

# Problem 1: Evolution of Universe with general Energy of Particles

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The Friedmann equation is given by

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

where

$$\rho = \frac{\rho_0 R_0^3}{R^3}.$$

Problem 1. Rewrite the Friedmann equation by replacing derivatives with respect to the time variable  $t$  by derivatives with respect to the conformal time variable  $\eta$  defined by

$$dt = R d\eta$$

Solution 1. By the chain rule, we have

$$\frac{dR}{dt} = \frac{dR}{d\eta} \frac{d\eta}{dt} \implies \frac{dR}{dt} = \frac{dR}{d\eta} \frac{1}{R},$$

and from this, the Friedmann equation is rewritten as

$$\left(\frac{dR}{d\eta} \frac{1}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi G R^4}{3}\rho + \frac{2E R^4}{mR^2},$$

but  $\rho$  is a function of  $R$ , thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi\rho_0 R_0^3 G}{3}R + \frac{2E}{m}R^2.$$

and by making

$$\alpha = \frac{8\pi\rho_0 R_0^3 G}{3}, \beta = \frac{2E}{m},$$

we have

$$\left(\frac{dR}{d\eta}\right)^2 = \alpha R + \beta R^2.$$

Problem 2. Use separation of variables to re-write the Friedmann equation as an expression for  $\eta$  in terms of an integral of  $R$ .

Solution 2. By separation of variables we have

$$\frac{dR}{\sqrt{\alpha R + \beta R^2}} = d\eta,$$

thus, the problem now resides on solving the previous integral.

Problem 3. Carry out the integral and invert to obtain  $R(\eta)$ ; simplify for  $E > 0$ ,  $E = 0$ , and  $E < 0$ .

Solution 3. In the variables that I'm using  $E = 0 \implies \beta = 0$ ,  $E > 0 \implies \beta > 0$ , and  $E < 0 \implies \beta < 0$ , So let's proceed accordingly.

**Case 1**,  $\beta = 0$ : in this case, the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha}{2} \eta^2$$

**Case 2**,  $\beta < 0$ : Here, I'm going to add the sign of  $\beta$  by hand (not elegant at all, but just to be more explicit),

$$\int \frac{dR}{\sqrt{\alpha R - \beta R^2}} = \int d\eta,$$

and here the idea is to make a change of variables (I attach the notes with the algebra at the end of the document). Having done that, the solution reads

$$R(\eta) = \frac{\alpha}{\beta} \cos^2 \left( \frac{\sqrt{\beta}}{2} \eta \right)$$

**Case 3**,  $\beta > 0$ : Here the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R + \beta R^2}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha e^{\alpha \eta}}{1 - \beta e^{\alpha \eta}},$$

and again, details about the algebra are attached at the end of the document.

Problem 4. Determine  $t(\eta)$ ; again simplify for  $E > 0$ ,  $E = 0$ , and  $E < 0$ .

Solution 4. We know that

$$dt = R d\eta,$$

and in the previous bullet we found exactly  $R(\eta)$ , thus we only need to integrate, this is

$$t(\eta) = \int d\eta R(\eta).$$

**Case 1**,  $\beta = 0$ : in this case we have

$$t(\eta) = \frac{\alpha}{2} \int d\eta (\eta^2),$$

with solution given by

$$t = \frac{\alpha}{6} \eta^3.$$

**Case 2**,  $\beta < 0$ : here we have

$$t(\eta) = \frac{\alpha}{\beta} \int d\eta \cos^2 \left( \frac{\sqrt{\beta}}{2} \eta \right),$$

with solution

$$t = \frac{\alpha}{\beta} \frac{2 \sin \left( \frac{\sqrt{\beta}}{2} \eta \right)}{\sqrt{\beta}}$$

**Case 3**,  $\beta > 0$ : and finally, we have

$$t(\eta) = \alpha \int d\eta \frac{e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}}$$

with solution given by

$$t(\eta) = -\frac{\log(1 - \beta e^{\alpha x})}{\beta}.$$

Problem 5. Using the above pair of parametric equations, plot  $R(t)$  (use some plotting software, such as Mathematica) for 3 different choices of  $E$ , namely  $E > 0$ ,  $E = 0$ , and  $E < 0$ .

Solution 5. By combining the two previous parts we can write a parametric equation

$$\gamma(\eta) = (t(\eta), R(\eta)),$$

therefore.

**Case 1**,  $\beta = 0$ : in this case we have

$$\gamma(\eta) = \frac{\alpha}{2} (\eta^3, \eta^2)$$

**Case 2**,  $\beta < 0$ : again, we have

$$\gamma(\eta) = \frac{\alpha}{\beta} \left( \frac{2}{\sqrt{\beta}} \sin \left( \frac{\sqrt{\beta}}{2} t \right), \cos^2 \left( \frac{\sqrt{\beta}}{2} t \right) \right)$$

**Case 3**,  $\beta > 0$ : and finally

$$\gamma(\eta) = \left( -\frac{\log(1 - \beta e^{\alpha x})}{\beta}, \frac{\alpha e^{\alpha \eta}}{1 - \beta e^{\alpha \eta}} \right)$$

The plots are shown in the next page.