HW5 POINT SET TOPOLOGY

EMMANUEL FLORES

1. Consider $\mathbb{R}$ with the standard topology. Let C be a compact subset of $\mathbb{R}$. Prove that C as a maximum, that is a point $m \in C$ such that $x \leq m$ for all $x \in C$.

Proof. This follows from the Heine-Borel theorem, which states that $C \subset \mathbb{R}$, with the standard topology, is compact if and only if C is bounded and closed.

Now let's suppose that $C \subset \mathbb{R}$ is compact, then it follows that C is bounded and closed, but being bounded means that C is bounded from above and bounded from below, in particular let's focus on being bounded by above, this implies that there is a point $m \in \mathbb{R}$ such that $x \leq m$ for all $x \in C$. Now, because C is bounded it follows that it has a least upper bound, and let's call it m. Now, let's prove that $m \in C$; because m is the least upper bound it follows that for any $\epsilon > 0$ there is an element $x \in C$ such that $m - \epsilon < x \leq m$, which implies that m is a limit point of C and because C is closed, this implies that C contains all its limit points, therefore $m \in C$

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Date: January 30, 2025.

2. If A and B are compact subspaces of a separated topological space $(X, \mathcal{T})$, prove that $A \cup B$ is a compact subspace of X.

Proof. Let A and B be compact subspaces of a separated topological space $(X, \mathcal{T})$. Thus let $\{O^A_{\alpha \in \lambda_A}\}$ and $\{O^B_{\beta \in \lambda_B}\}$ be open covers of A and B respectively, this is

$$A \subset \cup\{O^A_{\alpha \in \lambda_A}\}, \ \ B \subset \cup\{O^B_{\beta \in \lambda_B}\}.$$

From this if follows that $\{O^A_{\alpha \in \lambda_A}\} \cup \{O^B_{\beta \in \lambda_B}\}$ is an open cover for $A \cup B$, this is

$$A \cup B \subset \{O^A_{\alpha \in \lambda_A}\} \cup \{O^B_{\beta \in \lambda_B}\}.$$

Now, because A and B are compact it follows that every open cover has a finite subcover, this is $\exists n, k$ such that $\{O^A_1, \cdots, O^A_n\}$ is subcover for A and $\{O^B_1, \cdots, O^B_k\}$ is subcover for B, this is

$$A \subset \cup\{O^A_1, \cdots, O^A_n\}, \ \ B \subset \cup\{O^B_1, \cdots, O^B_k\}.$$

And again, from this it follows that $\{O^A_1, \cdots, O^A_n\} \cup \{O^B_1, \cdots, O^B_k\}$ is a finite subcover for $A \cup B$. Because the covers were arbitrary it follows that every open cover for $A \cup B$ has a finite subcover, therefore, $A \cup B$ is compact.

$\square$

3. A $(X, \mathcal{T})$ topological space is said to be normal if for every disjoint pair of closed subsets A and B there exist two disjoint open set U and V with $A \subset U$ and $B \subset V$.
3.1 Prove that $(X, \mathcal{T})$ is a normal topological space if and only if for each closed set A in X and open set U containing A, there exists an open set V such that $A \subset V$ and $V \subset U$.
3.2 Prove that $(X, \mathcal{T})$ is a normal topological space if and only if for every disjoint pair of closed subsets A and B there exist two disjoint open sets U and V with $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.
3.3 Prove that if $(X, \mathcal{T})$ is a (Hausdorff) compact topological space, then (X, T) is normal.

Proof. 3.1 ($\Longrightarrow$) Let $(X, \mathcal{T})$ is a normal topological space, A be a closed set in X, and U be an open set containing A, then it follows that $X \setminus$ are disjoint closed sets in X. Now, because X is normal, it follows that there exists disjoint open sets V and W such that $A \subset V$ and $X \setminus U \subset W$. Since V and W are disjoint, it follows that $V \subset X \setminus W$, but $X \setminus W \subset U$ then $A \subset V \subset X \setminus W \subset U$, which implies that $A \subset V \subset U$.

($\Longleftarrow$) Now, let's suppose that for each closed set $A \subset X$ and open set U containing A, there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$. And let A and B be two disjoint closed sets in X, then,$X \setminus B$ is an open set containing A. By supposition, there is an open set V such that $A \subset V$ and $\overline{V} \subset X \setminus B$, then it follows that $B \subset X \setminus \overline{V}$. Because $X \setminus \overline{V}$ is open and disjoint from V, we have found disjoint open sets V and $X \setminus \overline{V}$ containing A and B respectively, therefore $(X, \mathcal{T})$ is a normal topological space.

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4. Let $\{K_n\}_{n\geq 1}$ be a decreasing sequence of compact subspaces of a Hausdorff topological space $(X, \mathcal{T})$. Prove that $K = \cap_{k=1}^{\infty} K_n$ is nonempty, and that for every open set O containing K, there exists a K_n contained in O.

Proof. First, let's prove that $K = \cap_{k=1}^{\infty} K_n$ is nonempty, and for that, let's do it by contradiction, that is, let's suppose that there's no point $x \in X$ that belongs to K_n for all n. Now, let's look at the complements of each one of the K_n, this is $U_n = X \setminus K_n$. Now, because each one of the K_n is compact and is also Hausdorff, it follows that K_n is compact for all n, which implies that U_n is open, and because no point belongs to all K_n, it follows that the collection $\{U_n\}_{n\leq 1}$ is an open cover of X. On the other hand, because each K_n is compact for each K_n there is finite subcover, in particular, let's focus on K_1, and the finite subcover $\{U_{n_1}, \cdots, U_{n_k}\}$. Now, let's suppose that $n_1 < n_2 < \cdots < n_k$, and because $\{K_n\}_n$ is decreasing, it follows that

$$U_{n_1} \subset U_{n_2} \subset \cdots U_{n_k} \subset .$$

But then it follows that U_{n_k} by itself covers K_1, which implies that $K_{n_k} \cap K_1 = \emptyset$ but this contradicts the fact that by supposition the sequence is decreasing, therefore $K = \cap_{k=1}^{\infty} K_n$ is nonempty.

Now, let's prove that for every open set O containing K, there exists a K_n contained in O. And let O be an open set containing K, thus $X \setminus O$ is closed. Now, let's define the closed sets $F_n = K_n \cap (X \setminus O)$, and each one is a subset of the compact K_n which implies that each F_n is also compact. On the other hand, by construction, the sequence $\{F_n\}_{n\leq 1}$ is also a decreasing sequence of compact sets. And even more, let's notice that

$$\cap_{n=1}^{\infty} F_n = (\cap_{n=1}^{\infty} K_n) \cap (X \setminus O) = K \cap (X \setminus O) = \emptyset$$

because $K \subset O$. By the same argument as before, a decreasing sequence of non-empty compact sets cannot have an empty intersection, therefore it follows that there must exist an n such that $F_n = \emptyset$, and from this it follows that $K_n \cap (X \setminus O) = \emptyset$, which implies that $K_n \subset O$ for each n, and this concludes the proof. $\square$

5. Consider the rational $\mathbb{Q}$ with the subspace topology from the standard topology on $\mathbb{R}$. Find a set A in $\mathbb{Q}$ that is closed and bounded but not compact.

Proof. Let's consider the following subset of $\mathbb{Q}$,

$$A = \{q \in \mathbb{Q} | \sqrt{2} < q < \sqrt{3}\}.$$

Now, let's prove that A is bounded and closed but it's not compact. Indeed, A is bounded from below by $\sqrt{2}$ and bounded from above by $\sqrt{3}$, it follows that A is bounded. On the other hand let's consider the complement of A and let's prove that is open. Indeed

$$\mathbb{Q} \setminus A = \{q \in \mathbb{Q} | q \leq \sqrt{2} \text{ or } q \geq \sqrt{3}\}.$$

It follows that every rational number in $\mathbb{Q} \setminus A$ will have an open internal around it that is entirely contained in $\mathbb{Q} \setminus A$, and it follows that A is closed. Now, let's prove that A is not compact, thus for each rational number $q \in A$, choose a small open interval (a_q, b_q) in $\mathbb{Q}$ centered at q such that

$$\sqrt{2} < a_q < q < b_q < \sqrt{3}.$$

It follows that the collection of all these intervals $\{(a_q, b_q | q \in A)\}$ is an open cover of A. Now, let's try to find a finite subcover; because $\mathbb{Q}$ is dense in $\mathbb{R}$ it follows that there are infinitely many rational numbers in between $\sqrt{2}$ and $\sqrt{3}$, and therefore for any finite subcover there will always be rational numbers in A that are not covered by this finite family, thus there is no finite subcover, and A is not compact. $\square$