Problem 1: Evolution of Universe with general Energy of Particles

Emmanuel Flores

January 31, 2025

The Friedmann equation is given by

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

where

$$\rho = \frac{\rho_0 R_0^3}{R^3}.$$

Problem 1. Rewrite the Friedmann equation by replacing derivatives with respect to the time variable t by derivatives with respect to the conformal time variable η defined by

$$dt = Rd\eta$$

Solution 1. By the chain rule, we have

$$\frac{dR}{dt} = \frac{dR}{d\eta}\frac{d\eta}{dt} \implies \frac{dR}{dt} = \frac{dR}{d\eta}\frac{1}{R},$$

and from this, the Friedmann equation is rewriten as

$$\left(\frac{dR}{d\eta}\frac{1}{R^2}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi GR^4}{3}\rho + \frac{2ER^4}{mR^2},$$

but ρ is a function of R, thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi \rho_0 R_0^3 G}{3}R + \frac{2E}{m}R^2.$$

and by making

$$\alpha = \frac{8\pi \rho_0 R_0^3 G}{3}, \beta = \frac{2E}{m},$$

we have

$$\left(\frac{dR}{d\eta}\right)^2 = \alpha R + \beta R^2.$$

Problem 2. Use separation of variables to re-write the Friedmann equation as an expression for η in terms of an integral of R.

Solution 2. By separation of variables we have

$$\frac{dR}{\sqrt{\alpha R + \beta R^2}} = d\eta,$$

thus, the problem now resides on solving the preovious integral.

Problem 3. Carry out the integral and invert to obtain $R(\eta)$; simplify for $E > 0$, $E = 0$, and $E < 0$.

Solution 3. In the variables that I'm using $E = 0 \implies \beta = 0$, $E > 0 \implies \beta > 0$, and $E < 0 \implies \beta < 0$, So let's proceed accordingly.

Case 1, $\beta = 0$: in this case, the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha}{2}\eta^2$$

Case 2, $\beta < 0$: Here, I'm going to add the sign of β by hand (not elegant at all, but just to be more explicit),

$$\int \frac{dR}{\sqrt{\alpha R - \beta R^2}} = \int d\eta,$$

and here the idea is to make a change of variables (I attach the notes with the algebra at the end of the document). Having done that, the solution reads

$$R(\eta) = \frac{\alpha}{\beta} \cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)$$

Case 3, $\beta > 0$: Here the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R + \beta R^2}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}},$$

and again, details about the algebra are attached at the end of the document.

Problem 4. Determine $t(\eta)$; again simplify for $E > 0$, $E = 0$, and $E < 0$.

Solution 4. We know that
$$dt = R\,d\eta,$$
and in the previous bullet we found exactly $R(\eta)$, thus we only need to integrate, this is
$$t(\eta) = \int d\eta\, R(\eta).$$

Case 1, $\beta = 0$: in this case we have
$$t(\eta) = \frac{\alpha}{2} \int d\eta \left(\eta^2\right),$$
with solution given by
$$t = \frac{\alpha}{6}\eta^3.$$

Case 2, $\beta < 0$: here we have
$$t(\eta) = \frac{\alpha}{\beta} \int d\eta \cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right),$$
with solution
$$t = \frac{\alpha}{\beta} \frac{2\sin\left(\frac{\sqrt{\beta}}{2}\eta\right)}{\sqrt{\beta}}$$

Case 3, $\beta > 0$: and finally, we have
$$t(\eta) = \alpha \int d\eta \frac{e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}}$$
with solution given by
$$t(\eta) = -\frac{\log\left(1 - \beta e^{\alpha x}\right)}{\beta}.$$

Problem 5. Using the above pair of parametric equations, plot $R(t)$ (use some plotting software, such as Mathematica) for 3 different choices of E, namely $E > 0$, $E = 0$, and $E < 0$.

Solution 5. By combining the two previous parts we can write a parametric equation

$$\gamma(\eta) = (t(\eta), R(\eta)),$$

therefore.

Case 1, $\beta = 0$: in this case we have

$$\gamma(\eta) = \frac{\alpha}{2} \left(\eta^3, \eta^2 \right)$$

Case 2, $\beta < 0$: again, we have

$$\gamma(\eta) = \frac{\alpha}{\beta} \left(\frac{2}{\sqrt{\beta}} \sin \left(\frac{\sqrt{\beta}}{2} t \right), \cos^2 \left(\frac{\sqrt{\beta}}{2} t \right) \right)$$

Case 3, $\beta > 0$: and finally

$$\gamma(\eta) = \left(-\frac{\log \left(1 - \beta e^{\alpha x} \right)}{\beta}, \frac{\alpha e^{\alpha \eta}}{1 - \beta e^{\alpha \eta}} \right)$$

The plots are shown in the next page.