Point Set Topology: HW7

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Problem 1 *Let X be a topological space and D be a dense subset of X. Suppose $f : X \to Y$ is a continuous surjective function. Prove that $f(D)$ is dense in Y.*

Proof 1 *Let $f : X \to Y$ be a continuous function and $D \subset Y$ be a dense subset of Y. The continuity of f means that the image inverse of any open set in Y is an open set in X.*

On the other hand, the density of D means that $\overline{D} = X$, or equivalently, that for any open set in $U \in X$, we have $U \cap D \neq \emptyset$.

We want to prove that $f(D)$ is dense in Y, which means

$$V \cap f(D) \neq \emptyset,$$

for any open set $V \in Y$.

So, let V be an open set in Y; because f is surjective, this means that $f(X) = Y$, or in other words, V contains points of the image of f. On the other hand, because V is open, $f^{-1}(V)$ is open for the continuity of f. And now, let's use the density of D in this open set, this is

$$f^{-1}(V) \cap D \neq \emptyset,$$

and because it is non-empty, let $x \in f^{-1}(V) \cap D$, which implies that

$$x \in f^{-1}(V) \implies f(x) \in V,$$

and

$$x \in D \implies f(x) \in D,$$

thus

$$f(x) \in V \cap f(D),$$

or in other words

$$V \cap f(D) \neq \emptyset$$

for any open set $V \in Y$; this is, $f(D)$ is dense in Y.

Problem 2 1. *Find an open function that is not continuous.*

2. *Find a closed function that is not continuous.*
3. *Find a continuous function that is neither open nor closed.*
4. *Find a continuous function that is open but not closed.*
5. *Find a continuous function that is closed but not open.*

Proof 2 1. *Let's consider the following function, $f : \mathbb{R} \to \mathbb{R}$ defined as follows*

$$\begin{cases} 2x & x < 0, \\ x+1 & x \geq 0, \end{cases}$$

clearly f is not continous at $x = 0$, however, it is open. Indeed, let $a, b \in \mathbb{R}$, if $(a, b) \subset (-\infty, 0)$, then the image of this interval is $(2a, 2b)$, which is an open set, on the other hand, if $(a, b) \subset [0, \infty)$ then the image will be $(a + 1, b + 1)$ which again, is an open set. Finally, if $0 \in (a, b)$ then the image will be $(2a, 0) \cup (1, b + 1)$ which again, is open.

2. *Now, let's consider the following function $f : \mathbb{R} \to \mathbb{R}$, also defined piece-wise;*

$$\begin{cases} 0 & x \leq 0, \\ 1 & x > 0, \end{cases}$$

this function is closed. Indeed, let $E \subset \mathbb{R}$ be a closed subset; then $f(E)$ only has 0 and/or 1, thus $f(E)$ can only be $\emptyset, \{0\}, \{1\}$ or $\{0, 1\}$, and all these sets are closed in $\mathbb{R}$, however, f is not continuous at $x = 0$.

3. —
4. —
5. *Finally, let $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = 1$ for all $x \in \mathbb{R}$, clearly f is continuous and is also closed because for any closed set $E \subset \mathbb{R}$ we have $f(E) = \{1\}$ which is closed, but it's not open.*

Problem 3 *Let X be a compact topological space and Y be a Hausdorff space. Suppose that $f : X \to Y$ is continuous. Prove that f is closed.*

Proof 3 *We want to prove that $f(E)$ is closed in Y for any closed set E in X.*

Let E be a closed set in X; because X is compact, every closed set is also compact; thus E is also compact.

On the other hand, f maps compact sets on compact sets by the continuity of f. Thus, $f(E)$ is compact.

But Y is Hausdorff, and in any Hausdorff space, any compact set is also closed; therefore $f(E)$ is closed.

Problem 4 *Let $\{X_\alpha\}_\alpha \in \Lambda$ be a collection of topological spaces. Prove that the product topology on the cartesian product $\Pi_\alpha \in \Lambda X_\alpha$ is the coarsest topology that each of the projection maps $\pi_\beta : \Pi_\alpha \in \Lambda X_\alpha X_\alpha \to X_\beta$ continuous.*

Proof 4 *Let T be the topology on the product space Π_α with $\alpha \in \Lambda$, we need to show that 1) all the projections are continuous in T and 2) for any other topology T' making any of the projection operators continuous we have $T \subset T'$.*

Let's prove 1); the product topology is generated by the basic elements

$$\pi_{\beta_1}^{-1}(U_1) \cap \pi_{\beta_2}^{-1}(U_2) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_n),$$

where each one of U_i being open in X_i. Now, let π_β be any projection map, and let $U \subset X_\beta$ an open set, then it follows that $\pi_\beta^{-1}(U)$ is open; thus π_β is continuous.

Now, let's prove that it is the coarsest topology; let T' be any topology that makes all the projection maps continuous, and let B any open set in the product topology T, then, we can express B as follows

$$B = \pi_{\beta_1}^{-1}(U_1) \cap \pi_{\beta_2}^{-1}(U_2) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_n),$$

and because π_{beta_i} is continuous in T' by assumption, thus $\pi_{\beta_i}^{-1}(U_i)$ is open in T', but because T' is a topology, it follows that all finite intersections are open, which implies that $B \in T'$, which implies that every basic open set in T is also an element in T', and T is generated by these elements; thus $T \subset T'$, just as we wanted.

Problem 5 *Let (E,d) be a complete metric space and $f : E \to E$ be a function such that there exists $c \in (0,1)$ such that $d(f(x), f(y)) \leq cd(x,y)$ for all $x, y \in E$.*

1. *Prove that f is uniformly continuous*
2. *Let $x_1 \in E$ and define the sequence $\{x_n\}$ in E be $x_{n+1} = f(x_n)$ for $n \geq 1$. Prove that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in E. Denote x_0 the limit of this sequence.*
3. *Prove that $f(x_0) = x_0$ and that x_0 is the unique point of E with this property.*

Proof 5

1. *Being f uniformly continuous on E means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all x and y in E, if $d(x,y) < \delta$, then $d(f(x), f(y) < \epsilon$.*

   So, let $\delta > 0$ and $x, y \in E$, and such that $d(x,y) < \delta$, but we know that

   $$d(f(x), f(y)) \leq cd(x,y) \implies d(f(x), f(y)) < c\delta,$$

   thus, making $\epsilon = \delta/c$ we get the desired inequality; therefore, f is uniformly continous.

2. *Now, let's prove that the given sequence is a Cauchy sequence. Indeed, let's look at the first few terms*

   $$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq cd(x_1, x_2),$$

   and also

   $$d(x_3, x_3) = d(f(x_2), f(x_3)) \leq cd(x_2, x_3) \leq c^2 d(x_1, x_2),$$

   thus, we can see that for any $k \geq 1$ we have

   $$d(x_k, x_{k+1}) \leq c^{k+1} d(x_1, x_2).$$

   Now, for any $p > 0$ and $n \geq 1$, let's look at $d(x_n, x_{n+p})$,

   $$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{n+p-1}, x_{n+p})$$

   but from the previous result, we have that

   $$d(x_n, x_{n+p}) \leq c^{n-1} d(x_1, x_2) + c^n d(x_1, x_2) + ... + c^{n+p-2} d(x_1, x_2)$$

and the right-hand side is equal to

$$d(x_1, x_2)[c^{n-1} + c^n + ... + c^{n+p-2}] = d(x_1, x_2)c^{n-1}[1 + c + ... + c^{p-1}],$$

thus

$$d(x_n, x_{n+p}) \leq d(x_1, x_2)c^{n-1}[(1 - c^p)/(1 - c)],$$

but because $c \in (0, 1)$*, thus* $c^{n-1} \to 0$ *as* $n \to \infty$*, and* $(1 - c^p)/(1 - c)$ *is bounded for all* $p \in \mathbb{R}$*, therefore* $d(x_n, x_{n+p}) \to 0$ *as* $n \to \infty$*, which means that for any* $\epsilon > 0$*, there exists* N *such that for all* $n \geq N$ *and all* $p > 0$*, we have* $d(x_n, x_{n+p}) < \epsilon$*, which is the definition of Cauchy sequence. Since* E *is complete, we know that* $\{x_n\}$ *converges, and let's call this limit* x_0*.*

3. *Let's prove that* $f(x_0) = x_0$*. Indeed, we know that* x_0 *is the limit of* $\{x_n\}$*, thus or any* $\epsilon > 0$*, there exists* N *such that for all* $n \geq N$ *we have* $d(x_n, x_0) < \epsilon$*. Now, let's consider* $d(f(x_0), x_0)$ *and prove that this distance is zero; indeed*

4.

$$d(f(x_0), x_0) \leq d(f(x_0), f(x_n)) + d(f(x_n), x_0) \leq cd(x_0, x_n) + d(x_{n+1}, x_0),$$

thus for any $\epsilon > 0$*, we can choose* N *large enough such that for* $n \geq N$ *we have* $d(x_0, x_n) < \epsilon$ *and* $d(x_{n+1}, x_0) < \epsilon$*, thus* $d(f(x_0), x_0) \leq c\epsilon + \epsilon = c(1 + \epsilon)$*, which is true for any* $\epsilon > 0$*, therefore* $d(f(x_0), x_0) = 0$*.*