{0 < 1 < \ldots < n\}$. The category Δ is the category with objects $\{\mathbf{n} : n \in \mathbb{N}\}$ and morphisms the set of non-decreasing maps $f : \mathbf{n} \to \mathbf{m}$. The faces $\delta_i : \mathbf{n\text{-}1} \to \mathbf{n}$ and the degeneracies $\sigma_j : \mathbf{n\text{+}1} \to \mathbf{n}$ such that δ_i is the injection

$$\delta_i(t) = \begin{cases} t & \text{if } t < i \\ t+1 & \text{if } t \geq i, \end{cases}$$

and σ_j is the surjection

$$\sigma_j(t) = \begin{cases} t & \text{if } t \leq j \\ t-1 & \text{if } t > j. \end{cases}$$

Lemma 1.28 (Lemma 8.1.2 [19]). *For any non-decreasing morphism $\phi : \mathbf{n} \to \mathbf{m}$ there is a unique decomposition*

$$\phi = \delta_{i_1} \delta_{i_2} \ldots \delta_{\sigma_{i_r}} \sigma_{j_1} \ldots \sigma_{j_s}$$

such that $i_1 \leq i_2 \leq \ldots \leq i_r$ and $j_1 < j_2 < \ldots < j_s$ with $m = n - s + r$.

Corollary 1.29 (Corollary B.3. [17]). *The category Δ is presented by the generators δ_i, $0 \leq i \leq n$ and σ_j, $0 \leq j \leq n$ (one for each n) subject to the simplicial relations*

$$(1.9) \qquad \delta_j \delta_i = \delta_i \delta_{j-1} \ \text{if } i < j$$
$$\sigma_j \sigma_i = \sigma_j \sigma_{j+1} \ \text{if } i \leq j$$
$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \\ 1 & \text{if } i = j \ \text{or } i = j + 1 \\ \delta_{i-1} \sigma_j & \text{if } i > j + 1. \end{cases}$$

Definition 1.30. A simplicial object in a category $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$. Equivalently, we can define a simplicial object X as a sequence of objects $\{X_n\}_{n \in \mathbb{N}}$ together with face maps $d_i : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_n \to X_{n+1}$ $(i = 0, 1, \ldots, n)$, which satisfies the following simplicial identities:

$$d_i d_j = d_{j-1} d_i \ \text{if } i < j$$
$$s_i s_j = s_{j+1} s_j \ \text{if } i \leq j$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ 1 & \text{if } i = j \ \text{or } i = j + 1 \} \\ s_j d_{i-1} & \text{if } i > j + 1. \end{cases}$$

A simplicial object will be denoted by $(X_\bullet, d_i, s_i)$, where $X_\bullet$ refers to the sequence of objects, and d_i and s_i refer to the face and degeneracy maps, respectively.

A simplicial object in the category of sets is called a **simplicial set**, analogously a simplicial object in the category of modules is called a **simplicial module**.

Example 1.31. Let $\Gamma \rightrightarrows X$ be a groupoid with set of objects X and set of arrows Γ, we can consider the simplicial set determined by Γ as a small category (see [17]). Hence, we define the simplicial set $(\Gamma_\bullet, d_i, s_i)$ as follows: set $\Gamma_0 := X$, $\Gamma_1 := \Gamma$, and $\Gamma_n := \{(\gamma_1, \gamma_2, \ldots, \gamma_n) : s(\gamma_i) = t(\gamma_{i+1}) \text{ for all } 1 \leq i \leq n-1\}$ for all $n \leq 2$, where $s : \Gamma \to X$ and $t : \Gamma \to X$ are the respective source and target maps of the groupoid. Hence, for all $n \leq 1$, we have $n+1$ face maps d_i, $0 \leq i \leq n$, where the map d_i is defined by "skipping the i-position object". In other words, an n-chain is a chain of composable morphisms

$$x_0 \xrightarrow{\gamma_1} x_1 \longrightarrow \cdots \longrightarrow x_{i-1} \xrightarrow{\gamma_i} x_i \xrightarrow{\gamma_{i+1}} x_{i+1} \longrightarrow \cdots \longrightarrow x_{n-1} \xrightarrow{\gamma_n} x_n.$$

The face map d_0 leaves out the first arrow and d_n leaves out the last arrow, and d_i works by composing the morphisms γ_i and γ_{i+1}

$$x_0 \xrightarrow{\gamma_1} x_1 \longrightarrow \cdots \longrightarrow x_{i-1} \xrightarrow{\gamma_{i+1} \gamma_i} x_{i+1} \longrightarrow \cdots \longrightarrow x_{n-1} \xrightarrow{\gamma_n} x_n.$$

For the degeneracy maps, $s_i : \Gamma_n \to \Gamma_{n+1}$ we put the identity map of $x_i \xrightarrow{1} x_i$ in the $i+1$-position of the chain

$$x_0 \longrightarrow \cdots \longrightarrow x_{i-1} \xrightarrow{\gamma_i} x_i \xrightarrow{1} x_i \xrightarrow{\gamma_{i+1}} x_{i+1} \longrightarrow \cdots \longrightarrow x_n.$$

Explicitly, for $n = 1$ set $d_0 := s$ and $d_1 := t$. For $n \geq 2$, we set

$$d_0(\gamma_1, \ldots, \gamma_n) := (\gamma_2, \ldots, \gamma_n), \quad d_n(\gamma_1, \ldots, \gamma_n) := (\gamma_1, \ldots, \gamma_{n-1}),$$

for $1 \leq i \leq n-1$

$$d_i(\gamma_1, \ldots, \gamma_n) := (\gamma_1, \ldots, \gamma_{i+1} \gamma_i, \ldots, \gamma_n),$$

and for the face maps
$$s_i(\gamma_1, \ldots, \gamma_n) := (\gamma_1, \ldots, \gamma_i, 1, \gamma_{i+1}, \ldots, \gamma_n).$$

Let $A = (A_\bullet, d_i, s_i)$ be a simplicial module. Then, A determines a chain-complex $C_\bullet(A) = (A_\bullet, \partial_\bullet)$ such that $\partial_n := \sum_{i=0}^{n} (-1)^n d_i$. We define the homology of the simplicial module A as the homology of its associated chain-complex. Thus, we set $H_\bullet(A) := H_\bullet(C_\bullet(A))$. Dually, the simplicial module A determines a cochain-complex $\mathscr{C}^\bullet(A) = (\hom_K(A_\bullet, K), \partial^\bullet)$, where $\partial^n := \hom_K(\partial_n, K)$, and the cohomology of the simplicial module A is defined as the cohomology of its associated cochain-complex. Thus, we set $H^\bullet(A) := H^\bullet(\mathscr{C}^\bullet(A))$.

Example 1.32. Let M be a G-module. Then, we define $C_n(G, M) := KG^{\otimes n} \otimes M$ and the face maps For $n \geq 1$ we define the face maps $d_i : C_n(G, M) \to C_{n-1}(G, M)$ by
$$d_i(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m) := \begin{cases} g_2 \otimes \ldots \otimes g_n \otimes m & \text{if } i = 0; \\ g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_n \otimes m & \text{if } 0 < i < n; \\ g_1 \otimes \ldots \otimes g_{n-1} \otimes \theta_{g_n}(m) & \text{if } i = n. \end{cases}$$
We set the degeneracy maps $s_i : C_n(G, M) \to C_{n+1}(G, M)$ by
$$s_i(g_1 \otimes \ldots \otimes g_n \otimes m) = g_1 \otimes \ldots \otimes g_i \otimes 1 \otimes g_{i+1} \otimes \ldots \otimes g_n \otimes m.$$
Then, $C_\bullet(G, M) := (C_\bullet(G, M), d_i, s_i)$ is a simplicial module. Furthermore, its respective associated chain complex $(C_\bullet(G, M), \partial_\bullet)$ is the standard complex that computes the group homology of G with coefficients in M. Dually, $\mathscr{C}^\bullet\big(C_\bullet(G, M)\big)$ is the cochain-complex that computes the group cohomology of G with coefficients in the right G-module M^*.

Any simplicial set $X = (X_\bullet, d_i, s_i)$ determines a simplicial module $KX = (KX_\bullet, d_i, s_i)$ by extending the face and degeneracy maps linearly. Consequently, we can define the homology of the simplicial set X as the homology of the simplicial module KX.

It is possible to construct chains and cochains of a simplicial set using a more general structure called a *coefficient system*. This construction will help us to compare both partial group cohomology theories: the one based on partial representations of groups [1] and the other based on partial group actions on commutative inverse monoids [7].

Definition 1.33. A **homological coefficient system** $\mathscr{A}$ on a simplicial set $X : \Delta \to \mathbf{Set}$ is a family of Abelian groups $\{\mathscr{A}_x\}$, one for each simplex $x \in X_n$, and a family of homomorphisms $\mathscr{A}(\phi, x) : \mathscr{A}_x \to \mathscr{A}_{X(\phi)(x)}$, one for each pair $x \in X_n$, $\phi : \mathbf{m} \to \mathbf{n}$, such that the following conditions are satisfied:

(1.10) $$\mathscr{A}(1, x) = 1 \text{ and } \mathscr{A}(\phi \circ \psi, x) = \mathscr{A}(\psi, X(\phi)(x))\mathscr{A}(\phi, x).$$

Dually, a **cohomological coefficient system** $\mathscr{A}$ on a simplicial set X is a family of Abelian groups $\{\mathscr{A}_x\}$, one for each simplex $x \in X_n$, and a family of homomorphisms $\mathscr{A}(\phi, x) : \mathscr{A}_{X(\phi)(x)} \to \mathscr{A}_x$, one for each pair $x \in X_n$, $\phi : \mathbf{m} \to \mathbf{n}$, such that the following conditions are satisfied:

(1.11) $$\mathscr{A}(1, x) = 1 \text{ and } \mathscr{A}(\phi \circ \psi, x) = \mathscr{A}(\phi, x)\mathscr{A}(\psi, X(\phi)(x)).$$

The **category of simplices** of a simplicial set X is denoted by $\Delta(X)$. This has as objects the pairs $(\mathbf{n}, x)$ where $\mathbf{n} \in \Delta$ and $x \in X(\mathbf{n})$, and a morphism $(\phi, x) : (\mathbf{m}, y) \to (\mathbf{n}, x)$ is a pair $\phi : \mathbf{m} \to \mathbf{n} \in \mathrm{Mor}(\Delta)$ and $x \in X(\mathbf{n})$, such that $X(\phi)(x) = y \in X(\mathbf{m})$. By a direct computation one verifies that a cohomological coefficient system of a simplicial set X is just a functor $\mathscr{A} : \Delta(X) \to \mathbf{Ab}$ (see [13, Apendix II]).

Remark 1.34. Let $X = (X_\bullet, d_i, s_j)$ be a simplicial set. Suppose that we have a family of Abelian groups $\{\mathscr{A}_x\}_{x \in X}$ and a for any pair $x \in X_n$ and $0 \le i \le n$, morphisms $\delta_{i,x} : \mathscr{A}_{d_i(x)} \to \mathscr{A}_x$ and $\sigma_{i,x} : \mathscr{A}_{s_i(x)} \to \mathscr{A}_x$ such that

$$(1.12) \qquad \delta_{j,x}\delta_{i,d_j(x)} = \delta_{i,x}\delta_{j-1,d_i(x)} \text{ if } i < j$$

$$\sigma_{j,x}\sigma_{i,s_j(x)} = \sigma_{j,x}\sigma_{j+1,s_j(x)} \text{ if } i \le j$$

$$\sigma_{j,x}\delta_{i,s_j(x)} = \begin{cases} \delta_{i,x}\sigma_{j-1,d_i(x)} & \text{if } i < j \\ 1_{\mathscr{A}_x} & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1,x}\sigma_{j,d_{i-1}(x)} & \text{if } i > j+1. \end{cases}$$

Let $\phi : \mathbf{m} \to \mathbf{n}$ be any morphism in Δ, then by Lemma 1.28 there exist a unique decomposition $\phi = \delta_{i_1}\delta_{i_2}\ldots\delta_{i_r}\sigma_{j_1}\ldots\sigma_{j_s}$. We define

$$\mathscr{A}(\phi, x) := \delta_{i_1,x_1}\delta_{i_2,x_2}\ldots\delta_{i_r,x_r}\sigma_{j_1,x_{r+1}}\ldots\sigma_{j_s,x_{r+s}},$$

where

$$x_k = \begin{cases} x & \text{if } x = 1 \\ d_{i_1}\ldots d_{i_{k-1}}(x) & \text{if } 2 \le k \le r+1 \\ d_{i_1}\ldots d_{i_r}s_{j_1}\ldots s_{j_{k-r-1}}(x) & \text{if } r+2 \le k \le r+s \end{cases}$$

Thus, by Corollary 1.29 and the relations (1.12) we conclude that $\{\mathscr{A}_x\}$ together with $\{\mathscr{A}(\phi, x)\}$ defines a functor $\mathscr{A} : \Delta(X) \to \mathbf{Ab}$, and therefore a cohomological coefficient system on X. We can dualize this argument to obtain a version of this construction for homological coefficient systems, i.e., the family of morphisms must satisfy the dual version of the relations (1.12).

Let (X, d_i, s_i) be a simplicial set, and let $\mathscr{A}$ be a homological coefficient system on X. An n-chain of X with coefficients in $\mathscr{A}$ is a formal linear combination $\sum_{x \in X_n} a(x)x$, where $a(x) \in \mathscr{A}_x$. The set of n-chains forms an Abelian group under addition, denoted by $C_n(X, \mathscr{A})$. For a basic n-chain $a(x)x \in C_n(X, \mathscr{A})$ we define its boundary by

$$(1.13) \qquad \partial_n(a(x)x) := \sum_{i=0}^{n}(-1)^i\mathscr{A}(\delta_i, x)(a(x))d_i(x).$$

Therefore, we obtain a differential $\partial_n : C_n(X, \mathscr{A}) \to C_{n-1}(X, \mathscr{A})$ and a chain-complex $(C_\bullet(X, \mathscr{A}), \partial_\bullet)$. The homology groups of the complex $C_\bullet(X, \mathscr{A})$ are called the homology groups of the simplicial set X with coefficients in $\mathscr{A}$, denoted by $H_n(X, \mathscr{A})$.

Analogously, let $\mathscr{A}$ be a cohomology coefficient system on X. Let

$$(1.14) \qquad C^n(X, \mathscr{A}) := \{f : X_n \to \cup_{x \in X_n}\mathscr{A}_x : f(x) \in \mathscr{A}_x\}.$$

The differential $\partial^n : C^n(X, \mathscr{A}) \to C^{n+1}(X, \mathscr{A})$ is given by

$$(1.15) \qquad \partial^n(f)(x) := \sum_{i=0}^{n+1}(-1)^i\mathscr{A}(\delta_i, x)\big(f(d_i(x))\big), \ x \in X_{n+1}.$$

We define the cohomology groups of X with coefficients in $\mathscr{A}$, denoted by $H^n(X, \mathscr{A})$, as the cohomology groups of the cochain-complex $C^\bullet(X, \mathscr{A})$.

2. Partial tensor product

Since we have introduced the concept of left and right G_{par}-modules, it is natural to ask how to define a tensor product version for this kind of partial structures. Let $\beta : X \curvearrowleft G$ and $\alpha : G \curvearrowright Y$ be partial actions on modules. Define $\mathcal{K}_{\beta,\alpha}$ as the K-submodule of $X \otimes_K Y$ generated by the set

$$\left\{ (x)\beta_g \otimes y - x \otimes \alpha_g(y) : g \in G,\ x \in X_{g^{-1}} \text{ and } y \in Y_{g^{-1}} \right\}.$$

Hence, we define the **partial tensor product** of the right G_{par}-module X by the left G_{par}-module Y as

$$(2.1) \qquad\qquad X \otimes_{G_{par}} Y := \frac{X \otimes_K Y}{\mathcal{K}_{\beta,\alpha}}.$$

Notice that we obtain a bilinear map $\otimes_{G_{par}} : X \times Y \to X \otimes_{G_{par}} Y$. Thus, we write $x \otimes_{G_{par}} y$ to denote $\otimes_{G_{par}}(x, y)$.

Proposition 2.1. *Let $\beta : X \curvearrowleft G$ and $\alpha : G \curvearrowright Y$ be partial actions on modules, and Z be a K-module. Suppose that $\phi : X \times Y \to Z$ is a bilinear map such that $\phi((x)\beta_g, y) = \phi(x, \alpha_g(y))$ for all $g \in G$, $x \in X_{g^{-1}}$, $y \in Y_{g^{-1}}$. Then, there exists a unique K-linear map $\tilde{\phi} : X \otimes_{G_{par}} Y \to Z$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\ \ \phi\ \ } & Z \\
{\scriptstyle \otimes_{G_{par}}}\big\downarrow & \nearrow_{\tilde{\phi}} & \\
X \otimes_{G_{par}} Y & &
\end{array}
$$

Proof. Since ϕ is bilinear we have a linear map $\phi' : X \otimes Y \to Z$, and by the hypotheses we have that $\mathcal{K}_{\beta,\alpha} \subseteq \ker \phi'$, thus such morphism exists. The uniqueness is a consequence of the fact $\tilde{\phi}(x \otimes_{G_{par}} y) = \phi(x, y)$ and that $\{ x \otimes_{G_{par}} y : x \in X \text{ and } y \in Y \}$ generates $X \otimes_{G_{par}} Y$ as K-module. $\qquad\square$

By a classical proof of uniqueness give by universal properties one proves that $X \otimes_{G_{par}} Y$ is the unique module (up to isomorphism) that satisfies the universal property mentioned in Proposition 2.1.

Remark 2.2. If M is a right G-module and N a left G-module then $M \otimes_{G_{par}} N$ is naturally isomorphic to $M \otimes_{KG} N$.

Let $\nu : M \curvearrowleft G$, $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be partial actions on K-modules, and let $\varphi : A \to B$ a left G-equivariant map. Then, the map $f_* : M \otimes_{G_{par}} A \to M \otimes_{G_{par}} B$ such that $f_*(m \otimes_{G_{par}} a) := m \otimes_{G_{par}} f(a)$ is a well-defined map of K-modules. Henceforth, we obtain a functor

$$(2.2) \qquad\qquad M \otimes_{G_{par}} - : G_{par}\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod},$$

analogously, if $\nu : G \curvearrowright M$ is a partial action on M, we obtain a functor

$$(2.3) \qquad\qquad - \otimes_{G_{par}} M : \mathbf{Mod}\text{-}G_{par} \to K\text{-}\mathbf{Mod},$$

Remark 2.3. Let $\beta : M \curvearrowleft G$ be a partial action on the module M. By abuse of notation we define the fuctor

$$M \otimes_{G_{par}} - : K_{par}G\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$$

as the composition of the functor Λ and the functor (2.2). That is, if X is a left $K_{par}G$-module, we consider the induced partial action $\theta : G \curvearrowright X$ of G on X, as in Example 1.14. Thus, $M \otimes_{G_{par}} X$ is

the partial tensor product of the right G_{par}-module M and the left G_{par}-module X. In this case, the module $\mathcal{K}_{\beta,\theta}$ is the K-submodule generated by the set

$$\left\{(m)\beta_g \otimes e_{g^{-1}} \cdot x - m \otimes [g] \cdot x : g \in G, m \in M_{g^{-1}} \text{ and } x \in X\right\}.$$

Analogously, if $\alpha : G \curvearrowright M$ is a partial action, we obtain a functor

$$- \otimes_{G_{par}} M : \mathbf{Mod}\text{-}K_{par}G \to K\text{-}\mathbf{Mod}.$$

Proposition 2.4. *Let $\mathcal{A}$ and $\mathcal{R}$ be K-algebras, and let $\beta : X \curvearrowleft G$ be a right partial group action of left $\mathcal{A}$-modules, and let $\alpha : G \curvearrowright Y$ be a left partial group action of right $\mathcal{R}$-modules. Then, $X \otimes_{G_{par}} Y$ is an $\mathcal{A}$-$\mathcal{R}$-bimodule.*

Proof. We know that $X \otimes_K Y$ is an $\mathcal{A}$-$\mathcal{R}$-bimodule. Observe that for all $a \in \mathcal{A}$ and $b \in \mathcal{R}$ we have that

$$a\big(x \otimes \alpha_g(y) - (x)\beta_g \otimes y\big)b = ax \otimes \alpha_g(yb) - (ax)\beta_g \otimes yb,$$

therefore $\mathcal{K}_{\beta,\alpha}$ is an $\mathcal{A}$-$\mathcal{R}$-subbimodule of $X \otimes_K Y$. Whence we conclude that $X \otimes_{G_{par}} Y$ is an $\mathcal{A}$-$\mathcal{R}$-bimodule. $\qquad\square$

Proposition 2.5. *Let $\beta : M \curvearrowleft G$ be a right partial group action on the K-module M. Then, the functor $M \otimes_{G_{par}} - : K_{par}G\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$ is right exact. Similarly, if $\alpha : G \curvearrowright M$ is a left partial action on M, then the functor $- \otimes_{G_{par}} M : \mathbf{Mod}\text{-}K_{par}G \to K\text{-}\mathbf{Mod}$ is right exact.*

Proof. Let $A \xrightarrow{f} B \xrightarrow{h} C \to 0$ be an exact sequence in $K_{par}G\text{-}\mathbf{Mod}$. Then, we obtain a sequence $M \otimes_{G_{par}} A \xrightarrow{f_*} M \otimes_{G_{par}} B \xrightarrow{h_*} M \otimes_{G_{par}} C \to 0$, since h is surjective we obtain that h_* is surjective. Furthermore, since $h_* f_* = 0$, we have that $\operatorname{im} f_* = \ker h_*$ if, and only if, the linear map

$$\overline{h_*} : (M \otimes_{G_{par}} B)/\operatorname{im} f_* \to M \otimes_{G_{par}} C$$

is an isomorphism, where $\overline{h_*}$ is the map induced by h_*. For all $c \in C$, choose $b_c \in B$ such that $h(b_c) = c$, and define the map $\eta : M \otimes C \to M \otimes_{G_{par}} B/\operatorname{im} f_*$ such that $\eta(m \otimes c) = \overline{m \otimes_{G_{par}} b_c}$ for all $m \in M$ and $c \in C$. To verify that this map is well-defined we have to check that η does not depend on the choice of b_c. Let b such that $h(b) = h(b_c) = c$, then $b - b_c \in \ker h = \operatorname{im} f$. Consequently, $m \otimes_{G_{par}} (b - b_c) \in \operatorname{im} f_*$, whence we have that η is well-defined.

Now, suppose that $m \in M_{g^{-1}}$ and $c \in C_{g^{-1}} = e_{g^{-1}} \cdot C$. Since h is a morphism of $K_{par}G$-module, by (iii) of Lemma 1.25, there exists $b \in B_{g^{-1}} = e_{g^{-1}} \cdot B$ such that $h(b) = c$. Then,

$$\eta\big((m)\beta_g \otimes c\big) = \overline{(m)\beta_g \otimes_{G_{par}} b_c} = \overline{(m)\beta_g \otimes_{G_{par}} b} = \overline{m \otimes_{G_{par}} [g] \cdot b} = \overline{m \otimes_{G_{par}} b_{[g] \cdot c}} = \eta\big(m \otimes [g] \cdot c\big)$$

Henceforth, by Proposition 2.1, there exists a K-linear map $\overline{\eta} : M \otimes_{G_{par}} C \to M \otimes_{G_{par}} B/\operatorname{im} f_*$ such that $\overline{\eta}(m \otimes_{G_{par}} c) = \overline{m \otimes_{G_{par}} b_c}$. Finally, note that $\overline{h_*}^{-1} = \overline{\eta}$. One shows in the same fashion that $- \otimes_{G_{par}} M$ is right exact. $\qquad\square$

Proposition 2.6. *Let $\beta : X \curvearrowleft G$ be a partial action, and let Y be a $K_{par}G$-$\mathcal{A}$-module. Then the functors*

$$\big(X \otimes_{G_{par}} Y\big) \otimes_{\mathcal{A}} - : \mathcal{A}\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod},$$

and

$$X \otimes_{G_{par}} \big(Y \otimes_{\mathcal{A}} -\big) : \mathcal{A}\text{-}\mathbf{Mod} \to K\text{-}\mathbf{Mod}$$

are isomorphic. Analogously, if $\alpha : G \curvearrowright Y$ is a partial action, X is an $\mathcal{A}$-$K_{par}G$-module, then the functors $- \otimes_{\mathcal{A}} \big(X \otimes_{G_{par}} Y\big)$ and $\big(- \otimes_{\mathcal{A}} X\big) \otimes_{G_{par}} Y$ are isomorphic.

Proof. Let Z be a left $\mathcal{A}$-module. For a fixed $z \in Z$ consider the map

$$X \times Y \to X \otimes_{G_{par}} (Y \otimes_{\mathcal{A}} Z)$$
$$(x, y) \mapsto x \otimes_{G_{par}} (y \otimes_{\mathcal{A}} z)$$

It is clear that this map is bilinear and that satisfies the conditions of Proposition 2.1, thus for all $z \in Z$ we obtain a morphism of K-modules

$$X \otimes_{G_{par}} Y \to X \otimes_{G_{par}} (Y \otimes_{\mathcal{A}} Z)$$
$$x \otimes_{G_{par}} y \mapsto x \otimes_{G_{par}} (y \otimes_{\mathcal{A}} z).$$

Furthermore, the map

$$(X \otimes_{G_{par}} Y) \times Z \to X \otimes_{G_{par}} (Y \otimes_{\mathcal{A}} Z)$$
$$(x \otimes_{G_{par}} y, z) \mapsto x \otimes_{G_{par}} (y \otimes_{\mathcal{A}} z)$$

is an $\mathcal{A}$-bilinear map, thus we obtain a K-linear map

$$\eta : (X \otimes_{G_{par}} Y) \otimes_{\mathcal{A}} Z \to X \otimes_{G_{par}} (Y \otimes_{\mathcal{A}} Z)$$
$$(x \otimes_{G_{par}} y) \otimes_{\mathcal{A}} z \mapsto x \otimes_{G_{par}} (y \otimes_{\mathcal{A}} z).$$

On the other hand, we have the following commutative diagram given by Proposition 2.1

$$
\begin{array}{ccc}
X \otimes_K Y \otimes_{\mathcal{A}} Z & \longrightarrow & (X \otimes_{G_{par}} Y) \otimes_{\mathcal{A}} Z \\
{\scriptstyle \otimes_{G_{par}}} \downarrow & & \nearrow \\
X \otimes_{G_{par}} (Y \otimes_{\mathcal{A}} Z) & \cdots\cdots & {\scriptstyle \psi}
\end{array}
$$

where the horizontal morphism is given by $x \otimes_K y \otimes_{\mathcal{A}} z \mapsto (x \otimes_{G_{par}} y) \otimes_{\mathcal{A}} z$. Hence, $\psi(x \otimes_{G_{par}} (y \otimes_{\mathcal{A}} z)) = (x \otimes_{G_{par}} y) \otimes_{\mathcal{A}} z$. Note that ψ and η are mutually inverses. Finally, it is easy to see that η determines a natural isomorphism. $\qquad\square$

2.1. Globalization of partial group actions on modules. Analogously to set-theoretical partial group actions (see [12] for more details), we can obtain partial group actions on modules by restricting global group actions on a K-module W to a submodule M of W. Indeed, let $\theta : G \curvearrowright W$ be a global group action, and let M be a K-submodule of W, define $M_g := \theta_g(M) \cap M$ and $\alpha_g := \theta_g|_{M_{g^{-1}}}$. Then, $\alpha := (G, M, \{M_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ is a partial group action of G on M. We say that α is the **restriction** of θ to M.

Definition 2.7. Let $\alpha : G \curvearrowright M$ be a partial action of a K-module M, the **universal global action** of α is a pair (θ, ι) where $\theta : G \curvearrowright W$ is a global group action and $\iota : M \to W$ is a G-equivariant map, such that for any global group action $\beta : G \curvearrowright X$ and a G-equivariant map $\psi : M \to X$ there exists a unique morphism $\tilde{\psi} : W \to X$ of G-modules such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & X \\
{\scriptstyle \iota} \downarrow & \nearrow & \\
W & {\scriptstyle \tilde{\psi}} &
\end{array}
$$

Remark 2.8. Note that by a classical argument of universal property, the universal global action (if there exists) is unique up to unique isomorphism.

Definition 2.9. Let $\alpha : G \curvearrowright M$ be a partial group action and $\theta : G \curvearrowright W$ a global group action. If there exists an injective G-equivariant map $\iota : M \to W$ such that α is isomorphic to the restriction of θ to $\iota(M)$ via ι, and $W = \sum_{g \in G} \theta_g(\iota(M))$ we say that θ is a **globalization** of α.

Contrary to the set-theoretical case, a globalization of partial group action on a module (if it exists) is not necessarily unique.

Example 2.10. Let $G := \langle g : g^3 = 1 \rangle$, and let $\alpha := (G, \mathbb{Z}, \{X_h\}_{h \in G}, \{\alpha_h\}_{h \in G})$ be the partial group action such that $X_1 = \mathbb{Z}$, $X_g = X_{g^2} = \{0\}$, and $\alpha_1 = 1_{\mathbb{Z}}$, $\alpha_g(0) = \alpha_{g^2}(0) = 0$. Now, we will-construct two non-isomorphic globalizations of α. $V := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, define the global action of G on V such that $g \cdot (x, y, z) := (z, x, y)$, thus the function $\iota : \mathbb{Z} \to V$ given by $\iota(n) := (n, 0, 0)$ is clearly a G-equivariant map. Moreover, note that V together with ι is a globalization of α.

For the second globalization consider the subgroup $N := \{(2n, 2n, 2n) : n \in \mathbb{Z}\}$, note that $g \cdot (2n, 2n, 2n) = (2n, 2n, 2n)$, then $W := V/N$, is a G-module such that $g \cdot \overline{(x, y, z)} = \overline{(z, x, y)}$, let $\phi : V \to W$ be the canonical projection. Then, $(V, \phi \circ \iota)$ is a globalization of M. Indeed, first observe that ψ is injective. Suppose that $\psi(n) = \overline{(0, 0, 0)}$ for some $n \in \mathbb{Z}$, then $(n, 0, 0) \in N$, thus $n = 0$. Now, we have to verify that $g \cdot \overline{(n, 0, 0)} \in \psi(\mathbb{Z})$ if, and only if, $n = 0$, and $g^2 \cdot \overline{(n, 0, 0)} \in \psi(\mathbb{Z})$ if and only if, $n = 0$. Note that

$$g \cdot \overline{(n, 0, 0)} \in \psi(\mathbb{Z}) \Leftrightarrow \text{ there exists } m \in \mathbb{Z} \text{ such that } \overline{(0, n, 0)} = \overline{(m, 0, 0)}$$
$$\Leftrightarrow (-m, n, 0) \in N$$
$$\Leftrightarrow m = n = 0,$$

analogously we have that $g^2 \cdot \overline{(n, 0, 0)} \in \psi(\mathbb{Z})$ if, and only if, $n = 0$. Hence, the restriction of the global action on W to $\psi(\mathbb{Z})$ is isomorphic to α. Finally, observe that V and W are not isomorphic as G-modules since W has torsion elements, but V does not.

Remark 2.11. Let $\alpha : G \curvearrowright M$ be a partial action on the module M. Then, by Proposition 2.4 $KG \otimes_{G_{par}} M$ is a left G-module. Let $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ be the associated global action of G on $KG \otimes_{G_{par}} M$. Define

$$\iota : M \to KG \otimes_{G_{par}} M$$
$$m \mapsto 1 \otimes_{G_{par}} m.$$

Then, ι is a well-defined G-equivariant map. Indeed, observe that

$$\iota(\alpha_g(m)) = 1 \otimes_{G_{par}} \alpha_g(m) = g \otimes_{G_{par}} m = \Theta_g(\iota(m)),$$

for all $g \in G$ and $m \in M_{g^{-1}}$.

Theorem 2.12. *Let $\alpha : G \curvearrowright M$ be a partial group action on the K-module M, and let $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ and $\iota : M \to KG \otimes_{G_{par}} M$ be as in Remark 2.11. Then, (Θ, ι) is the universal global action of α.*

Proof. Let X be a left G-module and $\psi : M \to X$ a G-equivariant map, then $\psi_* : KG \otimes_{G_{par}} M \to KG \otimes_{G_{par}} X$ is a map of G-modules. Note that, by Remark 2.2, $KG \otimes_{G_{par}} X \cong KG \otimes_{KG} X \cong X$, hence we obtain a morphism of G-modules $\tilde{\psi} : KG \otimes_{G_{par}} M \to X$ such that $\tilde{\psi}(g \otimes_{G_{par}} m) := g \cdot \psi(m)$. It is clear that $\psi = \tilde{\psi} \circ \iota$. Finally, the uniqueness of $\tilde{\psi}$ is due to the universal property that satisfies $KG \otimes_{G_{par}} M$ (see Definition 2.7). $\square$

Remark 2.13. Theorem 2.12 and the functor (2.2) guarantee the existence of a functor

$$(2.4) \qquad KG \otimes_{G_{par}} - : G_{par}\text{-}\mathbf{Mod} \to G\text{-}\mathbf{Mod}.$$

Furthermore, by universal property in Definition 2.7 the functor $KG \otimes_{G_{par}} -$ is a left adjoint to the embedding functor $G\text{-}\mathbf{Mod} \to G_{par}\text{-}\mathbf{Mod}$ defined in Example 1.13, i.e., for any G_{par}-module M and G-module X there exists a natural isomorphism

$$\hom_G(KG \otimes_{G_{par}} M, X) \cong \hom_{G_{par}}(M, X).$$

Notation. Let $\alpha : G \curvearrowright M$ be a partial group action on the module M. For the sake of simplicity, if there is no ambiguity, a basic element $g \otimes_{G_{par}} m$ of $KG \otimes_{G_{par}} M$ will be denoted by $[g, m]$. Thus, for any $h \in G$, we have that $h \cdot [g, x] = [hg, x]$. This notation is motivated by the notation used in [12] for the elements of the globalization of a set-theoretical partial group action. It is important to take care not to confuse this notation with the notation used for the basic elements of the partial group algebra $K_{par}G$.

Let $\alpha : G \curvearrowright X$ be a partial group action on the K-module M, and let θ be the natural global action of G on KG, we will denote the module $\mathcal{K}_{\theta,\alpha}$ just by $\mathcal{K}_\alpha$. From the definition of $\mathcal{K}_\alpha$ we know that

$$(2.5) \qquad \mathcal{K}_\alpha := \left\langle g \otimes x - gh^{-1} \otimes \alpha_h(x) : g, h \in G, \ x \in X_{h^{-1}} \right\rangle$$

$$= \left\langle g \otimes x - h \otimes y : x \in X_{g^{-1}h} \text{ and } \alpha_{h^{-1}g}(x) = y \right\rangle.$$

Note that in Definition 2.7, we do not require ι to be an injective map. In fact, ι may not be injective.

Example 2.14. Let M be the free K-module with basis consisting of the set symbols $\{x, y, z, u, v, w\}$, and let $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For simplicity, we set $e = (0, 0, 0)$, $a := (1, 0, 0)$, $b := (0, 0, 1)$ and $c := (0, 0, 1)$, and we use product notation instead of sum notation. Define:

- $M_e := M$ and $\alpha_e := 1_M$;
- $M_a := \langle x + y \rangle$, and $\alpha_a := 1|_{M_a}$;
- $M_b := \langle z - u \rangle$, and $\alpha_b := 1|_{M_b}$;
- $M_c := \langle v + w \rangle$, and $\alpha_c := 1|_{M_c}$;
- $M_{abc} := \{0\}$, and $\alpha_{abc} := 1|_{M_{abc}}$;
- $M_{ab} := \langle x, u \rangle$, and $\alpha_{ab} : M_{ab} \to M_{ab}$, such that $\alpha_{ab}(x) = u$ and $\alpha_{ab}(u) = x$;
- $M_{ac} := \langle v, y \rangle$, and $\alpha_{ac} : M_{ac} \to M_{ac}$, such that $\alpha_{ac}(y) = v$ and $\alpha_{ac}(v) = y$;
- $M_{bc} := \langle z, w \rangle$, and $\alpha_{bc} : M_{bc} \to M_{bc}$, such that $\alpha_{bc}(z) = w$ and $\alpha_{bc}(w) = z$.

It is clear that α_g is a K-linear isomorphism for all $g \in G$. To verify that α is a partial group action observe that

$$(2.6) \qquad M_g \cap M_h = \begin{cases} M_g & \text{if } h \in \{1, g\} \\ \{0\} & \text{otherwise.} \end{cases}$$

Then,

$$\alpha_g(M_g \cap M_h) = \begin{cases} M_g & \text{if } h \in \{1, g\} \\ \{0\} & \text{otherwise,} \end{cases}$$

what verifies (ii) of Definition 1.9. Observe that (iii) of Definition 1.9 is satisfied for $x = 0$. Otherwise, if there exists $x \in M_{g^{-1}} \cap M_{g^{-1}h^{-1}}$ such that $x \neq 0$, then by Equation (2.6) we know that $g^{-1} = 1$ or $h^{-1} = 1$ or $g^{-1}h^{-1} = 1$, then

- if $g^{-1} = 1$, then $x \in M_{h^{-1}}$ and $\alpha_h\alpha_1(x) = \alpha_h(x)$,

- if $h^{-1} = 1$, then $x \in M_{g^{-1}}$ and $\alpha_1 \alpha_g(x) = \alpha_g(x)$,
- if $g^{-1} h^{-1} = 1$, then $x \in M_{g^{-1}}$, $g^{-1} = h$ and $\alpha_{g^{-1}} \alpha_g(x) = x = \alpha_1(x)$.

Therefore, α is a well-defined partial group action of G on M. Now define

$$t := x + y + z - u - v - w \in M.$$

It is clear that $t \neq 0$. Note that we can rewrite t as follows

$$\begin{aligned}
t &= (x + y) + (z - u) - (v + w) \\
&= \alpha_a(x + y) + \alpha_b(z - u) - \alpha_c(v + w) \\
&= \alpha_a(x + y) + \alpha_b(z - \alpha_{ab}(x)) - \alpha_c(\alpha_{ac}(y) + \alpha_{bc}(z)).
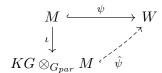\end{aligned}$$

Then,

$$\begin{aligned}
[1, t] &= [1, \alpha_a(x + y)] + [1, \alpha_b(z - \alpha_{ab}(x))] - [1, \alpha_c(\alpha_{ac}(y) + \alpha_{bc}(z))] \\
&= [a, x + y] + [b, z - \alpha_{ab}(x)] - [c, \alpha_{ac}(y) + \alpha_{bc}(z)] \\
&= [a, x] + [a, y] + [b, z] - [b, \alpha_{ab}(x)] - [c, \alpha_{ac}(y)] - [c, \alpha_{bc}(z)] \\
&= [a, x] + [a, y] + [b, z] - [bab, x] - [cac, y] - [cbc, z] \\
&= [a, x] + [a, y] + [b, z] - [a, x] - [a, y] - [b, z] \\
&= 0.
\end{aligned}$$

Hence, the map $\iota : M \to KG \otimes_{G_{par}} M$ is not injective.

Proposition 2.15. *Let $\alpha : G \curvearrowright M$ be a partial group action on the K-module M. Suppose that there exists a global action $\theta : G \curvearrowright W$ and an injective G-equivariant map $\psi : M \to W$ such that α is the restriction of θ to M. Then, the map ι is injective and the canonical global action of G on $KG \otimes_{G_{par}} M$ is a globalization of α, we call this globalization as the **universal globalization***

Proof. By the universal property of $KG \otimes_{G_{par}} M$ the following diagram is commutative

$$\begin{array}{ccc}
M & \overset{\psi}{\hookrightarrow} & W \\
\iota \downarrow & \nearrow & \\
KG \otimes_{G_{par}} M & \overset{\hat{\psi}}{\dashleftarrow} &
\end{array}$$

Note that the injectivity of ψ implies the injectivity of ι. Let $\beta := (G, \iota(M), \{D_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ be the restriction of the global action $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ to $\iota(M)$, since ι is a G-equivaliant map, it is clear that $\iota|_{\iota(M)} : M \to \iota(M)$ is a bijective G-equivariant map, thus we only have to verify that $D_g \subseteq \iota(M_g)$ for all $g \in G$. Let $g \in G$ and $m \in M$ such that $[g, m] \in \iota(M)$, then $\theta_g(\psi(m)) = \hat{\psi}([g, m]) \in \psi(M) \cap \theta_g(\psi(M))$, whence we conclude that $m \in M_{g^{-1}}$ since α is restriction of θ, thus $[g, m] = [1, \theta_g(m)] \in \iota(M_g)$. Therefore, $KG \otimes_{G_{par}} M$ is a globalization of $\alpha : G \curvearrowright M$. $\square$

Corollary 2.16. *Let $\alpha : G \curvearrowright M$ be a partial group action. If the map $\iota : M \to KG \otimes_{G_{par}} M$ is not injective, then α is not globalizable.*

Proposition 2.17. *Let $\alpha : G \curvearrowright M$ be a partial group action. Then, the universal global action $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ is a globalization if, and only if, for all $x, y \in M$ and $g \in G$ such that $[g, x] = [1, y]$ then $x \in M_{g^{-1}}$ and $\alpha_g(x) = y$.*

Proof. Recall that $\iota : M \to KG \otimes_{G_{par}} M$ is the canonical morphism defined in Remark 2.11. Let $\beta := (G, \iota(M), \{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ be the restriction of Θ to $\iota(M)$. Set $\hat{\iota} := \iota|_{\iota(M)} : M \to \iota(M)$. Suppose that Θ is a globalization of α, then $\hat{\iota}$ is an isomorphism of partial group actions. Let $x, y \in M$ and $g \in G$ such that $[g, x] = [1, y] \in \iota(M)$, since β is the restriction of Θ and $\hat{\iota}$ is an isomorphism we conclude that $x \in M_{g^{-1}}$. Then, $[1, y] = [g, x] = [1, \alpha_g(x)]$, by the injectivity of ι we obtain that $y = \alpha_g(x)$. For the converse we want to verify that $\hat{\iota}$ is an isomorphism of partial actions. Let $x \in M$ such that $\iota(x) = 0$, then $[1, x] = 0 = [1, 0]$, thus $x = \alpha_1(0) = 0$. Therefore, the map ι is injective. Hence, to verify that $\hat{\iota}$ is an isomorphism of partial actions we only have to show that $\iota(M_g) = D_g$. Let $[g, x] \in D_g \subseteq \iota(M)$, since $[g, x] \in \iota(M)$ there exists $z \in M$ such that $[g, x] = [1, z]$, thus by the hypotheses we get that $x \in M_{g^{-1}}$, and therefore $[g, x] = [1, \alpha_g(x)] = \iota(\alpha_g(x))$. Hence, $\iota(M_g) = D_g$. $\square$

Remark 2.18. For any $K_{par}G$-module we can define a K-linear map $\tau_0 : KG \otimes X \to X$ such that $\tau_0(g \otimes x) := [g] \cdot x$, note that $g \otimes [h] \cdot x - gh \otimes e_{h^{-1}} \cdot x \in \ker \tau_0$ for all $x \in X$ and $g, h \in G$. Then, by the universal property of the partial tensor product (Proposition 2.1) there exists a K-linear map $\tau : KG \otimes_{G_{par}} X \to X$ such that $\tau(g \otimes_{G_{par}} x) = [g] \cdot x$. Furthermore, note that $\tau \circ \iota = 1_X$.

Theorem 2.19. *Let* $\pi : G \to \text{End}_K(M)$ *be a partial group representation, and let* $\alpha : G \curvearrowright M$ *be the partial group action induced by* π. *Then,* $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ *the universal globalization of* α.

Proof. Let $g \in G$, $x, y \in M$ such that $[g, x] = [1, z]$. By Remark 2.18, $[g] \cdot x = \tau([g, x]) = \tau([1, z]) = z$. Therefore, $[g, x] = [1, [g] \cdot x] = [g, e_{g^{-1}} \cdot x]$. Thus, $[1, x] = [1, e_{g^{-1}} \cdot x]$. Applying τ again we obtain $x = e_{g^{-1}} \cdot x$. Then, $x \in M_{g^{-1}}$ and $\alpha_g(x) = [g] \cdot x = z$. Hence, by Proposition 2.17, Θ is a globalization of $\alpha : G \curvearrowright M$. $\square$

Note that the fact that a partial group action is determined by a partial representation does not imply uniqueness of globalization. In fact, the partial action defined in Example 2.10 arises from a partial representation.

Example 2.20. We will see in Lemma 3.20 and Remark 3.21 that the universal globalization of the partial group algebra $K_{par}G$ is isomorphic as KG-$K_{par}G$-bimodule to $KG \otimes \mathcal{B}$, where $KG \otimes \mathcal{B}$ is a KG-$K_{par}G$-module such that

$$g \cdot (h \otimes w) = gh \otimes w \text{ and } (h \otimes w) \cdot [g] = hg \otimes (w \triangleleft [g]).$$

Furthermore, $KG \otimes \mathcal{B}$ have a structure of algebra, with the product given by

$$(g \otimes w) * (h \otimes u) := gh \otimes (w \triangleleft [h])u.$$

One can verify that $[1, 1] * KG \otimes \mathcal{B} \cong \mathcal{B} \rtimes G \cong K_{par}G$.

3. Homology of partial group actions

3.1. **Homology of set-theoretical partial group actions.** In this section, we propose a homology theory for partial group actions based on their associated simplicial structure. We show that this homology theory aligns with the homology theory for small categories (see [19]) of the groupoid associated to the partial group action, as well as with the homology theory based on partial representation (see [1] or [2]). Furthermore, we show that the (co)homology of a partial group action coincides with the (co)homology of its globalization.

Let G be a group, X be a set and $\theta : G \curvearrowright X$ a partial group action. We define the simplicial set $C_\bullet^{par}(G, X) := (C_\bullet^{par}(G, X), d_i, s_i)$ associated to θ as follows:

(i) $C_0^{par}(G, X) := X$

(ii) $C_n^{par}(G, X) := \{(g_1, g_2, \ldots, g_n, x) : x \in \operatorname{dom} \theta_{g_1} \theta_{g_2} \ldots \theta_{g_n}\}$

For $n \geq 1$ we define the face maps $d_i : C_n^{par}(G, X) \to C_{n-1}^{par}(G, X)$ by

$$(3.1) \qquad d_i(g_1, g_2, \ldots, g_n, x) := \begin{cases} (g_2, \ldots, g_n, x) & \text{if } i = 0; \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n, x) & \text{if } 0 < i < n; \\ (g_1, \ldots, g_{n-1}, \theta_{g_n}(x)) & \text{if } i = n. \end{cases}$$

For $n \geq 0$ we set the degeneracy maps $s_i : C_n^{par}(G, X) \to C_{n+1}^{par}(G, X)$

$$(3.2) \qquad s_i(g_1, \ldots, g_n, x) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n, x) \text{ for } 1 \leq i \leq n.$$

Through direct computations, one can verify that $C_\bullet^{par}(G, X)$ is indeed a simplicial set. Therefore, we can define the homology of a partial group action based on the associated simplicial structure that it determines.

Definition 3.1. Let $\theta : G \curvearrowright X$ a partial group action on X. We define the **homology of the partial group action** θ as the homology of the simplicial set $C_\bullet^{par}(G, X)$. Hence, we set $H_\bullet^{par}(G, X) := H_\bullet(C_\bullet^{par}(G, X))$. Dually, we define the **cohomology of the partial group action** θ as the cohomology of the simplicial set $C_\bullet^{par}(G, X)$, we set $\mathscr{H}_{par}^\bullet(G, X) := H^\bullet(C_\bullet^{par}(G, X))$.

Remark 3.2. Let $\theta : G \curvearrowright X$ be a partial group action. We will see that $H_\bullet^{par}(G, X)$ is isomorphic to the partial group homology $H_\bullet^{par}(G, KX) = \operatorname{Tor}_\bullet^{K_{par}G}(\mathcal{B}, KX)$, where KX is a left $K_{par}G$-module as in Proposition 1.17. Dually, we have that $\mathscr{H}_{par}^\bullet(G, X) \cong H_{par}^\bullet(G, (KX)^*) = \operatorname{Ext}_{K_{par}G}^\bullet(\mathcal{B}, (KX)^*)$, where $(KX)^\bullet$ is the right $K_{par}G$-module $\hom_K(KX, K)$. This justifies our use of the notation $\mathscr{H}_{par}^\bullet$ instead of a usual $H_{par}^\bullet$.

The elements of $C_n^{par}(G, X)$ are called **partial n-chains**, the cycles and boundaries of the complex determined by $C_n^{par}(G, X)$ are called **partial n-cycles** and **partial n-boundaries** respectively.

Remark 3.3. If $\theta : G \curvearrowright X$ is a global action it is immediately that $KC_\bullet^{par}(G, X)$ coincides with the canonical simplicial K-module that computes the group homology of G with coefficients in KX, see Section 1.2. Therefore, $H_\bullet^{par}(G, X) = H_\bullet(G, KX)$ and $\mathscr{H}_{par}^\bullet(G, X) = H^\bullet(G, (KX)^*)$.

Proposition 3.4. *Let $\theta : G \curvearrowright X$ be a partial group action, and let $\Gamma \rightrightarrows X$ be associated partial group action groupoid. Then, the simplicial set $C_\bullet^{par}(G, X)$ is isomorphic to the nerve of Γ.*

Proof. Recall that $\Gamma \rightrightarrows X$ is the groupoid whose set of objects is X, and whose set of morphisms is $\{(y, g, x) : g \in G, x \in X_{g^{-1}} \text{ and } y = \theta_g(x)\}$. Through direct computations, one can verify that the mapping

$$(g_1, \ldots, g_n, x) \mapsto \big((y_1, g_1, y_2), \ldots, (y_i, g_i, y_{i+1}), \ldots, (y_{n-1}, g_{n-1}, y_n), (y_n, g_n, x)\big),$$

where $y_n := \theta_{g_n}(x)$ and $y_i := \theta_{g_i}(y_{i+1})$ for all $1 \leq i \leq n - 1$, is an isomorphism of simplicial sets. $\qquad \square$

Corollary 3.5. *Let $\alpha : G \curvearrowright X$ and $\beta : S \curvearrowright Y$ be partial group actions such that the associated groupoids are equivalent (as categories). Then, $H_\bullet^{par}(G, X) \cong H_\bullet^{par}(S, Y)$ and $\mathscr{H}_{par}^\bullet(G, X) \cong \mathscr{H}_{par}^\bullet(S, Y)$.*

Proof. By Proposition 3.4 we only have to verify that the (co)homology of the associated groupoids are isomorphic. However, this follows as a consequence of the equivalence of the groupoids (see for example [4, Theorem 1.11]). $\qquad \square$

Let $\alpha : G \curvearrowright X$ be a partial group action, recall that the globalization of α is a global action $\theta : G \curvearrowright Z$ together with an injective G-equivariant map $\iota : X \to Z$ such that $Z = \bigcup_{g \in G} \theta_g(\iota(X))$ and α is isomorphic to the restriction of θ to $\iota(X)$, i.e., if we identify X with $\iota(X)$, then we have that

(i) $X_g := \theta_g(X) \cap X$,
(ii) $\alpha_g := \theta_g|_{X_{g^{-1}}}$.

Furthermore, the globalization always exists and is unique (see [11, Theorem 3.5]).

Proposition 3.6. *Consider a partial group action $\alpha : G \curvearrowright X$, and let $\theta : G \curvearrowright Z$ together with $\iota : X \to Z$ be the globalization of α. Let $\Gamma^\alpha \rightrightarrows X$ and $\Gamma^\theta \rightrightarrows Z$ be the associated partial action groupoids of α and θ respectively. Then, ι determines a full faithful essentially surjective functor from Γ^α to Γ^θ.*

Proof. Define the functor $I : \Gamma^\alpha \to \Gamma^\theta$ given by:

• $I(x) := \iota(x)$ for all $x \in X$,
• $I(y, g, x) := (I(y), g, I(x))$.

Let $x, y \in X$, then $\hom_{\Gamma^\alpha}(x, y) = \{(y, g, x) : g \in G,\ x \in X_g^{-1}$ and $\alpha_g(x) = y\}$ and

$$\hom_{\Gamma^\theta}(I(x), I(y)) = \{(I(y), g, I(x)) : g \in G,\ x \in X_g^{-1} \text{ and } \theta_g(I(x)) = I(y)\}$$
$$= \{(\iota(y), g, \iota(x)) : g \in G,\ x \in X_g^{-1} \text{ and } \alpha_g(\iota(x)) = \iota(y)\}.$$

Since (θ, ι) is the globalization of α is clear that I is full. As ι is injective, we can conclude that I is faithful. Finally, I is essentially surjective due to the fact that $Z = \bigcup_{g \in G} \theta_g(\iota(X))$. $\square$

We recall that a consequence of the Axiom of Choice is that full, faithful, essentially surjective functors determine equivalence of categories (see for example [15, Proposition 1.4]), then Proposition 3.6 implies that the associated groupoids of a partial group action and its globalization are equivalent. Hence, by Corollary 3.5 and Remark 3.3 we obtain the following proposition.

Proposition 3.7. *Let $\alpha : G \curvearrowright X$ be a partial group action, and let $\theta : G \curvearrowright Z$ be the globalization of α. Then, $H_\bullet^{par}(G, X) \cong H_\bullet(G, KZ)$ and $\mathscr{H}_{par}^\bullet(G, X) \cong H^\bullet(G, (KZ)^*)$.*

3.2. **Comparing the partial group cohomology theories.** During the development of a *proper* cohomology theory for partial group actions, two seemingly distinct theories emerged: one based on unital partial group actions on semigroups (see [7]), and the other based on partial representations (see [1]). Despite sharing numerous similarities, these theories lack an explicit means of interconnection. Our approach to a cohomology theory for partial group actions from simplicial sets enables us to establish the connection between both theories. Specifically, we show that both theories correspond to the simplicial cohomology of the nerve of the group G with coefficients in a suitable cohomological coefficient system.

In Example 1.31 if we replace the arbitrary groupoid for a group G we obtain the *nerve* $B_\bullet G$ of the group G, where $B_n G = G^n$. For $n \geq 1$, the face maps $d_i : B_n G \to B_{n-1} G$ are given by:

(3.3) $$d_i(g_1, g_2, \ldots, g_n, x) := \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0; \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n; \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$

For $n \geq 0$, the degeneracy maps $s_i : B_n G \to B_{n+1} G$:

(3.4) $$s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n) \text{ for } 1 \leq i \leq n.$$

Definition 3.8 ([9])**.** Let A be a commutative semigroup. A unital partial action of G on A is a set-theoretical partial action $\theta : G \curvearrowright A$ such that each A_g is an unital ideal A, i.e., A_g is generated as ideal by an idempotent element 1_g, and each map $\theta_g : A_{g^{-1}} \to A_g$ is an isomorphism of semigroups.

Let $\theta : G \curvearrowright A$ be a unital partial group action on the commutative semigroup A. Define:

($\mathscr{A}1$) For $x \in B_n G$, $x = (g_1, g_2, \ldots, g_n)$, we set

$$\mathscr{A}_x := \{\text{invertible elements of the ideal } A_{g_1} A_{g_1 g_2} \ldots A_{g_1 g_2 \ldots g_n}\};$$

($\mathscr{A}2$) for a face map $\delta_i : \mathbf{n} \to \mathbf{n+1}$ and a n-simplex $x \in B_{n+1}G$, $x = (g_1, \ldots, g_{n+1})$, we define $\mathscr{A}(\delta_i, x) : \mathscr{A}_{d_i(x)} \to \mathscr{A}_x$ such that

$$\mathscr{A}(\delta_i, x)(a) := \begin{cases} \theta_{g_1}(1_{g_1^{-1}}a) & \text{if } i = 0; \\ 1_x a & \text{if } 0 < i \leq n+1, \end{cases}$$

for all $a \in \mathscr{A}_{d_i(x)}$, where $1_x := 1_{g_1} 1_{g_1 g_2} \ldots 1_{g_1 g_2 \ldots g_{n+1}}$ is the identity element of $\mathscr{A}_x$.

($\mathscr{A}3$) for a degeneracy $\sigma_i : \mathbf{n+1} \to \mathbf{n}$ we set $\mathscr{A}(\sigma_i, x) : \mathscr{A}_{s_i(x)} \to \mathscr{A}_x$ by

$$\mathscr{A}(\sigma_i, x)(a) := a.$$

Remark 3.9. Observe that the axioms in Definition 3.8 implies that for all $x = (g_1, g_2, \ldots, g_n) \in B_n G$ and $a \in \mathscr{A}_{g_2, \ldots, g_n}$ we have that $\mathscr{A}(\delta_0, x)(x) = \theta_{g_1}(1_{g_1^{-1}}a) \in \mathscr{A}_{(g_1, \ldots, g_n)}$, and $\theta_{g_1}(1_{g_1^{-1}} 1_{d_0(x)}) = 1_x$.

We need to verify that the maps in ($\mathscr{A}2$) and ($\mathscr{A}3$) are well-defined. Let $x = (g_1, \ldots, g_{n+1})$. Suppose $i = 0$. Then $d_0(x) = (g_2, \ldots, g_{n+1})$. By Remark 3.9 we conclude that $\mathscr{A}(\delta_0, x)$ is well-defined for all $x \in X_n$. For $i \geq 1$, let $a \in \mathscr{A}_{d_i(x)}$, then a is an invertible element of the ideal generated by $1_{d_i(x)}$, since $1_x 1_{d_i(x)} = 1_x$ we have that $1_x a \in \mathscr{A}_x$. Thus, the maps in ($\mathscr{A}2$) are well-defined. For ($\mathscr{A}3$) note that $\mathscr{A}_{s_i(x)} = \mathscr{A}_x$ for all $x \in X_n$ and $0 \leq i \leq n$, hence it is clear that $\mathscr{A}(\sigma_i, x)$ is well-defined.

Proposition 3.10. *Let $\theta : G \curvearrowright A$ be a unital partial group action on the semigroup A. Then, the set $\{\mathscr{A}_x\}$ together with the morphisms $\mathscr{A}(\delta_i, x)$ and $\mathscr{A}(\sigma_i, x)$ determines a cohomological coefficient system on the simplicial set $B_\bullet G$. Furthermore, the cohomology of the simplicial set $B_\bullet G$ with coefficient in $\mathscr{A}$ is the partial group cohomology of G with coefficient in the **partial G-module** A, as defined in [7].*

Proof. To verify that $\mathscr{A}$ is a cohomological coefficient system on $B_\bullet G$, we only have to prove that the maps $\delta_{i,x} := \mathscr{A}(\delta_i, x)$ and $\sigma_{j,x} := \mathscr{A}(\sigma_j, x)$ satisfy the relations (1.12). Let $x = (g_1, \ldots, g_n)$ For $0 < i < j$

$$\delta_{j,x}\delta_{i,d_j(x)}(a) = \delta_{j,x}(1_{d_j(x)}a) = 1_x 1_{d_j(x)}a = 1_x a = 1_x 1_{d_{j-1}(x)}a = \delta_{i,x}\delta_{j-1 d_i(x)}(a).$$

For $i = 0$ and $j = 1$

$$\begin{aligned} \delta_{1,x}\delta_{0,d_1(x)}(a) &= 1_x \theta_{g_1 g_2}(1_{g_2^{-1} g_1^{-1}}a) \\ &= 1_x 1_{g_1} 1_{g_1 g_2} \theta_{g_1 g_2}(1_{g_2^{-1} g_1^{-1}}a) = 1_x \theta_{g_1 g_2}(1_{g_2^{-1}} 1_{g_2^{-1} g_1^{-1}}a) \\ &= 1_x \theta_{g_1} \theta_{g_2}(1_{g_2^{-1}} 1_{g_2^{-1} g_1^{-1}}a) = 1_x \theta_{g_1}(1_{g_1^{-1}} \theta_{g_2}(1_{g_2^{-1}}a)) \\ (\flat) &= \theta_{g_1}(1_{g_1^{-1}} \theta_{g_2}(1_{g_2^{-1}}a)) \\ &= \delta_{0,x}\delta_{0,d_0(x)}(a), \end{aligned}$$

the equality $(\flat)$ holds since $a \in \mathscr{A}_{d_0 d_1(x)} = \mathscr{A}_{(g_3,\ldots,g_n)}$, and therefore $\theta_{g_1}(1_{g_1^{-1}}\theta_{g_2}(1_{g_2^{-1}}a)) \in \mathscr{A}_x$. For $i = 0$ and $j > 1$

$$\delta_{j,x}\delta_{0,d_j(x)}(a) = 1_x\theta_{g_1}(1_{g^{-1}}a)$$
$$(\text{By Remark 3.9}) = \theta_{g_1}(1_{g^{-1}}1_{d_0(x)})\theta_{g_1}(1_{g^{-1}}a)$$
$$= \theta_{g_1}(1_{g^{-1}}1_{d_0(x)}a)$$
$$= \delta_{0,x}(1_{d_0(x)}a)$$
$$= \delta_{0,x}\delta_{j-1,d_0(x)}(a).$$

Thus, we have verified the first relation of (1.12). The second relation is easy to see from $(\mathscr{A}3)$ in the construction of $\mathscr{A}$. For the third relation, note that if $0 = i < j$, then

$$\sigma_{j,x}\delta_{0,s_j(x)}(a) = \theta_{g_1}(1_{g_1^{-1}}a) = \delta_{0,x}\sigma_{j-1,d_0(x)}(a),$$

If $0 = i = j$, it is immediate that

$$\sigma_{0,x}\delta_{0,s_0(x)}(a) = \sigma_{0,x}(\theta_1(1a)) = a.$$

Observe that if $i > 0$, then

$$\sigma_{j,x}\delta_{i,s_j(x)}(a) = 1_{s_j(x)}a = 1_x a,$$

for all $j \in \{0,1,\ldots,n\}$. Recall that $a \in \mathscr{A}_{d_i s_j(x)}$, thus if $i = j$ or $i = j + 1$, then $a \in \mathscr{A}_x$. Therefore, $1_x a = a$. Thus, $\sigma_{j,x}\delta_{i,s_j(x)} = 1_{\mathscr{A}_x}$. If $i < j$ then

$$\delta_{i,x}\sigma_{j-1,d_i(x)}(a) = \delta_{i,x}(a) = 1_x a,$$

or if $i > j + 1$

$$\delta_{i-1,x}\sigma_{j,d_{i-1}(x)}(a) = \delta_{i-1,x}(a) = 1_x a.$$

Hence, the third relation holds. Then, by Remark 1.34 we conclude that $\mathscr{A}$ is a cohomological coefficient system on $B_\bullet G$. Thus, the n-cochains are

(3.5) $$C^n(B_\bullet G, \mathscr{A}) := \{f : G^n \to A : f(x) \in \mathscr{A}_x\}.$$

The differential $\partial^n : C^n(B_\bullet G, \mathscr{A}) \to C^{n+1}(B_\bullet G, \mathscr{A})$ written in product notation takes the form

$$\partial^n(f)(x) := \theta_{g_1}(1_{g_1^{-1}}f(g_2,\ldots,g_{n+1}))$$
$$\prod_{i=1}^{n+1} f(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1})^{(-1)^i}$$
$$f(g_1,\ldots,g_{n+1}).$$

Hence, we have $C^\bullet(B_\bullet G, \mathscr{A}) = C^\bullet(G, A)$, wherein the latter complex is the complex defined in [7] for the purpose of define the partial group cohomology of G with coefficients in A. $\square$

Let $\pi : G \to \mathrm{End}_K(M)$ be a partial representation. Define:

(i) For $x \in B_n G$, $x = (g_1, g_2, \ldots, g_n)$, we set

$$\mathscr{M}_x := \mathrm{im}\,\pi_{g_1}\pi_{g_2}\ldots\pi_{g_n} = e_{g_1}e_{g_1 g_2}\ldots e_{g_1 g_2 \ldots g_n} \cdot M;$$

(ii) for a face map $\delta_i : \mathbf{n} \to \mathbf{n+1}$ and a n-simplex $x \in B_{n+1}G$, $x = (g_1, \ldots, g_{n+1})$, we define $\mathscr{M}(\delta_i, x) : \mathscr{M}_{d_i(x)} \to \mathscr{M}_x$ such that

$$\mathscr{M}(\delta_i, x)(m) := \begin{cases} \pi_{g_1}(m) & \text{if } i = 0; \\ e_{g_1} e_{g_1 g_2} \ldots e_{g_1 \ldots g_{n+1}} \cdot m & \text{if } 0 < i \le n+1, \end{cases}$$

for all $m \in \mathscr{M}_{d_i(x)}$.

(iii) for a degeneracy $\sigma_i : \mathbf{n+1} \to \mathbf{n}$ we define

$$\mathscr{M}(\sigma_i, x)(m) := m.$$

Analogously to Proposition 3.10 on can prove the following proposition.

Proposition 3.11. *Let $\pi : G \to \mathrm{End}_K(M)$ be a partial representation. Then, the set $\{\mathscr{M}_x\}$ together with the morphisms $\mathscr{M}(\delta_i, x)$ and $\mathscr{M}(\sigma_i, x)$ determines a cohomological coefficient system on the simplicial set $B_\bullet G$. The complex $C^\bullet(B_\bullet, \mathscr{M})$ is isomorphic to the complex $C^\bullet_{par}(G, M)$ defined in [8, Definition 2.15]. Therefore, the cohomology of the simplicial set $B_\bullet G$ with coefficient in $\mathscr{M}$ is the partial group cohomology of G with coefficient in the left $K_{par}G$-module M, as defined in [1].*

Dually, in reference to Proposition 3.11, we can obtain the partial group homology of G with coefficients in a left $K_{par}G$-module M as the simplicial homology of $B_\bullet G$ with coefficients in a suitable homological coefficient system arising from M. Consequently, Proposition 3.10, Proposition 3.11, along with this commentary, highlight that the nerve $B_\bullet G$ of the group G does not merely serve a central role in the *global* (co)homology group theory but also in the (co)homology theory for partial group actions.

3.3. (Co)Homology of partial actions on modules.

The construction of the simplicial set obtained from a set-theoretical partial group action can be extended to partial group actions on modules. Utilizing such a simplicial K-module, we can define a homology theory for partial group actions on modules.

Definition 3.12. Let $\theta := \big(G, M, \{M_g\}, \{\theta_g\}\big)$ be a partial action of G on the K-module M, we define the **partial n-chains** $C_n^{par}(G, M)$ as the K-submodule of $KG^{\otimes n} \otimes_K M$ generated by the set of elements

$$\{g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m : g_1, \ldots, g_n \in G \text{ and } m \in \mathrm{dom}\, \theta_{g_1} \theta_{g_2} \ldots \theta_{g_n}\}$$

For $n \ge 1$ we define the face maps $d_i : C_n^{par}(G, M) \to C_{n-1}^{par}(G, M)$ by

$$(3.6) \qquad d_i(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m) := \begin{cases} g_2 \otimes \ldots \otimes g_n \otimes m & \text{if } i = 0; \\ g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes g_n \otimes m & \text{if } 0 < i < n; \\ g_1 \otimes \ldots \otimes g_{n-1} \otimes \theta_{g_n}(m) & \text{if } i = n. \end{cases}$$

We set the degeneracy maps $s_i : C_n^{par}(G, M) \to C_{n+1}^{par}(G, M)$ by

$$(3.7) \qquad s_i(g_1 \otimes \ldots \otimes g_n \otimes m) = g_1 \otimes \ldots \otimes g_i \otimes 1 \otimes g_{i+1} \otimes \ldots \otimes g_n \otimes m.$$

Proposition 3.13. *$(C_\bullet^{par}(G, M), d_i, s_i)$ is a simplicial K-module.*

Proof. Direct computations. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Let $\alpha = (G, M, \{M_g\}, \{\alpha_g\})$ and $\beta = (H, N, \{N_h\}, \{\beta_h\})$ be partial group actions and $\phi := (f, \varphi) : \alpha \to \beta$ a morphism of partial actions, then ϕ induces a morphism $\phi_\bullet : C_\bullet^{par}(G, M) \to C_\bullet^{par}(H, N)$ of chain-complexes. Indeed, define $\phi_n : C_n^{par}(G, M) \to C_n^{par}(H, N)$ such that

$$(3.8) \qquad \phi_n(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m) = \varphi(g_1) \otimes \varphi(g_2) \otimes \ldots \otimes \varphi(g_n) \otimes f(m).$$

Note that $f(m) \in \mathrm{dom}\, \beta_{\varphi(g_1)}\beta_{\varphi(g_2)}\ldots\beta_{\varphi(g_n)}$, since $f(M_g) \subseteq N_{\varphi(g)}$ and $f(\alpha_g(m)) = \beta_{\varphi(g)}(f(t))$ for all $g \in G$ and $t \in M_{g^{-1}}$. Then, $\phi_n : C_n^{par}(G, M) \to C_n^{par}(H, N)$ is a well-defined K-linear map. Moreover, by direct computations one verifies that $\phi_\bullet$ commutes with the face maps and degeneracy maps. Therefore, we obtain a functor $\mathcal{C} : \textbf{PA-Mod} \to K\textbf{-Mod}^{\Delta^{op}}$ such that for any partial group action $\theta : G \curvearrowright M$ we have $\mathcal{C}(\theta) := C^{par}(G, M)$ and $\mathcal{C}(\phi) := \phi_\bullet$ as in Equation (3.8).

Observe that if $\theta : G \curvearrowright X$ is a set-theoretical then the simplicial module $KC_\bullet^{par}(G, X)$ is isomorphic to the simplicial module $C_\bullet^{par}(G, KX)$, where the partial action associated to KX is that of Proposition 1.17.

Definition 3.14. Let $\theta : G \curvearrowright M$ be a partial action of G on the K-module M. We define the homology groups of the partial group action θ by

$$H_\bullet^{par}(G, M) := H_\bullet(C_\bullet^{par}(G, M)).$$

Dually, we set the partial group cohomology as

$$\mathscr{H}_{par}^\bullet(G, M) := H^\bullet(C_\bullet^{par}(G, M)).$$

In Definition 3.14, we use the same notation as that used to define the partial homology groups of G with coefficients in a $K_{par}G$-module M. In fact, the following proposition shows that if $\theta : G \curvearrowright M$ is the partial group action induced by a partial representation $\pi : G \to \mathrm{End}_K(M)$, we have that both homology theories are, in fact, the same.

Proposition 3.15. *Let $\pi : G \to \mathrm{End}_K(M)$ be a partial group representation, and θ the respective induced partial group action of G on M. Let $C'_\bullet \to \mathcal{B}$ be the projective resolution of right $K_{par}G$-modules defined in Section 1.2. Then, $C'_\bullet \otimes_{K_{par}G} M \cong C_\bullet^{par}(G, M)$ as chain-complexes. Consequently, the partial group homology of G with coefficients in the partial representation (M, π) and the homology of the partial group action of G on M induced by π are the same.*

Proof. Let $\zeta : K_{par}G \to KG$ be such that $\zeta([g]) := g$. Observe that $K_{par}G^{\otimes_{\mathcal{B}} n+1} \otimes_{K_{par}G} M \cong K_{par}G^{\otimes_{\mathcal{B}} n} \otimes_{\mathcal{B}} M$. Define $\psi_n : K_{par}G^{\otimes_{\mathcal{B}} n} \otimes_{\mathcal{B}} M \to C_n^{par}(G, M)$ such that

$$\psi_n(z_1 \otimes_{\mathcal{B}} z_2 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} z_n \otimes_{\mathcal{B}} m) := \zeta(z_1) \otimes \zeta(z_2) \otimes \ldots \otimes \zeta(z_n) \otimes \varepsilon(z_1 z_2 \ldots z_n) \cdot m,$$

for all $z_i \in K_{par}G$ and $m \in M$. In particular, we have that

$$\psi([g_1] \otimes_{\mathcal{B}} [g_2] \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} [g_n] \otimes_{\mathcal{B}} m) := g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes \varepsilon([g_1][g_2]\ldots[g_n]) \cdot m,$$

since by Lemma 1.25 we know that $\mathrm{dom}\, \theta_{g_1}\theta_{g_2}\ldots\theta_{g_n} := \varepsilon([g_1][g_2]\ldots[g_n]) \cdot M$, thus is a well-defined K-linear morphism. By brute force computations, one can verify that $\psi_\bullet$ is a morphism of simplicial K-modules. Finally, note that the map

$$C_n^{par}(G, M) \ni g_1 \otimes g_2 \otimes \ldots g_n \otimes m \mapsto [g_1] \otimes_{\mathcal{B}} [g_2] \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} [g_n] \otimes_{\mathcal{B}} m \in K_{par}G^{\otimes_{\mathcal{B}} n} \otimes_{\mathcal{B}} M,$$

is the inverse map of ψ_n for all $n \in \mathbb{N}$. $\qquad\square$

Corollary 3.16. *Let $\pi : G \to \mathrm{End}_K(M)$ be a partial group representation. Then, $\mathscr{H}_{par}^\bullet(G, M) \cong H_{par}^\bullet(G, M^*)$.*

Proof. By Proposition 3.15 cochain-complexes $\hom_K(C'_\bullet \otimes_{K_{par}G} M, K)$ and $\hom_K(C_\bullet^{par}(G, M), K)$ are isomorphic. On the other hand, by the tensor-hom adjunction we obtain

$$H^\bullet(\hom_K(C'_\bullet \otimes_{K_{par}G} M, K)) = H^\bullet(\hom_{K_{par}G}(C'_\bullet, M^*)) = \mathrm{Ext}_{K_{par}G}^\bullet(\mathcal{B}, M^*) = H_{par}^\bullet(\mathcal{B}, M^*).$$

$\qquad\square$

3.4. **Globalization problem for partial group homology.** In Proposition 3.7 we proved that the partial group homology coincides with the group homology of the globalization of the partial group action. Such a result suggests a strong homological relation between partial group actions on modules and their globalization. In this section we prove that the partial group homology of a $K_{par}G$-module M coincides with the group homology of the universal globalization $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$.

Recall that in Example 1.14 we obtained a functor $\Lambda : K_{par}G$-**Mod** $\to G_{par}$-**Mod**. Thus, by Theorem 2.19, when we compose Λ with the functor $KG \otimes_{G_{par}} - : G_{par}$-**Mod** $\to G$-**Mod** (as outlined in Remark 2.3), we obtain a functor $KG \otimes_{G_{par}} - : K_{par}G$-**Mod** $\to G$-**Mod** that maps a $K_{par}G$-module M to the universal globalization $\Theta : G \curvearrowright KG \otimes_{G_{par}} M$ of the G_{par}-module M. We refer to this functor as the **globalization functor**.

Lemma 3.17. *Let X be a left $K_{par}G$-module, suppose that $\{z_i\}_{i=0}^n \subseteq X$ and $\{g_i\}_{i=1}^n \subseteq G$ are such that*

$$1 \otimes_{G_{par}} z_0 + \sum_{i=1}^n g_i \otimes_{G_{par}} z_i = 0.$$

Then, $z_0 = -\sum_{i=1}^m [g_i] \cdot z_i$.

Proof. Note that, by Remark 2.18, we have

$$0 = \tau\left(1 \otimes_{G_{par}} z_0 + \sum_{i=1}^n g_i \otimes_{G_{par}} z_i\right) = z_0 + \sum_{i=1}^n [g_i] \cdot z_i,$$

whence we obtain the desired conclusion. $\qquad\square$

Theorem 3.18. *The globalization functor $KG \otimes_{G_{par}} - : K_{par}G$-**Mod** $\to G$-**Mod** is exact.*

Proof. By Proposition 2.5 we know that $KG \otimes_{G_{par}} -$ is right exact, thus we only have to prove that $KG \otimes_{G_{par}} -$ preserve injective morphisms. Let X and Y be left $K_{par}G$-modules, $\psi : X \to Y$ an injective map of $K_{par}G$-modules, and let $\psi_* : KG \otimes_{G_{par}} X \to KG \otimes_{G_{par}} Y$ be the morphism of G-modules determined by ψ. It is worth to note that ψ_* is given by $\psi_*([g,x]) = [g, \psi(x)]$. Recall that for any $z \in KG \otimes_{G_{par}} X$ there exists $\{g_i\}_{i=0}^n \subseteq G$ and $\{x_i\}_{i=0}^n \subseteq X$ such that $z = \sum_{i=0}^n [g_i, x_i]$. Then, the injectivity of ψ_* is equivalent to:

(♭) For all $n \in \mathbb{N}$, $\{g_i\}_{i=0}^n \subseteq G$ and $\{x_i\}_{i=0}^n \subseteq X$ such that $\sum_{i=0}^n [g_i, \psi(x_i)] = 0$ we have that $\sum_{i=0}^n [g_i, x_i] = 0$.

We proceed to prove (♭) with an induction argument over $n \in \mathbb{N}$. For $n = 0$, note that if there exists $x \in X$ and $g \in G$ such that $[g, \psi(x)] = 0$, then $[1, \psi(x)] = 0$. Since $\iota : Y \to KG \otimes_{G_{par}} Y$ is injective, we have $\psi(x) = 0$, consequently $x = 0$ and $[g, x] = 0$, this give us the base of the induction. Let $n \in \mathbb{N}$, suppose that for all $\{h_i\}_{i=0}^n \subseteq G$ and $\{y_i\}_{i=0}^n \subseteq X$ such that $\psi_*\left(\sum_{i=0}^n [h_i, y_i]\right) = \sum_{i=0}^n [h_i, \psi(y_i)] = 0$ we have that $\sum_{i=0}^n [h_i, y_i] = 0$. Let $z = \sum_{i=0}^{n+1} [g_i, x_i]$ be such that $\psi_*(z) = 0$, then

$$0 = g_{n+1}^{-1} \cdot \psi_*(z) = [1, \psi(x_{n+1})] + \sum_{i=0}^n [g_{n+1}^{-1} g_i, \psi(x_i)].$$

Then, by Lemma 3.17 we have that

$$\psi(x_{n+1}) = -\sum_{i=0}^n [g_{n+1}^{-1} g_i] \cdot \psi(x_i) = \psi\left(-\sum_{i=0}^n [g_{n+1}^{-1} g_i] \cdot x_i\right).$$

Since ψ is injective we conclude that $x_{n+1} = -\sum_{i=0}^{n}[g_{n+1}^{-1}g_i]\cdot x_i$. Therefore,

$$[1, x_{n+1}] = -\sum_{i=0}^{n}\Big[1,\, [g_{n+1}^{-1}g_i]\cdot x_i\Big]$$

$$= -\sum_{i=0}^{n}\Big[g_{n+1}^{-1}g_i,\, e_{g_i^{-1}g_{n+1}}\cdot x_i\Big].$$

Then,

$$g_{n+1}^{-1}\cdot z = [1, x_{n+1}] + \sum_{i=0}^{n}\Big[g_{n+1}^{-1}g_i,\, x_i\Big] = \sum_{i=0}^{n}\Big[g_{n+1}^{-1}g_i,\, x_i - e_{g_i^{-1}g_{n+1}}\cdot x_i\Big].$$

Observe that $\psi_*(g_{n+1}^{-1}\cdot z) = g_{n+1}^{-1}\cdot\psi_*(z) = 0$, then by the induction hypotheses we have that $g_{n+1}^{-1}\cdot z = 0$, which implies $z = 0$. Therefore, ψ_* is injective. $\square$

Corollary 3.19. $KG\otimes_{G_{par}} K_{par}G$ is flat as right $K_{par}G$-module.

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of left $K_{par}G$-modules. By Proposition 2.6, the complex

$$0 \to KG\otimes_{G_{par}} K_{par}G\otimes_{K_{par}G} A \to KG\otimes_{G_{par}} K_{par}G\otimes_{K_{par}G} B \to KG\otimes_{G_{par}} K_{par}G\otimes_{K_{par}G} C \to 0$$

is isomorphic to

$$0 \to KG\otimes_{G_{par}} A \to KG\otimes_{G_{par}} B \to KG\otimes_{G_{par}} C \to 0,$$

and, by Theorem 3.18, this complex is exact. $\square$

Lemma 3.20. *The modules $KG\otimes_{G_{par}} K_{par}G$ and $KG\otimes\mathcal{B}$ are isomorphic as left KG-modules. In particular, $KG\otimes_{G_{par}} K_{par}G$ is free as left KG-module.*

Proof. Recall that $K_{par}G$ is the K-algebra generated by the Exel's semigroup $\mathcal{S}(G)$ of G and any element $z \in \mathcal{S}(G)$ has a unique representation $z = [g]u$, where $g \in G$ and u is an idempotent element of $\mathcal{S}(G)$ (see [11] for more details). Thus, we can define a K-linear map $f_0 : KG\otimes K_{par}G \to KG\otimes\mathcal{B}$ such that $f_0(g\otimes[h]u) = gh\otimes e_{h^{-1}}u$. Moreover, note that by the definition of f_0 we have that $f_0(gh\otimes e_{h^{-1}}u) = f_0(g\otimes[h]u)$. Therefore by Proposition 2.1 there exists a K-linear map $f : KG\otimes_{G_{par}} K_{par}G \to KG\otimes\mathcal{B}$ such that $f(g\otimes_{G_{par}}[h]u) = gh\otimes e_{h^{-1}}u$, it is clear that this function is also a morphism of G-modules. Conversely, recall that the idempotent elements of $\mathcal{S}(G)$ form a K-basis for $\mathcal{B}$ as K-module. Thus, there exists a K-linear map $f' : KG\otimes\mathcal{B} \to KG\otimes_{G_{par}} K_{par}G$ such that $f'(g\otimes u) = g\otimes_{G_{par}} u$. It is easy to see that $f\circ f' = 1_{KG\otimes\mathcal{B}}$, on the other hand observe that

$$f'\circ f(g\otimes_{G_{par}}[h]u) = f'(gh\otimes e_{h^{-1}}u) = gh\otimes_{G_{par}} e_{h^{-1}}u = g\otimes_{G_{par}}[h]u.$$

Thus, f' and f are mutually inverses. $\square$

Remark 3.21. Consider the isomorphism of left KG-modules $f : KG\otimes_{G_{par}} K_{par}G \to KG\otimes\mathcal{B}$ defined in the proof of Lemma 3.20. Note that the right $K_{par}G$-module structure on $KG\otimes\mathcal{B}$ induced by f is given by:

$$(g\otimes w)\cdot[h] = gh\otimes(w\triangleleft[h]),$$

with that $K_{par}G$-module structure on $KG\otimes\mathcal{B}$ in mind and considering K a trivial KG-module we obtain the following isomorphisms of right $K_{par}G$-modules

$$K\otimes_{G_{par}} K_{par}G \cong K\otimes_{KG} KG\otimes_{G_{par}} K_{par}G \overset{f_*}{\cong} K\otimes_{KG} KG\otimes\mathcal{B} \cong K\otimes\mathcal{B} \cong \mathcal{B}.$$

Explicitly, this is isomorphism is given by $r \otimes_{G_{par}} z \mapsto r\varepsilon(z)$ for all $r \in K$ and $z \in K_{par}G$. Let $\hat{\varepsilon} : KG \to K$ be such that $\hat{\varepsilon}(g) = 1$, then applying the functor $- \otimes_{KG} KG \otimes_{G_{par}} K_{par}G$ to $\hat{\varepsilon}$ we obtain a surjective morphism of right $K_{par}G$-modules $\varepsilon' : KG \otimes_{G_{par}} K_{par}G \to \mathcal{B}$ such that $\varepsilon'([g,z]) = \varepsilon(z)$.

Theorem 3.22. *Let $\pi : G \to \mathrm{End}_K(M)$ be a partial group representation. Then,*
$$H_\bullet^{par}(G, M) \cong H_\bullet(G, KG \otimes_{G_{par}} M).$$

Proof. Let $P_\bullet \to K$ be a projective resolution of K in G-**Mod**. Since $KG \otimes_{G_{par}} K_{par}G$ is free as left KG-module we obtain a resolution $P_\bullet \otimes_{KG} (KG \otimes_{G_{par}} K_{par}G) \to K \otimes_{KG} (KG \otimes_{G_{par}} K_{par}G)$. By Remark 3.21 and the fact that $KG \otimes_{G_{par}} K_{par}G$ is flat as right $K_{par}G$-module, we have that $P_\bullet \otimes_{KG} (KG \otimes_{G_{par}} K_{par}G) \to \mathcal{B}$ is a flat resolution of right $K_{par}G$-modules of $\mathcal{B}$. By [19, Lemma 3.2.8] we can compute $H_\bullet^{par}(G, M) = \mathrm{Tor}_\bullet^{K_{par}G}(\mathcal{B}, M)$ using a flat resolution of $\mathcal{B}$. Then,
$$\begin{aligned}
H_\bullet^{par}(G, M) &\cong H_\bullet(P_\bullet \otimes_{KG} (KG \otimes_{G_{par}} K_{par}G) \otimes_{K_{par}G} M) \\
&\cong H_\bullet(P_\bullet \otimes_{KG} (KG \otimes_{G_{par}} M)) \\
&\cong H_\bullet(G, KG \otimes_{G_{par}} M).
\end{aligned}$$
$\square$

The following remark is a well-known technique that can be proven using a classical argument of spectral sequences of bicomplexes or by utilizing the Acyclic Assembly Lemma [19, Lemma 2.7.3] as in the proof of [19, Theorem 2.7.2]

Remark 3.23. Let $\mathcal{A}$ be a K-algebra, and let $(X_\bullet, d) \xrightarrow{f} M$ and $(Y_\bullet, b) \xrightarrow{g} N$ be flat resolutions of right $\mathcal{A}$-modules and left $\mathcal{A}$-modules respectively. Consider the bicomplex $(B_{\bullet,\bullet}, \partial^h, \partial^v)$ such that $B_{p,q} := X_p \otimes_{\mathcal{A}} Y_q$ with horizontal differential $\partial_{p,q}^h := d_p \otimes_{\mathcal{A}} 1$ and vertical differential $\partial_{p,q}^v := (-1)^p(1 \otimes_{\mathcal{A}} b_q)$. Consider M and N as complexes concentrated in degree 0, thus f and g can be seen as chain maps of complexes. Consequently, $X_\bullet \otimes_{\mathcal{A}} N$ forms a one-row bicomplex, and $M \otimes_{\mathcal{A}} Y_\bullet$ forms a one-column bicomplex, and the maps f and g induce morphisms of bicomplexes
$$1_{X_\bullet} \otimes_{\mathcal{A}} g : B_{\bullet,\bullet} \to X_\bullet \otimes_{\mathcal{A}} N \text{ and } f \otimes_{\mathcal{A}} 1_{Y_\bullet} : B_{\bullet,\bullet} \to M \otimes_{\mathcal{A}} Y_\bullet.$$
Then, $1_{X_\bullet} \otimes_{\mathcal{A}} g$ and $f \otimes_{\mathcal{A}} 1_{Y_\bullet}$ are quasi-isomorphism of bicomplex, i.e., the maps
$$\mathrm{Tot}(1_{X_\bullet} \otimes_{\mathcal{A}} g) : \mathrm{Tot}(B) \to \mathrm{Tot}(X_\bullet \otimes_{\mathcal{A}} N) = X_\bullet \otimes_{\mathcal{A}} N$$
and
$$\mathrm{Tot}(f \otimes_{\mathcal{A}} 1_{X_\bullet}) : \mathrm{Tot}(B) \to \mathrm{Tot}(M \otimes_{\mathcal{A}} Y_\bullet) = M \otimes_{\mathcal{A}} Y_\bullet$$
are quasi-isomorphism.

Let M be a $K_{par}G$-module, and let $\theta : G \curvearrowright M$ be the associated partial group action. Then, the G-equivariant map $\iota : M \to KG \otimes_{G_{par}} M$ determines an inclusion of simplicial modules $\iota_\bullet : C_\bullet^{par}(G, M) \to C_\bullet(G, KG \otimes_{G_{par}} M)$ such that
$$(3.9) \qquad\qquad \iota_n(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m) = g_1 \otimes \ldots \otimes g_n \otimes [1, m].$$

Thus, we can view the complex of partial n-chains $C_\bullet^{par}(G, M)$ as a subcomplex of the complex of n-chains of the universal globalization of M. Therefore, a natural question arises: is $\iota_\bullet$ a quasi-isomorphism? To answer that question consider the projective resolution $C_\bullet' \xrightarrow{\varepsilon} \mathcal{B}$ of $\mathcal{B}$, and let $C_\bullet'' \xrightarrow{\hat{\varepsilon}} K$

be the standard resolution of K of right KG-modules as mentioned in Section 1.2. Let

$$\varphi : C''_n \otimes_{KG} KG \otimes_{G_{par}} K_{par}G \to KG^{\otimes n} \otimes (KG \otimes_{G_{par}} K_{par}G)$$

be the canonical isomorphism. We set $F_\bullet$ as the simplicial K-module such that $F_n := KG^{\otimes n} \otimes (KG \otimes_{G_{par}} K_{par}G)$ with the face and degeneracy maps induced by the isomorphism φ. Thus, the face maps of $F_\bullet$ have the form:

$$d_i(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes [g_{n+1}, z]) := \begin{cases} g_2 \otimes \ldots \otimes g_n \otimes [g_{n+1}, z] & \text{if } i = 0; \\ g_1 \otimes \ldots \otimes g_i g_{i+1} \otimes \ldots \otimes [g_{n+1}, z] & \text{if } 0 < i < n; \\ g_1 \otimes \ldots \otimes \ldots \otimes g_{n-1} \otimes [g_n g_{n+1}, z] & \text{if } i = n \end{cases}$$

for all $g_1, \ldots, g_n, g_{n+1} \in G$ and $z \in K_{par}G$. Now, we define $\phi_\bullet : C'_\bullet \to F_\bullet$ such that

(i) $\phi_0(1_{K_{par}G}) := 1_{KG} \otimes_{G_{par}} 1_{K_{par}G} = [1, 1]$
(ii) $\phi_n(z_1 \otimes_{\mathcal{B}} z_2 \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} z_{n+1}) = \zeta(z_1) \otimes \zeta(z_2) \otimes \ldots \otimes \zeta(z_n) \otimes \phi_0(\varepsilon(z_1 z_2 \ldots z_n) z_{n+1})$,

where $\zeta : K_{par}G \to KG$ is the natural morphism of K-algebras such that $\zeta([g]) = g$. In particular,

$$\begin{aligned} \phi_n([g_1] \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} [g_{n+1}]u) &= g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes \phi_0(\varepsilon([g_1][g_2] \ldots [g_n])[g_{n+1}]u) \\ &= g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes (g_{n+1} \otimes_{G_{par}} \varepsilon([g_1][g_2] \ldots [g_n])[g_{n+1}]u) \\ &= g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes [g_{n+1}, \varepsilon([g_1][g_2] \ldots [g_n])[g_{n+1}]u]. \end{aligned}$$

By direct computations one verifies that $\phi_\bullet$ commutes with the respective face maps of $C'_\bullet$ and $F_\bullet$. Hence, we obtain a chain-map $\phi_\bullet : C'_\bullet \to F_\bullet$. Let $\varepsilon' : KG \otimes_{G_{par}} K_{par}G \to \mathcal{B}$ such that $\varepsilon'([g, z]) = \varepsilon(z)$. By Remark 3.21 we know that $F_\bullet \xrightarrow{\varepsilon'} \mathcal{B}$ is a flat resolution of right $K_{par}G$-modules. Furthermore, we have that the following diagram commutes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_{par}G^{\otimes_{\mathcal{B}} 3} & \longrightarrow & K_{par}G^{\otimes_{\mathcal{B}} 2} & \longrightarrow & K_{par}G & \xrightarrow{\varepsilon} & \mathcal{B} \\ & & \downarrow{\phi_2} & & \downarrow{\phi_1} & & \downarrow{\phi_0} & & \| 1 \\ \cdots & \longrightarrow & KG^{\otimes 3} \otimes_{G_{par}} K_{par}G & \longrightarrow & KG^{\otimes 2} \otimes_{G_{par}} K_{par}G & \longrightarrow & KG \otimes_{G_{par}} K_{par}G & \xrightarrow{\varepsilon'} & \mathcal{B} \end{array}$$

Let $P_\bullet \xrightarrow{\eta} M$ be a projective resolution of left $K_{par}G$-modules. Consider $C'_\bullet \otimes_{K_{par}G} M$, as one-row bicomplexes concentrated in the 0-row and $\mathcal{B} \otimes_{K_{par}G} P_\bullet$ as a one-column bicomplex concentrated in the 0-column. Then, we have a commutative diagram of bicomplexes

$$\begin{array}{ccc} & \mathcal{B} \otimes_{K_{par}G} P_\bullet & \\ {}^{\varepsilon_* \otimes_{K_{par}G} 1_{P_\bullet}} \nearrow & & \nwarrow {}^{\varepsilon'_* \otimes_{K_{par}G} 1_{P_\bullet}} \\ C'_\bullet \otimes_{K_{par}G} P_\bullet & \xrightarrow{\phi_\bullet \otimes_{K_{par}G} 1_{P_\bullet}} & F_\bullet \otimes_{K_{par}G} P_\bullet \\ {}^{1_{C'_\bullet} \otimes_{K_{par}G} \eta} \downarrow & & \downarrow {}^{1_{F_\bullet} \otimes_{K_{par}G} \eta} \\ C'_\bullet \otimes_{K_{par}G} M & \xrightarrow{\phi_\bullet \otimes_{K_{par}G} 1_M} & F_\bullet \otimes_{K_{par}G} M \end{array}$$

By Remark 3.23 we know that $\varepsilon_* \otimes_{K_{par}G} 1_{P_\bullet}$, $\varepsilon'_* \otimes_{K_{par}G} 1_{P_\bullet}$, $1_{C'_\bullet} \otimes_{K_{par}G} \eta$ and $1_{F_\bullet} \otimes_{K_{par}G} \eta$ are quasi-isomorphism, consequently $\phi_\bullet \otimes_{K_{par}G} 1_{P_\bullet}$ and $\phi_\bullet \otimes_{K_{par}G} 1_M$ are quasi-isomorphism. Note that

by Proposition 3.15, we have an isomorphism $\psi : C'_\bullet \otimes_{K_{par}G} M \to C^{par}_\bullet(G, M)$. Moreover, by Proposition 2.6, the map $\hat\psi : F_\bullet \otimes_{K_{par}G} M \to C_\bullet(G, KG \otimes_{G_{par}} M)$ such that

$$\hat\psi(g_1 \otimes \ldots g_n \otimes (g \otimes_{G_{par}} z) \otimes_{K_{par}G} m) = g_1 \otimes \ldots g_n \otimes (g \otimes_{G_{par}} z \cdot m), \forall g_i \in G, z \in K_{par}G, m \in M$$

is an isomorphism of complexes. Via these isomorphisms, we obtain a commutative diagram

$$\begin{array}{ccc}
C'_\bullet \otimes_{K_{par}G} M & \xrightarrow{\phi_\bullet \otimes_{K_{par}G} 1_M} & F_\bullet \otimes_{K_{par}G} M \\
\psi \downarrow & & \downarrow \hat\psi \\
C^{par}_\bullet(G, M) & \xrightarrow{\iota_\bullet} & C_\bullet(G, KG \otimes_{G_{par}} M)
\end{array}$$

Indeed, note that

$$\hat\psi \circ (\phi_\bullet \otimes_{K_{par}G} 1_M) \circ \psi^{-1}(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m)$$
$$= \hat\psi \circ (\phi_\bullet \otimes_{K_{par}G} 1_M)([g_1] \otimes_{\mathcal{B}} \ldots \otimes_{\mathcal{B}} [g_n] \otimes_{\mathcal{B}} 1 \otimes_{K_{par}G} m)$$
$$= \hat\psi(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes (1 \otimes_{G_{par}} \varepsilon([g_1] \ldots [g_n]) \otimes_{K_{par}G} m))$$
$$= g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes (1 \otimes_{G_{par}} \varepsilon([g_1] \ldots [g_n]) \cdot m)$$
$$(\flat) = g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes [1, m]$$
$$= \iota_n(g_1 \otimes g_2 \otimes \ldots \otimes g_n \otimes m).$$

Equality $(\flat)$ holds since $m \in \operatorname{dom}\theta_{g_1} \ldots \theta_{g_n}$, where θ is the partial group action of G on M induced by the $K_{par}G$-module structure of M, and by Lemma 1.25 we know that $m \in \varepsilon([g_1] \ldots [g_n]) \cdot M$. Thus, we have that $\iota_\bullet$ is a quasi-isomorphism. Consequently, we obtain the following proposition.

Proposition 3.24. *Let $\pi : G \to \operatorname{End}_K(M)$ be a partial representation. Then, the inclusion map $\iota_\bullet : C^{par}_\bullet(G, M) \to C_\bullet(G, KG \otimes_{G_{par}} M)$ is a quasi-isomorphism.*

This provides a more technical proof of Theorem 3.22, with the advantage of offering an explicit quasi-isomorphism between the canonical complexes of the partial group homology complex of a partial representation and the group homology complex of its universal globalization.

A direct application of Theorem 3.22 give us the following two corollaries:

Corollary 3.25 (Lyndon-Hochschild-Serre spectral sequence)**.** *Let $\pi : G \to \operatorname{End}_K(M)$ be a partial representation, and let N be a normal subgroup of G. Then, there exists a spectral sequence*

$$H_p(G/N, H_q(N, KG \otimes_{G_{par}} M)) \Rightarrow H^{par}_{p+q}(G, M).$$

Proof. The spectral sequence is obtained using the Lyndon-Hochschild-Serre spectral sequence for $KG \otimes_{G_{par}} M$ and applying Theorem 3.22. $\square$

Corollary 3.26 (Shapiro's lemma)**.** *Let G be a group, let S be a subgroup of G, and let $\pi : S \to \operatorname{End}_K(M)$ be a partial representation. Then,*

$$H^{par}_\bullet(S, M) \cong H_\bullet(G, KG \otimes_{S_{par}} M).$$

Proof. Using the classical Shapiro's lemma and Theorem 3.22 we obtain the following natural isomorphisms

$$H_\bullet^{par}(S, M) \cong H_\bullet(S, KS \otimes_{S_{par}} M)$$
$$\text{(by Shapiro's lemma)} \cong H_\bullet(G, KG \otimes_{KS} (KS \otimes_{S_{par}} M))$$
$$\text{(by Proposition 2.6)} \cong H_\bullet(G, (KG \otimes_{KS} KS) \otimes_{S_{par}} M)$$
$$\cong H_\bullet(G, KG \otimes_{S_{par}} M).$$

$\square$

3.5. Globalization problem for partial group cohomology.
We now study the cohomological framework, extending the implications of Theorem 3.22 by dualizing it into a cohomological spectral sequence that converge to the partial group cohomology.

Let M be a right $K_{par}G$-module, by the tensor-hom adjunction and Remark 3.21 we obtain the following isomorphisms:

$$\hom_{K_{par}G}(\mathcal{B}, M) \cong \hom_{K_{par}G}(K \otimes_{G_{par}} K_{par}G, M)$$
$$\cong \hom_{K_{par}G}\left(K \otimes_{KG} (KG \otimes_{G_{par}} K_{par}G), M\right)$$
$$\cong \hom_{KG}\left(K, \hom_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, M)\right).$$

Recall that by Lemma 3.20 we have that $KG \otimes_{G_{par}} K_{par}G$ is free as left KG-module. Thus, if Q is an injective right $K_{par}G$-module by [16, Lemma 3.5] the right KG-module $\hom_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, Q)$ is injective. Therefore, we have that $\hom_{K_{par}G}(\mathcal{B}, -) : \mathbf{Mod\text{-}}K_{par}G \to K\text{-}\mathbf{Mod}$ is isomorphic to $\hom_{KG}(K, -) \circ \hom_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, -)$, and that the functor $\hom_{K_{par}G}, (KG \otimes_{G_{par}} K_{par}G, -)$ sends injective right $K_{par}G$-modules to injective right KG-modules. Hence, by [19, Theorem 5.8.3] we obtain the following

Proposition 3.27. *Let M be a right $K_{par}G$-module. Then, there exist a cohomological spectral sequence*

$$(3.10) \qquad E_2^{p,q} = H^p(G, \mathrm{Ext}_{K_{par}G}^q(KG \otimes_{G_{par}} K_{par}G, M)) \Rightarrow H_{par}^{p+q}(G, M).$$

To understand when the cohomological spectral sequence (3.10) collapses, we first need to establish under what conditions $KG \otimes_{G_{par}} K_{par}G$ is projective. With this purpose in mind, we will identify $KG \otimes_{G_{par}} K_{par}G$ with $KG \otimes \mathcal{B}$ as in Remark 3.21.

Lemma 3.28. *Let G be a group, for all $g \in G$ we define $\nu_g := 1 - e_g$. Then,*

(i) $\nu_g \nu_h = \nu_h \nu_g$ for all $g, h \in G$,
(ii) $[g]\nu_h = \nu_{gh}[g]$ for all $g, h \in G$,
(iii) $\nu_g[g] = [g]\nu_{g^{-1}} = 0$ for all $g \in G$,
(iv) $K_{par}G = e_g K_{par}G \oplus \nu_g K_{par}G = K_{par}Ge_g \oplus K_{par}G\nu_g$ for all $g \in G$.

Proof. Items (i), (ii), and (iii) are direct consequences of Proposition 1.4. Finally, (iv) follows from (iii) and the obvious fact $1 = e_g + \nu_g$. $\square$

Lemma 3.29. *Let $\mathcal{N}$ be the $K_{par}G$-submodule of $KG \otimes \mathcal{B}$ generated by the set $\{g \otimes \nu_g : g \in G\}$. Then, $KG \otimes \mathcal{B} \cong K_{par}G \oplus \mathcal{N}$ as right $K_{par}G$-modules.*

Proof. Consider the maps of right $K_{par}G$-modules $\psi_0 : KG \otimes \mathcal{B} \to K_{par}G$ such that $\psi_0(g \otimes u) = [g]u$ and $\phi_0 : K_{par}G \to KG \otimes \mathcal{B}$ such that $\phi_0([g]u) = (1 \otimes 1) \triangleleft [g]u = g \otimes e_{g^{-1}}u$. Observe that

$$\psi_0\phi_0([g]u) = \psi_0(g \otimes e_{g^{-1}}u) = [g]e_{g^{-1}}u = [g]u \text{ for all } g \in G \text{ and } u \in E(\mathcal{S}(G)).$$

Then, $\psi_0\phi_0 = 1_{K_{par}G}$. Note that $\operatorname{im}\phi_0$ is generated as right $K_{par}G$-submodule by the set $\{\phi_0([g]) = g \otimes e_{g^{-1}} : g \in G\}$. By Lemma 3.28 we know that $g \otimes u = g \otimes e_{g^{-1}}u + g \otimes \nu_{g^{-1}}u$ for all $g \in G$ and $u \in \mathcal{B}$, then $KG \otimes \mathcal{B} = \operatorname{im}\phi_0 + \mathcal{N}$. On the other hand, it is clear that $\mathcal{N} \subseteq \ker \psi_0$. Therefore, $\operatorname{im}\phi_0 \cap \mathcal{N} = 0$, thus $KG \otimes \mathcal{B} = \mathcal{N} \oplus \operatorname{im}\phi_0$. Furthermore, since ϕ_0 is injective we have that $K_{par}G \overset{\phi_0}{\cong} \operatorname{im}\phi_0$ as right $K_{par}G$-modules. $\square$

Lemma 3.30. *Consider $KG \otimes K_{par}G$ as the free right $K_{par}G$-module with the usual right action inherited from $K_{par}G$. Define $\delta : KG \otimes K_{par}G \to \mathcal{N}$ as the morphism of right $K_{par}G$-modules such that $\delta(g \otimes z) := (g \otimes \nu_{g^{-1}}) \triangleleft [z]$. Then, the following statements are equivalent:*

(i) $KG \otimes \mathcal{B}$ is projective as right $K_{par}G$-module,
(ii) $\mathcal{N}$ is projective as right $K_{par}G$-module,
(iii) there exists a morphism of right $K_{par}G$-modules $\phi : \mathcal{N} \to KG \otimes K_{par}G$ such that $\delta\phi = 1_{\mathcal{N}}$.

Proof. By Lemma 3.29 we have that $KG \otimes \mathcal{B} \cong K_{par}G \oplus \mathcal{N}$, then $KG \otimes \mathcal{B}$ is projective if, and only if $\mathcal{N}$ is projective. For the item (iii), consider the following exact sequence of right $K_{par}G$-modules

$$0 \to \ker \delta \to KG \otimes K_{par}G \overset{\delta}{\to} \mathcal{N} \to 0.$$

Then, it is clear that (ii) implies (iii). On the other hand, the existence of the map ϕ implies that the exact sequence splits and thus $\mathcal{N}$ is a direct summand of the right free $K_{par}G$-module $KG \otimes K_{par}G$. $\square$

Lemma 3.31. *Let G be a group, and $S \subseteq G$ such that $1 \notin S$. Then, the ideal of $\mathcal{B}$ generated by $\{e_s : s \in G\} \neq \mathcal{B}$.*

Proof. Note that the ideal generated by $\{e_s : s \in G\}$ has the basis $\{e_s e_{g_1} \ldots e_{g_m} : s \in S, m \in \mathbb{N}, \text{ and } g_i \in G\}$ as a K-module. Finally, observe that such a set is linearly independent to 1. $\square$

Lemma 3.32. *Let G be an infinite group and $z \in K_{par}G$. If there exists $S \subseteq G$ infinite such that $ze_g = 0$ for all $g \in S$ then $z = 0$.*

Proof. Suppose that we have already proven the lemma for the case $z \in \mathcal{B}$. Let $x \in K_{par}G$, since $K_{par}G = \oplus_{h \in G}[h]\mathcal{B}$, then there exists $\{w_h\}_{h \in G} \subseteq \mathcal{B}$ such that $x = \sum_{h \in G}[h]w_h$. Then, $ze_s = 0$ if, and only if, $[h]w_h e_s = 0$ for all $h \in G$. Note that $[h]w_h e_s = 0$ if, and only if, $e_{h^{-1}}w_h e_s = 0$. Suppose that there exists S infinite such that $ze_s = 0$ for all $s \in G$. Then, $e_{h^{-1}}w_h e_s = 0$ for all $s \in S$ and $h \in G$, thus by hypotheses $e_{h^{-1}}w_h = 0$ for all $h \in G$. Therefore, $z = 0$.

Thus, it is enough to prove the lemma for the case where $z \in \mathcal{B}$. Recall that $\mathcal{B}$ as K-module has basis $E(\mathcal{S}(G)) = \{e_{g_0}e_{g_1}\ldots e_{g_n} : n \in \mathbb{N}, g_i \in G\}$. Moreover, set $\mathcal{P} := \{U \subseteq G : 1 \in U, |U| < \infty\}$, then the function

$$\xi : \mathcal{P} \to E(\mathcal{S}(G))$$
$$U \to \prod_{g \in U} e_g$$

is bijective. If $z = 0$, then the proof is trivial. Let $z \in \mathcal{B} \setminus \{0\}$, then there exists $\{a_i\}_{i=0}^n \subseteq K \setminus \{0\}$ and $\{U_i\}_{i=0}^n \subseteq \mathcal{P}$ such that

$$z = \sum_{i=0}^n a_i \xi(U_i),$$

and the set $\{\xi(U_i) : 0 \leq i \leq n\}$ is linear independent. Furthermore, observe that the set $\{\xi(U_i \cup \{g\}) : 0 \leq i \leq n\}$ is linear independent for all $g \in G \setminus \cup_{i=0}^n U_i$. Therefore,

$$ze_g = \sum_{i=0}^n a_i \xi(U_i \cup \{g\}) \neq 0 \text{ for all } g \in G \setminus \cup_{i=0}^n U_i.$$

Thus, we conclude that if $ze_s = 0$, then $s \in \cup_{i=0}^n U_i$. Now suppose that there exists $S \subseteq G$ infinite such that $ze_s = 0$ for all $s \in S$. Then, $S \subseteq \cup_{i=0}^n U_i$, but this is a contradiction since the $\cup_{i=0}^n U_i$ is finite. Thus, if $z \neq 0$ it only can be annihilated by finitely many e_s's. $\qquad \square$

Proposition 3.33. *Let G be an uncountable infinite group, then $KG \otimes_{G_{par}} K_{par}G$ is not projective.*

Proof. Let $KG \otimes K_{par}G$ be the right $K_{par}G$-module where $(h \otimes z) \triangleleft [g] := h \otimes z[g]$, note that with that structure we have that $KG \otimes K_{par}G$ is a free right $K_{par}G$-module with basis $\{g \otimes 1 : g \in G\}$. Consider the surjective map of right $K_{par}G$-modules $\delta : KG \otimes K_{par}G \rightarrow \mathcal{N}$ such that $\delta(g \otimes 1) := g \otimes v_{g^{-1}}$. Consider the canonical maps $\pi_g : KG \otimes K_{par}G \rightarrow K_{par}G$ such that

$$\pi_g(h \otimes z) = \begin{cases} z & \text{if } h = g, \\ 0 & \text{otherwise} \end{cases}$$

for all $h \in G$ and $z \in K_{par}G$. Suppose that $\mathcal{N}$ is projective, then there exists a morphism of right $K_{par}G$-modules $\phi : \mathcal{N} \rightarrow KG \otimes K_{par}G$ such that $\delta \circ \phi = 1_{\mathcal{N}}$. Under the above hypotheses, we divide the rest of the proof into four steps:

Step 0 We note that the set $A_t := \{s \in G : \pi_s(\phi(t \otimes \nu_{t^{-1}})) \neq 0\}$ is finite for all $t \in G$.

Step 1 We show that $x_g := \pi_g(\phi(g \otimes \nu_{g^{-1}})) \neq 0$ for all $g \in G$.

Step 2 We prove that the set $Z^g := \{t \in G : \pi_g(\phi(t \otimes \nu_{t^{-1}})) = 0\}$ is finite for all $g \in G$.

Step 3 We show that for all $S \subseteq G$ infinite and countable, there exists $t \in G$ such that $S \subseteq A_t$, which leads to a contradiction.

Step 0. Observe that $\phi(t \otimes \nu_{t^{-1}}) = \sum_{g \in G} g \otimes \pi_g(\phi(t \otimes \nu_{t^{-1}}))$. Therefore, A_t is finite.

Step 1. Let $g \in G$, recall that $K_{par}G = \oplus_{h \in G}[h]\mathcal{B}$, then there exists $\{u_{s,h}\}_{s,h \in G} \subseteq \mathcal{B}$ such that and $\phi(g \otimes \nu_{g^{-1}}) = \sum_{s,h \in G} s \otimes [h]u_{s,h}$. Then,

$$g \otimes \nu_{g^{-1}} = \delta(\phi(g \otimes \nu_{g^{-1}})) = \delta \left(\sum_{s,h \in G} s \otimes [h]u_{s,h} \right)$$

$$= \sum_{s,h \in G} (s \otimes \nu_{s^{-1}}) \triangleleft [h]u_{s,h} = \sum_{s,h \in G} sh \otimes \nu_{h^{-1}s^{-1}}e_{h^{-1}}u_{s,h}.$$

Thus, $\nu_{g^{-1}} = \sum_{h \in G} \nu_{g^{-1}}e_{h^{-1}}u_{gh^{-1},h}$. Therefore,

$$\nu_{g^{-1}}u_{g,1} = \nu_{g^{-1}} - \sum_{\substack{h \in G \\ h \neq 1}} \nu_{g^{-1}}u_{gh^{-1},h}e_{h^{-1}} = 1 - e_{g^{-1}} - \sum_{\substack{h \in G \\ h \neq 1}} \nu_{g^{-1}}u_{gh^{-1},h}e_{h^{-1}}.$$

Observe that $e_{g^{-1}} + \sum_{\substack{s \in G \\ s \neq 1}} \nu_{g^{-1}} u_{gh^{-1},h} e_{h^{-1}}$ is in the ideal generated by the set $\{e_{g^{-1}} : g \in G \setminus \{1\}\}$, hence by Lemma 3.31 we conclude that $\nu_{g^{-1}} u_{g,1} \neq 0$, and therefore $u_{g,1} \neq 0$. Note that

$$x_g := \pi_g \phi(g \otimes \nu_{g^{-1}}) = \sum_{h \in G} [h] u_{g,h},$$

thus $\pi_g \phi(g \otimes \nu_{g^{-1}}) = 0$ if, and only if, $[h] u_{g,h} = 0$ for all $h \in G$, but $[1] u_{g,1} = u_{g,1} \neq 0$. Therefore, $x_g \neq 0$ for all $g \in G$.

Step 2. Since ϕ is a morphism of right $K_{par}G$-modules we get

$$(3.11) \qquad \phi(t \otimes \nu_{t^{-1}})[t^{-1}g] = \phi((t \otimes \nu_{t^{-1}}) \triangleleft [t^{-1}g]) = \phi(g \otimes \nu_{g^{-1}} e_{g^{-1}t}) = \phi(g \otimes \nu_{g^{-1}}) e_{g^{-1}t},$$

for all $g, t \in G$. Let $g \in G$, define $Z^g := \{t \in G : \pi_g(\phi(t \otimes \nu_{t^{-1}})) = 0\}$. By Equation (3.11) we have

$$x_g e_{g^{-1}t} = \pi_g(\phi(g \otimes \nu_{g^{-1}})) e_{g^{-1}t} = \pi_g(\phi(t \otimes \nu_{t^{-1}}))[t^{-1}g] = 0 \text{ for all } t \in Z^g.$$

By *Step 1* we know that $x_g \neq 0$, then, by Lemma 3.32, we conclude that Z^g is finite for all $g \in G$.

Step 3. Let S be an infinite countable subset of G. Thus, by *Step 2*, $T = G \setminus \cup_{s \in S} Z^s$ is an infinite uncountable set. Let $t \in T$, then $\pi_s(\phi(t \otimes \nu_{t^{-1}})) \neq 0$ for all $s \in S$, thus $S \subseteq A_t$, but by *Step 0* the set A_t is finite. Therefore, $KG \otimes \mathcal{B}$ cannot be projective as right $K_{par}G$-module. $\qquad\square$

Lemma 3.34. *Let $G = \{g_n\}_{n \in \mathbb{N}}$ be a countable group, where $g_0 = 1$. If there exists $\{x_n\}_{n \geq 1} \subseteq KG \otimes K_{par}G$ such that:*

(i) $\delta(x_n) = g_n \otimes \nu_{g_n^{-1}}$,
(ii) for all $n \in \mathbb{N}$, $x_n[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n}$ for all $r \leq n$,

Then, the map $\phi : \mathcal{N} \to KG \otimes K_{par}G$ given by $\phi(g_n \otimes \nu_{g_n^{-1}}u) := x_n \nu_{g_n^{-1}}u$ is a well-defined morphism of right $K_{par}G$-modules such that $\delta\phi = 1_{\mathcal{N}}$.

Proof. It is clear that ϕ is a well-defined K-linear map since $\{g_n \otimes \nu_{g_n^{-1}}u : n \geq 1, u \in E(\mathcal{S}(G))\}$ is a basis of $\mathcal{N}$ as K-module. Note that in (ii) if we apply $[g_r^{-1}g_n]$ to the right we obtain

$$x_n e_{g_n^{-1}g_r} = x_n[g_n^{-1}g_r][g_r^{-1}g_n] = x_r e_{g_r^{-1}g_n}[g_r^{-1}g_n] = x_r[g_r^{-1}g_n],$$

whence we conclude that (ii) is valid not only for $r \leq n$ but for all $r, n \in \mathbb{N}$. Let $u \in \mathcal{B}$ and $t \in G$, observe that for any $n \geq 0$ there exists $r \geq 0$ such that $t = g_n^{-1} g_r$, then

$$
\begin{aligned}
\phi\big((g_n \otimes \nu_{g_n^{-1}} u) \triangleleft [t]\big) &= \phi\big((g_n \otimes \nu_{g_n^{-1}} u) \triangleleft [g_n^{-1} g_r]\big) \\
&= \phi\big(g_r \otimes \nu_{g_r^{-1}} e_{g_r^{-1} g_n}(u \triangleleft [g_n^{-1} g_r])\big) \\
&= x_r e_{g_r^{-1} g_n} \nu_{g_r^{-1}}(u \triangleleft [g_n^{-1} g_r]) \\
\text{by } (ii) &= x_n[g_n^{-1} g_r]\nu_{g_r^{-1}}(u \triangleleft [g_n^{-1} g_r]) \\
&= x_n \nu_{g_n^{-1}}[g_n^{-1} g_r](u \triangleleft [g_n^{-1} g_r]) \\
&= x_n \nu_{g_n^{-1}}[g_n^{-1} g_r]([g_r^{-1} g_n] u [g_n^{-1} g_r]) \\
&= x_n \nu_{g_n^{-1}} e_{g_n^{-1} g_r} u [g_n^{-1} g_r] \\
&= x_n \nu_{g_n^{-1}} u [g_n^{-1} g_r] \\
&= \phi(g_n \otimes \nu_{g_n^{-1}} u)[g_n^{-1} g_r] \\
&= \phi(g_n \otimes \nu_{g_n^{-1}} u)[t].
\end{aligned}
$$

Then, ϕ is a well-defined map of right $K_{par}G$-modules. Finally, note that

$$
\delta\big(\phi(g_n \otimes \nu_{g_n^{-1}})\big) = \delta(x_n \nu_{g_n^{-1}}) = \delta(x_n)\nu_{g_n^{-1}} = g_n \otimes \nu_{g_n^{-1}}.
$$

Hence, $\delta \circ \phi = 1_{\mathcal{N}}$. □

Proposition 3.35. *Let G be an infinite countable group, then $KG \otimes \mathcal{B}$ is projective.*

Proof. Let $\{g_n\}_{n \in \mathbb{N}}$ be an enumeration of G such that $g_0 = 1$. Define $x_1 := g_1 \otimes \nu_{g_1^{-1}}$, we define $\{x_n\}_{n \geq 1} \subseteq KG \otimes K_{par}G$ recursively. Suppose that we have already defined x_r for all $r < n$. Set

$$
x_{n,1} := g_n \otimes \nu_{g_n^{-1}} + x_1[g_1^{-1} g_n] - g_n \otimes \nu_{g_n^{-1}} e_{g_n^{-1} g_1},
$$

and recursively define:

$$
x_{n,r} := x_r[g_r^{-1} g_n] + x_{n,r-1}\nu_{g_n^{-1} g_r} \text{ for all } r \in \{2, 3, \dots n - 1\}.
$$

Set $x_n := x_{n,n-1}$. We affirm that the set $\{x_n\}_{n \geq 1}$ satisfies the hypotheses of Lemma 3.34. To prove this affirmation, we need to employ nested induction arguments. For the sake of clarity, we will explicitly write the begining and end of each induction argument.

◄ **Induction A**. Note that for $n = 1$ we have that x_1 trivially satisfies conditions (i) and (ii) of Lemma 3.34, thus establishing the base of our induction argument. Let $n > 1$, suppose that we already proved that x_r satisfies (i) and (ii) of Lemma 3.34 for all $r < n$. First observe that

$$
\begin{aligned}
x_{n,1}[g_n^{-1} g_1] &= \big(g_n \otimes \nu_{g_n^{-1}} + x_1[g_1^{-1} g_n] - g_n \otimes \nu_{g_n^{-1}} e_{g_n^{-1} g_1}\big)[g_n^{-1} g_1] \\
&= g_n \otimes \nu_{g_n^{-1}}[g_n^{-1} g_1] + g_1 \otimes \nu_{g_1^{-1}}[g_1^{-1} g_n][g_n^{-1} g_1] - g_n \otimes \nu_{g_n^{-1}} e_{g_n^{-1} g_1}[g_n^{-1} g_1] \\
&= g_n \otimes \nu_{g_n^{-1}}[g_n^{-1} g_1] + g_1 \otimes \nu_{g_1^{-1}} e_{g_1^{-1} g_n} - g_n \otimes \nu_{g_n^{-1}}[g_n^{-1} g_1] \\
&= g_1 \otimes \nu_{g_1^{-1}} e_{g_1^{-1} g_n} = x_1 e_{g_1^{-1} g_n}.
\end{aligned}
$$

For $1 < r < n$, observe that

$$
x_{n,r}[g_n^{-1} g_r] = x_r[g_r^{-1} g_n][g_n^{-1} g_r] + x_{n,r-1}\nu_{g_n^{-1} g_r}[g_n^{-1} g_r] = x_r e_{g_r^{-1} g_n} + 0 = x_r e_{g_r^{-1} g_n}.
$$

Then,

$$(3.12) \qquad x_{n,r}[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n} \text{ for all } r \in \{1, 2, \ldots n-1\}.$$

We want to prove that

$$(3.13) \qquad x_{n,r+m}[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n},$$

for all $r \in \{1, 2, \ldots, n-1\}$ and $m \in \{0, 1, 2, \ldots, n-1-r\}$.

We will prove Equation (3.13) by induction.

◄◄ **Induction A1**. Fix $r \in \{1, 2, \ldots, n-1\}$. Note that (3.12) is the base of the induction. For the general case take $m \in \{1, 2, \ldots, n-1-r\}$ and suppose that $x_{n,r+m-1}[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n}$. Then,

$$\begin{aligned}
x_{n,r+m}[g_n^{-1}g_r] &= x_{r+m}[g_{r+m}^{-1}g_n][g_n^{-1}g_r] + x_{n,r+m-1}\nu_{g_n^{-1}g_{r+m}}[g_n^{-1}g_r] \\
&= x_{r+m}[g_{r+m}^{-1}g_r]e_{g_r^{-1}g_n} + x_{n,r+m-1}[g_n^{-1}g_r]\nu_{g_r^{-1}g_{r+m}} \\
(\flat) &= x_r e_{g_r^{-1}g_{r+m}}e_{g_r^{-1}g_n} + x_r e_{g_r^{-1}g_n}\nu_{g_r^{-1}g_{r+m}} \\
&= x_r e_{g_r^{-1}g_n}\left(e_{g_r^{-1}g_{r+m}} + \nu_{g_r^{-1}g_{r+m}}\right) \\
&= x_r e_{g_r^{-1}g_n},
\end{aligned}$$

where the equality $(\flat)$ holds since $x_{r+m}[g_{r+m}^{-1}g_r] = x_r e_{g_r^{-1}g_{r+m}}$ by the hypotheses of the **Induction A**, and $x_{n,r+m-1}[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n}$ by the induction hypotheses of the **Induction A1**. **End of Induction A1** ►►. Therefore, Equation (3.13) holds.

In particular, when $m = n-1-r$ in Equation (3.13) we obtain $x_n[g_n^{-1}g_r] = x_{n,n-1}[g_n^{-1}g_r] = x_r e_{g_r^{-1}g_n}$ for all $r \leq n$. Thus, the set $\{x_r\}_{r\leq n}$ satisfies the condition (ii) of Lemma 3.34.

To verify that $\{x_r\}_{r\leq n}$ also satisfies the condition (i) of Lemma 3.34 we have to perform another induction argument.

◄◄ **Induction A2**. Observer that

$$\begin{aligned}
\delta(x_{n,1}) &= \delta(g_n \otimes \nu_{g_n^{-1}} + x_1[g_1^{-1}g_n] - g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1}) \\
&= \delta(g_n \otimes \nu_{g_n^{-1}} + (g_1 \otimes \nu_{g_1^{-1}})[g_1^{-1}g_n] - g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1}) \\
&= g_n \otimes \nu_{g_n^{-1}} + (g_1 \otimes \nu_{g_1^{-1}}) \triangleleft [g_1^{-1}g_n] - g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1} \\
&= g_n \otimes \nu_{g_n^{-1}} + g_n \otimes [g_n^{-1}g_1]\nu_{g_1^{-1}}[g_1^{-1}g_n] - g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1} \\
&= g_n \otimes \nu_{g_n^{-1}} + g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1} - g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_1} \\
&= g_n \otimes \nu_{g_n^{-1}}.
\end{aligned}$$

This give us the base of the induction. Let $k \in \{2, \ldots, n-1\}$, suppose that we already have proven that $\delta(x_{n,k-1}) = g_n \otimes \nu_{g_n^{-1}}$, then

$$\begin{aligned}
\delta(x_{n,k}) &= \delta\left(x_k[g_k^{-1}g_n] + x_{n,k-1}\nu_{g_n^{-1}g_k}\right) \\
(\flat) &= (g_k \otimes \nu_{g_k^{-1}}) \triangleleft [g_k^{-1}g_n] + g_n \otimes \nu_{g_n^{-1}}\nu_{g_n^{-1}g_k} \\
&= g_n \otimes \nu_{g_n^{-1}}e_{g_n^{-1}g_k} + g_n \otimes \nu_{g_n^{-1}}\nu_{g_n^{-1}g_k} \\
&= g_n \otimes \nu_{g_n^{-1}},
\end{aligned}$$

where ($\flat$) holds since $\delta(x_k) = g_k \otimes \nu_{g_k^{-1}}$ by the hypotheses of **Induction A** and $\delta(x_{n,k-1}) = g_n \otimes \nu_{g_n^{-1}}$ by the hypotheses of **Induction A2**. Therefore, $\delta(x_n) = \delta(x_{n,n-1}) = g_n \otimes \nu_{g_n^{-1}}$. Hence, $\{x_n\}_{r \leq n}$ satisfies (i) of Lemma 3.34. **End of Induction A2 ▶▶**.

From **Induction A1** and **Induction A2** we conclude that the set $\{x_r\}_{r \leq n}$ satisfies the hypotheses of Lemma 3.34. **End of Induction A ▶**.

By **Induction A** we know that $\{x_n\}_{n \geq 1}$, satisfies the hypotheses of Lemma 3.34. Therefore, there exists a map of right $K_{par}G$-modules $\phi : \mathcal{N} \to KG \otimes K_{par}G$ such that $\delta\phi = 1_{\mathcal{N}}$. Hence, by Lemma 3.30 we conclude that $K_{par}G \otimes \mathcal{B}$ is projective. $\square$

Remark 3.36. If G is a finite group, then the construction made in Proposition 3.35 also holds, implying that $KG \otimes \mathcal{B}$ is projective as a right $K_{par}G$-module. An alternative way to verify this is to consider the following exact sequence:

$$0 \to \mathcal{K} \to KG \otimes K_{par}G \to KG \otimes_{G_{par}} K_{par}G \to 0,$$

where $\mathcal{K}$ is the module defined in Equation (2.5). Note that $\mathcal{K}$ is finitely generated, as it is generated as a right $K_{par}G$-module by the finite set $\{g \otimes e_{h^{-1}} - gh^{-1} \otimes [h] : g, h \in G\}$. Thus, $KG \otimes_{G_{par}} K_{par}G$ is a finitely presented right $K_{par}G$-module. By Corollary 3.19, we know that $KG \otimes_{G_{par}} K_{par}G$ is flat as a right $K_{par}G$-module. Therefore, by [18, Theorem 3.56], we conclude that $KG \otimes_{G_{par}} K_{par}G$ is projective.

Proposition 3.37. *Let M be a left $K_{par}G$-module, then $\mathrm{Ext}^n_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, M^*) = 0$ for all $n \geq 1$.*

Proof. Recall that $\hom_K(M, K)$ is a right $K_{par}G$-module with the action determined by $(f \triangleleft [g])(m) := f([g] \cdot m)$. Let $P_\bullet \to M$ be a projective resolution of left $K_{par}G$-modules. Then, by [16, Lemma 3.5], we conclude that $\hom_K(M, K) \to \hom_K(P_\bullet, K)$ is an injective resolution of right $K_{par}G$-modules. Then, for $n \geq 1$, we have

$$\mathrm{Ext}^n_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, M^*) \cong H^n\big(\hom_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, \hom_K(P_\bullet, K))\big)$$
$$(\flat) \cong H^n\big(\hom_K(KG \otimes_{G_{par}} K_{par}G \otimes_{K_{par}G} P_\bullet, K)\big)$$
$$(\flat\flat) \cong H^n\big(\hom_K(KG \otimes_{G_{par}} P_\bullet, K)\big) = 0.$$

Equality ($\flat$) holds because of the well-known tensor-hom adjunction, the equality ($\flat\flat$) holds by Proposition 2.6. Finally, the last equality holds since the functors $\hom_K(-, K)$ and $KG \otimes_{G_{par}} -$ are exact. $\square$

Combining Proposition 3.27, Proposition 3.35, Remark 3.36, and Proposition 3.37, we obtain the following theorem:

Theorem 3.38. *Let M be a right $K_{par}G$-module. Then, there exists a cohomological spectral sequence*

$$E_2^{p,q} := H^p\big(G, \mathrm{Ext}^q_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, M)\big) \Rightarrow H^{p+q}_{par}(G, M).$$

If G is finite, countable, or M is the dual module of a left $K_{par}G$-module, then the spectral sequence collapses and give rise to an isomorphism

$$H^n_{par}(G, M) \cong H^n(G, \hom_{K_{par}G}(KG \otimes_{G_{par}} K_{par}G, M)).$$

Remark 3.39. Note that by Proposition 3.33 the spectral sequence in Theorem 3.38 may not collapse when G is an infinite uncountable group.

Acknowledgments

References

[1] E. R. Alvares, M. M. Alves, and M. J. Redondo. Cohomology of partial smash products. *J. Algebra*, 482:204–223, 2017.

[2] M. M. Alves, M. Dokuchaev, and D. H. Kochloukova. Homology and cohomology via the partial group algebra. *(Preprint)*, 2020.

[3] E. Batista. Partial actions: what they are and why we care. *Bull. Belg. Math. Soc. Simon Stevin*, 24(1):35 – 71, 2017.

[4] H.-J. Baues and G. Wirsching. Cohomology of small categories. *Journal of Pure and Applied Algebra*, 38(2):187–211, 1985.

[5] M. Dokuchaev. Recent developments around partial actions. *São Paulo J. Math. Sci.*, 13(1):195–247, 2019.

[6] M. Dokuchaev and E. Jerez. (co)homology of partial smash products. *(Preprint)*, 2023.

[7] M. Dokuchaev and M. Khrypchenko. Partial cohomology of groups. *J. Algebra*, 427:142–182, 2015.

[8] M. Dokuchaev, M. Khrypchenko, and J. J. Simón. Globalization of group cohomology in the sense of Alvares-Alves-Redondo. *J. Algebra*, 546:604–640, 2020.

[9] M. Dokuchaev and B. Novikov. Partial projective representations and partial actions. *J. Pure Appl. Algebra*, 214(3):251–268, 2010.

[10] M. Dokuchaev, B. Novikov, and H. Pinedo. The partial Schur multiplier of a group. *J. Algebra*, 392:199–225, 2013.

[11] R. Exel. Partial actions of groups and actions of inverse semigroups. *Proc. Am. Math. Soc.*, 126(12):3481–3494, 1998.

[12] R. Exel. *Partial Dynamical Systems, Fell Bundles and Applications*. American Mathematical Society, 2017.

[13] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Springer, Berlin, Germany, 1967.

[14] S. I. Gelfand and Y. I. Manin. *Methods of Homological Algebra*. Springer, Berlin, 2003.

[15] N. Jacobson. *Basic Algebra II: Second Edition*. Dover Books on Mathematics. Dover Publications, 2012.

[16] T. Y. Lam. Lectures on modules and rings. In *Lectures on Modules and Rings*, Graduate Texts in Mathematics. Springer New York, NY, 1998.

[17] J. L. Loday. *Cyclic Homology*, volume 301. Springer Science & Business Media, 2013.

[18] J. Rotman. *An Introduction to Homological Algebra*. Universitext. Springer New York, 2008.

[19] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

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