Electricity and Magnetism

Tufts University

Graduate School of Arts and Sciences

Long Assignment 2

J. Emmanuel Flores

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Statement of the problem.

Consider a spherical conductor of radius R centered on the origin of a coordinate system. A point charge q (i.e. a spherical charge distribution of radius $a << R$) is placed at a distance $r > R$ from the center of the conducting sphere.

1. Show that the total electrostatic energy of the conducting sphere is given by:

$$U = \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{2l+1} \left[\frac{R^3 \sigma_l}{2} \frac{4\pi}{2l+1} + \frac{q R^{l+2}}{r^{l+1}} \right]$$

   where the σ_l are the Legendre's expansion coefficient of the charge distribution $\sigma(\theta)$. Justify your assumptions.

2. Use Thomson's theorem and the above formula for $U(\sigma_l)$ to find $\sigma(\theta)$ in terms of Legendre's polynomials. Interpret your result.

3. A net charge Q is produced on the conducting sphere if it is grounded before the point charge q is removed. This procedure will generate an inflow of particles from Earth that will cancel the induced charges on one side of the sphere, producing a net charge Q on the conductor. Find Q using Green's reciprocity relation

4. What would be the total charge Q_1 induced on the grounded conductor if, rather than a point charge q, a pure dipole $\overrightarrow{p}$ was placed at a point $\overrightarrow{r}$ outside the conductor?.

Solutions.

1. We know that the electrostatic energy of a charge distribution $\sigma(\mathbf{r})$ inmersed in an external potential is given by the following expression

$$U_E = U_S + U_I,$$

or, more explicitily

$$U_E = \frac{1}{8\pi\epsilon_0} \int dS \int dS' \frac{\rho(\mathbf{r}) \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int dS' \sigma(\mathbf{r}') \phi_{ext}(\mathbf{r}'),$$

and in the above expression, the first integral is the term called self-energy, whereas the second is callded the interaction energy between the external field and the charge distribution $\sigma(\mathbf{r})$, which we know is given by the following expression

$$\phi_{ext}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}.$$

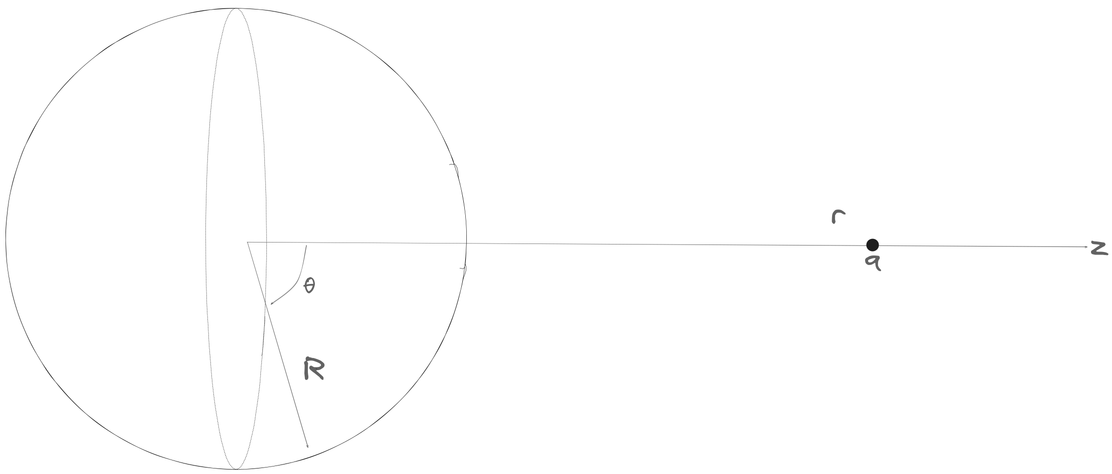

Figure 0.1: Configuration of the problem.

Now, let's focus on the interaction energy, but before, let's plot a figure of the problem at hand . Going back to the interaction energy, we have that

$$U_I = \int dS' \sigma\left(\mathbf{r}'\right) \phi_{ext}\left(\mathbf{r}'\right) = \frac{q}{4\pi\epsilon_0} \int dS' \frac{\sigma\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|},$$

but $dS = R^2 \sin\theta d\theta d\phi$, and even more, because of the symmetry of the problem, we're going to assume that $\sigma\left(\mathbf{r}'\right) = \sigma\left(\theta\right)$, thus

$$U_I = \frac{q}{4\pi\epsilon_0} \int R^2 \sin\theta d\theta d\phi \frac{\sigma\left(\theta\right)}{\left|\mathbf{r}-\mathbf{r}'\right|},$$

$$\implies U_I = \frac{q}{4\pi\epsilon_0} \int R^2 \sin\theta d\theta d\phi \frac{\sigma\left(\theta\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}$$

on the other hand, we can expand

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right),$$

where $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' = \cos\theta$, in which θ is the usual polar angle, and, in addition, we have $\mathbf{r}' = \mathbf{R}$, thus

$$\frac{1}{\left|\mathbf{r}-\mathbf{R}\right|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l\left(\cos\theta\right),$$

then we have that the integral for the interaction energy is

$$U_I = \frac{q}{4\pi\epsilon_0} \int_S R^2 \sin\theta d\theta d\phi \left[\sigma\left(\theta\right)\left(\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l\left(\cos\theta\right)\right)\right],$$

but

$$\int_S \to \int_0^{2\pi} d\phi \int_0^{\pi} d\theta,$$

thus

$$U_I = \frac{qR^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \left[\sigma\left(\theta\right)\left(\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l\left(\cos\theta\right)\right)\right],$$

$$\implies U_I = \frac{2\pi qR^2}{4\pi\epsilon_0} \int_0^{\pi} d\theta \sin\theta \left[\sigma\left(\theta\right)\left(\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l\left(\cos\theta\right)\right)\right],$$

but $d\theta \sin\theta = d\left(\cos\theta\right)$, then

$$U_I = \frac{qR^2}{2\epsilon_0}\int_{-1}^{1} d\left(\cos\theta\right)\sigma\left(\theta\right)\left(\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l\left(\cos\theta\right)\right),$$

$$\implies U_I = \frac{q}{2\epsilon_0}\sum_{l=0}^{\infty}\frac{R^{l+2}}{r^{l+1}}\int_{-1}^{1} d\left(\cos\theta\right)\sigma\left(\theta\right)\left(P_l\left(\cos\theta\right)\right),$$

now, if we assume that we can expand the surface charge density $\sigma\left(\theta\right)$ in the following way

$$\sigma\left(\theta\right) = \sum_m \sigma_m P_m\left(\cos\theta\right),$$

i.e, as an expansion in terms of Legendre polinomials, then the expression for the energy becomes

$$U_I = \frac{q}{2\epsilon_0}\sum_{l=0}^{\infty}\frac{R^{l+2}}{r^{l+1}}\left(\sum_m \sigma_m \int_{-1}^{1} d\left(\cos\theta\right)P_m\left(\cos\theta\right)\left(P_l\left(\cos\theta\right)\right)\right),$$

but, one property of this set of complete set of functions is the orthogonality, which reads

$$\int_{-1}^{1} d\left(\cos\theta\right)P_m\left(\cos\theta\right)\left(P_l\left(\cos\theta\right)\right) = \frac{2}{2l+1}\delta_{lm},$$

thus

$$U_I = \frac{q}{2\epsilon_0}\sum_{l=0}^{\infty}\frac{R^{l+2}}{r^{l+1}}\left(\sum_m \sigma_m \frac{2}{2l+1}\delta_{lm}\right),$$

therefore, the interaction energy becomes

$$U_I = \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\frac{\sigma_l}{2l+1}\frac{R^{l+2}}{r^{l+1}}. \tag{0.1}$$

Now, let's move to the self-energy part, and in this case the energy is given by

$$U_S = \frac{1}{8\pi\epsilon_0}\int dS\int dS'\frac{\sigma\left(\mathbf{r}\right)\sigma\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|},$$

in which

$$\sigma\left(\mathbf{r}\right) = \sum_{m_1}\sigma_{m_1}P_{m_1}\left(\cos\theta_1\right), \quad \sigma\left(\mathbf{r}'\right) = \sum_{m_2}\sigma_{m_2}P_{m_2}\left(\cos\theta_2\right),$$

then we have that

$$U_S = \frac{1}{8\pi\epsilon_0}\int dS\int dS'\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}\left(\sum_{m_1}\sum_{m_2}\sigma_{m_1}P_{m_1}\left(\cos\theta_1\right)\sigma_{m_2}P_{m_2}\left(\cos\theta_2\right)\right),$$

and on the other hand, let's expand the inverse of the distance in terms of the spherical harmonics, i.e.

$$\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|} = \frac{1}{r}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left(\frac{r'}{r}\right)\sum_{m=-l}^{l}Y_{lm}^*\left(\theta_1,\phi_1\right)Y_{lm}\left(\theta_2,\phi_2\right),$$

but in this case we're considering the case in which $\mathbf{r}\to\mathbf{r}'$, and even more, in our notation, $\mathbf{r}' = \mathbf{R}$, thus

$$\frac{1}{\left|\mathbf{r}-\mathbf{R}\right|} = \frac{1}{R}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}^*\left(\theta_1,\phi_1\right)Y_{lm}\left(\theta_2,\phi_2\right),$$

then the integral for the self-energy becomes

$$U_S = \frac{1}{8\pi\epsilon_0 R}\sum_{m_1,m_2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\int dS\int dS'\left(\frac{4\pi}{2l+1}Y_{lm}^*\left(\theta_1,\phi_1\right)Y_{lm}\left(\theta_2,\phi_2\right)\right)\left(\sigma_{m_1}P_{m_1}\left(\cos\theta_1\right)\sigma_{m_2}P_{m_2}\left(\cos\theta_2\right)\right),$$

where we're using this notation, $\sum_{m_1,m_2} = \sum_{m_1}\sum_{m_2}$, and moreover, we also know that

$$dS = R^2 \sin\theta_1 d\theta_1 d\phi_1, \quad dS' = R^2 \sin\theta_2 d\theta_2 d\phi_2,$$

then, the integral becomes

$$U_S = \frac{1}{8\pi\epsilon_0 R} \sum_{m_1,m_2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \int_1 \int_2 R^4 \left(\sin\theta_1 d\theta_1 d\phi_1\right)\left(\sin\theta_2 d\theta_2 d\phi_2\right)\left(Y^*_{lm}\left(\theta_1,\phi_1\right) Y_{lm}\left(\theta_2,\phi_2\right)\right) \times$$
$$\times \left(\sigma_{m_1} P_{m_1}\left(\cos\theta_1\right) \sigma_{m_2} P_{m_2}\left(\cos\theta_2\right)\right),$$

in which $\int_1$ and $\int_2$ refers to the variables in which the integration must be performed, and more explicitly

$$\int_1 \to \int d\theta_1 \int d\phi_1, \quad \int_2 \to \int d\theta_2 \int d\phi_2,$$

and the intervas in which we're performing the integrals are $\theta \in (0,\pi)$ and $\phi \in (0,2\pi)$. Now, it's important to notice the following: inside the integration sign, the only dependence on ϕ_1 or ϕ_2 is in the spherical harmonics, therefore, in the integral for the self-energy we can do

$$U_S = \frac{R^3}{8\pi\epsilon_0} \sum_{m_1,m_2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \int\int \left(\sin\theta_1 d\theta_1\right)\left(\sin\theta_2 d\theta_2\right)\left(\sigma_{m_1} P_{m_1}\left(\cos\theta_1\right) \sigma_{m_2} P_{m_2}\left(\cos\theta_2\right)\right) \times$$
$$\times \left(\int d\phi_1 Y^*_{lm}\left(\theta_1,\phi_1\right) \int d\phi_2 Y_{lm}\left(\theta_2,\phi_2\right)\right),$$

but there's the following relationship for the spherical harmonics

$$Y_{l-m}\left(\theta_1,\phi_1\right) = (-1)^m Y^*_{lm}\left(\theta_1,\phi_1\right),$$

therefore, we have the following integrals

$$\frac{1}{(-1)^m} \int d\phi_1 Y_{l-m}\left(\theta_1,\phi_1\right), \quad \int d\phi_2 Y_{lm}\left(\theta_2,\phi_2\right),$$

and even more, the spherical harmonics also have the following property

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi Y_{lm}\left(\theta,\phi\right) = \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta\right) \delta_{m0},$$

and with this information at hand, the integrals become

$$\frac{1}{(-1)^m} \int d\phi_1 Y_{l-m}\left(\theta_1,\phi_1\right) = \frac{2\pi}{(-1)^m} \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta\right) \delta_{m0} = 2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_1\right) \delta_{m0},$$

$$\implies \int d\phi_1 Y^*_{lm}\left(\theta_1,\phi_1\right) = 2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_1\right) \delta_{m0}$$

and

$$\int d\phi_2 Y_{lm}\left(\theta_2,\phi_2\right) = 2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_2\right) \delta_{m0} = 2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_2\right),$$

$$\implies \int d\phi_2 Y_{lm}\left(\theta_2,\phi_2\right) = 2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_2\right) \delta_{m0},$$

then the integral for the self energy becomes

$$U_S = \frac{R^3}{8\pi\epsilon_0} \sum_{m_1,m_2} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \int\int \left(\sin\theta_1 d\theta_1\right)\left(\sin\theta_2 d\theta_2\right)\left(\sum_{m_1,m_2} \sigma_{m_1} P_{m_1}\left(\cos\theta_1\right) \sigma_{m_2} P_{m_2}\left(\cos\theta_2\right)\right) \times$$
$$\left(2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_1\right)\right)\left(2\pi \sqrt{\frac{2l+1}{4\pi}} p_l\left(\cos\theta_2\right)\right),$$

it's important to notice that with the previous manipulation, we've killed the summation for the m index. Now, moving on with the calculations, we have that $\sin\theta_1 d\theta_1 = d(\cos\theta_1)$ and $\sin\theta_2 d\theta_2 = d(\cos\theta_2)$, then

$$U_S = \frac{R^3}{8\pi\epsilon_0}\sum_{m_1,m_2}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left[4\pi^2\left(\frac{2l+1}{4\pi}\right)\right]\sigma_{m_1}\sigma_{m_2}\int d(\cos\theta_1)\,P_{m_1}(\cos\theta_1)\,p_l(\cos\theta_1)\times$$

$$\times\int d(\cos\theta_2)\,P_{m_2}(\cos\theta_2)\,p_l(\cos\theta_2),$$

but remember, the interval of integration for this variable is $\phi \in (0, 2\pi)$, and with the change of variable, the interval of integration changes to $(-1, 1)$, and even more, we know the following property of the Legendre polinominals

$$\int_{-1}^{1} d(\cos\theta)\,P_l(\cos\theta)\,p_m(\cos\theta) = \frac{2}{2l+1}\delta_{lm},$$

then for the two integrals we have

$$\int_{-1}^{1} d(\cos\theta_1)\,P_{m_1}(\cos\theta_1)\,p_l(\cos\theta_1) = \frac{2}{2m_1+1}\delta_{m_1 l},$$

$$\int_{-1}^{1} d(\cos\theta_2)\,P_{m_2}(\cos\theta_2)\,p_l(\cos\theta_2) = \frac{2}{2m_2+1}\delta_{m_2 l},$$

then the self energy becomes

$$U_S = \frac{R^3}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\sum_{m_1,m_2}\left[4\pi^2\left(\frac{2m+1}{4\pi}\right)\right]\sigma_{m_1}\sigma_{m_2}\left(\frac{2}{2m_1+1}\delta_{m_1 l}\right)\left(\frac{2}{2m_2+1}\delta_{m_2 l}\right),$$

and for the summation over m_1 and m_2, we have

$$\sum_{m_1,m_2}\sigma_{m_1}\sigma_{m_2}\left(\frac{2}{2m_1+1}\delta_{m_1 l}\right)\left(\frac{2}{2m_2+1}\delta_{m_2 l}\right) = \frac{4\sigma_l^2}{(2l+1)^2},$$

then

$$U_S = \frac{R^3}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\left[4\pi^2\left(\frac{2m+1}{4\pi}\right)\frac{4\sigma_l^2}{(2l+1)^2}\right],$$

$$\implies U_S = \frac{R^3}{8\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{16\pi^2}{2l+1}\right)\left(\frac{\sigma_l^2}{(2l+1)}\right),$$

$$\therefore U_S = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{2\pi R^3\sigma_l^2}{(2l+1)^2}, \tag{0.2}$$

then, the total energy will be

$$U_E = U_S + U_I = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{\sigma_l}{2l+1}\frac{R^{l+2}}{r^{l+1}} + \frac{q}{\epsilon_0}\sum_{l=0}^{\infty}\frac{2\pi R^3\sigma_l^2}{(2l+1)^2},$$

$$\implies U_E = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{\sigma_l}{2l+1}\left(\frac{2\pi R^3\sigma_l}{2l+1} + q\frac{R^{l+2}}{r^{l+1}}\right),$$

or

$$U_E = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{\sigma_l}{2l+1}\left(\frac{R^3\sigma_l}{2}\frac{4\pi}{2l+1} + q\frac{R^{l+2}}{r^{l+1}}\right), \tag{0.3}$$

just as we wanted.

2. Now, for this part we're going to make use of the Thompson's Theorem, which states that "the electrostatic energy of a body of fixed shape and size is minimized when its charge Q distributes itself to make the electrostatic potential constant throughout the body". Therefore, we need to minimize the revious expression with respect to σ_l which is the only variable unknown, then the minimization condition reads

$$\frac{\partial U_E}{\partial \sigma_l} = 0,$$

$$\iff \frac{\partial}{\partial \sigma_l}\left[\frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{\sigma_l}{2l+1}\left(\frac{R^3\sigma_l}{2}\frac{4\pi}{2l+1}+q\frac{R^{l+2}}{r^{l+1}}\right)\right] = 0,$$

$$\iff \sum_{l=0}^{\infty}\frac{1}{2l+1}\left(R^3\sigma_l\frac{4\pi}{2l+1}+q\frac{R^{l+2}}{r^{l+1}}\right) = 0,$$

but let's remember that the σ_l comes from the Legendre expansion of the surface charge densiity, and we know that those polinomials are a complete set, which in other things, mean that they form a basis, which implies that the σ_l must be independent of each other, then, th previous condition translates into

$$R^3\sigma_l\frac{4\pi}{2l+1}+q\frac{R^{l+2}}{r^{l+1}} = 0,$$

$$\iff \sigma_l = -q\frac{2l+1}{4\pi R^3}\frac{R^{l+2}}{r^{l+1}} = -q\frac{2l+1}{4\pi}\frac{R^{l-1}}{r^{l+1}},$$

$$\therefore \sigma_l = -q\frac{2l+1}{4\pi}\frac{R^{l-1}}{r^{l+1}}.$$

Now, if we go back to the original expansion for the surface charge density, we have

$$\sigma(\theta) = \sum_l \sigma_l P_l(\cos\theta),$$

then, with the previous result, we have

$$\sigma(\theta) = \sum_l\left(-q\frac{2l+1}{4\pi}\frac{R^{l-1}}{r^{l+1}}\right)P_l(\cos\theta),$$

$$\implies \sigma(\theta) = \frac{-q}{4\pi}\sum_l\left(\frac{R^{l-1}}{r^{l+1}}\right)P_l(\cos\theta),$$

but in order to make the expression more symmetrical let's write

$$\sigma(\theta) = \frac{-q}{4\pi R^2}\sum_l\left(\frac{R}{r}\right)^{l+1}P_l(\cos\theta).$$

3. Let's start by writting the Green's Reciprocity, which is given by

$$\int d^3r'\rho_2(\mathbf{r}')\,\phi_1(\mathbf{r}') = \int d^3r\rho_1(\mathbf{r})\,\phi_2(\mathbf{r}),$$

which says that the potential energy of the charge distribution ρ_2 in the field prodced by ϕ_1 is equal to the potential energy of the charge distribution ρ_1 in the field produced by ϕ_2. It important to make clear that when we say field produced by ϕ_1 we refer to the field produced by the charge distribution ρ_1 and the same thing for the ϕ_2 field. Then we have that

$$\rho_1 = q\delta(\mathbf{r}-\mathbf{r}') + \sigma\delta(\mathbf{r}-\mathbf{R}), \phi_1 = 0,$$

and

$$\rho_2 = \sigma\delta(\mathbf{r}-\mathbf{R}), \phi_2 = \frac{Q}{4\pi\epsilon_0 r},$$

then, we have that

$$0 = \int d^3r\rho_1(\mathbf{r})\,\phi_2(\mathbf{r}),$$

$$\implies 0 = \int d^3r \left[\left(q\delta\left(\mathbf{r}-\mathbf{r}'\right) + \sigma\delta\left(\mathbf{r}-\mathbf{R}\right) \right) \frac{Q}{4\pi\epsilon_0 r} \right],$$

$$\implies q\frac{Q}{4\pi\epsilon_0 r'} + \sigma\frac{Q}{4\pi\epsilon_0 R} 4\pi R^2 = 0,$$

where the $4\pi R^2$ comes from the integration of the solid angle, and moreover, we know that for point outside the sphere, we have $\sigma = \frac{Q}{4\pi}$,then we have that

$$q\frac{Q}{4\pi\epsilon_0 r'} + \frac{Q^2}{4\pi\epsilon_0 R} = 0,$$

which implies that

$$Q = -\frac{Rq}{r},$$

in which I've renamed the variable r' for r.

4. Now, for this part, we're still using Green's reciprocity relation, but in this case we're going to consider $\rho\left(r'\right) = -\mathbf{p}\cdot\nabla\delta\left(\mathbf{r}'-\mathbf{r}_0\right)$, and in this case, we're going to make the comparison with a system with zero volume charge density. Then, using Green's reciprocity we have that,

$$Q\phi_c' + \int d^3r\rho\left(\mathbf{r}\right)\phi'\left(\mathbf{r}\right) = Q'\phi_c + \int d^3r\rho'\left(\mathbf{r}\right)\phi\left(\mathbf{r}'\right),$$

but, in this case, we have that $\phi_c = 0$, $\rho' = 0$, $\phi_c' = \frac{Q'}{4\pi\epsilon_0 R}$ and $\phi_c = \frac{Q'}{4\pi\epsilon_0 r}$

$$\implies Q\phi_c' + \int d^3r\rho\left(\mathbf{r}\right)\phi'\left(\mathbf{r}\right) = 0,$$

$$\implies Q\frac{Q'}{4\pi\epsilon_0 R} + \int d^3r\rho\left(\mathbf{r}\right)\frac{Q'}{4\pi\epsilon_0 r} = 0,$$

$$\implies \frac{Q}{R} + \int d^3r\frac{\rho\left(\mathbf{r}\right)}{r} = 0,$$

but in this case we have that

$$\rho\left(\mathbf{r}\right) = -\mathbf{p}\cdot\nabla\delta\left(\mathbf{r}-\mathbf{r}_0\right),$$

$$\implies \frac{Q}{R} - \int d^3r\frac{\mathbf{p}}{r}\cdot\nabla\delta\left(\mathbf{r}-\mathbf{r}_0\right) = 0$$

but one property of the delta is this

$$\int dxf\left(x\right)\frac{d}{dx}\delta\left(x-x'\right) = -f'\left(x'\right),$$

then, we have that

$$\frac{Q}{R} = -\nabla\cdot\left(\frac{\mathbf{p}}{r}\right),$$

but

$$\nabla\cdot\left(\frac{\mathbf{p}}{r}\right) = \frac{1}{r}\nabla\cdot\mathbf{p} + \mathbf{p}\cdot\nabla\frac{1}{r},$$

and we know that

$$\nabla\cdot\mathbf{p} = 0, \quad \nabla\frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2},$$

then, we have that

$$\frac{Q}{R} = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r_0^2},$$

$$\therefore Q = \frac{R\hat{\mathbf{r}}\cdot\mathbf{p}}{r_0^2}.$$