Advanced Classical Mechanics

Tufts University

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Homework 2

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October 4th, 2023

1. (Derivation 1.10) Prove that the Lagrange equations are invariant under point-like transformations.

   Solution. Let's suppose that we have a set of generalized coordinates given by $\{q_i\}$, where $i = 1, \ldots, n$, therefore, we know that for this set of coordinates, the Euler-Lagrange equations are valid, i.e.

   $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

   But now, let's suppose that we have another set of generalized coordinates given by $\{s_i\}$, where $i = 1, \ldots, n$, and even more, let's assume that we have the following relationship between the generalized coordinates

   $$q_i = q_i\left(\{s_i\}, t\right), \forall i.$$

   Now, let's prove that for this new set of generalized coordinates the Euler-Lagrange equations are also valid. Let's begin by noticing the following

   $$dq_i = \sum_j \frac{\partial q_i}{\partial s_j} ds_i + \frac{\partial q_i}{\partial t} dt,$$

   $$\implies \frac{dq_i}{dt} = \sum_j \frac{\partial q_i}{\partial s_j}\frac{ds_i}{dt} + \frac{\partial q_i}{\partial t},$$

   $$\therefore \dot{q}_i = \sum_j \frac{\partial q_i}{\partial s_j}\dot{s}_i + \frac{\partial q_i}{\partial t}.$$

   Now we must remember that the Lagrangian its a function of $q_i, \dot{q}_i, t$, and with this, let's perform the following the derivative $\frac{\partial \mathcal{L}}{\partial \dot{s}_i}$, then we have

   $$\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \sum_j \left(\frac{\partial \mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial \dot{s}_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \dot{s}_i}\right),$$

   but, we know that $q_j \neq q_j(\dot{s}_i)$, and the $\frac{\partial \dot{q}_j}{\partial \dot{s}_i} = \frac{\partial q_j}{\partial s_i}$ therefore, the previous expression transforms into

   $$\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial q_j}{\partial s_i},$$

now, taking the time derivative of the previous equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \sum_j \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial q_j}{\partial s_i}\right) = \sum_j \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right)\frac{\partial q_j}{\partial s_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\left(\frac{d}{dt}\frac{\partial q_j}{\partial s_i}\right),$$

but because $\mathcal{L}$ in the coordinates $\{q_i\}$ follows Euler-Lagrange equation, we have that we can express the previous equation as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \sum_j \left(\frac{\partial \mathcal{L}}{\partial q_i}\right)\frac{\partial q_j}{\partial s_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\left(\frac{d}{dt}\frac{\partial q_j}{\partial s_i}\right). \tag{0.1}$$

And now, let's calculate the derivative $\frac{d}{dt}\frac{\partial q_j}{\partial s_i}$, thus

$$\frac{d}{dt}\frac{\partial q_j}{\partial s_i} = \sum_k \frac{\partial}{\partial q_k}\frac{\partial q_j}{\partial s_i} + \frac{\partial^2 q_j}{\partial t \partial s_i} = \sum_k \frac{\partial}{\partial s_i}\frac{\partial q_j}{\partial q_k} + \frac{\partial^2 q_j}{\partial s_i \partial t},$$

in the previous expression I've been changed the order of the partial derivatives, then, following with the calculation, we have

$$\frac{d}{dt}\frac{\partial q_j}{\partial s_i} = \frac{\partial}{\partial s_i}\left(\sum_k \frac{\partial q_j}{\partial q_k} + \frac{\partial q_j}{\partial \partial t}\right) = \frac{\partial \dot{q}_k}{\partial s_i},$$

$$\therefore \frac{d}{dt}\frac{\partial q_j}{\partial s_i} = \frac{\partial \dot{q}_k}{\partial s_i},$$

then, we have

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \sum_j \frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_j}{\partial s_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_k}{\partial s_i} = \frac{\partial \mathcal{L}}{\partial s_i},$$

$$\therefore \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{s}_i} = \frac{\partial \mathcal{L}}{\partial s_i},$$

therefore, we end with the Euler-Lagrange equations for the new system of generalized coordinates.

2. (Problem 1.14) Two points of mass m are joined by a rigid weightless rod of lenght l, the center of which is constrain to move on a circle of radius a. Express the kinetic energy in generalized coordinates.

   For this problem, we are going to use the help of the following diagram:

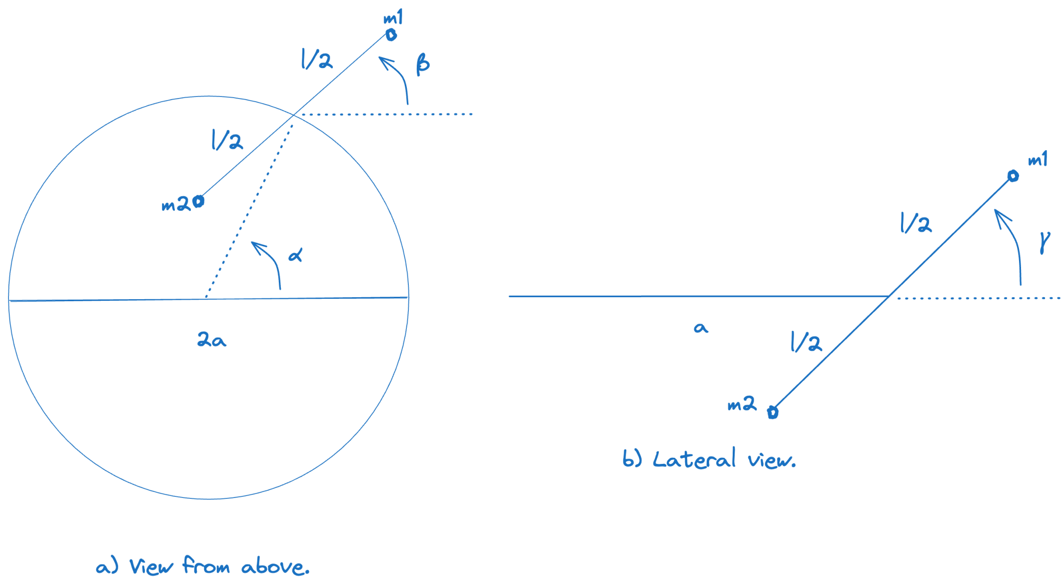

Figure 0.1:

And from the figure, it follows that we're analizing the movement in 3D. But firts, let's focus on the view from above, wich corresponds to the a) part in the previous figure, and from that it follows that the movement of each one of the masses in the rod is given by

$$x_1 = a\cos\alpha + \frac{l}{2}\cos\beta, y_1 = a\sin\alpha + \frac{l}{2}\sin\beta,$$

and, for the m_2 we havesomething very similar

$$x_2 = a\cos\alpha - \frac{l}{2}\cos\beta, y_2 = a\sin\alpha - \frac{l}{2}\sin\beta,$$

and from the b) view, we can see that the z component for both masses as

$$z_1 = \frac{l}{2}\sin\gamma, z_2 = -\frac{l}{2}\sin\gamma.$$

We also know that the kinetic energy is given by

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y_1}^2 + \dot{z_1}^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right),$$

so, let's calculate the derivatives. Let's start with the mass m_1

$$\dot{x_1} = -a\sin\alpha\dot{\alpha} - \frac{l}{2}\sin\beta\dot{\beta},$$

$$\implies \dot{x_1}^2 = a^2\sin^2\alpha\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\sin^2\beta\dot{\beta}^2 + la\sin\alpha\sin\beta\dot{\alpha}\dot{\beta},$$

$$\dot{y_1} = a\cos\alpha\dot{\alpha} + \frac{l}{2}\cos\beta\dot{\beta},$$

$$\implies \dot{y_1}^2 = a^2\cos^2\alpha\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\cos^2\beta\dot{\beta}^2 + la\cos\alpha\cos\beta\dot{\alpha}\dot{\beta},$$

$$\dot{z_1} = \frac{l}{2}\cos\gamma\dot{\gamma},$$

$$\dot{z_1}^2 = \left(\frac{l}{2}\right)^2 \cos^2\gamma\dot{\gamma}^2,$$

then, the energy for this particle will be

$$T_1 = \frac{1}{2}m_1\left(a^2\dot{\alpha}^2\left(\sin^2\alpha + \cos^2\alpha\right) + \left(\frac{l}{2}\right)^2\dot{\beta}^2\left(\sin^2\beta + \cos^2\beta\right) + la\dot{\alpha}\dot{\beta}\left(\sin\alpha\sin\beta + \cos\alpha\cos\beta\right) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right),$$

but we know that $\cos^2 a + \sin^2 b = 1$, and $\cos a\cos b + \sin a\sin b = \cos(a-b)$, then

$$\implies T_1 = \frac{1}{2}m_1\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 + la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right).$$

Now, let's move to the calculations for the m_2 particle

$$\dot{x}_2 = -a\sin\alpha\dot{\alpha} + \frac{l}{2}\sin\beta\dot{\beta},$$

$$\implies \dot{x}_2^2 = a^2\sin^2\alpha\dot{\alpha} + \left(\frac{l}{2}\right)^2\sin^2\beta\dot{\beta}^2 - la\sin\alpha\sin\beta\dot{\alpha}\dot{\beta},$$

$$\dot{y}_2 = a\cos\alpha\dot{\alpha} - \frac{l}{2}\cos\beta\dot{\beta},$$

$$\implies \dot{y}_2^2 = a^2\cos^2\alpha\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\cos^2\beta\dot{\beta}^2 - la\cos\alpha\cos\beta\dot{\alpha}\dot{\beta},$$

$$\dot{z_2} = -\frac{l}{2}\cos\gamma\dot{\gamma},$$

$$\implies \dot{z_2} = \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2,$$

and then, the kinetic energy will be

$$T_2 = \frac{1}{2}m_2\left(a^2\dot{\alpha}^2\left(\sin^2\alpha + \cos^2\alpha\right) + \left(\frac{l}{2}\right)^2\dot{\beta}^2\left(\sin^2\beta + \cos^2\beta\right) - la\dot{\alpha}\dot{\beta}\left(\sin\alpha\sin\beta + \cos\alpha\cos\beta\right) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right),$$

and we can perform the same simplifications as for the particle m_1, then we have

$$\implies T_2 = \frac{1}{2}m_2\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 - la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right),$$

then, the total energy will be

$$\begin{aligned} T = &\frac{1}{2}m_1\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 + la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right) \\ &+\frac{1}{2}m_2\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 - la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right) \end{aligned}$$

But we have that the two masses are equal, then this implies that the total kinetic energy will be

$$\begin{aligned} T = &\frac{1}{2}m\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 + la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right) \\ &+\frac{1}{2}m\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 - la\dot{\alpha}\dot{\beta}\cos(\alpha-\beta) + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right), \end{aligned}$$

$$\implies T = m\left(a^2\dot{\alpha}^2 + \left(\frac{l}{2}\right)^2\dot{\beta}^2 + \left(\frac{l}{2}\right)^2\cos^2\gamma\dot{\gamma}^2\right).$$

3. (Problem 1.19) Obtain the Lagrangian equations of motion for a spherical pendulum, i.e., a mass point suspended by a rigid weightless rod.

   For this problem we are going to use spherical coordinates as our generalized coordinates, with the restriction that the lenght, of the rodcan't change, which therefore reduces the degrees of freedom. Thus, we can analize the motion of the system by just looking at the two angles. Here I present a figure of the problem at hand.

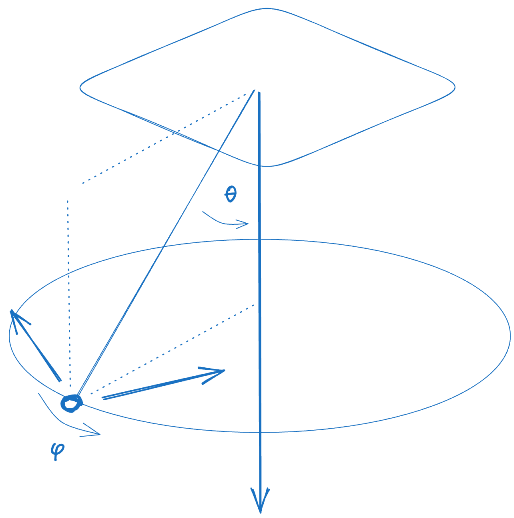

Figure 0.2: Spherical pendulum.

   Now, we know that we can express the cartesian coordinates in terms of the spherical ones, by the following tranformation

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta,$$

   and then it's time derivatives will be, for x

$$\dot{x} = r\frac{d}{dt}\left(\sin\theta\cos\phi\right) = r\left(\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi}\right),$$
$$\implies \dot{x} = r\left(\cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi}\right),$$

   now, for y,

$$\dot{y} = r\frac{d}{dt}\left(\sin\theta\sin\phi\right) = r\left(\cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi}\right),$$
$$\implies \dot{y} = r\left(\cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi\dot{\phi}\right),$$

   and finally, for z

$$\dot{z} = r\frac{d}{dt}\left(\cos\theta\right) = -r\sin\theta\dot{\theta},$$
$$\dot{z} = -r\sin\theta\dot{\theta}$$

   Now, we also know that the kinetic energy is given by

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right),$$

   and usign the previous expressions for $\dot{x}, \dot{y}$ and $\dot{z}$, we have

$$\dot{x}^2 = r^2\left(\cos^2\theta\cos^2\phi\dot{\theta}^2 + \sin^2\theta\sin^2\phi\dot{\phi}^2 - 2\cos\theta\cos\phi\sin\theta\sin\phi\dot{\theta}\dot{\phi}\right),$$

$$\dot{y}^2 = r^2\left(\cos^2\theta\sin^2\phi\dot{\theta}^2 + \sin^2\theta\cos^2\phi\dot{\phi}^2 + 2\cos\theta\sin\phi\sin\theta\cos\phi\dot{\theta}\dot{\phi}\right),$$
$$\dot{z}^2 = r^2\sin^2\theta\dot{\theta}^2,$$

if, we now, sum the three previous equations, we have

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\left(\cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2 + \sin^2\theta\dot{\theta}^2\right),$$
$$\implies \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right),$$

and therefore, the kinetic energy, will be

$$T = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right).$$

On the other hand, the potential energy will be

$$V = mgz = mgr\left(1 - \cos\theta\right),$$

and with this at hand, the Lagrangian will be

$$\mathcal{L} = \frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgr\cos\theta, \tag{0.2}$$

in which I've drop a constant term. Now, we know the lenght is fixed, but to make this even clear, let's change the variable $r \to l$, then we have

$$\mathcal{L} = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta, \tag{0.3}$$

and we also know that the Euler-Lagrange equation for this system are given by

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0, \text{ and } \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

Then, performing the calculations, we have

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta\right) = \frac{d}{dt}\left(ml\dot{\theta}\right),$$
$$\implies \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = ml\ddot{\theta},$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgl\cos\theta\right) = ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta,$$
$$\implies \frac{\partial\mathcal{L}}{\partial\theta} = ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta,$$

then, we have the following equation of motion for the coordinate θ

$$ml\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\phi}^2 + mgl\sin\theta = 0. \tag{0.4}$$

On the other hand, for the ϕ variable, from the Lagrangian, we can see that there's no depedence on the variable ϕ, only on the variable $\dot{\phi}$, therefore, we have that

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 \implies \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \alpha, \tag{0.5}$$

with α a constant, but still we've to perform the derivative

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial}{\partial\dot{\phi}}\left(\frac{1}{2}mr^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mgr\cos\theta\right) = mr^2\sin^2\theta\dot{\phi},$$
$$\implies \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\sin^2\theta\dot{\phi},$$

and finally, our equations of motion will be

$$ml\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\phi}^2 + mgl\sin\theta = 0, \tag{0.6}$$
$$mr^2\sin^2\theta\dot{\phi} = \alpha. \tag{0.7}$$

4. (Problem 1.20) A particle of mass m moves in one dimension such that the Lagrangian is given by

$$L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x), \tag{0.8}$$

where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

Solution.

Lets begin by writing the Euler-Lagrange equation considering only one degree of freedom

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0,$$

then, pluging the Lagrangian given in (0.8), we have that

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(m\dot{x}^2 V(x) - V^2(x)\right) = m\dot{x}^2\frac{dV}{dx} - 2V\frac{dV}{dx},$$

$$\implies \frac{\partial L}{\partial x} = m\dot{x}^2\frac{dV}{dx} - 2V\frac{dV}{dx},$$

with

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V\right) = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V,$$

$$\implies \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{m^2\dot{x}^3}{3} + 2m\dot{x}V\right) = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V + 2m\dot{x}\frac{dV}{dx}\frac{dx}{dt},$$

$$\therefore \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2m\dot{x}^2\frac{dV}{dx} + 2m\ddot{x}V,$$

thus, the Euler-Lagrange equation for this system it's given by

$$m^2\dot{x}^2\ddot{x} + 2m\dot{x}^2\frac{dV}{dx} + 2m\ddot{x}V - m\dot{x}^2\frac{dV}{dx} + 2V\frac{dV}{dx} = 0,$$

and now, let's simplify the previous expression

$$m\dot{x}^2\left(m\ddot{x} + 2\frac{dV}{dx} - \frac{dV}{dx}\right) + \left(2m\ddot{x}V + 2\frac{dV}{dx}\right) = 0,$$

$$\implies m\dot{x}^2\left(m\ddot{x} + \frac{dV}{dx}\right) + 2V\left(m\ddot{x} + \frac{dV}{dx}\right) = 0,$$

$$\implies \left(m\dot{x}^2 + 2V\right)\left(m\ddot{x} + \frac{dV}{dx}\right) = 0,$$

but we know that $T = \frac{m}{2}\dot{x}^2$, and if the system is conservative, then it follows that the force can be obtained via the gradient, or in this case $F = -\frac{dV}{dx}$, therefore, if we plug this information in the previous equation, we have that

$$\implies (2T + 2V)(m\ddot{x} - F) = 0,$$

$$\therefore (T + V)(m\ddot{x} - F) = 0.$$

5. (Problem 1.21) Two mass points of mass m_1 and m_2 are connected by a string passing through a hole in a smooth table so that m_1 rests on the table surface and m_2 hangs suspended. Assuming m_2 moves in a vertical line, what are the generalized coordinates for the system? Write the Lagrange equations for the system and if possible, discuss the physical significance any of them might have. Reduce the problem to a second-orer differential equation and obtain a first integral.

First of all let's write a diagram of the problem, which is given in the following figure

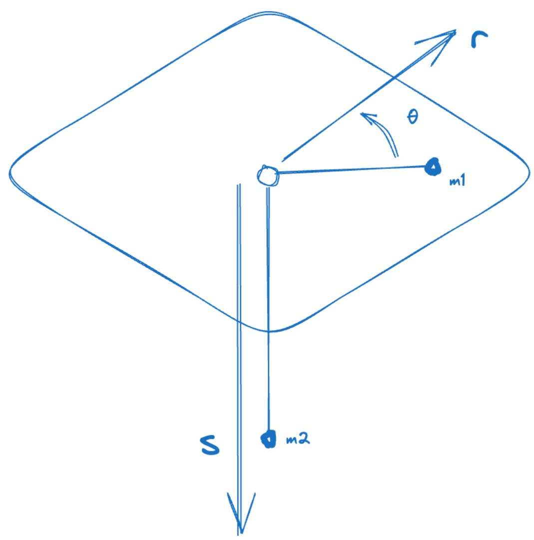

Figure 0.3: Diagram of Problem 1.21.

Now, for this problem, we have one constraint, which is the lenght of the string, and because of this constraint the number of degrees of fredom will reduce. Moreover, because the mass falling down, respect to the table, is just able to move in the vertical direction, i.e., we just need one coordinate to describe its motion.

For the motion of the mass m_1 we are going to use polar coordinates, i.e. $\{r, \theta\}$, but because the masses are connected, via the following equation

$$l = r + s,$$

where r is the coordinate of the particle on the table, s is the coordinate for the particle "falling" down, and l the lenght os the string.

Then, we have the following set of generalized coordinates for the system

$$m_1 :\rightarrow x = (l - s)\cos\theta, \quad y = (l - s)\sin\theta,$$

$$m_2 :\rightarrow s = s.$$

But as I mention before, the radial coordinate of the particle m_1 is not free, instead, it's related to the coordinate for m_2, and therefore, we can express it in terms of s thus, the coordinates for m_1 end up being

$$m_1 :\rightarrow x = (l - s)\cos\theta, \quad y = (l - s)\sin\theta.$$

And even more, we know that the kinetic energy, in terms of polar coordinates can be written as

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right),$$

therefore, for this problem we will have, for the particle m_1

$$T_1 = \frac{1}{2}m_1\left(\dot{s}^2 + (l - s)^2\,\dot{\theta}^2\right),$$

whereas, for the mass m_2, we just have

$$T_2 = \frac{1}{2}m_2\dot{s}^2,$$

therefore, the total kinetic energy will be

$$T = T_1 + T_2 = \frac{1}{2}m_1\left(\dot{s}^2 + (l-s)^2\,\dot{\theta}^2\right) + \frac{1}{2}m_2\dot{s}^2,$$

$$\therefore T = \frac{1}{2}(m_1+m_2)\,\dot{s}^2 + \frac{1}{2}m_1\,(l-s)^2\,\dot{\theta}^2.$$

And, for the potetial energy, we just have

$$V = -m_2 g s,$$

thus, the Lagrangian will be

$$\mathcal{L} = T - V,$$

$$\therefore \mathcal{L} = \frac{1}{2}(m_1+m_2)\,\dot{s}^2 + \frac{1}{2}m_1\,(l-s)^2\,\dot{\theta}^2 + m_2 g s.$$

Now, we have that the Euler-Lagrange equations for this system will be

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} - \frac{\partial\mathcal{L}}{\partial s} = 0, \quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0.$$

So, let's perform the calculations

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} = \frac{d}{dt}\left(\frac{1}{2}(m_1+m_2)\,\dot{s}^2 + \frac{1}{2}m_1\,(l-s)^2\,\dot{\theta}^2 + m_2 g s\right) = \frac{d}{dt}\left((m_1+m_2)\,\dot{s}\right),$$

$$\implies \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} = (m_1+m_2)\,\ddot{s},$$

$$\frac{\partial\mathcal{L}}{\partial s} = \frac{\partial}{\partial s}\left(\frac{1}{2}(m_1+m_2)\,\dot{s}^2 + \frac{1}{2}m_1\,(l-s)^2\,\dot{\theta}^2 + m_2 g s\right) = -m_1\,(l-s)\,\dot{\theta}^2 + m_2 g,$$

thus, we have

$$(m_1+m_2)\,\ddot{s} + m_1\,(l-s)\,\dot{\theta}^2 - m_2 g = 0, \tag{0.9}$$

whilr for the θ coordinate, we have that there is no dependence of the Lagrangian with respect to θ, just with respect to $\dot{\theta}$, therefore, we have that

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = 0,$$

$$\implies \frac{d}{dt}\frac{\partial}{\partial\dot{\theta}}\left(\frac{1}{2}(m_1+m_2)\,\dot{s}^2 + \frac{1}{2}m_1\,(l-s)^2\,\dot{\theta}^2 + m_2 g s\right) = \frac{d}{dt}\left(m_1\,(l-s)^2\,\dot{\theta}\right) = 0,$$

$$\implies m_1\,(l-s)^2\,\dot{\theta} = \alpha,$$

where α it's just a constant, and this also implies

$$\dot{\theta} = \frac{\alpha}{m_1\,(l-s)^2},$$

. And with this information at hand, we can write the equation (0.9) as

$$(m_1+m_2)\,\ddot{s} + m_1\,(l-s)\left(\frac{\alpha}{m_1\,(l-s)^2}\right)^2 - m_2 g = 0,$$

$$\implies (m_1+m_2)\,\ddot{s} + \frac{\alpha^2}{m_1\,(l-s)^3} - m_2 g = 0,$$

if we now multiply the previous equation by $\dot{s}$, we have that

$$\implies (m_1+m_2)\,\ddot{s}\dot{s} + \frac{\alpha^2\dot{s}}{m_1\,(l-s)^3} - m_2 g\dot{s} = 0,$$

which can be integrated

$$\frac{1}{2}\left(m_1+m_2\right)\dot{s}^2+\frac{\alpha^2}{m_1\left(l-s\right)^2}-m_2gs=\beta,$$

where β it's just a constant, and moreover, if we plug the expression for α in the previous equation, we have that

$$\frac{1}{2}\left(m_1+m_2\right)\dot{s}^2+\frac{\left(m_1\left(l-s\right)^2\dot{\theta}\right)^2}{m_1\left(l-s\right)^2}-m_2gs=\beta,$$

$$\Longrightarrow\ \frac{1}{2}\left(m_1+m_2\right)\dot{s}^2+m_1\left(l-s\right)^2\dot{\theta}^2-m_2gs=\beta,$$

butm the left hand side of the previous equiation it's just the total energy, therefore, we have

$$T+V=\beta,$$

i.e., the energy is conserved.

6. (Problem 1.22) Obtain the Lagrangian equations of motion for a double pendulum.

For this problem we can use the angles of the two angles as generalized coordinates, as shown in the figure

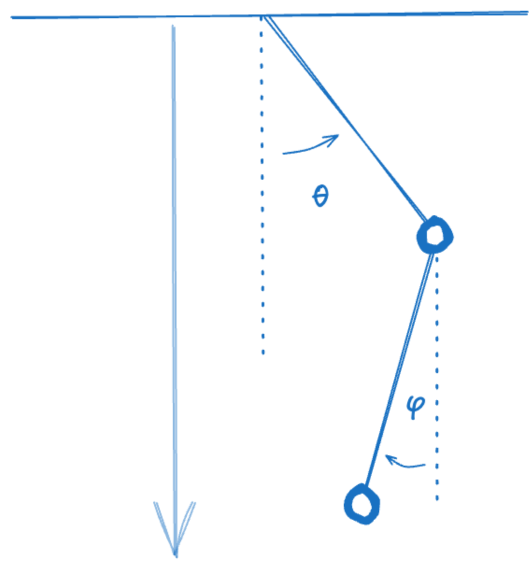

Figure 0.4: Diagram of Problem 1.22.

Then, we can express the cartesian coordinates of the positions of the masses in terms of these angles. So, let's do that, for the mass m_1, let's suppose lenght l_1 fixed, and for this, the motion will be described by the angle θ, and in the coordinates x, y we have that

$$x_1 = l_1 \sin\theta, \qquad y_1 = -l_1 \cos\theta,$$

whereas for the second pendulum, we're going to assume the lengh it's fixed to be l_2 and the coordinates will be

$$x_2 = x_1 + l_2 \sin\phi, \qquad y_2 = y_1 - l_2 \cos\phi,$$
$$\implies x_2 = l_1 \sin\theta + l_2 \sin\phi, \qquad y_2 = -l_1 \cos\theta - l_2 \cos\phi.$$

And with this at hand, we can calculate the kinetic energy for each one of the masses, thus, for m_1, we have

$$T_1 = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right),$$

where

$$\dot{x}_1 = l_1 \cos\theta\dot{\theta}, \qquad \dot{y}_1 = l_1 \sin\theta\dot{\theta},$$
$$\implies \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}^2,$$

and then, the kinetic energy will be

$$T_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}^2. \tag{0.10}$$

Now, for the mass m_2, we have

$$T_2 = \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right),$$

with

$$\dot{x}_2 = l_1 \cos\theta\dot{\theta} + l_2 \cos\phi\dot{\phi},$$
$$\dot{y}_2 = l_1 \sin\theta\dot{\theta} + l_2 \sin\phi\dot{\phi},$$

then

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \cos^2\theta\dot{\theta}^2 + l_2^2 \cos^2\phi\dot{\phi}^2 + 2 l_1 l_2 \cos\theta \cos\phi\dot{\theta}\dot{\phi} + l_1^2 \sin^2\theta\dot{\theta}^2 + l_2^2 \sin^2\phi\dot{\phi}^2 + 2 l_1 l_2 \sin\theta \sin\phi\dot{\theta}\dot{\phi},$$

$$\implies \dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\theta}^2\left(\cos^2\theta+\sin^2\theta\right)+l_2^2\dot{\phi}^2\left(\cos^2\phi+\sin^2\phi\right)+2l_1l_2\dot{\theta}\dot{\phi}\left(\cos\theta\cos\phi+\sin\theta\sin\phi\right),$$

but we know that $\cos^2 a+\sin^2 b=1$, and $\cos a\cos b+\sin a\sin b=\cos(a-b)$, then

$$\dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\theta}^2+l_2^2\dot{\phi}^2+2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi),$$

then, the kinetic energy for this particle will be

$$T_2=\frac{1}{2}m_2\left(l_1^2\dot{\theta}^2+l_2^2\dot{\phi}^2+2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right),$$

and therefore the kinetic energy of the system will be

$$T=\frac{1}{2}m_1l_1^2\dot{\theta}^2+\frac{1}{2}m_2\left(l_1^2\dot{\theta}^2+l_2^2\dot{\phi}^2+2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right).$$

And, the potential energy for the system will be

$$V=V_1+V_2,$$

where V_1 is the potential energy for the mass m_1 and V_2 for m_2, and for each one, we have

$$V_1=m_1gl_1(1-\cos\theta),$$

and

$$V_2=m_2g\left[l_1(1-\cos\theta)+l_2(1-\cos\phi)\right],$$

and then, the total potential will be

$$V=g\left\{m_1l_1(1-\cos\theta)+m_2\left[l_1(1-\cos\theta)+l_2(1-\cos\phi)\right]\right\},$$

and then, the Lagrangian is given by

$$\mathcal{L}=\frac{1}{2}m_1l_1^2\dot{\theta}^2+\frac{1}{2}m_2\left(l_1^2\dot{\theta}^2+l_2^2\dot{\phi}^2+2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$
$$-g\left\{m_1l_1(1-\cos\theta)+m_2\left[l_1(1-\cos\theta)+l_2(1-\cos\phi)\right]\right\},$$

and in this case, the Euler-Lagrange equations will be

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}-\frac{\partial\mathcal{L}}{\partial\theta}=0,$$
$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}-\frac{\partial\mathcal{L}}{\partial\phi}=0.$$

Now, let's performd the derivatives, for θ

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=\frac{d}{dt}\left(m_1l_1^2\dot{\theta}+m_2l_1^2\dot{\theta}+2l_1l_2\dot{\phi}\cos(\theta-\phi)\right),$$
$$\implies \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=m_1l_1^2\ddot{\theta}+m_2l_1^2\ddot{\theta}+2l_1l_2\ddot{\phi}\cos(\theta-\phi)-2l_1l_2\dot{\phi}\left(\dot{\theta}-\dot{\phi}\right)\sin(\theta-\phi),$$
$$\therefore \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=(m_1+m_2)\,l_1^2\ddot{\theta}++2l_1l_2\ddot{\phi}\cos(\theta-\phi)-2l_1l_2\dot{\phi}\left(\dot{\theta}-\dot{\phi}\right)\sin(\theta-\phi),$$

and for the other derivative we have

$$\frac{\partial\mathcal{L}}{\partial\theta}=-l_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)-m_1gl_1\sin\theta-m_2gl_2\sin\theta,$$
$$\implies \frac{\partial\mathcal{L}}{\partial\theta}=-l_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)-(m_1l_1+m_2l_2)\,g\sin\theta.$$

Theferore, we have that the Euler-Lagrange equation for θ is given by

$$(m_1+m_2)\,l_1^2\ddot{\theta}++2l_1l_2\ddot{\phi}\cos(\theta-\phi)-2l_1l_2\dot{\phi}\left(\dot{\theta}-\dot{\phi}\right)\sin(\theta-\phi)+l_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)+(m_1l_1+m_2l_2)\,g\sin\theta=0.$$

Now let's move on to the ϕ coordinate, and calculate

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\frac{d}{dt}\left(m_2l_2^2\dot{\phi}+m_2l_1l_2\dot{\theta}\cos(\theta-\phi)\right),$$

$$\implies\ \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=m_2l_2^2\ddot{\phi}+m_2l_1l_2\ddot{\theta}\cos(\theta-\phi)+m_2l_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi),$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi}=m_2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)+m_2gl_2\sin\theta,$$

and finally, the Euler-Lagrange equations for the coordinate ϕ are given by

$$m_2l_2^2\ddot{\phi}+m_2l_1l_2\ddot{\theta}\cos(\theta-\phi)+m_2l_1l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)-m_2l_1l_2\dot{\theta}\dot{\phi}\cos(\theta-\phi)-m_2gl_2\sin\theta=0.$$

7. (Problem 2.13) A heavy particle is placed at the top of a vertical hoop. Calculate the reaction of the hoop on the particle by means of the Lagrange's undetermined multipliers and Lagrande's equations. Find the height at which the particle falls of.

   For this problem let's use polar coordinates as generalized coordinates, and moreover, because of the nature of the problem, we can use lagrange undefined multipliers, which, of course, modifiy the Euler-Lagrange equations. So let'a begin, by definition, the Lagrangian is given by

$$\mathcal{L} = T - V,$$

   but in this case, as stated before, we can use polar coordinates, with the transformation from cartesian to polar given by

$$x = r\cos\theta, y = r\sin\theta,$$

   with the equation above, we can find the kinetec energy as follows

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right),$$

$$\implies T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right),$$

   now, for the potential energy, we can see that it'll be given by the following expression

$$V = mgr\cos\theta,$$

   therefore, the Lagrangian will be

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta,$$

   but, we also have the restriction giving by the fact that the particle is placed at the hoop, for now, let's assume that the particle is constrained to allways be moving on the surface of the hoop, and then the constraint, can be written as

$$g = R - r, \tag{0.11}$$

   where R is the radius of the hoop and r is the coordinate of the particle. Now, in order to consider the constraint we have to modify the Euler-Lagrange equations as follows

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = \lambda\frac{\partial g}{\partial r},$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = \lambda\frac{\partial g}{\partial\theta},$$

   but, as we can see from equation (0.11), there is no dependece on the variable θ, therefore, the Euler-Lagrage equations simplify to

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = \lambda\frac{\partial g}{\partial r},$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0,$$

   now lets work with Euler Lagrange equation for r

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = \frac{d}{dt}\left(\frac{\partial}{\partial\dot{r}}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta\right)\right) = \frac{d}{dt}\left(m\dot{r}\right) = m\ddot{r},$$

$$\implies \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = m\ddot{r},$$

   on the other hand, for the other derivative, we have

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta\right) = mr\dot{\theta}^2 - mgcos\theta,$$

$$\implies \frac{\partial\mathcal{L}}{\partial r} = mr\dot{\theta}^2 - mgcos\theta,$$

therefore, the Lagrangian for the r coodinate becomes

$$m\ddot{r} - mr\dot{\theta}^2 + mgcos\theta = \lambda,$$

but because the particle is contraint to move in the surface of the hoop, we have that $r = R$, and also it's time derivatives would be equal to zero, therefore, we have that the last equation reduces to

$$-mR\dot{\theta}^2 + mgcos\theta = \lambda, \tag{0.12}$$

and following the same line of though, for θ we have that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta\right)\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta},$$

with

$$\frac{\partial \mathcal{L}}{\partial \theta} = mgr\sin\theta,$$

and therefore, for θ we have that

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} - mgr\sin\theta = 0,$$

and as before, considering the constraint equations, $r = R$, and also the time derivatives of r will be equal to zero, thus, we have that

$$mR^2\ddot{\theta} - mgR\sin\theta = 0. \tag{0.13}$$

Now, let's focus on the previous equation, and rewrite in the following way

$$\ddot{\theta} = \frac{g}{R}\sin\theta,$$

and now, let's use the chain rule to rewrite the left hand side of the above equation as a derivative of θ, thus

$$\ddot{\theta} = \frac{d}{dt}\dot{\theta} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta},$$

$$\implies \ddot{\theta} = \frac{d\dot{\theta}}{d\theta}\dot{\theta},$$

and therefore, we have that

$$\frac{d\dot{\theta}}{d\theta}\dot{\theta} = \frac{g}{R}\sin\theta,$$

$$\implies \dot{\theta}d\dot{\theta} = \frac{g}{R}\sin\theta d\theta,$$

therefore, we can integrate the previous equation asuming that $\dot{\theta} = 0$ at $\theta = 0$, then

$$\int_0^{\dot{\theta}} \dot{\theta}d\dot{\theta} = \int_0^{\theta} \frac{g}{R}\sin\theta d\theta,$$

$$\implies \dot{\theta}^2 = \frac{2g}{R}\left(1 - \cos\theta\right).$$

Now, if we insert the previous equation into (0.12), we have that

$$-mR\frac{2g}{R}\left(1 - \cos\theta\right) + mgcos\theta = \lambda,$$

$$\implies \lambda = -2mg + 2mg\cos\theta + mgcos\theta = mg\left(3\cos\theta - 2\right),$$

$$\therefore \lambda = mg\left(3\cos\theta - 2\right). \tag{0.14}$$

and, from the previous equation we can see that $\lambda = 0$ when $\cos\theta = \frac{2}{3}$, or $\theta = \arccos\frac{2}{3}$. Now, if we consider that the surface cannot hold the mass anymore at some angle, then the mass will fly, when the next condition becames true

$$\theta > \arccos\frac{2}{3} \approx 48^\circ,$$

and therefore the height at which the particle will fly, becomes

$$\frac{2}{3}R.$$

8. Consider the one-dimensional power law force $F(x) = -k|x|^n$ where $k > 0$ and n is a positive integer.
   (a) What it is the corresponding potential $U(x)$?
   (b) Show that the corresponding potential energy E is conserved.
   (c) Express the turning point of the motion $|x_{max}|$ in terms of the maximum instantaneous velocity v_{max}.

Let's begin by plotting the force law for several n

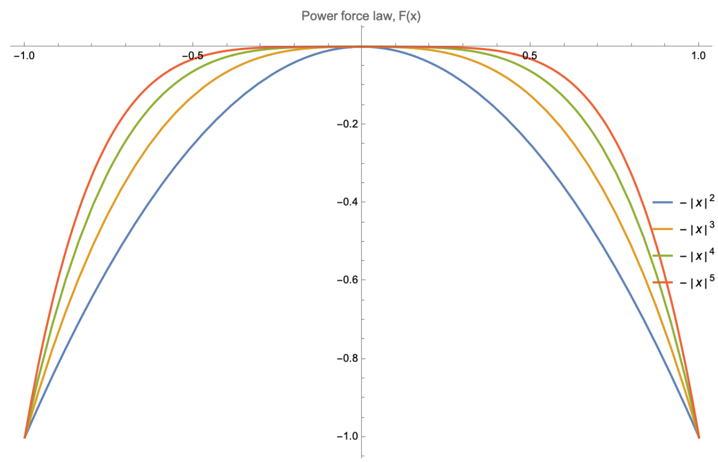

Figure 0.5: Plot of several power force law, $F(x)$, with $k = 1$.

We can see that because of the nature of the force, i.e., the dependence of the absolute value, the force doesn't take any negative values, but because it'a multiplied by the factor $-k$ and k is positive, then the force will always be negative. From newtonian mechanics we know that

$$\overrightarrow{F} = -\nabla U,$$

or, in 1D,

$$F(x) = -\frac{dU}{dx},$$

therefore, we can find the potential by simply taking the integral of the previous expresion, this is

$$U = -\int F dx.$$

On the other hand, we also know that the definition of the absolute value it's given by x if $x > 0$ and $-x$ if $x < 0$, therefore, we can split the previous integral in two parts, let's begin by considering $x > 0$, then we have

$$U = -\int (-k|x|^n)\, dx = k \int x^n dx = \frac{x^{n+1}}{n+1} + C,$$

$$\implies U = \frac{x^{n+1}}{n+1} + C,$$

it's important to notice that in the previous integration, we're only consdering the cases in which $x > 0$, on the other hand, if $x < 0$ we can make the following change of variable $y = -x$, and we also need to change the domain in which we're integrating the function, but we will end with the same integral, therefore we have that the potential is given by

$$U = \frac{|x|^{n+1}}{n+1} + C.$$

An even more, because we were able to write the force as the derivative of some potential function, we know that this implies that the system is conservative, i.e, the total energy is conserved. And finnaly, by the conservation of the energy we can choose two points in time, which will be one in which the potential energy is zero, and the energy is max, and other when the velocity iz zero, which it's the turning point, and the potential is max, then we have

$$
\begin{aligned}
\frac{1}{2}mv_{max}^2 &= \frac{|x_{max}|^{n+1}}{n+1}, \\
\implies |x_{max}|^{n+1} &= \left(\frac{n+1}{2}\right) mv_{max}^2, \\
\therefore |x_{max}| &= \sqrt[n+1]{\left(\frac{n+1}{2}\right) mv_{max}^2}.
\end{aligned}
$$