SOLUTION OF THE ASSIGNED PROBLEM

ZAHRA BAYAT, EMMANUEL FLORES

Problem 1. A spin 1/2 paramagnet in a magnetic field B can be modeled as a set of independent two-level systems with energy $-\mu_B B$ (where $\mu_B \equiv e\hbar/2m$ is the Bohr magneton).

(1) Show that for one magnetic ion, the partition function is

$$Z = 2\cosh(\beta\mu_B B)$$

(2) For N independent magnetic ions, the partition function Z_N is $Z_N = Z^N$. Show that the Helmholtz function is given by

$$F = -Nk_B T \ln[2\cosh(\beta\mu_B B)].$$

(3) Considering that the magnetic moment m is given by $m = -\left(\partial F/\partial B\right)_T$, show that

$$m = N\mu_B \tanh(\beta\mu_B B)$$

Sketch m as a function of B.

(4) Show further that for small fields, $\mu_B B \ll k_B T$,

$$m \simeq N\mu_B^2 B/k_B T$$

(5) The magnetic susceptibility is defined as $\chi \simeq \mu_0 M/B$ for small B. Hence show that $\chi \propto 1/T$, which is Curie's law.

Solution. Here we provide a detailed solution of the assignment problem.

(1) The energy sprectrum is given by the $-\mu_B B$ and $\mu_B B$, and we know that the partition function is defined as

$$Z = \sum_i \exp\left[-\beta E_i\right],$$

thus, from this we have

$$Z = \exp\left[-\beta\left(-\mu_B B\right)\right] + \exp\left[-\beta\left(\mu_B B\right)\right],$$
$$\implies Z = \exp\left[\mu_B \beta B\right] + \exp\left[-\mu_B \beta B\right],$$

but we know that

$$\cosh\left(x\right) = \frac{\exp\left(x\right) + \exp\left(-x\right)}{2},$$
$$\implies \exp\left(x\right) + \exp\left(-x\right) = 2\cosh\left(x\right)$$

Date: 05/06/2024.

and from this the partition function becomes

$$Z = 2\cosh\left(\mu_B \beta B\right),$$

(2) Now, for the free energy we have the following relation

$$F = -k_B T \ln Z,$$

and if we consider

$$Z = Z_1^N,$$

where

$$Z_1 = 2\cosh\left(\mu_B \beta B\right),$$

then it follows that

$$Z = \left[2\cosh\left(\mu_B \beta B\right)\right]^N,$$

thus,

$$\ln Z = \ln\left\{\left[2\cosh\left(\mu_B \beta B\right)\right]^N\right\},$$
$$\implies \ln Z = N \ln\left[2\cosh\left(\mu_B \beta B\right)\right],$$

and with this the free energy becomes

$$F = -N k_B T \ln\left[2\cosh\left(\mu_B \beta B\right)\right],$$

(3) Considering

$$m = -\left(\partial F/\partial B\right)_T,$$

we have

$$\left(\frac{\partial F}{\partial B}\right)_T = \left(\frac{\partial}{\partial B}\left(-N k_B T \ln\left[2\cosh\left(\mu_B \beta B\right)\right]\right)\right)_T,$$
$$\implies \left(\frac{\partial F}{\partial B}\right)_T = -N k_B T\left(\frac{\partial}{\partial B}\left(\ln\left[2\cosh\left(\mu_B \beta B\right)\right]\right)\right)_T,$$

and for the derivative we're going to make use of the chain rule, with the fact that

$$\frac{d}{dx}\cosh\left(x\right) = \sinh\left(x\right),$$

thus

$$\left(\frac{\partial F}{\partial B}\right)_T = -N k_B T\left(\frac{1}{2\cosh\left(\mu_B \beta B\right)} 2\mu_B \beta \sinh\left(\mu_B \beta B\right)\right),$$
$$\implies \left(\frac{\partial F}{\partial B}\right)_T = -N k_B \mu_B \beta T\left(\frac{\sinh\left(\mu_B \beta B\right)}{\cosh\left(\mu_B \beta B\right)}\right),$$

but we also know that

$$\tanh\left(x\right) = \frac{\sinh\left(x\right)}{\cosh\left(x\right)},$$

thus

$$\left(\frac{\partial F}{\partial B}\right)_T = -N k_B \mu_B \beta T \tanh\left(\mu_B \beta B\right),$$
$$\implies -\left(\partial F/\partial B\right)_T = N k_B \mu_B \beta T \tanh\left(\mu_B \beta B\right),$$

therefore

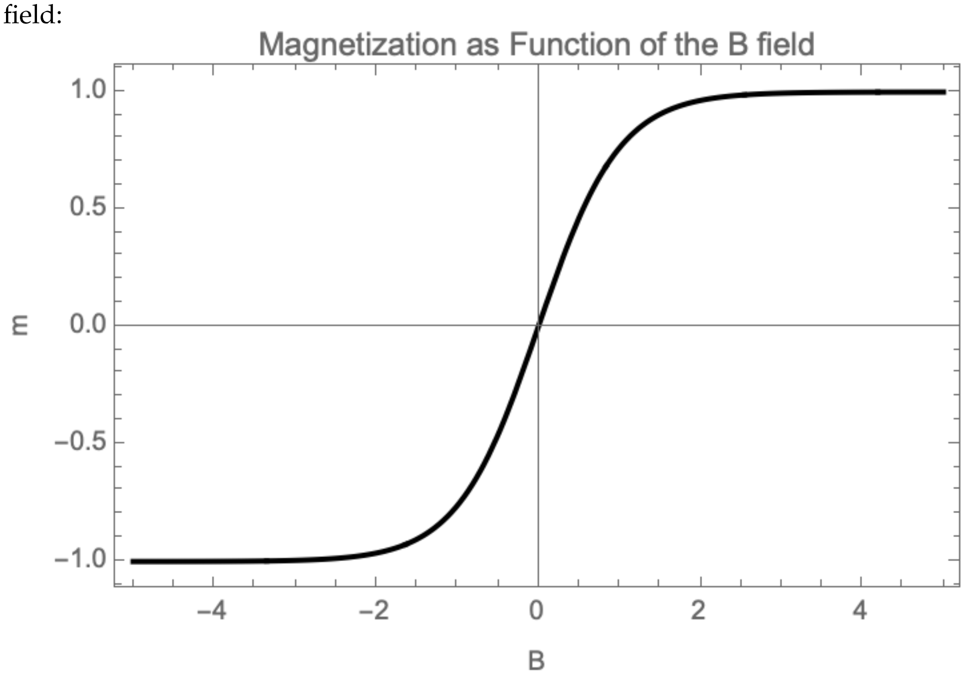
$$m = Nk_B\mu_B\beta T \tanh\left(\mu_B\beta B\right),$$

but we have to remember that $\beta = 1/k_BT$, thus

$$m = Nk_B\mu_B\left(\frac{1}{k_BT}\right)T \tanh\left(\mu_B\beta B\right),$$

$$\implies m = N\mu_B \tanh\left(\mu_B\beta B\right)$$

just as we wanted. Finally, we present a plot for the magnetization as function of the magnetic field:



(4) Now if $\mu_B B \ll k_BT$, then we can Taylor expand the hyperbolic tangent as

$$\tanh\left(x\right) \approx x,$$

which means that

$$m \approx N\mu_B\left(\mu_B\beta B\right),$$

and

$$m \approx N\mu_B\left(\mu_B\beta B\right),$$

and again, using $\beta = 1/k_BT$, we have

$$m \approx \frac{N\mu_B^2 B}{k_BT},$$

just as we wanted.

(5) Finally, considering

$$\chi \simeq \frac{\mu_0 M}{B},$$

and using our calculated m, we have

$$\chi \simeq \mu_0 \frac{\frac{N\mu_B^2 B}{k_B T}}{B},$$

$$\implies \chi \simeq \mu_0 \frac{N\mu_B^2}{k_B T},$$

and making

$$\gamma = \frac{\mu_0 N \mu_B^2}{k_B},$$

we have

$$\implies \chi \simeq \frac{\gamma}{T},$$

and from this it follows that

$$\chi \propto \frac{1}{T},$$

which, as stated in the problem is Curie's law.