Structure Formation, Statistics, and Scalar Field

J. Emmanuel Flores

April 18, 2025

Problem 1. Dark Matter and Baryon Density Growth.
a. Let's start by defining

$$\epsilon = \delta_b(t) - \delta_d(t),$$

and we're given

$$\ddot{\delta}_d(t) + 2H\dot{\delta}_d(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

$$\ddot{\delta}_b(t) + 2H\dot{\delta}_b(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

by taking the difference of the two previous equations, we have the following

$$\ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H\dot{\delta}_b(t) - 2H\dot{\delta}_d(t) = 0 \implies \ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H(\dot{\delta}_b(t) - \dot{\delta}_d(t)) = 0$$

but we know that $H = 12/3t$, thus

$$\ddot{\epsilon} + \frac{4}{3t}\dot{\epsilon} = 0,$$

b. Let's seek solutions of the previous equation with the following form

$$\epsilon = t^n,$$

then, we have

$$t^{n-2}\left(n(n-1) + \frac{4}{3}n\right) = 0 \iff n\left(n + \frac{1}{3}\right) = 0$$

which implies that

$$n \in \{0, -\frac{1}{3}\},$$

and with this, the general solution will be

$$\epsilon(t) = \epsilon_0 + \frac{\epsilon_1}{t^{1/3}},$$

and from the form of the solution, we can see that a late times $\epsilon \to \epsilon_0$, which implies that at late times

$$\delta_b - \delta_d \to \epsilon_0 \implies \frac{\delta_b - \delta_d}{\delta_b} \to \frac{\epsilon_0}{\delta_b} \implies 1 - \frac{\delta_d}{\delta_b} \to \frac{\epsilon_0}{\delta_b},$$

which is valid for all ϵ_0, and in particular is valid for $\epsilon_0 = 0$, thus we have

$$\frac{\delta_d}{\delta_b} \to 1.$$

c. I append the Mathematica notebook with the solution and the corresponding plots.
d. From the numerical plot we can see that a late times δ_b becomes almost equal to δ_d which is in agreement with that I found in b. And finally,

$$t_{dec} = 3.17432 \times 10^7 \text{ years.}$$

Problem 2. Matter Growth with Dark Energy

a. Let's take derivatives

$$\frac{d\delta}{dt} = \frac{da}{dt}\frac{d\delta}{da} \implies \dot{\delta} = \delta'\dot{a},$$

where $'$ means derivative with respect to a. On the other hand, we also have

$$\ddot{\delta} = \frac{d}{dt}(\dot{\delta}) = \frac{d}{dt}(\delta'\dot{a}) = \dot{a}\frac{d\delta'}{dt} + \delta'\frac{d}{dt}(\dot{a}) \implies \ddot{\delta} = \dot{a}^2\delta'' + \delta'\ddot{a},$$

just as we wanted.

b. The first Friedmann equation with just matter and dark matter is given by

$$\frac{H^2}{H_0^2} = \Omega_{M0}a^{-3} + \Omega_{V0}$$

and from this we have

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right),$$

which implies that

$$\dot{a}^2 = a^2H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right).$$

Now, if we take the time derivative of the previous expression we have

$$2\dot{a}\ddot{a} = 2a\dot{a}H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) + a^2H_0^2(-3\Omega_{M0}a^{-2}\dot{a}),$$

which implies that

$$2\dot{a}\ddot{a} = 2a\dot{a}H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) - 3\dot{a}a^{-2}H_0^2\Omega_{M0},$$

and by cancelind the $\dot{a}$ we have

$$2\ddot{a} = 2aH_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) - 3a^{-2}H_0^2\Omega_{M0},$$

and from this we have

$$2\ddot{a} = -a^{-2}H_0^2\Omega_{M0} + 2aH_0^2\Omega_{V0},$$

and by factorizing some terms

$$2\ddot{a} = -aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0}),$$

which leads us to

$$\ddot{a} = -aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0})/2$$

just as we wanted.

c. Starting with

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_{M0}\delta = 0,$$

and using the previous results we have that

$$a^2H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right)\delta'' + \delta'(-aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0})/2) + 2H\delta'\dot{a} - \frac{3}{2}H^2\Omega_{M0}\delta = 0$$

d. Let's look at both limit cases

- By considering a small and dark matter dominating, we have that Ω_{M0}/a^3 dominates, and therefore, the equation becomes

$$a^2(\Omega_{M0}/a^3)\delta'' + \frac{3}{2}(\Omega_{M0}/a^3)\delta' - \frac{3}{2}\Omega_{M0}/a^3\delta = 0,$$

  which implies that

$$a^2\delta'' + \frac{3}{2}\delta' - \frac{3}{2}\delta = 0,$$

  and if we seek solutions of the form $\delta = a^n$, we'll have the following equation

$$a^n(n^2 + \frac{1}{2}n - \frac{3}{2}) = 0,$$

  with solutions $n = -3/2$ and $n = 1$, therefore $\delta \propto a$ is a valis solution.

- On the other hand a is large and dark energy dominates, we have that $\Omega_{M0}/a^3 \ll 1$, which implies that terms without $\Omega_{M0}/a^3 \ll 1$ will dominate, and in limit the equation will take the form

$$a^2(1 - \Omega_{M0})\delta'' + 3a(1 - \Omega_{M0})\delta' = 0,$$

  and from here we can see that δ proportional to a constant is a valid solution.

Therefore such possible scenarios have either $\delta \propto a$ or $\delta \propto$ a constant.
e. I append a Mathematica notebook with the graph.
f. In the plot I made I include a fit taken into account the firts points, wich indeed resemble a linear function in a, whereas for higer values of a one can see that the solution reached a plateau, which is consistent witht the previous analysis in both scenarios/limits.

Problem 3. Power Spectrum

By definition $\langle \delta(\mathbf{k})\delta^*(\mathbf{k})' \rangle$ is given by

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k})' \rangle = \int d^3x d^3x' \exp(-i\mathbf{k}\cdot\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}') \langle \delta(\mathbf{x})\delta(\mathbf{x}') \rangle,$$

and if we make $\mathbf{r} = |\mathbf{x} - \mathbf{x}'|$ we have

$$\exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}\cdot\mathbf{x}' - i\mathbf{k}\cdot\mathbf{x}' + i\mathbf{k}'\cdot\mathbf{x}'),$$

then we have

$$\exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp[-i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}') - i\mathbf{x}\cdot(\mathbf{k} - \mathbf{k}')],$$
$$\implies \exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp(-i\mathbf{k}\cdot\mathbf{r}) \exp(-i\mathbf{x}\cdot(\mathbf{k} - \mathbf{k}')).$$

With this information, we can arrive at the following expression

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k})' \rangle = \int d^3r d^3x' e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')} \xi(r),$$

where

$$\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x}') \rangle.$$

And if now we integrate over the x' variables we have

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k})' \rangle = (2\pi)^3 \delta^3(\mathbf{x} - \mathbf{x}') \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \xi(r),$$

and by making

$$\mathcal{P}(k) = \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \xi(r),$$

we arrive at the following result

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k})' \rangle = (2\pi)^3 \delta^3(\mathbf{x} - \mathbf{x}') \mathcal{P}(k)$$

just as we wanted.

Problem 4. Scalar Field (Inflaton) in Expanding Universe

a. Let's begin with the following action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right],$$

where

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2),$$

and with this metric, we have

$$\sqrt{-g} = a^3,$$

whereas

$$g^{\mu\nu} = \text{diag}(1, -1/a^2, -1/a^2, -1/a^2),$$

therefore, the action takes the form

$$S = \int dtd^3xa^3\left[\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)\right]$$

which can be simplified to

$$S = \int dtd^3x\left[\frac{1}{2}a^3(\dot{\phi})^2 - \frac{1}{2}a(\nabla\phi)^2 - a^3V(\phi)\right].$$

And if we perform the variation, we have

$$\delta S = \int dtd^3x\left[a^3\dot{\phi}\delta\dot{\phi} - a\nabla\phi\cdot\nabla\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi\right],$$

for the first two terms, we can perform integration by parts as follows

$$\int dt(a^3\dot{\phi})\delta\dot{\phi} = -\int\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + \text{boundary terms},$$

and

$$\int d^3xa\nabla\phi\cdot\nabla\delta\phi = -\int d^3x(a\nabla^2\phi)\delta\phi + \text{boundary terms},$$

thus, if we neglect the boundary terms, the variation becomes

$$\delta S = \int dtd^3x\left[-\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + (a\nabla^2\phi)\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi\right],$$

and impossing the condition $\delta S = 0$ we have

$$-\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + (a\nabla^2\phi)\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi = 0,$$

which can be simplified to

$$-3a^2\dot{a}\dot{\phi} - a^3\ddot{\phi} + a\nabla^2\phi - a^3\frac{V(\phi)}{d\phi} = 0,$$

since $a \neq 0$ we have

$$-3\frac{\dot{a}}{a}\dot{\phi} - \ddot{\phi} + \frac{\nabla^2\phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

and by using the fact $H = \dot{a}/a$ we have

$$-3H\dot{\phi} - \ddot{\phi} + \frac{\nabla^2\phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

which can also be writen as

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{V(\phi)}{d\phi} = 0,$$

just as we wanted.
b. By assuming ϕ is homogeneous in space and that $V = \frac{1}{2}m^2\phi^2$ we have

$$\frac{dV}{d\phi} = m^2\phi, \nabla^2\phi = 0,$$

thus the previous equation becomes:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$$

just as we wanted.
c. Finally, by assuming H a constant we seek solutions of the form

$$\phi = \phi_0 \exp(rt),$$

we have the following equation

$$\phi_0 \exp(rt)\left(r^2 + 3Hr + m^2\right) = 0$$

and solving the characteristic equation lead us to the following expression

$$r = \frac{-3H \pm \sqrt{1 - \frac{2m^2}{9H^2}}}{2},$$

which can be simplified to

$$r = \frac{3H}{2}\left(1 \pm \sqrt{1 - \left(\frac{2m}{3H}\right)^2}\right)$$

or we can also make

$$r = \frac{m}{\alpha}(1 \pm \sqrt{1 - \alpha^2}),$$

where $\alpha = \frac{2m}{3H}$. Therefore, the general solution takes the form

$$\phi(t) = \phi_+ \exp\left[\frac{m}{\alpha}(1 + \sqrt{1 - \alpha^2})t\right] + \phi_- \exp\left[\frac{m}{\alpha}(1 - \sqrt{1 - \alpha^2})t\right].$$

Now, the solution will oscillate when the following condition hold

$$1 - \left(\frac{2m}{3H}\right)^2 < 0 \iff 2m > 3H.$$

On the other hand, the solution will not oscillate when

$$1 - \left(\frac{2m}{3H}\right)^2 \geq 0 \iff 3H \geq 2m.$$

And finally, if $m \ll H$, this implies that $\frac{m}{H} \ll 1$, thus the solution will take the form

$$r \approx 3H \implies \phi(t) \propto e^{3Ht}$$

therefore, ϕ will evolve quickly.

Cosmology PSET 6:

Problem 1: Dark Matter and Baryon Density Growth

In[2]:=
```
vMap = ResourceFunction["ViridisColor"];
```

In[3]:=
```
ode = D[ϵ[t], {t, 2}] + (4 / (3 t)) D[ϵ[t], t] == 0;
ϵSol = DSolveValue[ode, ϵ[t], t]
```

Out[4]= $-\frac{3\,\mathbb{c}_1}{t^{1/3}} + \mathbb{c}_2$

In[5]:=
```
params = {Ωd → 5/6, Ωb → 1/6};
```

In[6]:=
```
odeD = D[δd[t], {t, 2}] + 4/(3 t) D[δd[t], t] - 2/(3 t^2) (Ωd δd[t] + Ωb δb[t]) == 0
```

Out[6]= $-\frac{2\,(\Omega b\,\delta_b[t] + \Omega d\,\delta_d[t])}{3\,t^2} + \frac{4\,\delta_d'[t]}{3\,t} + \delta_d''[t] == 0$

In[7]:=
```
odeD /. params
```

Out[7]= $-\frac{2\left(\frac{\delta_b[t]}{6} + \frac{5\,\delta_d[t]}{6}\right)}{3\,t^2} + \frac{4\,\delta_d'[t]}{3\,t} + \delta_d''[t] == 0$

In[8]:=
```
odeB = D[δb[t], {t, 2}] + 4/(3 t) D[δb[t], t] - 2/(3 t^2) (Ωd δd[t] + Ωb δb[t]) == 0
```

Out[8]= $-\frac{2\,(\Omega b\,\delta_b[t] + \Omega d\,\delta_d[t])}{3\,t^2} + \frac{4\,\delta_b'[t]}{3\,t} + \delta_b''[t] == 0$

In[9]:=
```
odeB /. params
```

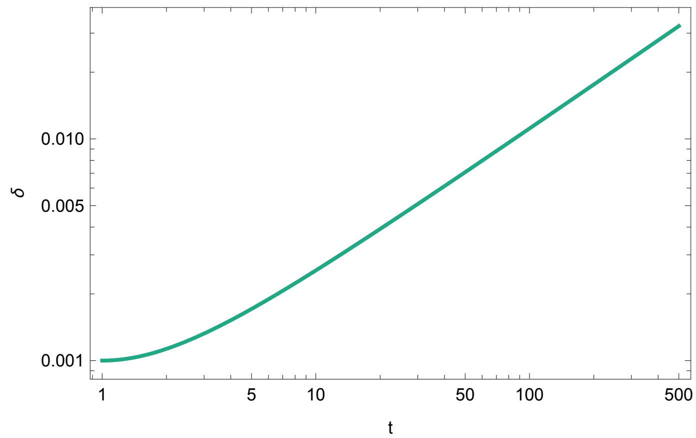
Out[9]= $-\frac{2\left(\frac{\delta_b[t]}{6} + \frac{5\,\delta_d[t]}{6}\right)}{3\,t^2} + \frac{4\,\delta_b'[t]}{3\,t} + \delta_b''[t] == 0$

In[10]:=
```
sysSolution = NDSolveValue[
  {odeD, odeB, δd[1] == 1*^-3, δb[1] == 1*^-4, δd'[1] == 0, δb'[1] == 0} /. params,
  {δd, δb}, {t, 1, 500}]
```

Out[10]=
```
{InterpolatingFunction[ Domain: {{1., 500. }} Output: scalar ],
 InterpolatingFunction[ Domain: {{1., 500. }} Output: scalar ]}
```

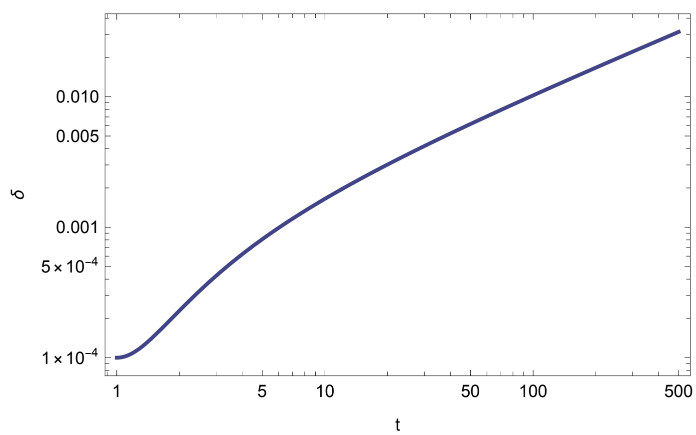
In[11]:=
```
δDPlot = Plot[sysSolution[[1]][t], {t, 1, 500}, ScalingFunctions → {"Log", "Log"},
  PlotStyle → vMap[0.6], Frame → True, FrameLabel → {"t", "δ"}]
```

Out[11]=



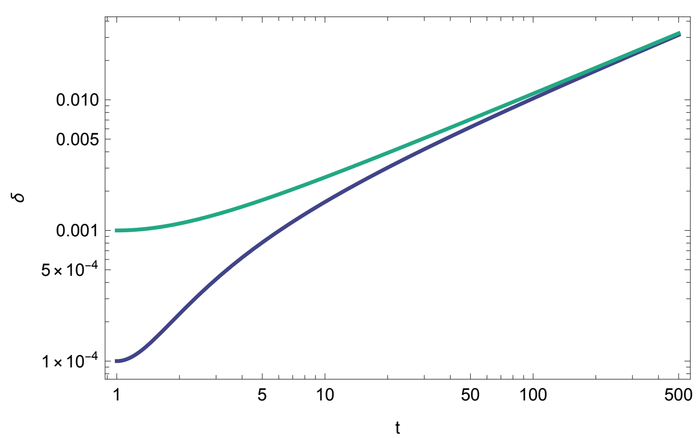
In[12]:=
```
δBPlot = Plot[sysSolution[[2]][t], {t, 1, 500}, ScalingFunctions → {"Log", "Log"},
  PlotStyle → vMap[0.2], Frame → True, FrameLabel → {"t", "δ"}]
```

Out[12]=



In[13]:=
```
Show[δBPlot , δDPlot]
```

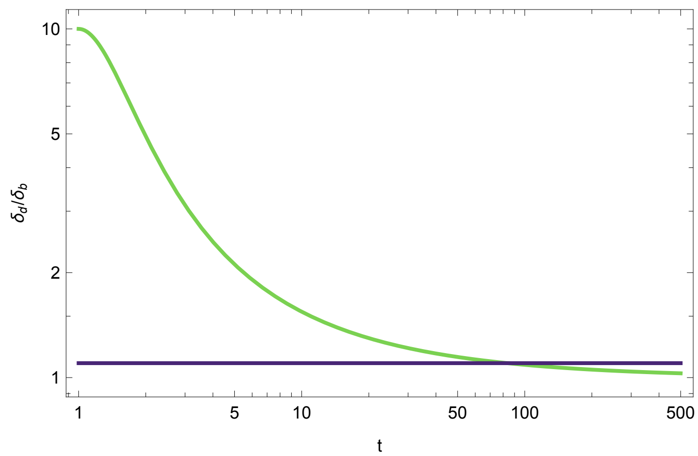
Out[13]=



In[14]:=
```
ratio = sysSolution[[1]][t] / sysSolution[[2]][t];
```

In[15]:=
```
ratioPlot = Plot[{ratio, 1.1}, {t, 1, 500},
  ScalingFunctions → {"Log", "Log"}, PlotStyle → {vMap[0.8], vMap[0.1]} ,
  PlotRange → All, Frame → True, FrameLabel → {"t", "δd/δb"}]
```

Out[15]=



In[16]:=
```
tPerturbation = (t /. FindRoot[ratio == 1.1, {t, 1, 500}]) × 380 000
```

Out[16]=

3.17432×10^7

Problem 2: Matter Growth with Dark Energy

In[17]:=
```
ode = a^2 (1 - ΩM0 + ΩM0 / (a^3)) f''[a] + (3 a / 2)
  (2 - 2 ΩM0 + ΩM0 / (a^3)) f'[a]  - (3 / 2) (ΩM0 / (a^3)) f[a] == 0
```

Out[17]=

$$-\frac{3\,\Omega M0\, f[a]}{2\,a^3} + \frac{3}{2}\,a\left(2 - 2\,\Omega M0 + \frac{\Omega M0}{a^3}\right) f'[a] + a^2\left(1 - \Omega M0 + \frac{\Omega M0}{a^3}\right) f''[a] == 0$$

In[18]:=
```
params2 = {ΩM0 → 0.3};
ai = 1 / 3600;
```

In[20]:=
```
sol = NDSolveValue[{ode /. params2, f[ai] == 2*^-3, f'[ai] == f[ai] / ai}, f, {a, ai, 5}]
```

Out[20]=

InterpolatingFunction[Domain: {{0.000278, 5.}} Output: scalar]

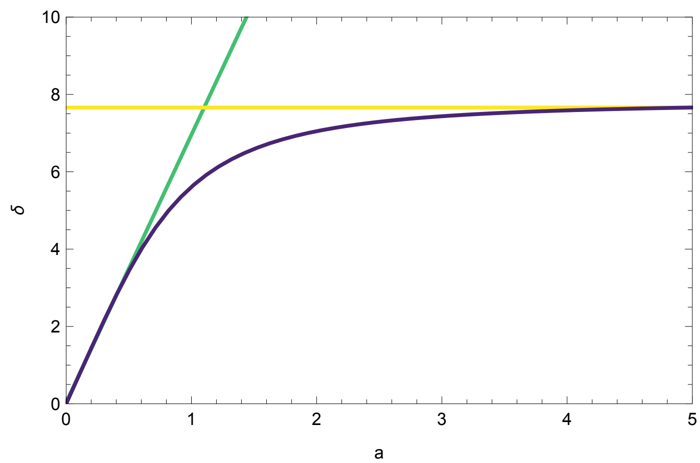
In[21]:=
```
data = Table[{a, sol[a]}, {a, ai, 0.5, 0.001}];
model = NonlinearModelFit[data, x0 + α a, {x0, α}, a]["BestFit"]
```

Out[22]=

0.034241 + 6.922 a

In[23]:=
```
Plot[{model, sol[5], sol[a]}, {a, ai, 5},
 Frame → True, PlotStyle → {vMap[0.7], vMap[1], vMap[0.1]},
 PlotRange → {{0, 5}, {0, 10}}, FrameLabel → {"a", "δ"}]
```

Out[23]=



Problem 3: Scalar Field (Inflaton) in Expanding Universe

In[24]:=
```
odeInflaton = D[ϕ[t], {t, 2}] + 3 H D[ϕ[t], t] + m^2 ϕ[t] == 0;
```

In[25]:=
```
solInflaton = DSolveValue[{odeInflaton}, ϕ[t], t, Assumptions → {H > 0, m > 0}]
```

Out[25]=

$$e^{\frac{1}{2}\left(-3H-\sqrt{9H^2-4m^2}\right)t}\,\mathbb{C}_1+e^{\frac{1}{2}\left(-3H+\sqrt{9H^2-4m^2}\right)t}\,\mathbb{C}_2$$