GR-HW-06

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Problem 1 (Transformation Rule for Inverse Metric)

Let's prove that the transformed $g^{\mu\nu}$ is the inverse of the transformed $g_{\mu\nu}$. Indeed, the transformation rule for a covariant tensor is given by

$$\bar{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu}\frac{\partial x^\sigma}{\partial \bar{x}^\mu} g_{\rho\sigma} \tag{1}$$

Whereas for a contravariant one, we have

$$\bar{g}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} g^{\alpha\beta}, \tag{2}$$

then, we have

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\alpha}{\partial x^\sigma} g^{\rho\sigma}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\alpha} g_{\delta\gamma}, \tag{3}$$

and from this we can see that, by moving the partials, we have a δ_σ^γ in the expression, thus

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\delta_\sigma^\gamma g^{\rho\sigma} g_{\delta\gamma}, \tag{4}$$

but we know that

$$\delta_\sigma^\gamma g^{\rho\sigma} g_{\delta\gamma} = g^{\rho\sigma} g_{\delta\rho} = \delta_\delta^\rho, \tag{5}$$

which implies

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\delta_\delta^\rho = \frac{\partial \bar{x}^\mu}{\partial \bar{x}^\nu}, \tag{6}$$

therefore, we have

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \delta_\nu^\mu, \tag{7}$$

which proves that indeed, the transformed $g^{\mu\nu}$ is the inverse of the transformed $g_{\mu\nu}$.

Problem 2 (Covariant Derivative of Vector)

1. Let's prove that $\partial_\mu V_\nu$ does not transform as a tensor. Indeed, from the transformation definition, we have that

$$\bar{\partial}_\mu \bar{V}_\nu = \bar{\partial}_\mu \left(\frac{\partial x^\beta}{\partial \bar{x}^\nu} V_\beta \right) = \frac{\partial x^\beta}{\partial \bar{x}^\nu} \bar{\partial}_\mu V_\beta + \frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\nu} V_\beta, \tag{8}$$

and by the chain rule, we have

$$\bar{\partial}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta, \tag{9}$$

and as we can see, we have an extra term, therefore, $\partial_\mu V_\nu$ does not transform as 2-rank tensor.

2. We need to prove that the covariant derivative transforms as a tensor, this is

$$\nabla_\mu V_\nu \to \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \bar{\nabla}_\alpha \bar{V}_\beta. \tag{10}$$

Indeed, by definition, we have that

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda, \tag{11}$$

thus $\bar{\nabla}_\mu \bar{V} \nu$ is given by

$$\bar{\nabla}_\mu \bar{V}_\nu = \bar{\partial}_\mu \bar{V}_\nu - \bar{\Gamma}^\lambda_{\mu\nu} \bar{V}_\lambda. \tag{12}$$

The only term that we need to be more careful is with the Christoffel symbol, since this does not transform like a tensor, in fact, the transformation rule for this object is given by

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\alpha_{\beta\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x^\mu}} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} + \frac{\partial \bar{x}^\lambda}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu}, \tag{13}$$

from this we have

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \left(\Gamma^\alpha_{\beta\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x^\mu}} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} + \frac{\partial \bar{x}^\lambda}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \right) \frac{\partial x^\delta}{\partial \bar{x}^\lambda} V_\delta, \tag{14}$$

from this we can see that inside the third term we have two Kronecker deltas, δ^δ_α and δ^δ_γ, respectively, thus

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \Gamma^\alpha_{\beta\gamma} \delta^\delta_\alpha \frac{\partial x^\beta}{\partial \bar{x^\mu}} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} V_\delta - \delta^\delta_\gamma \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\delta, \tag{15}$$

thus we have

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \frac{\partial x^\beta}{\partial \bar{x^\mu}} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} \Gamma^\alpha_{\beta\gamma} V_\alpha - \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\gamma, \tag{16}$$

and from this we can see that the second term cancels with the fourth one, whereas for the third, we can relabel the indices as follows

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta - \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \Gamma^\lambda_{\alpha\beta} V_\lambda = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \left(\nabla_\mu V_\nu \right), \tag{17}$$

this is

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \left(\nabla_\mu V_\nu \right), \tag{18}$$

therefore, the covariant derivative is indeed a good 2-rank tensor.