Electricity and Magnetism

Tufts University

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Long Assignment 1

Tufts UNIVERSITY | School of Arts and Sciences

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Statement of the problem.
A small mass of the photon would be responsible for changing the Coulomb's law from its classical form to

$$\overrightarrow{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\left(1+\frac{r}{\lambda}\right)\exp\left(\frac{-r}{\lambda}\right)\hat{r} \tag{0.1}$$

where λ is a new constant of nature proportional to the inverse of the photon mass (mg^{-1}). Assuming that the superposition principle still holds, this formula can be easily generalized to any charge distribution ρ. We will use this to reformulate Plimton & Lawton's experiments in terms of the small-photon-mass potential.

1. Find the potential of a point charge q situated at the origin of the coordinate system, if we set the reference to be null at infinity, assuming this new version of Coulomb's law.

2. If this charge q is rather distributed on a conducting shell of radius R maintained at a constant potential value by a battery, show that the potential at any point inside the shell at a distance $r < R$ from the center satisfies:

$$\frac{V(r) - V(R)}{V(R)} \approx \frac{-1}{6\lambda^2}\left[R^2 - r^2\right]$$

3. The other approach, subject of this problem, does not assume any specific expression for the deviation of the electrostatic potential from Coulomb's law. It is therefore more general, but less explicative. It only assumes that the deviation to the Coulomb's potential is very small, i.e. that:

$$V(r) = kqr^{-(1+\epsilon)}, \text{ with } \epsilon \ll 1.$$

Measuring the potential difference between two concentric spherical shells, the outer shell of a radius R being maintained at a constant potential $V(R)$, and the inner shell of radius r being initially grounded, would test this hypothesis. Show that a small deviation with respect to Coulomb's potential would yield a ratio of the potential difference to the applied voltage of:

$$\frac{V(r) - V(R)}{V(R)} = \frac{\epsilon}{2}\left[\ln\left(\frac{R-r}{R+r}\right) + \ln\left(\frac{4R^2}{R^2-r^2}\right)\right].$$

4. The two approaches have been formulated in similar terms and can now be compared. In 1970, Barnett et al. obtained a limit of $|e| < 1.3 \times 10^{-13}$ using the generic deviation approach. How would Barnett limit on ϵ convert to a limit on $1/\lambda$? To answer this, assume that the inner sphere has a radius of 60 cm, 2/3 of the radius of the large sphere. Compare this result with the best limits on λ. Discuss your comparison. This comparison shows that while being very general, the second approach is not the most precise way to test all possible assumptions about deviations to $1/r^2$.

Solutions.

1. Let's start by write the equation (0.1) as

$$\overrightarrow{F} = q_1 \overrightarrow{E},$$

where

$$\overrightarrow{E} = \frac{1}{4\pi\epsilon_0} \frac{q_2}{r^2} \left(1 + \frac{r}{\lambda}\right) \exp\left(\frac{-r}{\lambda}\right) \hat{r}, \tag{0.2}$$

in the previous expresion let's make the following change of variables

$$q_2 \to q, \frac{r}{\lambda} \to x \implies \frac{1}{r^2} = \frac{1}{\lambda^2 x^2}, dr = \lambda dx$$

then we have that the electric field given in equation (0.2) becomes

$$\overrightarrow{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{\lambda^2 x^2} (1 + x) \exp(-x) \hat{r}.$$

On the other hand, we know that the potential can be obtain by the following expression

$$V(r) = -\int_{\mathcal{O}}^{r} \overrightarrow{E} \cdot d\overrightarrow{l},$$

but because $\overrightarrow{E}$ is a central force, the previous integral can be written as

$$V(r) = -\int_{\mathcal{O}}^{r} E dr,$$

then, we have that

$$V = -\frac{1}{4\pi\epsilon_0} \frac{q}{\lambda} \int_{\mathcal{O}}^{r} \frac{1}{x^2} (1 + x) \exp(-x)\, dx,$$

then if we rename

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{q}{\lambda},$$

we have that the previous integral, can be written as

$$V = -\alpha \int \left(\frac{1}{x^2} + \frac{1}{x}\right) \exp(-x)\, dx,$$

$$V = -\alpha \left(\int \frac{1}{x^2} \exp(-x)\, dx + \int \frac{1}{x} \exp(-x)\, dx\right), \tag{0.3}$$

and the problem nos transforms in solving the following two integrals

$$\int \frac{1}{x^2} \exp(-x)\, dx \quad , \int \frac{1}{x} \exp(-x)\, dx,$$

but now, let's focus on the first integral,

$$\int \frac{1}{x^2} \exp(-x)\, dx, \tag{0.4}$$

and the plan will be to attack it using integration by parts, and choose

$$dv = \frac{1}{x^2} dx \implies v = -\frac{1}{x},$$

and

$$u = \exp(-x) \implies du = -\exp(-x)\,dx,$$

then, we have that the equation (0.4) will be

$$\int \frac{1}{x^2}\exp(-x)\,dx = -\frac{1}{x}\exp(-x) - \int\left(-\frac{1}{x}\right)(-\exp(-x)\,dx),$$

$$\implies \int \frac{1}{x^2}\exp(-x)\,dx = -\frac{1}{x}\exp(-x) - \int \frac{1}{x}\exp(-x)\,dx.$$

And if we know plug this new information in the equation (0.3), we have that

$$V = -\alpha\left(-\frac{1}{x}\exp(-x) - \int\frac{1}{x}\exp(-x)\,dx + \int\frac{1}{x}\exp(-x)\,dx\right),$$

and as we can see, the integrals cancel, then, we have that

$$V = \alpha\frac{1}{x}\exp(-x),$$

or in the original coordinates, we have that

$$V(r) = \frac{\alpha}{\left(\frac{r}{\lambda}\right)}\exp\left(-\frac{r}{\lambda}\right) = \frac{\alpha\lambda}{r}\exp\left(-\frac{r}{\lambda}\right),$$

$$\implies V(r) = \frac{\alpha\lambda}{r}\exp\left(-\frac{r}{\lambda}\right),$$

but in the previous expression we need to evaluate on the limits of integration, which are the observation point $\mathcal{O}$ and r, which I intentionally droped from the calculation, just to make the notiation a little bit cleaner. Even more, because we are considering our reference point at ∞, we have that the exponential goes to 0, and then, we have

$$V(r) = \frac{\alpha\lambda}{r}\exp\left(-\frac{r}{\lambda}\right),$$

and using the actual representation of α, we have

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\exp\left(-\frac{r}{\lambda}\right).$$

2. For this part we're going to generalize the previous result to a continuous distribution, in this particular case, a spherical shell, and for that, we're going to use the following diagram.

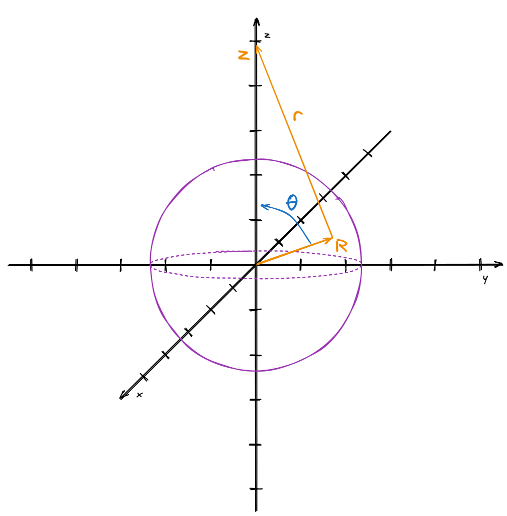

Figure 0.1: Spherical shell with constant surface density.

Therefore, we have that

$$dq = \sigma dS,$$

where σ it's a surface charge density, thus

$$dV = \frac{1}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda}\right) \sigma dS,$$

$$\implies V = \int_S \frac{1}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda}\right) \sigma dS,$$

in this case, given the geometry, we have that

$$V = \frac{\sigma}{4\pi\epsilon_0} \int_S \frac{1}{r} \exp\left(-\frac{r}{\lambda}\right) dS,$$

now let's write the explicit integral and also the differential surface element

$$V = \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^{\pi} \frac{1}{r} \exp\left(-\frac{r}{\lambda}\right) R^2 \sin\theta d\theta d\phi,$$

but because there's no dependence of ϕ in the integrand, that integration becomes immediate

$$V = \frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi} \frac{1}{r} \exp\left(-\frac{r}{\lambda}\right) \sin\theta d\theta,$$

but given the geometry of this problem, we have that

$$r = \sqrt{R^2 + z^2 - 2Rz\cos\theta},$$

therefore, we can write the integral as

$$V = \frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi} \frac{\exp\left(-\frac{1}{\lambda}\sqrt{R^2 + z^2 - 2Rz\cos\theta}\right)}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} \sin\theta d\theta.$$

So let's try to write the previous equation in a better way, and for so, let's make the following change of variable

$$u = \sqrt{R^2 + z^2 - 2Rz\cos\theta},$$

$$\implies \frac{du}{d\theta} = \frac{2Rz\sin\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}},$$

$$\implies du = \frac{2Rz\sin\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} d\theta,$$

and then we have that

$$\frac{du}{2Rz} = \frac{\sin\theta}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} d\theta,$$

therefore, the whole integral tranforms into

$$V = \frac{\sigma R^2}{2\epsilon_0} \int_0^{\pi} \frac{\exp\left(-\frac{1}{\lambda}\sqrt{R^2 + z^2 - 2Rz\cos\theta}\right)}{\sqrt{R^2 + z^2 - 2Rz\cos\theta}} \sin\theta d\theta = \frac{\sigma R^2}{2\epsilon_0} \frac{1}{2Rz} \int_b^b \exp\left(-\frac{u}{\lambda}\right) du,$$

$$\therefore V = \frac{\sigma}{4\epsilon_0} \frac{R}{z} \int_b^b \exp\left(-\frac{u}{\lambda}\right) du,$$

and the above integral it's immediate, thus, we have

$$V = \frac{\sigma}{4\epsilon_0} \frac{R}{z} \left[-\lambda \exp\left(-\frac{1}{\lambda}\sqrt{R^2 + z^2 - 2Rz\cos\theta}\right)\right]_0^{\pi},$$

$$\implies V = -\lambda \frac{\sigma}{4\epsilon_0} \frac{R}{z} \left[\exp\left(-\frac{1}{\lambda} \sqrt{R^2 + z^2 + 2Rz} \right) - \exp\left(-\frac{1}{\lambda} \sqrt{R^2 + z^2 - 2Rz} \right) \right],$$

$$\implies V = -\lambda \frac{\sigma}{4\epsilon_0} \frac{R}{z} \left[\exp\left(-\frac{1}{\lambda} \sqrt{(R+z)^2} \right) - \exp\left(-\frac{1}{\lambda} \sqrt{(R-z)^2} \right) \right],$$

and because we're considering the case in which we're inside the sphere, we have that $R - z > 0$, then

$$\implies V(z) = -\lambda \frac{\sigma}{4\epsilon_0} \frac{R}{z} \left[\exp\left(-\frac{R+z}{\lambda} \right) - \exp\left(-\frac{R-z}{\lambda} \right) \right],$$

and before move on to more calculations let's rename make the following change of variable

$$\alpha = \frac{\sigma}{4\epsilon_0},$$

therefore, the potential field becomes

$$V(z) = \alpha \frac{\lambda R}{z} \left[\exp\left(\frac{z-R}{\lambda} \right) - \exp\left(\frac{-z-R}{\lambda} \right) \right].$$

On the other hand, with this result we can obtain $V(R)$, which will be

$$V(R) = \alpha\lambda \left[1 - \exp\left(-\frac{2R}{\lambda} \right) \right].$$

And now, let's Taylor Expand the exponentials inside the potential field, but before doing any math, let's write the first three terms of the Taylor expansion for an exponential function

$$\exp(x) \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

So, for the $\exp\left(\frac{z-R}{\lambda}\right)$ we have

$$\exp\left(\frac{z-R}{\lambda} \right) \approx 1 + \left(\frac{z-R}{\lambda} \right) + \frac{1}{2!} \left(\frac{z-R}{\lambda} \right)^2 + \frac{1}{3!} \left(\frac{z-R}{\lambda} \right)^3 + \dots,$$

and, for the other exponential, we have

$$\exp\left(\frac{-z-R}{\lambda} \right) \approx 1 + \left(\frac{-z-R}{\lambda} \right) + \frac{1}{2!} \left(\frac{-z-R}{\lambda} \right)^2 + \frac{1}{3!} \left(\frac{-z-R}{\lambda} \right)^3 + \dots,$$

and now, let's take the rest of the two previous equations, and let's do it term by term separatedly

$$\left(\frac{z-R}{\lambda} \right) - \left(\frac{-z-R}{\lambda} \right) = \frac{2z}{\lambda},$$

$$\frac{1}{2!} \left[\left(\frac{z-R}{\lambda} \right)^2 - \left(\frac{-z-R}{\lambda} \right)^2 \right] = \frac{1}{2!} \left(\frac{-4zR}{\lambda^2} \right),$$

$$\frac{1}{3!} \left[\left(\frac{z-R}{\lambda} \right)^3 - \left(\frac{-z-R}{\lambda} \right)^3 \right] = \frac{1}{3!} \left[\left(\frac{z-R}{\lambda} \right)^3 + \left(\frac{z+R}{\lambda} \right)^3 \right] = \frac{1}{3!} \left(\frac{6R^2 z}{\lambda^3} + \frac{2z^3}{\lambda^3} \right),$$

and now, let's add the previous three equations together

$$\mathcal{O}_3 = \frac{2z}{\lambda} + \frac{1}{2!} \left(\frac{-4zR}{\lambda^2} \right) + \frac{1}{3!} \left(\frac{6R^2 z}{\lambda^3} + \frac{2z^3}{\lambda^3} \right),$$

$$\implies \mathcal{O}_3 = \frac{2z}{\lambda} - \frac{2zR}{\lambda^2} + \frac{R^2 z}{\lambda^3} + \frac{2z^3}{6\lambda^3},$$

$\Longrightarrow$ therefore the potential field will be

$$V\left(z\right)_{\mathcal{O}_3}=\alpha\frac{\lambda R}{z}\left[\frac{2z}{\lambda}-\frac{2zR}{\lambda^2}+\frac{R^2z}{\lambda^3}+\frac{z^3}{3\lambda^3}\right]=\alpha\left[2R-\frac{2R^2}{\lambda}+\frac{R^3}{\lambda^3}+\frac{2Rz^2}{6\lambda^3}\right],$$

$$\Longrightarrow\; V\left(z\right)_{\mathcal{O}_3}=\alpha\left[2R-\frac{2R^2}{\lambda}+\frac{R^3}{\lambda^2}+\frac{2Rz^2}{6\lambda^2}\right],$$

on the other, hand, we can also consider an aproximation for the potential $V\left(R\right)$, and we have that

$$V\left(R\right)_{\mathcal{O}_3}=\alpha\lambda\left[1-\left(1+\left(-\frac{2R}{\lambda}\right)+\frac{1}{2!}\left(-\frac{2R}{\lambda}\right)^2+\frac{1}{3!}\left(-\frac{2R}{\lambda}\right)^3\right)\right],$$

$$\Longrightarrow\; V\left(R\right)_{\mathcal{O}_3}=\alpha\lambda\left[\frac{2R}{\lambda}-\frac{4R^{\,2}}{2\lambda^2}+\frac{8R^3}{6\lambda^3}\right],$$

$$\therefore V\left(R\right)_{\mathcal{O}_3}=\alpha\left[2R-\frac{2R^{\,2}}{\lambda}+\frac{4R^3}{3\lambda^2}\right],$$

and with this result, we have that

$$V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}=\alpha\left[2R-\frac{2R^2}{\lambda}+\frac{R^3}{\lambda^2}+\frac{2Rz^2}{6\lambda^2}-2R+\frac{2R^2}{\lambda}-\frac{4R^3}{3\lambda^2}\right],$$

$$\Longrightarrow\; V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}=\alpha\left[-\frac{R^3}{3\lambda^2}+\frac{2Rz^2}{6\lambda^2}\right]=\alpha R\left[\frac{2z^2}{6\lambda^2}-\frac{2R^2}{6\lambda^2}\right],$$

$$\therefore V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}=\frac{2\alpha R}{6\lambda^2}\left[z^2-R^2\right],$$

and, with this result we have

$$\frac{V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}}{V\left(R\right)_{\mathcal{O}_3}}=\frac{\frac{2\alpha R}{6\lambda^2}\left[z^2-R^2\right]}{\alpha\left[2R-\frac{2R}{\lambda}^2+\frac{4R^3}{3\lambda^2}\right]}=\frac{\frac{2\alpha R}{6\lambda^2}\left[z^2-R^2\right]}{2\alpha R\left[1-\frac{R}{\lambda}+\frac{2R^2}{3\lambda^2}\right]},$$

$$\Longrightarrow\; \frac{V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}}{V\left(R\right)_{\mathcal{O}_3}}=\frac{1}{6\lambda^2}\frac{\left[z^2-R^2\right]}{\left[1-\frac{R}{\lambda}+\frac{2R^2}{3\lambda^2}\right]},$$

in the previous equation we can perform another aproximation for the denominator, we can perform a binomial-like expansion in order to transform the denominator into the numerator, but with this we're going to have something like

$$\frac{1}{\left[1-\frac{R}{\lambda}+\frac{2R^2}{3\lambda^2}\right]}\approx 1+\frac{R}{\lambda}+\mathcal{O}\left(\frac{1}{\lambda^3}\right),$$

then, we have

$$\frac{V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}}{V\left(R\right)_{\mathcal{O}_3}}=\frac{1}{6\lambda^2}\left[z^2-R^2\right]\left[1+\frac{R}{\lambda}+\mathcal{O}\left(\frac{1}{\lambda^3}\right)\right],$$

but in the previous expression we have terms that behave like $1/\lambda^3$, therefore, we drop those terms, and we keep up to order $1/\lambda^2$, then we finnlay have

$$\frac{V\left(z\right)_{\mathcal{O}_3}-V\left(R\right)_{\mathcal{O}_3}}{V\left(R\right)_{\mathcal{O}_3}}=\frac{1}{6\lambda^2}\left[z^2-R^2\right]=-\frac{1}{6\lambda^2}\left[R^2-z^2\right],$$

$$\therefore \frac{V(r)-V\left(R\right)}{V\left(R\right)}\approx-\frac{1}{6\lambda^2}\left[R^2-z^2\right] \tag{0.5}$$

3. For this part, we should have that the potential for a point charge is given as

$$V_p\left(r\right)=kqr^{-(1+\epsilon)},$$

where $\epsilon \ll 1$, and the sub index p s just to make even more explicit that this potential it's for a point charge. So the first thing is to generalize the given potential for a continous distribution, in this case, let's generalize it to a spherical shell. And before doing any calculation, let's use the following diagram as a guide

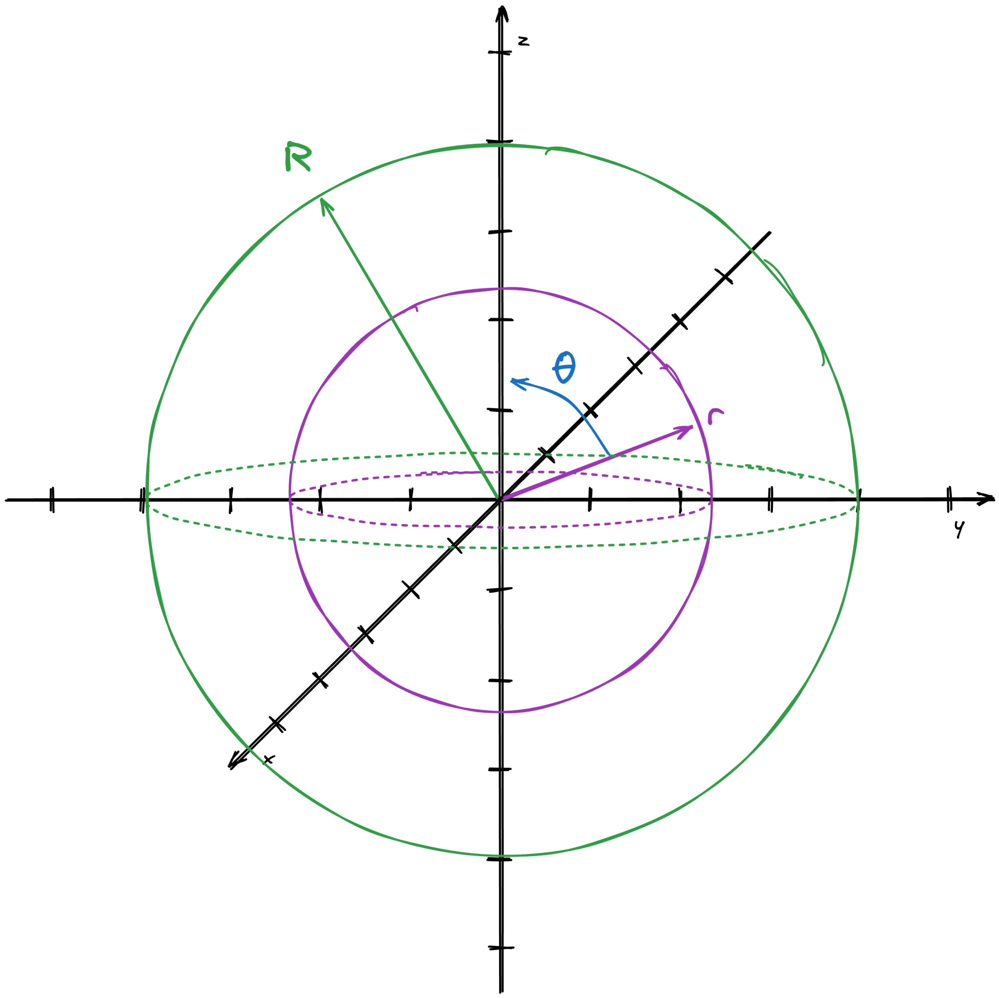

Figure 0.2: Concentric shells.

What we're going to do is to integrate the previous expression for a surface, i.e,

$$V = \int_S V_p dS,$$

where $dS = R^2 \sin\theta d\theta d\phi$, and we also have to consider this $dq = \sigma dS$, therefore, we have

$$V = \int_S k\sigma r^{-(1+\epsilon)} R^2 \sin\theta d\theta d\phi,$$

and from the figure we can see that

$$r^2 = z^2 + R^2 - 2zR\cos\theta,$$

therefore, we have that

$$V = k\sigma \int_0^{2\pi} d\phi \int_0^{\pi} \frac{R^2 \sin\theta d\theta}{(z^2 + R^2 - 2zR\cos\theta)^{(1+\epsilon)}},$$

and now, for this integral we're going to use the following change of variables

$$u^2 = z^2 + R^2 - 2zR\cos\theta,$$

$$\implies 2udu = 2zR\sin\theta d\theta,$$
$$\implies udu = zR\sin\theta d\theta,$$
$$\implies \sin\theta d\theta = \frac{1}{zR}udu$$

and with change of variables we have that $\theta = 0$, implies $u = R - z$, and $u = \pi$, implies $u = R + z$, thus, we have that

$$V = k\sigma\frac{R^2}{zR}\int_0^{2\pi} d\phi \int_{R-z}^{R+z} \frac{udu}{u^{(1+\epsilon)}},$$
$$\implies V = k\sigma\frac{2\pi R}{z}\int_{R-r}^{R+r} \frac{udu}{u^{(1+\epsilon)}} = \frac{k\sigma}{zR}\frac{1}{1-\epsilon}\left(u^{(1-\epsilon)}\right)|_{R-r}^{R+r},$$
$$\implies V = k\sigma\frac{2\pi R}{z}\frac{1}{1-\epsilon}\left(u^{(1-\epsilon)}\right)|_{R-r}^{R+r},$$
$$\therefore V(r) = \frac{\alpha R}{r(1-\epsilon)}\left((R+r)^{(1-\epsilon)} - (R-r)^{(1-\epsilon)}\right),$$

where $\alpha = 2\pi k\sigma$, and if we know evaluate $V(R)$, we have that

$$V(R) = \alpha\frac{R}{R}\frac{1}{1-\epsilon}\left((R+R)^{(1-\epsilon)} - (R-R)^{(1-\epsilon)}\right) = \frac{\alpha}{1-\epsilon}(2R)^{1-\epsilon},$$
$$\implies V(R) = \frac{\alpha}{1-\epsilon}(2R)^{1-\epsilon}.$$

Therefore

$$\frac{V(r)}{V(R)} = \frac{\frac{\alpha R}{r(1-\epsilon)}\left((R+r)^{(1-\epsilon)} - (R-r)^{(1-\epsilon)}\right)}{\frac{\alpha}{1-\epsilon}(2R)^{1-\epsilon}},$$
$$\implies \frac{V(r)}{V(R)} = \frac{R}{r}\frac{\left((R+r)^{(1-\epsilon)} - (R-r)^{(1-\epsilon)}\right)}{(2R)^{1-\epsilon}}.$$

Now, if we use the following approximation $x^{1-\epsilon} = x(1-\ln x)$, then we have that

$$(2R)^{1-\epsilon} = 2R(1-\ln 2R),$$

and also

$$(R+r)^{(1-\epsilon)} = (R+r)(1-\epsilon\ln(R+r)),$$
$$(R-r)^{1-\epsilon} = (R-r)(1-\epsilon\ln(R-r)),$$

then we can express

$$\frac{V(r)}{V(R)} = \frac{R}{r}\frac{((R+r)(1-\epsilon\ln(R+r)) - (R-r)(1-\epsilon\ln(R-r)))}{2R(1-\epsilon\ln 2R)},$$
$$\implies \frac{V(r)}{V(R)} = \frac{1}{2r}\frac{((R+r)(1-\epsilon\ln(R+r)) - (R-r)(1-\epsilon\ln(R-r)))}{(1-\epsilon\ln 2R)},$$

now, let's play a little with the numerator of the previous equation, which, for a better reading, I'm going to call it D, then

$$D = R + r - R\epsilon\ln(R+r) - r\epsilon\ln(R+r) - R + r + R\epsilon\ln(R-r) - r\epsilon\ln(R-r),$$

then, we have

$$D = 2r + R\epsilon(\ln(R+r) - \ln(R-r)) - r\epsilon(\ln(R+r) + \ln(R-r)),$$

but using properties of the logatimic function, we can write the previous equation as

$$D = 2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) - r\epsilon(\ln[(R+r)(R-r)]),$$

$$\implies D = 2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) - r\epsilon\left(\ln\left(R^2-r^2\right)\right),$$

$$\implies D = 2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) + r\epsilon\left(\ln\left(R^2-r^2\right)^{-1}\right),$$

$$\therefore D = 2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) + r\epsilon\left(\ln\frac{1}{R^2-r^2}\right)$$

then the expression, becomes

$$\frac{V(r)}{V(R)} = \frac{1}{2r}\frac{2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) + r\epsilon\left(\ln\frac{1}{R^2-r^2}\right)}{(1-\epsilon\ln 2R)},$$

and now, in order to get rid of the denominator, we're going to perform the following approximation, for x small enough, we have

$$\frac{1}{1-x} \approx 1+x,$$

then, we can express the denominator as

$$\frac{1}{1-\epsilon\ln 2R} \approx 1+\epsilon\ln 2R,$$

therefore, we have that

$$\frac{V(r)}{V(R)} \approx \frac{1}{2r}\left(2r + R\epsilon\left(\ln\frac{R+r}{R-r}\right) + r\epsilon\left(\ln\frac{1}{R^2-r^2}\right)\right)(1+\epsilon\ln 2R),$$

$$\implies \frac{V(r)}{V(R)} \approx \left(1 + \frac{R\epsilon}{2r}\left(\ln\frac{R+r}{R-r}\right) + \frac{\epsilon}{2}\left(\ln\frac{1}{R^2-r^2}\right)\right)(1+\epsilon\ln 2R)$$

but becasue $\epsilon \ll 1$ we are not going to consider the terms that go as ϵ^2, then, we have that the previous expression can be written as

$$\frac{V(r)}{V(R)} \approx 1 + \frac{R\epsilon}{2r}\left(\ln\frac{R+r}{R-r}\right) + \frac{\epsilon}{2}\left(\ln\frac{1}{R^2-r^2}\right) + \epsilon\ln 2R,$$

but we can write the last term as

$$\epsilon\ln 2R = \frac{\epsilon}{2}2\ln 2R = \frac{\epsilon}{2}\ln(2R)^2 = \frac{\epsilon}{2}\ln\left(4R^2\right),$$

$$\implies \epsilon\ln 2R = \frac{\epsilon}{2}\ln\left(4R^2\right),$$

therefore, we have that

$$\frac{V(r)}{V(R)} \approx 1 + \frac{R\epsilon}{2r}\left(\ln\frac{R+r}{R-r}\right) + \frac{\epsilon}{2}\left(\ln\frac{1}{R^2-r^2}\right) + \frac{\epsilon}{2}\ln\left(4R^2\right),$$

and again, using the properties of the logaritmic function, we have

$$\frac{V(r)}{V(R)} \approx 1 + \frac{R\epsilon}{2r}\left(\ln\frac{R+r}{R-r}\right) + \frac{\epsilon}{2}\left(\ln\frac{4R^2}{R^2-r^2}\right),$$

$$\implies \frac{V(r)}{V(R)} - 1 \approx \frac{\epsilon}{2}\left[\frac{R}{r}\left(\ln\frac{R+r}{R-r}\right) + \left(\ln\frac{4R^2}{R^2-r^2}\right)\right],$$

$$\therefore \frac{V(r)-V(R)}{V(R)} \approx \frac{\epsilon}{2}\left[\frac{R}{r}\left(\ln\frac{R+r}{R-r}\right) + \left(\ln\frac{4R^2}{R^2-r^2}\right)\right].$$

4. For this final question we have to compare the two approximations, and one way to do it is by comparing the expressions for the potentials, on one hand we have

$$\frac{V(r)-V(R)}{V(R)} \approx \frac{-1}{6\lambda^2}\left[R^2-r^2\right],$$

and on another

$$\frac{V(r)-V(R)}{V(R)}=\frac{\epsilon}{2}\left[\ln\left(\frac{R-r}{R+r}\right)+\ln\left(\frac{4R^2}{R^2-r^2}\right)\right],$$

and now, if we have that $r=\frac{2}{3}R$, then, form the first equation we have

$$\frac{V(r)-V(R)}{V(R)}\approx\frac{-1}{6\lambda^2}\left[R^2-\frac{4}{9}R^2\right]=-\frac{1}{6\lambda^2}\left(\frac{5}{9}R^2\right),$$

and on the other hand, for the second approximation we have that

$$\frac{V(r)-V(R)}{V(R)}=\frac{\epsilon}{2}\left[\ln\left(\frac{R}{5R}\right)+\ln\left(\frac{4R^2}{\frac{5}{9}R^2}\right)\right]=\frac{\epsilon}{2}\left[\ln\left(\frac{R}{5R}\right)+\ln\left(\frac{36R}{5R}\right)\right]=$$

$$\implies\frac{V(r)-V(R)}{V(R)}=\frac{\epsilon}{2}\left[\ln\left(\frac{1}{5}\right)+\ln\left(\frac{36}{5}\right)\right].$$