Cosmology: Problem Set 2, (Emmanuel Flores)

Problem 1: $T^{\mu}_{\ \nu} = g_{\nu\alpha} \sum_n m_n \int d\tau \frac{dx^{\mu}_n}{d\tau} \frac{dx^{\alpha}_n}{d\tau} \frac{\delta^4(x - x_n)}{\sqrt{-g}}$

a) By definition g is the determinant of the metric, thus:

$\Rightarrow$ By using $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$

and since the metric is diagonal, we have

$$g = \det(\mathrm{diag}(1, -a(t)^2, -a(t)^2, -a(t)^2)) = -a(t)^6,$$

$\Rightarrow$ $-g = a(t)^6$, and from this, we have:

$$\sqrt{-g} = a(t)^3$$

b) In general we have

$$T^{\mu}_{\ \nu} = g_{\nu\alpha} \sum_n m_n \int d\tau \frac{dx^{\mu}_n}{d\tau} \frac{dx^{\alpha}_n}{d\tau} \frac{\delta^4(x - x_n)}{\sqrt{-g}}$$

And by separating the δ^3 into a product of spatial and temporal "components", we found:

$$T^{\mu}_{\ \nu} = g_{\nu\alpha} \sum_n m_n \frac{dx^{\mu}_n}{dt} \frac{dx^{\alpha}_n}{d\tau} \frac{\delta^3(\vec{x} - \vec{x}_n)}{\sqrt{-g(t)}}$$

but $\sqrt{-g(t)} = a(t)^3$, thus

$$T^{\mu}_{\ \nu} = g_{\nu\alpha} \sum_n m_n \frac{dx^{\mu}_n}{dt} \frac{dx^{\alpha}_n}{d\tau} \frac{\delta^3(\vec{x} - \vec{x}_n)}{a(t)^3}.$$

And by the chain rule, we have

$$T^{\mu}{}_{\nu} = g_{\nu\alpha} \sum_n m_n \frac{dx_n^{\mu}}{dt} \frac{dx_n^{\alpha}}{dt} \frac{dt}{d\tau} \frac{\delta^3(\vec{x} - \vec{x}_n)}{a(t)^3}$$

but $\frac{dt}{d\tau} = \gamma_n$, thus:

$$T^{\mu}{}_{\nu} = g_{\nu\alpha} \sum_n m_n \gamma_n \frac{dx_n^{\mu}}{dt} \frac{dx_n^{\alpha}}{dt} \frac{\delta^3(\vec{x} - \vec{x}_n)}{a(t)^3}$$

And, for T^0_0, we have:

$$T^0_0 = g_{0\alpha} \sum_n m_n \gamma_n \frac{dx_n^0}{dt} \frac{dx_n^{\alpha}}{dt} \frac{\delta^3(\vec{x} - \vec{x}_n)}{a(t)^3}$$

but $x_n^0 = t$, and since $g_{\mu\nu}$ is diagonal for $g_{0\alpha}$, the only non-zero component is when $\alpha = 0$, which corresponds to 1 _ therefore

$$T^0_0 = \sum_n m_n \gamma_n \frac{dt}{dt} \frac{dt}{dt} \frac{\delta^3(\vec{x} - \vec{x}_n)}{a(t)^3}$$

$$\therefore \quad T^0_0 = \sum_n m_n \gamma_n \delta^3(\vec{x} - \vec{x}_n) / a(t)^3$$

just as we wanted.

-> I think the factor of $1/a(t)^3$ takes into account for the fact that we're assuming the universe is expanding; so it "normalizes" in some sense.

c) For $T^1{}_1$, we follow the same procedure

In general:

$$T^\mu{}_\nu = g_{\nu\alpha} \sum_n m_n \gamma_n \frac{dx^\mu_n}{dt} \frac{dx^\alpha_n}{dt} \frac{\delta^3(\vec{x}-\vec{x}_n)}{a(t)^3}$$

$$\Rightarrow T^1{}_1 = g_{1\alpha} \sum_n m_n \gamma_n \frac{dx^1_n}{dt} \frac{dx^\alpha_n}{dt} \frac{\delta^3(\vec{x}-\vec{x}_n)}{a(t)^3}$$

and again, since $g_{\mu\nu}$ is diagonal, for $g_{1\alpha}$, the only non-zero component is when $\alpha = 1$, and in that case $g_{11} = -a(t)^2$,

thus:

$$T^1{}_1 = -a(t)^2 \sum_n m_n \gamma_n (v^1_n)^2 \delta^3(\vec{x}-\vec{x}_n)/a(t)^3,$$

where $v^1_n = \frac{dx^1_n}{dt}$, thus:

$$T^1{}_1 = -\sum_n m_n \gamma_n (v^1_n)^2 \delta(\vec{x}-\vec{x}_n)/a(t)$$

just as we wanted.

→ Again, since we're considering an expanding universe, and in this case we are working with the spatial coordinates, in particular with $(v^1_n)^2$, this in some sense accounts for two factors of $a(t)$.

d) In general, for this problem we have:

$$T^{\mu}{}_{\nu} = n \left(\begin{array}{c|c} \langle E_n \rangle & -\langle E_n v_n^i \rangle \\ \hline -\langle E_n v_n^i \rangle & -\langle E_n v_n^i v_n^j \rangle \end{array} \right)$$

where $\langle E_n \rangle n = \rho$ and $\langle E_n (v_n^i)^2 \rangle n = P$.

- But, by assuming isotropy, that requires $T^{\mu}{}_{\nu}$ to be diagonal.
- On the other hand in the comoving frame, there's no γ factor.

$$T = \mathrm{diag}\left(n\langle E_n \rangle, -n\langle E_n (v_n^x)^2 \rangle, -n\langle E_n (v_n^y)^2 \rangle, -n\langle E_n (v_n^z)^2 \rangle \right)$$

$$\Rightarrow \quad T = \mathrm{diag}(\rho, -P, -P, -P)$$

Problem 2.

a). $$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_v) \qquad (c=1)$$

But, we know that $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$, thus:

$$\frac{8\pi G}{3H_0^2} = \frac{1}{\rho_{crit,0}},$$

and from this, it follows:

$$H^2 = \frac{8\pi G}{3H_0^2} H_0^2 (\rho_m + \rho_v) = \frac{H_0^2}{\rho_{crit,0}} (\rho_m + \rho_v)$$

$$\Rightarrow \quad H^2 = \frac{H_0^2}{\rho_{crit,0}} (\rho_m + \rho_v).$$

On the other hand, we know: $\rho_m = \rho_{m,0} a^{-3}$ $(a_0 = 1)$ and $\rho_v = \rho_{v,0}$, thus:

$$H^2 = H_0^2 \left(\frac{\rho_{m,0}}{\rho_{crit,0}} \frac{1}{a^3} + \frac{\rho_{v,0}}{\rho_{crit,0}} \right)$$

By defining: $\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{crit,0}}$ and $\Omega_{v,0} = \frac{\rho_{v,0}}{\rho_{crit,0}}$, we have:

$$H^2 = H_0^2 \left(\Omega_{m,0}/a^3 + \Omega_{v,0} \right)$$ just as we wanted.

On the other hand, flat space means $\Omega - 1 = 0$, where

$$\Omega = \frac{\rho}{\rho_{crit,0}} = \Omega_{m,0} + \Omega_{v,0} \Rightarrow \Omega_{v,0} = 1 - \Omega_{m,0}$$

b) Starting with $H^2 = H_0^2 \left(\Omega_{m,0}/a^3 + \Omega_{v,0} \right)$

$$\Rightarrow \quad H = H_0 \left(\Omega_{m,0}/a^3 + \Omega_{v,0} \right)^{1/2}$$

But we know that $H = \frac{\dot{a}}{a}$, thus

$$\frac{1}{a}\frac{da}{dt} = H_0 \left(\Omega_{m,0}/a^3 + \Omega_{v,0} \right)^{1/2}$$

$$\Rightarrow \quad dt = \frac{da}{H_0\, a \left(\Omega_{m,0}/a^3 + \Omega_{v,0} \right)^{1/2}}$$

$$\Rightarrow \quad \int_0^{t_0} dt = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_{m,0}/a^3 + \Omega_{v,0}}}$$

$$\therefore \quad t_0 = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_{m,0}/a^3 + \Omega_{v,0}}}$$

c) The solution of the integral is:

$$t_0 = 2\,\mathrm{ArcSinh}\left[\left(\frac{\Omega_{v,0}}{\Omega_{m,0}} \right)^{1/2} \right] \Big/ 3 H_0 \sqrt{\Omega_{v,0}}$$

d I append a Mathematica file with the plot, and the solution to t_0 = age of the universe is

$$\Omega_{m,0} = 0.297893$$