MATH 171: HOMEWORK 3

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Problem 1. Prove that $\mathcal{B} = \{(a,b) \subset R : a,b \in \mathbb{Q}\}$ is a basis for the standard topology on R.

Solution. Let's start with the definition of basis for a topology. Given a TS (topological space) $(X, \mathcal{T})$, and a family $\mathcal{B} \subset \mathcal{T}$, we say that $\mathcal{B}$ is a basis for the TS $(X, \mathcal{T})$ if every open set can be written as the union of elements of $\mathcal{B}$. On the other hand, we know that the rationals are dense in the reals, this is

$$\forall x, y \in \mathbb{R} \text{ with } x < y, \exists r \in \mathbb{Q}, \text{ s.t } x < r < y.$$

Now, let $x, y \in \mathbb{R}$ we know that the standard topology of $\mathbb{R}$ is given by open intervals such as (x, y), with this in mind we have that

$$x < \frac{x+y}{2} < y,$$

and let's call $z_1 = (x+y)/2$, thus we have $z_1 \in \mathbb{R}$, and because the rationals are dense in the reals, we have that there exists $r_1, r_1' \in \mathbb{Q}$ such that

$$x < r_1 < z_1 \quad \& \quad z_1 < r_1' < y,$$

and from this let's consider $\mathcal{B}_1 = (r_1, r_1')$. Now let's repeat the same process, we have $x < z_1$ and $z_1 < y$, thus we have

$$x < \frac{x+z_1}{2} < z_1 \quad \& \quad z_1 < \frac{z_1+y}{2} < y,$$

and let's call $z_2 = (x+z_1)/2$ and $z_2' = (z_1+y)/2$. Again, by the property of $\mathbb{Q}$ being dense in $\mathbb{R}$ there exists $r_2, r_2' \in \mathbb{Q}$ such that

$$x < r_2 < z_2 \quad \& \quad z_2' < r_2' < y,$$

and from this we consider $\mathcal{B}_2 = (r_2, r_2')$. We repeat this process recursively and we'll have a family of open intervals of the form

$$\mathcal{B} = \{(a,b)\,; a, b \in \mathbb{Q}\},$$

and by construction we have that the union of this family is (x, y).

Therefore, we've just proved that $\mathcal{B}$ is a basis for the usual topology of the real line.

Problem 2. Give an example of two topologies on $\mathbb{R}$ such that neither is finer than the other space topology.

Solution. Let's also consider the lower limit topology on $\mathbb{R}$, $\mathbb{R}_l$, which is generated by the basis

$$[a, b) = \{x \in \mathbb{R}; a \leq x < b\},$$

and in this topology $\mathcal{U} \subset \mathbb{R}$ is open if $\forall x \in \mathcal{U}$, there is an $\epsilon > 0$ such that $[x, x + \epsilon) \subset \mathcal{U}$. And let's also consider the K-Topology on $\mathbb{R}$, $\mathbb{R}_k$, which is obtained by taking as a base the family of open intervals (a, b) together with sets of the form

$$(a, b) / K,$$

where $a, b \in \mathbb{R}$ and K is defined as

$$K = \left\{ \frac{1}{n}; n = 1, 2, \cdots \right\}.$$

Now, let's prove that these topologies are not comparable; clearly $[1, 2)$ is open in $\mathbb{R}_l$ but it's not open in $\mathbb{R}_k$. Indeed, any open interval that contains 1 must contain an interval given by $(x - \epsilon, x + \epsilon)$. On the other hand $(-1, 1) / K$ is clearly open in $\mathbb{R}_k$, but is not open in $\mathbb{R}_l$. Indeed any open set in $\mathbb{R}_l$ that contains 0 must contain an interval $[0, \epsilon)$ for some $\epsilon > 0$, and therefore contain elements of K.

Problem 3. Consider the following:

3.1 Consider sequences in $\mathbb{R}$ with the finite complement topology. Which sequences converge? To what value(s) to they converge?

3.2 Consider sequences in $\mathbb{R}$ with the countable complement topology. Which sequences converge? To what value(s) to they converge?

Solution. Let $X = \mathbb{R}$.

3.1 The finite complement topology $\mathcal{T}_{fc}$ over $\mathbb{R}$ is defined as: $\mathcal{U} \in \mathcal{T}_{fc}$ if $\mathbb{R}/\mathcal{U}$ is finite or $\mathcal{U} = \emptyset$. Now, let's assume that $x_i \to x$, with $x \in \mathbb{R}$, then this implies that for each open set $\mathcal{U} \ni x$ there exist some integer $N \in \mathbb{N}$ such that for each $i > N$ we have $x_i \in \mathcal{U}$.

But being open in this topology means that the complement is finite, this is $V = \mathbb{R}/\mathcal{U}$ is finite, and because for each $i > N$ all elements belong to the open set, this implies that $x_i \notin V$ for each $i > N$. Moreover, any open set in this topology is finite, thus this implies that after $i > N$ we have $x_i = x$. This means that after some label i the sequence is constant.

3.2 On the other hand, the countable complement topology $\mathcal{T}_{cc}$ is defined as $\mathcal{U} \in \mathcal{T}_{cc}$ if $\mathbb{R}/\mathcal{U}$ is countable or $\mathcal{U} = \emptyset$. Here, the reasoning is pretty similar to the previous problem. Again, let's assume that $x_i \to x$, with $x \in \mathbb{R}$, then this implies that for each open set $\mathcal{U} \ni x$ there exist some integer $N \in \mathbb{N}$ such that for each $i > N$ we have $x_i \in \mathcal{U}$. In this case $\mathcal{U}$ being open means that its complement is countable. Now, let's consider

$$\mathcal{U} = \{x_i \in \mathbb{R} : i \in \mathbb{N}\},$$

it follows that $\mathbb{R}/\mathcal{U}$ is open. Now, let's suppose that $x_i \neq x$ for all $i \in \mathbb{N}$, but this implies that $x \in \mathbb{R}/\mathcal{U}$ but this a contradiction with the assumption that the sequence converges, because we found an open set that contains x and does not contain any of the members of x_i. Thus it follows that if $x_i \to x$, then $x_i = x$ for at least one $i \in \mathbb{N}$. Now let's consider the following set

$$A = \{x_i \in \mathbb{R}; i \in \mathbb{N} \text{ and } x_i \neq x\},$$

it follows that this set is countable, therefore $\mathbb{R}/A$ is open, but $x_i \to x$, thus $x_i = n$ for all i sufficiently large.

Problem 4. Let X be a totally ordered set by $\leq$. Let $\mathcal{S}$ be the collection of sets of the form

$$\mathcal{S} = \{x \in X : x < a\} \text{ or } \{x \in X : a < x\}$$

for $a \in X$. Prove that $\mathcal{S}$ is a sub-basis for a topology on X called the order topology.

Solution. The order topology is defined as follows: let X be a set totally ordered by $\leq$ and let $a, b \in X$, then let $\mathcal{B}$ be the family of subsets of X that are of the form

$$\{x \in X; x < a\} \text{ or } \{x \in X; a < x\} \text{ or } \{x \in X; a < x < b\}.$$

Then $\mathcal{B}$ is a basis for the topology $\mathcal{T}$, called the order topology on X.

On the other hand we know that a family of open sets $\mathcal{S}$ is sub-basis if all finite intersections of $\mathcal{S}$ form a basis for the TS.

So let X be a totally ordered set, and $a, b \in X$, with $a \leq b$, and let's consider

$$S_{a<} = \{x \in X : x < a\}, \quad S_{a>} = \{x \in X : a < x\},$$

$$S_{b<} = \{x \in X : x < b\}, \quad S_{b>} = \{x \in X : b < x\},$$

and let's look at the intersections. It's clear that if $a \leq b$ it follows that $S_{a<} \subset S_{b<}$ and $S_{b>} \subset S_{a>}$, which implies that

$$S_{a<} = S_{a<} \cap S_{b<}, \quad \& \quad S_{b>} = S_{b>} \cap S_{a>}$$

and on the other hand,

$$\{x \in X; a < x < b\} = S_{a>} \cap S_{b<}$$

therefore, as we can see all finite (and allowed) intersections of $\mathcal{S}$ form a basis for the order topology of X, which implies that $\mathcal{S}$ is a sub-basis.