Point Set Topology

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Problem 1 *Recall that* $\mathbb{R}$ *with the standard topology is a Hausdorff space. A subset* $S \subset \mathbb{R}$ *is said to be sequentially compact provided that every sequence in* S *has a subsequence that converges to a point in* S.

1. *Prove that* S *is sequentially compact if and only if* S *is closed and bounded. (This is known as the Bolzano-Weierstrass Theorem).*

2. *Prove that if* S *is compact in the standard topology of* $\mathbb{R}$, *then* S *is closed and bounded, hence sequentially compact. (Note: This has now established the Heine-Borel Theorem on* $\mathbb{R}$ *with the standard topology: Every closed bounded subset of* $\mathbb{R}$ *is compact.)*

3. *Prove that if S is sequentially compact then it is compact in the standard topology of* $\mathbb{R}$.

Problem 2 *For two points* $x = (x_k)_{k=1}^n$, $y = (y_k)_{k=1}^n \in \mathbb{R}$, *consider the following three functions:*

1. *Verify that each of these functions defines a metric on* $\mathbb{R}^n$.
2. *Prove that the three distances generate the same topology on* $\mathbb{R}^n$.

Problem 3 *A topological space $(E, \mathcal{T})$ is said to be locally compact provided that it is Hausdorff and every point in E has a least on compact neighborhood.*

1. *Prove that every compact space is locally compact.*
2. *Prove that E equipped with the discrete topology is locally compact.*
3. *Every closed subspace of a locally compact space is locally compact.*

Problem 4 *Let d, d' be two metrics on a set E, and let $\psi : [0,\infty][0,\infty]$ be an increasing function whose derivative $\psi : [0,\infty)[0,\infty]$ is also increasing with $\psi(0) = \psi'(0) = 0$. Suppose that for all $x, y \in E$*

$$d'(x,y) \leq \varphi(d(x,y)) \text{ and } d(x,y) \leq \varphi'(d'(x,y))$$

Prove that these two distances generate the same topology on E.

Problem 5 *Let (A_n)be a decreasing sequence of subsets of R, each of which is a finite union of pairwise disjoint closed intervals. We also assume that each of the intervals making up A_n contains exactly two of the intervals which make up A_{n+1}, and that the diameter of these intervals tends to 0 with $1/n$. Show that the set $A = \cap_n A_n$ is a compact set without any isolated points.*