

Quantization

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April 21, 2025

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Canonical Quantization

Poisson Bracket and Commutator

Quantization: transition from a classical description to a quantum version, "procedure for constructing quantum mechanics from classical mechanics."

Dirac notice the following connection

$$\{q, p\} = 1, -\frac{i}{\hbar} [Q, P] = 1,$$

together with

$$\frac{df}{dt} = \{f, h\}, \frac{d\mathcal{O}}{dt} = -\frac{i}{\hbar} [\mathcal{O}, H]$$

Procedure

Given any classical system, we can quantize it by finding a rule as follows: for each function f defined in the phase space, we associate a self-adjoint operator \mathcal{O}_f , acting on a state space \mathcal{H} , such that

$$O_{\{f,g\}} = -\frac{i}{\hbar} [\mathcal{O}_f, \mathcal{O}_g]$$

Linear Functions and Schrodinger Representation

"The Heisenberg Lie algebra is isomorphic to the threedimensional subalgebra of functions on phase space given by linear combinations of the constant function, the function q and the function p ." We have

$$O_1 = \mathbf{1}, O_q = Q, O_p = P,$$

with

$$\Gamma'_S(1) = -i\mathbf{1},$$

$$\Gamma'_S(q) = -iQ = -iq,$$

$$\Gamma'_S(p) = -P = \frac{d}{dq}.$$

Quadratic Polynomials

Quadratic polynomials can be quantized as follows

$$O_{p^2/2} = \frac{p^2}{2}, O_{q^2/2} = \frac{q^2}{2}$$

but, we need to work a more for pq since the order here matters. It turns out that

$$O_{pq} = \frac{1}{2} (PQ + QP).$$

And the issue here is: " Γ'_S has the same sort of problem as the spinor representation of $su(2) = so(3)$, which was not a representation of $SO(3)$, but only of its double cover $SU(2) = Spin(3)$ "

The Groenwold-van Hove no-go theorem

Main challenges

The issue lies here: how can we quantize polynomial functions on phase space with a degree greater than two?

- Operator Ordering Ambiguities: Ordering matters.
- Inconsistency with Poisson Bracket Relation.
- Lowest Order Approximation.
- Limited Lie Algebra Representation.

And from the physics point of view, different ways of ordering the P and Q operators will lead to different operators O_f for the same function f , with physically different observables.

Main challenges

For polynomials of degree greater than two there is no possible way to do this consistent with the following relation:

$$Q_{\{f,g\}} = -\frac{i}{\hbar} [O_f, O_g].$$

"Whatever method one devises for quantizing higher-degree polynomials, it can only satisfy that relation to lowest order in \hbar , and there will be higher-order corrections, which depend upon one's choice of quantization scheme."

Groenwold-van Hove no-go theorem

Theorem

There is no map $f \rightarrow O_f$ from polynomials on \mathbf{R}^2 to self-adjoint operators on $L^2(\mathbf{R})$ satisfying

$$O_{\{f,g\}} = -\frac{i}{\hbar} [O_f, O_g]$$

and

$$O_p = P, O_q = Q,$$

or any Lie subalgebra of the functions on \mathbf{R}^2 for which the subalgebra of polynomials of degree less than or equal to two is a proper subalgebra.

Canonical quantization in d dimensions

Generalization

Moving on to d dimensions, we have

$$\Gamma'_S(q_j) = -iQ_j, \Gamma'_S(p_j) = -iP_j,$$

wich satisfy the Heisenberg relations

$$[Q_j, P_k] = i\delta_{jk}$$

And for quadratic polynomials

$$\Gamma'_S(q_j q_k) = -iQ_j Q_k, \Gamma'_S(p_j p_k) = -iP_j P_k$$

$$\Gamma'_S(q_j p_k) = -i\frac{i}{2} (Q_j P_k + P_k Q_j)$$

Quantization and Symmetries

Example: Angular Momentum

"The observables that commute with the Hamiltonian operator H will make up a Lie algebra of symmetries of the quantum system and will take energy eigenstates to energy eigenstates of the same energy."

Example: The group $SO(3)$.

The following operators provide a basis for the Lie algebra representation

$$-i(Q_2P_3 - Q_3P_2), -i(Q_3P_1 - Q_1P_3), -i(Q_1P_2 - Q_2P_1)$$

General Ways of Quantization

Feynman Path Integral

The key ideas are:

- The quantum amplitude is calculated by summing over all possible paths a system can take between two states.
- It naturally incorporates the principle of least action from classical mechanics.
- The method is particularly useful in quantum field theory and for systems with many degrees of freedom.