

Rotations and the Spin $1/2$ Particle in a Magnetic Field

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Advanced Mathematical Methods

Spinor Representation

Some Motivation

- The existence of spin $1/2$ particles shows that is $Spin(3)$ rather than $SO(3)$ that is the symmetry group of corresponding of rotations of fundamental quantum systems.
- The idea is to study $\mathcal{H} = \mathbb{C}^2$ with the group action given by rotations in $3D$.

Definition

The spinor representation of $Spin(3) = SU(2)$ is the representation on \mathbb{C}^2 given by

$$g \in SU(2) \rightarrow \pi_{spinor}(g) = g,$$

and elements of the representation space \mathbb{C}^2 are called spinors.

Spin $\frac{1}{2}$ in a Magnetic Field

Elements of the Lie algebra

- We will consider only the $SU(2)$ subgroup of $U(2)$.
- “When it occurs in its role as double cover of the rotational group, the quantum system is said to carry “spin”, in particular “spin 1/2” for the two dimensional irreducible representation.”
- Elements of the Lie algebra

$$X_j = -i\frac{\sigma_j}{2},$$

with commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

- Making contact with physics

$$S_j = i\hbar X_j,$$

we like this as observables because the eigenvalues are real $\pm 1/2$ (experimental measures).

- Elements of the group are given by

$$\Omega(\theta, \mathbf{w}) = \exp\left(-\frac{i}{\hbar} \mathbf{w} \cdot \mathbf{S}\right) \in SU(2).$$

- States in \mathcal{H} that have a well-defined value of the observable S_j will be eigenvectors of S_j with eigenvalues $\pm 1/2$.

- Let $|\psi\rangle \in \mathcal{H}$, thus we have

$$|\psi\rangle \rightarrow \Omega |\psi\rangle .$$

- Baker-Hausdorff lemma

$$\begin{aligned} \exp(iG\lambda) A \exp(-iG\lambda) &= A + i\lambda [G, A] \\ &\quad + \left(\frac{i^2\lambda^2}{2!}\right) [G, [G, A]] + \dots \end{aligned}$$

- The spin degree of freedom that we are describing by \mathcal{H} has a dynamics given by

$$H = -\mu \cdot B,$$

where

$$\mu = -\frac{ge}{2mc}S,$$

is the magnetic moment operator.

With Schrödinger Equation

The Schrödinger equation is given by

$$\frac{d}{dt} |\psi\rangle = -i (-\mu \cdot B) |\psi\rangle$$

and solution

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle ,$$

where

$$U(t) = \exp(it\mu \cdot B) .$$

- The Zeeman effect,
- Stern Gerlach experiment,
- Nuclear magnetic resonance spectroscopy,
- Quantum computing.

Heisenberg Picture

Schrödinger and Heisenberg Pictures

	Heisenberg Picture	Schrödinger Picture
State Ket	No Change	Evolution Given by H
Observable	Evolution Given by H	No Change

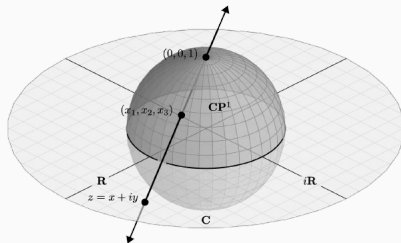
Complex Projective Space

Trying Another Characterization

- Multiplication on \mathcal{H} by non-zero complex number do not change eigenvectors \implies no physical effect.
- The relevant part is the quotient space $(\mathbb{C}^2 - \{0\}) / \mathbb{C}^*$, and constructed by: taking all non-zero elements of \mathbb{C}^2 and identifying those related by multiplication by a non-zero complex number.
- In some sense the space CP^1 is the complex plane, but with a “point at infinity” added.

Riemann Sphere

- CP^1 : “Riemann sphere” with the relation to the plane and the point at infinity given by stereographic projection.



- Relation between coordinates on the sphere (x_1, x_2, x_3) and complex coordinates $z_1/z_2 = z = x + iy$ is given by

$$x = \frac{x_1}{1 - x_3}, y = \frac{x_2}{1 - x_3},$$

and

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, x_2 = \frac{2y}{x^2 + y^2 + 1}, x_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

The Bloch Sphere

- The unit sphere $S^2 \subset \mathbb{R}^3$ can be mapped to operators by

$$x \rightarrow \sigma \cdot x,$$

and for each point $x \in S^2$, $\sigma \cdot x$ has eigenvalues ± 1 .

Eigenvectors with eigenvalue $+1$ are solutions to

$$\sigma \cdot x |\psi\rangle = |\psi\rangle.$$

Interpretation in terms of spin operators

- One can characterize the $\mathbb{C} \subset \mathcal{H}$ corresponding to $\mathbf{x} \in S^2$ as the solutions to

$$\mathbf{S} \cdot \mathbf{x} |\psi\rangle = \frac{1}{2} |\psi\rangle ,$$

thus, the North pole of the sphere is a “spin-up” state and the South pole is a “spin down” state.

- Along the equator one finds two points corresponding to states with definite values for S_1 , as well as two for states that have definite values for S_2 .