Cosmology | Problem Set 4 | Emmanuel Flores

Problem 1

Assuming $-1 < w < 0$, with

$$a(t) = (t/t_0)^{\frac{2}{3(1+w)}} \qquad (a_0 = 1)$$

And t_0 @ x_0:

a) Analytical expression for x_{phys} vs t.

We have;

$$x_{phys} = a\, x;$$

→ ligth travels along null geodesics, this is

$$ds^2 = 0,$$

And in FRW we have;

$$ds^2 = -c^2 dt^2 + a(t)^2 dx^2 = 0$$

$$\Rightarrow \quad dx = -\frac{c\,dt}{a(t)},$$

thus, in our case we have:

$$dx = -c\left(\frac{t}{t_0}\right)^{-\frac{2}{3(1+w)}} \frac{dt}{t_0},$$

and from here we have;

$$\int_{x_0}^{x} dx' = -c\, t_0^{\frac{2}{3(1+w)}} \int_{t_0}^{t} t'^{-\frac{2}{3(1+w)}} \frac{dt'}{t_0}$$

$$\Rightarrow \quad x - x_0 = -c\, t_0^{\frac{2}{3(1+w)}} \int_{t_0}^{t} t'^{-\frac{2}{3(1+w)}} \frac{dt'}{t_0}$$

thus,

$$\Rightarrow \quad \boxed{x = x_0 - \frac{3c(1+w)t_0}{1+3w}\left[\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} - 1\right]}$$

And the x_{phys} is given by:

$$\boxed{x_{phys} = a(t)\, x = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}\left[x_0 - \frac{3c(1+w)}{1+3w}\left[\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} - 1\right]\right]}$$

b) We want $x_{phys}(t) = 0$ for some t, thus:

$$0 = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}\left[x_0 - \frac{3c}{1+3w}\left(\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} - 1\right)\right]$$

but $t/t_0 > 0$, thus:

$$x_0 = \frac{3c}{1+3w}\left[\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} - 1\right]$$

And for ligth to always reach the origin, we need the term inside the brackets diverge as $t \to \infty$, And this happens when; $\frac{1+3w}{3(1+w)} > 0$, but

we know that $1+w > 0$, thus

$$\Rightarrow 1+3w > 0 \Rightarrow w > -1/3$$

therefore, the range is $w \in (-1/3, 0)$

c) for this part, we need to find x_0 max such that light doesn't always reach the origin:

→ We're in the case $w \in (-1/3, 0)$,

→ And again, by considering the limit $t \to \infty$, we have

$$\frac{1+3w}{3(1+w)} < 0 \quad \text{(condition for not reaching origin)}$$

since this would imply

$$\left(\frac{t}{t_0}\right)^{\frac{1+3w}{3(1+w)}} \to 0 \quad \text{as } t \to \infty$$

$$\therefore \; x_{0,max} = \frac{3c(1+w)t_0}{1+3w}(0-1) = -\frac{3c(1+w)t_0}{1+3w}$$

$$\Rightarrow x_{0,max} = -\frac{3c(1+w)t_0}{1+3w}$$

since $-1 < w < 0$ and $1+3w < 0 \Rightarrow x_{0,max} > 0$ ■

Problem 2.

(1) Case 1: inside the bulge: $r < R_B$,

The mass enclosed by a sphere is given by

$$M = \frac{4}{3}\pi r^3 \rho,$$

On the other hand, the gravitational force is given by

$$F_g = -\frac{GM}{r^2}m$$

whereas for a circular motion the centripetal force is given by

$$F_c = m v^2 / r$$

If we set $|F_g| = |F_c|$, we have;

$$\left| -\frac{GMm}{r^2} \right| = |m v^2/r|,$$

and using the M computed previously, we have;

$$mG\left(\frac{4}{3}\pi r^3 \rho\right)/r^2 = mv^2/r$$

and solving for v, we have;

$$v^2 = \frac{4}{3}\pi\rho G r^4/r^2$$

$$\Rightarrow \quad v = \sqrt{\frac{4}{3}\pi\rho G}\ r \qquad \boxed{v(r) \propto r}$$

Case 2:

If $r > R_B$, then we have;

$$M = \frac{4}{3}\pi R_B^3$$

$\hookrightarrow$ radius of bulge.

And again, the gravitational force is given by

$$F_g = -\frac{GMm}{r^2},$$

whereas the centripetal force

$$F_c = mv^2/r,$$

and by making

$|F_g| = |F_c|$, we have.

$$G\frac{Mm}{r^2} = mv^2/r \Rightarrow \frac{4}{3}\pi G R_B^3/r^2 = v^2/r$$

Solving for v, we have;

$$v^2 = \frac{4}{3}\pi R_B^3 G/r$$

$$\Rightarrow v = \sqrt{\frac{4}{3}\pi G R_B^3}/r^{1/2}$$

$$\therefore v \propto 1/r^{1/2}$$

2. Now, the profile is different

$$r < R_B : \rho(r) = \rho$$
$$r > R_B : \rho(r) = \rho\left(\frac{R_B}{r}\right)^2$$

Case 1: inside the bulge, the approach is the same as before, thus:

$$\Rightarrow \quad v = \sqrt{\frac{4}{3}\pi\rho G}\; r$$

Case 2:

Now we need to integrate to find the total enclosed mass:

$$M = M_{R_B} + \int_{M}^{r} \rho(r') 4\pi r'^2 dr'$$

$$\Rightarrow \quad M = \frac{4}{3}\pi R_B^3 \rho + \int_{R_B}^{r} \rho\left(\frac{R_B}{r'}\right)^2 4\pi r'^2 dr'$$

$$\Rightarrow \quad M = \frac{4}{3}\pi R_B^3 \rho + 4\pi\rho R_B^2 \int_{R_B}^{r} dr$$

$$\Rightarrow \quad M = \frac{4}{3}\pi R_B^3 \rho + 4\pi\rho R_B^2 (r - R_B)$$

$$\therefore \quad M = 4\pi\rho R_B^2 r - \frac{8}{3}\pi\rho R_B^3$$

from this, we have can equate gravitational and centripetal forces, thus:

$$\frac{mv^2}{r} = \left(4\pi\rho R_B^2 r - \frac{8}{3}\pi\rho R_B^3\right)\frac{G}{r^2}m$$

$$\Rightarrow v^2 = \left(4\pi\rho R_B^2 r - \frac{8}{3}\pi\rho R_B^3\right)\frac{G}{r}$$

$$\Rightarrow v = \sqrt{4\pi\rho R_B^2 - \frac{8}{3}\pi\rho R_B^3/r}$$

And as $r \to \infty$, we have $v \to \sqrt{4\pi\rho R_B^2}$

-> Plots are included in the Mathematica notebook.

Problem 3

– Thermal Behavior of Massive Species –

a) let's find the number density.

In general, the number density is given by

$$n(T) = \frac{g}{(2\pi)^3}\int d^3p\, f(p,T)$$

where f can be Bose-Einstein or Fermi-Dirac statistics.

In this case we have:

$$f = n_p = \frac{1}{\exp(E_p/T) \pm 1}$$

thus

$$n(T) = \frac{g}{(2\pi)^3}\int d^3p \frac{1}{\exp\left[\sqrt{m^2+p^2}/T\right] \pm 1}, \quad \left(\begin{array}{c}\text{by} \\ \text{setting} \\ g=1\end{array}\right)$$

$$\Rightarrow n(T) = \frac{1}{2\pi^2}\int_0^\infty dp \frac{p^2}{\exp\left[\sqrt{p^2+m^2}/T\right] \pm 1}$$

Making the change of variables $x = m/T$, $\xi = p/T$

we have

$$p^2 = \xi^2 T^2, \quad m^2 = x^2 T^2$$

$$\Rightarrow p^2 + m^2 = T^2(\xi^2 + x^2) \Rightarrow \sqrt{p^2+m^2}/T = \xi^2 + x^2$$

and $\frac{d\xi}{dp} = \frac{1}{T} \Rightarrow dp = T\,d\xi$

and $p^2 = \xi^2 T^2 \Rightarrow \int dp\, p^2 \to \int d\xi\, T^3 \xi^2$

then, we have:

$$\Rightarrow n = \frac{1}{2\pi^2} T^3 \int_0^\infty d\xi \frac{\xi^2}{\exp\left[\sqrt{\xi^2 + x^2}\right] \pm 1}$$

And for the purpose of the plot, I will make

$$2\pi^2 \frac{n}{T^3} = \int_0^\infty d\xi \frac{\xi^2}{\exp\left[\sqrt{\xi^2 + x^2}\right] \pm 1}$$

$n' = \frac{2\pi^2 n}{T^3}$, thus:

$$n' = \int_0^\infty d\xi \frac{\xi^2}{\exp\left[\sqrt{\xi^2 + x^2}\right] \pm 1}$$

→ I append a Mathematica Notebook with the plots.

•) For the limit's, $T \gg m$ and $T \ll m$, in the variables I used, these translate to:

$T \gg m \Rightarrow 1 \gg \frac{m}{T} = x \Rightarrow$ $T \gg m \sim 1 \gg x$

and $T \ll m \sim 1 \ll x$

•) $1 \gg x$, and x is positive, so this is equivalent to $x \to 0$. Then we can neglect x in the integral as follows

$$n' \simeq n'(0) = \int_0^\infty d\xi \, \frac{\xi^2}{\exp(\xi) \pm 1} .$$

And for the integrand we have;

$$\frac{\xi^2}{e^\xi \pm 1} = \frac{e^{-\xi}}{1 \pm e^{-\xi}} \xi^2 \text{, but } \frac{e^{-\xi}}{1 \pm e^{-\xi}} = \sum_{i=1}^{\infty} (\mp 1)^{i-1} e^{-i\xi} ,$$

thus:

$$\frac{e^{-\xi}}{1 \pm e^{-\xi}} \xi^2 = \sum_{i=1}^{\infty} (\mp 1)^{i-1} e^{-i\xi} \xi^2 ,$$ Assuming the series converges we can interchange the sum and the integral

$$\Rightarrow n' = \sum_{i=1}^{\infty} (\mp 1)^{i-1} \int_0^\infty d\xi \, e^{-i\xi} \xi^2 \text{, but } \int_0^\infty d\xi \, e^{-i\xi} \xi^2 = \frac{2}{i^3} ,$$

thus:

$$n' = \sum_{i=1}^{\infty} (\mp 1)^{i-1} \frac{2}{i^3} ,$$ and now we need to evaluate either + or −

•) For bosons, we have;

$$n' = 2\sum_{i=1}^{\infty} \frac{1^{i-1}}{i^3}$$

$$= 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right)$$, but we know that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{(Riemann zeta function)}$$

so; $n' = 2\zeta(3)$, and $\zeta(3) = 1.20206$

$$\Rightarrow \quad n' = 2.40412 \quad \Rightarrow \quad \frac{2\pi^2 n}{T^3} = 2.40412$$

•) For fermions, we have:

$$n' = 2\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i^3}$$

$$\Rightarrow \quad n' = 2\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots\right)$$, and

by rearrangement of the terms

$$n' = 2\left(1 - \frac{1}{2^3} + \left(\frac{1}{2^3} - \frac{1}{2^3}\right) + \frac{1}{3^3} - \frac{1}{4^3} + \left(\frac{1}{4^3} - \frac{1}{4^3}\right) + \cdots\right)$$

$$= 2\left(1 + \frac{1}{2^3} - 2\left(\frac{1}{2^3}\right) + \frac{1}{3^3} - 2\left(\frac{1}{4^3}\right) + \cdots\right)$$

$$\Rightarrow \quad n' = 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots - 2\left(\frac{1}{2^3} + \frac{1}{4^3} + \cdots\right)\right)$$

$$\Rightarrow \quad n' = 2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right) - 2_1\left(\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \cdots\right)$$

$$\Rightarrow \quad n' = 2\zeta(3) - \frac{4}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots\right)$$

$$\Rightarrow \quad n' = 2\zeta(3) - \frac{4}{8}\zeta(3) = 2\zeta(3) - \frac{1}{2}\zeta(3)$$

$$\boxed{\Rightarrow \quad n' = \left(1 - \frac{1}{4}\right) 2\zeta(3)}$$

or numerically:

$$n' = 1.80309$$

$$\boxed{\Rightarrow \quad \frac{2\pi^2 n}{T^3} = 1.80309}$$

- On the other hand, if $x \gg 1$, then both integrals take the same functional form, this is:

$$n' = \int_0^\infty d\xi \, \frac{\xi^2}{\exp\left[\sqrt{x^2 + \xi^2}\right]} \, ,$$

but again since $x \gg 1$ and $x \gg \xi$, we can

Taylor expand the root:

$$\Rightarrow \quad n' = \int_0^\infty d\xi \, \frac{\xi^2}{e^{x + \xi^2/2x}}$$

$$= e^{-x} \int_0^\infty d\xi \, \xi^2 \, e^{-\xi^2/2x}$$

$$= e^{-x} \int_0^\infty d\xi \, \xi^2 \, e^{-\xi^2},$$ and if we

make the following change of variables:

$$y^2 = \frac{\xi^2}{2x} \quad \Rightarrow \quad y = \frac{\xi}{\sqrt{2x}}$$

and $d\xi = \sqrt{2x}\, dy$ with $\xi^2 = 2x\,y^2$,

thus:

$$n' = (2x)^{3/2} e^{-x} \int_0^\infty \underbrace{dy \, y^2 \, e^{-y^2}}_{\text{Gaussian } \to \ \frac{1}{4}\sqrt{\pi}}$$

$$\Rightarrow \quad n' = \frac{1}{4}\sqrt{\pi}\,\sqrt{(2x)^3}\, e^{-x}$$

$$\boxed{\Rightarrow \quad n' = \sqrt{\frac{\pi}{2}}\, x^{3/2}\, e^{-x}}$$

∎

b) For the density, we know that:

$$\rho = \frac{g}{2\pi^2}\int_0^\infty dp \, \frac{p^2\sqrt{p^2+m^2}}{\exp\left[\sqrt{p^2+m^2}/T\right] \pm 1}$$

and again, by making the same change of variables as before

$$x = \frac{m}{T} \quad \text{and} \quad \xi = p/T,$$

we have:

$$dp = T\,d\xi,$$

and

$$\frac{T^2}{T^2}(p^2)\sqrt{\frac{T^2}{T^2}(p^2+m^2)} = T^2\xi^2\sqrt{T^2(\xi^2+x^2)}$$

$$= T^3\xi^2\sqrt{\xi^2+x^2},$$

with

$$\exp\left(\sqrt{\frac{T^2}{T^2}(p^2+m^2)}\Big/T\right) = \exp\left(\frac{T}{T}\sqrt{\xi^2+x^2}\right)$$

$$= \exp\left(\sqrt{\xi^2+x^2}\right),$$

thus, we have:

$$\rho = \frac{g}{2\pi^2}\int_0^\infty d\xi\, T\,\frac{T^3\xi^2\sqrt{\xi^2+x^2}}{\exp\left(\sqrt{\xi^2+x^2}\right) \pm 1}$$

$$\therefore\ \rho = \frac{gT^4}{2\pi^2}\int_0^\infty d\xi\,\frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left(\sqrt{\xi^2+x^2}\right) \pm 1}$$

And, if we make $\rho' = \dfrac{2\pi^2 \rho}{gT^4}$, we have;

$$\rho' = \int_0^\infty d\xi \, \frac{\xi^2 \sqrt{\xi^2 + x^2}}{\exp\left[\sqrt{\xi^2 + x^2}\right] \pm 1}$$

→ The plot of the previous function is provided in the Mathematica Notebook.

Now, let's move on to the limits:

•) $x \to 0$

In this case we have;

$$\rho' \approx \int_0^\infty d\xi \, \frac{\xi^3}{\exp(\xi) \pm 1},$$

but, just as before:

$$\frac{1}{\exp(\xi) \pm 1} = \frac{\exp(-\xi)}{1 \pm \exp(-\xi)} = \sum_{n=1}^{\infty} (\mp 1)^{n-1} e^{-n\xi},$$

$$\Rightarrow \rho' = \int_0^\infty d\xi \sum_{n=1}^{\infty} (\mp 1)^{n-1} \xi^3 e^{-n\xi},$$

assuming we can interchange Σ and $\int$, we have;

$$\rho' = \sum_{n=1}^{\infty} (\mp 1)^{n-1} \int_0^{\infty} d\xi \; \xi^3 \; e^{-n\xi}$$ ↳ (Mathematica)

$$\Rightarrow \rho' = \sum_{n=1}^{\infty} (\mp 1)^{n-1} \frac{6}{n^4}$$

$$\therefore \; \rho' = 6 \sum_{n=1}^{\infty} \frac{(\mp 1)^{n-1}}{n^4}$$

→ For Bosons, we have:

$$\rho'_- = 6 \sum_{n=1}^{\infty} \frac{1^{n-1}}{n^4} = 6 \sum_{n=1}^{\infty} \frac{1}{n^4}$$ (Riemann zeta function)

$$\Rightarrow \rho'_- = 6\,\zeta(4)$$

→ For fermions, we have:

$$\rho'_+ = 6 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots\right)$$

$$\Rightarrow \rho'_+ = 6\left(1 - \frac{1}{2^4} + \left(\frac{1}{2^4} - \frac{1}{2^4}\right) + \frac{1}{3^4} - \frac{1}{4^4} + \left(\frac{1}{4^4} - \frac{1}{4^4}\right) + \cdots\right),$$

$$\Rightarrow \rho'_+ = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots - 2\left(\frac{1}{2^4} + \frac{1}{4^4} + \cdots\right)\right)$$

$$\Rightarrow \rho'_+ = 6\left(\zeta(4) - \frac{2}{2^4}\underbrace{\left(1 + \frac{1}{2^4} + \cdots\right)}_{\zeta(4)}\right)$$

which implies that:

$$\rho'_+ = 6\left(\zeta(4) - \frac{2}{2^4}\zeta(4)\right)$$

$$\Rightarrow \rho'_+ = 6\left(\zeta(4) - \frac{2}{16}\zeta(4)\right)$$

$$\Rightarrow \rho'_+ = \frac{7}{8}\left(6\zeta(4)\right)$$

$$\Rightarrow \rho'_+ = \frac{7}{8}\rho'_-$$

•) In the other limit, we have:

$$E(p) = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m},$$

thus, going back to the definition of ρ we have:

$$\rho = \frac{g}{2\pi^2}\int_0^\infty dp \frac{p^2(m + p^2/2m)}{\exp[(m + p^2/2m)/T]}$$

$$\Rightarrow \rho = \frac{g}{2\pi^2} e^{-m/T}\left[\int_0^\infty dp\, m p^2 e^{-p^2/2mT} + \int_0^\infty dp \frac{p^4}{2m} e^{-p^2/2mT}\right],$$

and again, by making a change of variables

$y^2 = \frac{p^2}{2mT}$, we have

$$\rho = \frac{g}{2\pi^2} e^{-m/T}\left[m\int_0^\infty dy\,\sqrt{2mT}\,(2mT)\,y^2 e^{-y^2} \right.$$

$$\left. + \frac{1}{2m}\int_0^\infty dy\,\sqrt{2mT}\,(2mT)^2 y^4 e^{-y^2} \right],$$

but $\int_0^\infty dy\, y^2 e^{-y^2} = \frac{1}{4}\sqrt{\pi}$ and $\int_0^\infty dy\, y^2 e^{-y^2} = \frac{3}{8}\sqrt{\pi}$,

thus:

$$\rho = \frac{g}{2\pi^2} e^{-m/T}\left[\frac{(2mT)^{3/2}}{4}\sqrt{\pi}\, m + \frac{1}{2m}(2mT)^{5/2}\left(\frac{3}{8}\sqrt{\pi}\right)\right]$$

$$\Rightarrow \rho = \frac{g e^{-mT}}{8}\left(\frac{2mT}{\pi}\right)^{3/2} m + \frac{g e^{-mT}}{32 m \pi^{3/2}}(3)(2mT)^{5/2}$$

$$\Rightarrow \rho = mg\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} + \frac{3}{2} g \left(\frac{mT}{2\pi}\right)^{3/2} T e^{-m/T}$$

In[233]:=

```
vMap = ResourceFunction["ViridisColor"];
```

Problem 1:

In[220]:=

```
Simplify[c × Integrate[y^(-2/(3 (1+ω))), {y, t0, t},
  Assumptions → {t0 → PositiveReals, t → PositiveReals, -1 < ω < 0}]]
```

Out[220]=

$$\frac{3\, c \left(t^{1-\frac{2}{3\,(1+\omega)}} - t0^{1-\frac{2}{3\,(1+\omega)}}\right)(1+\omega)}{1+3\,\omega} \quad \text{if} \quad \text{condition}$$

Problem 2:

Case 1

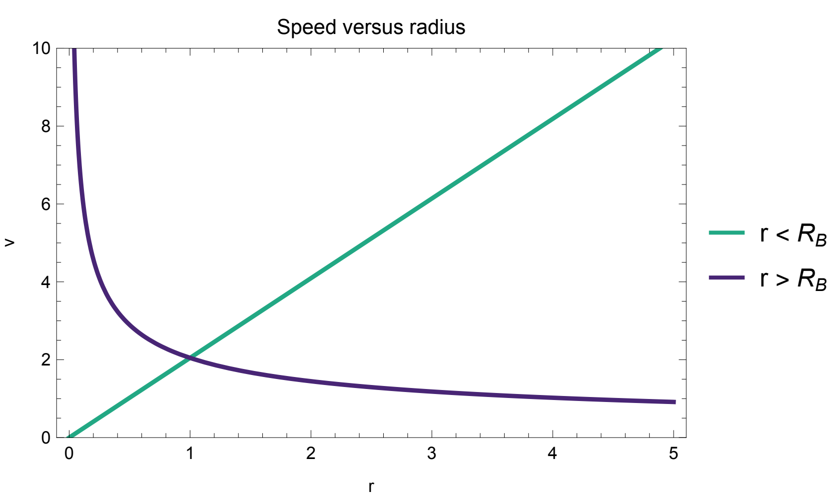
In[221]:=

```
params = {ρ → 1, G → 1, R_B → 1};
```

In[255]:=

```
Plot[{(Sqrt[4/3 π ρ G] r) /. params, (Sqrt[4/3 π ρ G R_B] r^(-1/2)) /. params}, {r, 0, 5},
 Frame → True,
 PlotLabel → "Speed versus radius",
 FrameLabel → {"r", "v"},
 PlotLegends → {"r < R_B", "r > R_B"},
 PlotRange → {{0, 10}},
 PlotStyle → {
   {Thickness[0.007], vMap[0.6]},
   {Thickness[0.007], vMap[0.1]}}]
```

Out[255]=

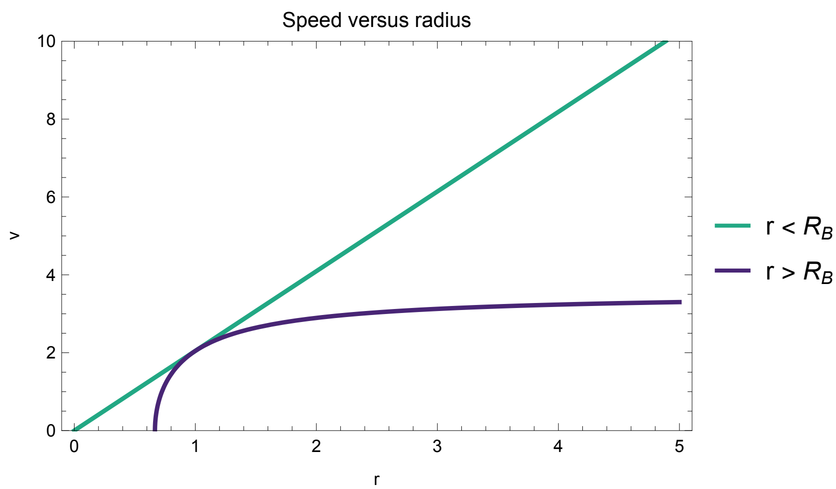


Case 2

In[254]:=

```
Plot[{(Sqrt[4/3 π ρ G] r) /. params,
  (Sqrt[4 π ρ G R_B^2 - 8/3 π ρ R_B^3 / r]) /. params}, {r, 0, 5},
 Frame → True,
 PlotLabel → "Speed versus radius",
 FrameLabel → {"r", "v"},
 PlotLegends → {"r < R_B", "r > R_B"},
 PlotRange → {{0, 10}},
 PlotStyle → {
   {Thickness[0.007], vMap[0.6]},
   {Thickness[0.007], vMap[0.1]}}]
```

Out[254]=



Problem 3:

In[185]:=

```
Integrate[ξ^2 Exp[-n ξ], {ξ, 0, Infinity}, Assumptions → {n ∈ PositiveReals}]
```

Out[185]=

$$\frac{2}{n^3}$$

In[186]:=

```
Integrate[ξ^3 Exp[-n ξ], {ξ, 0, Infinity}, Assumptions → {n ∈ PositiveReals}]
```

Out[186]=

$$\frac{6}{n^4}$$

In[175]:=

```
fPlus[x_] := NIntegrate[ξ^2 / (Exp[Sqrt[ξ^2 + x^2]] + 1), {ξ, 0, Infinity}];
fMinus[x_] := NIntegrate[ξ^2 / (Exp[Sqrt[ξ^2 + x^2]] - 1), {ξ, 0, Infinity}];
```

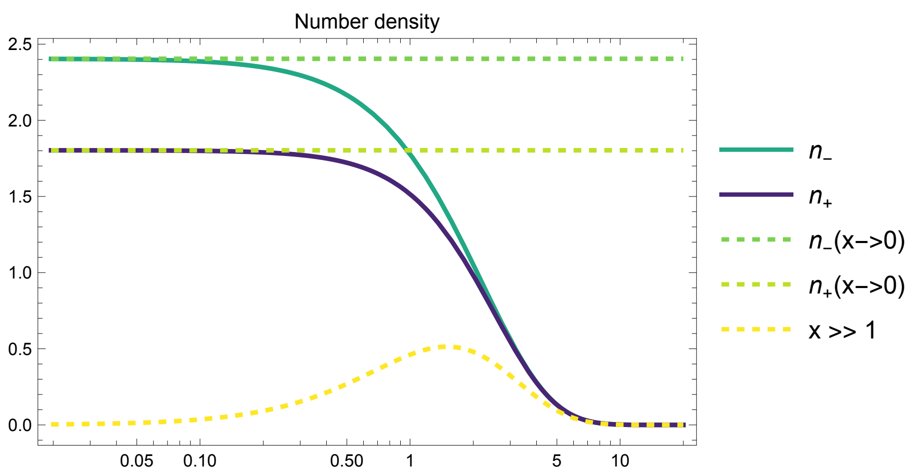
In[177]:=

```
fMinusInterp = Interpolation[Table[{x, fMinus[x]}, {x, 0, 20, 0.01}]];
fPlusInterp  = Interpolation[Table[{x, fPlus[x]}, {x, 0, 20, 0.01}]];
```

In[258]:=

```
fFullPlot = Plot[{
    fMinusInterp[x],
    fPlusInterp[x],
    2 N[Zeta[3]],
    (3/4) 2 N[Zeta[3]],
    Sqrt[π/2] x^(3/2) Exp[-x]}, {x, 0, 20},
   ScalingFunctions → {"Log"},
   Frame → True,
   PlotLabel → "Number density",
   PlotLegends → {"n₋", "n₊", "n₋(x->0)", "n₊(x->0)", "x >> 1"},
   PlotStyle → {
     {Thickness[0.007], vMap[0.6]},
     {Thickness[0.007], vMap[0.1]},
     {Thickness[0.007], Dashed, vMap[0.8]},
     {Thickness[0.007], Dashed, vMap[0.9]},
     {Thickness[0.007], Dashed, vMap[1.0]}}]
```

Out[258]=



In[180]:=

```
jPlus[x_] := NIntegrate[ξ^2 x Sqrt[ξ^2 + x^2] / (Exp[Sqrt[ξ^2 + x^2]] + 1), {ξ, 0, Infinity}]
```

In[181]:=

```
jMinus[x_] := NIntegrate[ξ^2 x Sqrt[ξ^2 + x^2] / (Exp[Sqrt[ξ^2 + x^2]] - 1), {ξ, 0, Infinity}]
```

In[182]:=

```
jMinusInterp = Interpolation[Table[{x, jMinus[x]}, {x, 0, 20, 0.01}]];
jPlusInterp = Interpolation[Table[{x, jPlus[x]}, {x, 0, 20, 0.01}]];
```

In[260]:=

```
jFullPlot = Plot[{
    jMinusInterp[x],
    jPlusInterp[x],
    6 N[Zeta[4]],
    7/8 × 6 N[Zeta[4]],
    Exp[-x]}, {x, 0, 20},
  ScalingFunctions → {"Log"},
  Frame → True,
  PlotLabel → "Energy density",
  PlotLegends → {"ρ₋", "ρ₊", "ρ₋(x->0)", "ρ₊(x->0)", "x >> 1"},
  PlotStyle → {
    {Thickness[0.007], vMap[0.6]},
    {Thickness[0.007], vMap[0.1]},
    {Thickness[0.007], Dashed, vMap[0.8]},
    {Thickness[0.007], Dashed, vMap[0.9]},
    {Thickness[0.007], Dashed, vMap[1.0]}}]
```

Out[260]=