

GR-HW-04: Spin and Electromagnetism and Linear Gravitational Field Equation

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Problem 1 (Spin 2 Polarization)

I did the whole problem in Mathematica and I append the notebook.

Problem 2 (Electromagnetism with Point Charges)

(i) The expression given by

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)), \quad (1)$$

is indeed a good 4-Lorentz vector, and let me explain why.

- First, q_i is a scalar, so it's fine.
- Proper time τ_i is a good Lorentz invariant.
- The 4-velocity, $dx_i^\mu/d\tau_i$ is a good Lorentz vector.
- And finally, the 4-dimensional delta is also a Lorentz invariant.

Therefore, the whole combination in the previous expression is a good 4-Lorentz vector, since is the combination of good Lorentz quantities.

(ii) Let's express J^μ in terms of a 3-dimensional delta function. Let's start with

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)), \quad (2)$$

then, by the chain rule, we have

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dt}{d\tau_i} \frac{dx_i^\mu}{dt} \delta(t - t_i(\tau_i)) \delta^3(x^\alpha - x_i^\alpha(\tau_i)), \quad (3)$$

which implies that

$$J^\mu(x) = \sum_i^N q_i \int dt \frac{dx_i^\mu}{dt} \delta(t - t_i(\tau_i)) \delta^3(x^\alpha - x_i^\alpha(\tau_i)), \quad (4)$$

and from this we can express perform the time integral. We have

$$J^\mu(x) = \sum_i^N q_i \frac{dx_i^\mu}{dt} \delta^3(x^\alpha - x_i^\alpha(t)), \quad (5)$$

(iii) And now let's compute

$$\int d^3x J^0(x), \quad (6)$$

from which the previous expression we have

$$\int d^3x J^0(x) = \int d^3x \left(\sum_i^N q_i \frac{dx_i^0}{dt} \delta^3(x^\alpha - x_i^\alpha(t)) \right), \quad (7)$$

and since everything under the integral sign is well behaved we can commute the summation and the integral

$$\int d^3x J^0(x) = \sum_i^N \int d^3x \left(q_i \frac{dx_i^0}{dt} \delta^3(x^\alpha - x_i^\alpha(t)) \right), \quad (8)$$

but $dx^0 = cdt$, thus the previous expression reduces to

$$\int d^3x J^0(x) = \sum_i^N cq_i, \quad (9)$$

or

$$\int d^3x J^0(x) = cQ, \text{ where } Q = \sum_i^N q_i. \quad (10)$$

(iii) The interaction term is given by

$$S_{int} = - \int d^4x A_\mu J^\mu \quad (11)$$

let's massage this term a little bit. So let's begin

$$S_{int} = - \int d^4x A_\mu J^\mu = - \int d^4x A_\mu \left[\sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)) \right], \quad (12)$$

and assuming the functions inside the integrand are well behaved, we can commute the sum and the integral operations, this yields

$$S_{int} = - \sum_i^N q_i \int d^4x A_\mu \left[\int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)) \right], \quad (13)$$

and if we unfold the 4-dimensional delta, we have

$$S_{int} = - \sum_i^N q_i \int d^4x A_\mu \left[\int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta(t - t_i(\tau_i)) \delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \quad (14)$$

and we can write the inner integral just as we did before, this is

$$S_{int} = - \sum_i^N q_i \int d^4x A_\mu \left[\int dt \frac{d\tau_i}{dt} \frac{dx_i^\mu}{d\tau_i} \delta(t - t_i(\tau_i)) \delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \quad (15)$$

it follows that

$$S_{int} = - \sum_i^N q_i \int d^4x A_\mu \left[\frac{dx_i^\mu}{dt} \delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \quad (16)$$

and from this, the 3-dimensional delta collapses the 4-dimensional integral to a 1-dimensional one as follows

$$S_{int} = - \sum_i^N q_i \int dt \frac{dx_i^\mu}{dt} A_\mu(x_i), \quad (17)$$

and again, by the chain rule, we have

$$S_{int} = - \sum_i^N q_i \int d\tau_i \frac{dt}{d\tau_i} \frac{dx_i^\mu}{dt} A_\mu(x_i), \quad (18)$$

thus, the final result is given by

$$S_{int} = - \sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} A_\mu(x_i), \quad (19)$$

just as we wanted.

(iv) Now, let's compute the equations of motion from the Euler–Lagrange equations. In this part, the Lagrangian is given by

$$\mathcal{L}_i = -m_i \sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu} - q_i A_\mu \dot{x}_i^\mu, \quad (20)$$

where

$$\dot{x}_i^\mu = \frac{dx_i^\mu}{d\tau} \quad (21)$$

and we know the Euler–Lagrange equation is given by

$$\frac{d}{d\tau} \left(\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial x_i^\alpha} = 0. \quad (22)$$

For the second term, we have

$$\frac{\partial \mathcal{L}}{\partial x_i^\alpha} = -q_i \dot{x}_i^\mu \partial_\alpha A_\mu, \quad (23)$$

now, for the first term we need to use the chain rule, which gives us

$$\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \frac{\eta_{\alpha\nu} \dot{x}_i^\nu}{\sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu}} - q_i A_\alpha, \quad (24)$$

but we know that

$$\sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu} = 1, \quad (25)$$

thus, we have

$$\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \dot{x}_i^\nu - q_i A_\alpha \implies \frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \ddot{x}_i^\nu - q_i \frac{dA_\alpha}{d\tau}, \quad (26)$$

but

$$\frac{dA_\alpha}{d\tau} = \frac{\partial A_\alpha}{\partial x_i^\mu} \frac{dx_i^\mu}{d\tau} = \partial_\mu A_\alpha \dot{x}_i^\mu, \quad (27)$$

which implies that

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \ddot{x}_i^\nu - q_i \partial_\mu A_\alpha \dot{x}_i^\mu \implies \frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \ddot{x}_{i\alpha} - q_i \partial_\mu A_\alpha \dot{x}_i^\mu \quad (28)$$

then, if we combine both terms in the Euler–Lagrange equations, we have

$$-m_i \ddot{x}_{i\alpha} - q_i \partial_\mu A_\alpha \dot{x}_i^\mu + q_i \dot{x}_i^\mu \partial_\mu A_\alpha = 0, \quad (29)$$

which implies that

$$m_i \ddot{x}_{i\alpha} = q_i \dot{x}_i^\mu (\partial_\alpha A_\mu - \partial_\mu A_\alpha), \quad (30)$$

and from this we have

$$m_i \ddot{x}_{i\alpha} = q_i \dot{x}_i^\mu F_{\alpha\mu}, \quad (31)$$

which can be written as

$$m_i \eta^{\sigma\alpha} \ddot{x}_{i\alpha} = q_i \eta^{\sigma\alpha} \dot{x}_i^\mu F_{\alpha\mu} \implies m_i \ddot{x}_i^\sigma = q_i F_\mu^\sigma \dot{x}_i^\mu, \quad (32)$$

and if we change $\sigma \rightarrow \alpha$, we have

$$m_i \ddot{x}_i^\alpha = q_i F_\mu^\alpha \dot{x}_i^\mu, \quad (33)$$

just as we wanted.

(v) Finally, if we set $\alpha = j$ in the previous expression, by the chain rule, we have

$$m_i \frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{dx_i^j}{d\tau} \right) = q_i F_\mu^j \frac{d\tau}{dt} \frac{dx_i^\mu}{d\tau}, \quad (34)$$

which implies that

$$\frac{d}{dt} \left(m_i \frac{dx_i^j}{d\tau} \right) = q_i F_\mu^j \frac{dx_i^\mu}{dt} \implies \frac{d}{dt} (\mathbf{p}_i) = q_i F_\mu^j \frac{dx_i^\mu}{dt}, \quad (35)$$

on the other hand, we know the matrix representation of the Faraday tensor, which is

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}, \quad (36)$$

and from this we know

$$F_\mu^\alpha = \eta_{\mu\beta} F^{\alpha\beta}, \quad (37)$$

or in matrix notation

$$F_\mu^\alpha = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \quad (38)$$

but since we're interested in the spatial index in α we have

$$F_\mu^i \frac{dx_i^\mu}{dt} = \begin{bmatrix} -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \cdot \begin{bmatrix} cdt/dt \\ -dx_i/dt \\ -dy_i/dt \\ -dz + i/dt \end{bmatrix} \quad (39)$$

then we have

$$\frac{d\mathbf{p}_i}{dt} = q_i \begin{bmatrix} E_x - B_z dy_i/dt + B_y dz_i/dt \\ E_y + B_z dx_i/dt - B_x dz_i/dt \\ E_z - B_y dx_i/dt + B_x dy_i/dt \end{bmatrix}, \quad (40)$$

which can also be written as

$$\frac{d\mathbf{p}_i}{dt} = q_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \quad (41)$$

which is the equation of motion for particle i in the presence of an electric and magnetic field, the RHS of the previous equation is the Lorentz force.

Problem 3 (Gravitational Field Equations)

The lagrangian is given by

$$\mathcal{L} = \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu} + \frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h\partial_\nu h^{\mu\nu} - h_{\mu\nu}\tau^{\mu\nu} \quad (42)$$

Let's compute the Euler-Lagrange equations of motion, which are given by

$$\partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha h^{\mu\nu})} \right) - \frac{\partial \mathcal{L}}{\partial h^{\mu\nu}} = 0, \quad (43)$$

and from here, probably the easiest term is the derivative with respect to $h^{\mu\nu}$, for that we have

$$\frac{\partial \mathcal{L}}{\partial h^{\mu\nu}} = \frac{\partial}{\partial h^{\mu\nu}} (h_{\mu\nu}\tau^{\mu\nu}) = \tau^{\mu\nu}. \quad (44)$$

Now, the hard work comes with the remainder terms, so I'll proceed one by one, but before that; in the derivation I'll be using heavily the following identity

$$\frac{\partial(\partial_\alpha h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\delta_k^\alpha [\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu], \quad (45)$$

and with this in mind, let's begin. For the first term, we have

$$\mathcal{L}_1 = \frac{1}{4}\partial_\lambda h_{\rho\sigma}\partial^\lambda h^{\rho\sigma}, \quad (46)$$

and from there we have

$$\frac{\partial(\mathcal{L}_1)}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4} \left[\frac{\partial(\partial_\lambda h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} \partial^\lambda h^{\rho\sigma} + \partial_\lambda h_{\rho\sigma} \frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} \right], \quad (47)$$

the first term in the previous expression is easy, as all the indices are lowered, and by using equation (45), we have that

$$\frac{1}{8}\delta_\lambda^\alpha [\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu] \partial^\lambda h^{\rho\sigma}, \quad (48)$$

by expanding the product we have

$$\frac{1}{8}\delta_\lambda^\alpha [\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu] \partial^\lambda h^{\rho\sigma} = \frac{1}{8}\delta_\lambda^\alpha [\delta_\rho^\mu\delta_\sigma^\nu \partial^\lambda h^{\rho\sigma} + \delta_\rho^\nu\delta_\sigma^\mu \partial^\lambda h^{\rho\sigma}] \quad (49)$$

and from this we can see that $\lambda = \alpha$, and in the first term we have $\rho = \mu$ and $\sigma = \nu$, whereas for the second one, we have $\rho = \nu$ and $\sigma = \mu$, thus

$$\frac{1}{8}\delta_\lambda^\alpha [\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu] \partial^\lambda h^{\rho\sigma} = \frac{1}{8}[\partial^\alpha h^{\mu\nu} + \partial^\alpha h^{\nu\mu}], \quad (50)$$

but $h_{\mu\nu}$ is symmetric, thus

$$\frac{1}{8}\delta_\lambda^\alpha [\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu] \partial^\lambda h^{\rho\sigma} = \frac{1}{4}h^\alpha h^{\mu\nu}, \quad (51)$$

and now, for the second term in equation (47), we need to work a little more since the indices are up, but we know the way to lower them is via the metric, thus

$$\partial^\lambda h^{\rho\sigma} = \eta^{\lambda\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}\partial_\beta h_{\gamma\delta}, \quad (52)$$

and from this we have

$$\frac{1}{4}\partial_\lambda h_{\rho\sigma} \frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4}\eta^{\lambda\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}\partial_\lambda h_{\rho\sigma} \frac{\partial(\partial_\beta h_{\gamma\delta})}{\partial(\partial_\lambda h_{\rho\sigma})}, \quad (53)$$

ana again, we can use the identity given in equation (45), which in this case reads

$$\frac{\partial(\partial_\beta h_{\gamma\delta})}{\partial(\partial_\lambda h_{\rho\sigma})} = \frac{1}{2}\delta_\beta^\alpha [\delta_\delta^\mu\delta_\gamma^\nu + \delta_\gamma^\nu\delta_\delta^\mu], \quad (54)$$

and from this, we can see that the first delta implies $\beta = \alpha$, the deltas in the first term also implies that $\delta = \mu$, $\gamma = \nu$ whereas for the second term we have $\gamma = \nu$ and $\delta = \mu$, then if we expand the product we have

$$\frac{1}{4}\partial_\lambda h_{\rho\sigma} \frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{8}\eta^{\lambda\alpha}\eta^{\rho\nu}\eta^{\sigma\mu}\partial_\lambda h_{\rho\sigma} + \frac{1}{8}\eta^{\lambda\alpha}\eta^{\rho\nu}\eta^{\sigma\mu}\partial_\lambda h_{\rho\mu}, \quad (55)$$

which can also be written as

$$\frac{1}{4}\partial_\lambda h_{\rho\sigma} \frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{8}\partial^\alpha h^{\mu\nu} + \frac{1}{8}\partial^\alpha h^{\nu\mu}, \quad (56)$$

but again, by symmetry, we have

$$\frac{1}{4}\partial_\lambda h_{\rho\sigma} \frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4}\partial^\alpha h^{\mu\nu} \quad (57)$$

therefore,

$$\frac{\partial(\mathcal{L}_1)}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4}\partial^\alpha h^{\mu\nu} + \frac{1}{4}\partial^\alpha h^{\nu\mu} \quad (58)$$

thus

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\partial^\alpha h^{\mu\nu}, \quad (59)$$

and from this

$$\partial_\alpha \left(\frac{\partial\mathcal{L}_1}{\partial(\partial_\alpha h_{\mu\nu})} \right) = \frac{1}{2}\partial_\alpha\partial^\alpha h^{\mu\nu} = \frac{1}{2}\square h^{\mu\nu}. \quad (60)$$

Now, let's move to the second term

$$\mathcal{L}_2 = -\frac{1}{2}\partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu}, \quad (61)$$

then we have

$$\frac{\partial\mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2} \left[\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho} + \partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h^{\sigma\rho})}{\partial(\partial_\alpha h_{\mu\nu})} \right] \quad (62)$$

but we know that

$$\partial^\lambda = \eta^{\lambda\beta}\partial_\beta, \quad (63)$$

thus, for the first term we have

$$\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho} = \eta^{\lambda\beta}\frac{\partial(\partial_\beta h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}, \quad (64)$$

and by making use of the equation (45), we have

$$\frac{\partial(\partial_\beta h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho} = \eta^{\lambda\beta} \left[\frac{1}{2}\delta_\beta^\alpha(\delta_\lambda^\mu\delta_\rho^\nu + \delta_\lambda^\nu\delta_\rho^\mu) \right] \partial_\sigma h^{\sigma\rho}, \quad (65)$$

which implies that

$$\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho} = \frac{1}{2}\eta^{\alpha\mu}\partial_\sigma h^{\sigma\nu} + \frac{1}{2}\eta^{\alpha\nu}\partial_\sigma h^{\sigma\mu}, \quad (66)$$

and for the second term is pretty much the same procedure, but we need to lower the indices of the $h^{\rho\sigma}$, this is

$$h^{\sigma\rho} = \eta^{\sigma\beta}\eta^{\rho\epsilon}h_{\beta\epsilon}, \quad (67)$$

then the second term becomes

$$\partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h^{\sigma\rho})}{\partial(\partial_\alpha h_{\mu\nu})} = \eta^{\sigma\beta} \eta^{\rho\epsilon} \partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} \quad (68)$$

and again, by making use of the equation (45), we have

$$\eta^{\sigma\beta} \eta^{\rho\epsilon} \partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2} \eta^{\sigma\beta} \eta^{\rho\epsilon} \partial^\lambda h_{\lambda\rho} \left[\delta_\sigma^\alpha (\delta_\beta^\mu \delta_\epsilon^\nu + \delta_\beta^\nu \delta_\epsilon^\mu) \right] \quad (69)$$

the previous expression implies that $\sigma = \alpha$, and for the first deltas we have $\beta = \mu$, $\epsilon = \nu$ whereas for the second deltas we have $\beta = \nu$, $\epsilon = \mu$, thus

$$\eta^{\sigma\beta} \eta^{\rho\epsilon} \partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2} \eta^{\alpha\mu} \eta^{\nu\rho} \partial^\lambda h_{\lambda\rho} + \frac{1}{2} \eta^{\alpha\nu} \eta^{\rho\mu} \partial^\lambda h_{\lambda\rho}, \quad (70)$$

now, from that expression we can see that each one of the η 's is going to mix the indices, this is; raise one index of h , and since we want the index in the derivative down (so we can match the other computed term), we can lower this and raise the other index in the field h , this results in

$$\eta^{\sigma\beta} \eta^{\rho\epsilon} \partial^\lambda h_{\lambda\rho} \frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2} \eta^{\alpha\mu} \partial_\lambda h^{\lambda\nu} + \frac{1}{2} \eta^{\alpha\nu} \partial_\lambda h^{\lambda\mu}, \quad (71)$$

and from this we have

$$\frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2} \left[\eta^{\alpha\mu} \partial_\lambda h^{\lambda\nu} + \eta^{\alpha\nu} \partial_\lambda h^{\lambda\mu} \right] \quad (72)$$

which is equivalent to

$$\frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2} [\eta^{\alpha\mu} \partial_\sigma h^{\sigma\nu} + \eta^{\alpha\nu} \partial_\sigma h^{\sigma\mu}], \quad (73)$$

and if we take ∂_α of the previous expression we can see that the η 's are going to raise the index in this partial, thus

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})} \right) = -\frac{1}{2} [\partial^\mu \partial_\sigma h^{\sigma\nu} + \partial^\nu \partial_\sigma h^{\sigma\mu}]. \quad (74)$$

Now, for the third term

$$\mathcal{L}_3 = -\frac{1}{4} \partial_\lambda h \partial^\lambda h, \quad (75)$$

we follow the same procedure as before, but before proceeding any further, we need to remember that

$$h = \eta^{\rho\sigma} h_{\rho\sigma}, \quad (76)$$

with this in mind we have

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{4} \left[\eta^{\rho\sigma} \frac{\partial(\partial_\lambda h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} + \eta^{\rho\sigma} \eta^{\lambda\delta} \partial_\lambda \frac{\partial(\partial_\delta h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} \right], \quad (77)$$

and again, using the identity in equation (45), we have

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{4} \left[\frac{1}{2} \eta^{\sigma\rho} \delta_\lambda^\alpha (\delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\nu \delta_\sigma^\mu) \partial^\lambda h + \frac{1}{2} \eta^{\rho\sigma} \eta^{\lambda\delta} \partial_\lambda h \delta_\delta^\alpha (\delta_\rho^\mu \delta_\sigma^\nu + \delta_\rho^\nu \delta_\sigma^\mu) \right], \quad (78)$$

and from this expression we have that $\lambda = \alpha$, and even more, the deltas in the first term impose $\rho = \nu$, $\sigma = \nu$ and $\rho = \nu$, $\sigma = \mu$, and for the second term the deltas impose the same relationship as well, which implies that

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{8} \eta^{\mu\nu} \partial^\alpha h - \frac{1}{8} \eta^{\nu\mu} \partial^\alpha h - \frac{1}{8} \eta^{\mu\nu} \partial^\alpha h - \frac{1}{8} \eta^{\nu\mu} \partial^\alpha h, \quad (79)$$

but η is symmetric, thus, the previous expression reduces to

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2}\eta^{\mu\nu}\partial^\alpha h, \quad (80)$$

and if we take ∂_α , we get the box operator, this is

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} \right) = -\frac{1}{2}\eta^{\mu\nu}\square h. \quad (81)$$

Finally, let's move to the last term

$$\mathcal{L}_4 = \frac{1}{2}\partial_\lambda h\partial_\rho h^{\lambda\rho}, \quad (82)$$

then we have

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2} \left[\frac{\partial(\partial_\lambda h)}{\partial(\partial_\alpha h_{\mu\nu})} \partial_\rho h^{\lambda\rho} + \partial_\lambda h \frac{\partial(\partial_\rho h^{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})} \right], \quad (83)$$

and for the first term we follow the same procedure as with \mathcal{L}_3 , whereas for the second term, we use two η 's to lower the indices in the field $h^{\lambda\rho}$, with this in mind we have

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2} \left[\eta^{\delta\sigma} \frac{\partial(\partial_\lambda h_{\delta\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} \partial_\rho h^{\lambda\rho} + \eta^{\lambda\delta} \eta^{\rho\gamma} \partial_\lambda h \frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})} \right], \quad (84)$$

and again, we use the identity given in equation (45), which again, is making $\lambda = \alpha$ in the first term and $\rho = \alpha$ in the second, for the first one we have

$$\eta^{\delta\sigma} \frac{\partial(\partial_\lambda h_{\delta\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} \partial_\rho h^{\lambda\rho} = \frac{1}{2}\eta^{\mu\nu}\partial_\rho h^{\alpha\rho} + \frac{1}{2}\eta^{\nu\mu}\partial_\rho h^{\alpha\rho} = \eta^{\mu\nu}\partial_\rho h^{\alpha\rho}, \quad (85)$$

whereas for the second term we have

$$\eta^{\lambda\delta} \eta^{\rho\gamma} \partial_\lambda h \frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\lambda\delta} \eta^{\rho\gamma} \partial_\lambda h (\delta_\delta^\mu \delta_\gamma^\nu + \delta_\delta^\nu \delta_\gamma^\mu), \quad (86)$$

which implies that

$$\eta^{\lambda\delta} \eta^{\rho\gamma} \partial_\lambda h \frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\lambda\mu} \eta^{\alpha\nu} \partial_\lambda h + \frac{1}{2}\eta^{\alpha\mu} \partial^\nu h = \frac{1}{2}(\eta^{\alpha\nu} \partial^\mu h + \eta^{\alpha\mu} \partial^\nu h), \quad (87)$$

then, the whole expression reduces to

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\mu\nu}\partial_\rho h^{\alpha\rho} + \frac{1}{4}(\eta^{\alpha\nu} \partial^\mu h + \eta^{\alpha\mu} \partial^\nu h), \quad (88)$$

and if we take ∂_α , we have

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} \right) = \frac{1}{2}\eta^{\mu\nu}\partial_\alpha \partial_\rho h^{\alpha\rho} + \frac{1}{4}\partial_\alpha(\eta^{\alpha\nu} \partial^\mu h + \eta^{\alpha\mu} \partial^\nu h), \quad (89)$$

now, in the second term, the η 's are going to raise the indec of ∂_α and since derivatives commute we have

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} \right) = \frac{1}{2}\eta^{\mu\nu}\partial_\alpha \partial_\rho h^{\alpha\rho} + \frac{1}{2}\partial^\mu \partial^\nu h, \quad (90)$$

then, if we put everything together, and define

$$K^{\mu\nu} = \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha h_{\mu\nu})} \right), \quad (91)$$

we have

$$K^{\mu\nu} = \frac{1}{2}\square h^{\mu\nu} - \frac{1}{2}[\partial^\mu\partial_\sigma h^{\sigma\nu} + \partial^\nu\partial_\sigma h^{\sigma\mu}] - \frac{1}{2}\eta^{\mu\nu}\square h + \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\partial_\rho h^{\alpha\rho} + \frac{1}{2}\partial^\mu\partial^\nu h, \quad (92)$$

now, the expression we got from G has its indices lowered, and the version with the indices up is given by

$$G^{\mu\nu} = \frac{1}{2}\left(\partial_\alpha\partial^\nu h^{\alpha\mu} + \partial_\alpha\partial^\mu h^{\alpha\nu} - \partial^\mu\partial^\nu h - \square h^{\mu\nu} - \eta^{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} + \eta^{\mu\nu}\square h\right), \quad (93)$$

and we can rearrange the terms in $K^{\mu\nu}$ as follows

$$K^{\mu\nu} = \frac{1}{2}(-\partial^\mu\partial_\sigma h^{\sigma\nu} - \partial^\nu\partial_\sigma h^{\sigma\mu} + \partial^\mu\partial^\nu h + \square h^{\mu\nu} + \eta^{\mu\nu}\partial_\alpha\partial_\rho h^{\alpha\rho} - \eta^{\mu\nu}\square h), \quad (94)$$

and from this it follows

$$K^{\mu\nu} = -G^{\mu\nu}, \quad (95)$$

going back to the original Euler-Lagrange equation, we have

$$-G^{\mu\nu} + \tau^{\mu\nu} = 0 \implies G^{\mu\nu} = \tau^{\mu\nu}, \quad (96)$$

just as we wanted.