Point Set Topology

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Problem 1 *Recall that $\mathbb{R}$ with the standard topology is a Hausdorff space. A subset $S \subset \mathbb{R}$ is said to be sequentially compact provided that every sequence in S has a subsequence that converges to a point in S.*

1. *Prove that S is sequentially compact if and only if S is closed and bounded. (This is known as the Bolzano-Weierstrass Theorem).*

2. *Prove that if S is compact in the standard topology of $\mathbb{R}$, then S is closed and bounded, hence sequentially compact. (Note: This has now established the Heine-Borel Theorem on $\mathbb{R}$ with the standard topology: Every closed bounded subset of $\mathbb{R}$ is compact.)*

3. *Prove that if S is sequentially compact, then it is compact in the standard topology of $\mathbb{R}$.*

Proof 1

1. *Let's suppose that S is sequentially compact; to prove, S is bounded and closed. Indeed let $\{x_n\}$ be a convergent sequence in S, this is $x_n \to x$, where $\in S$, and because S is sequentially compact, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that $\{x_{n_k}\} \to x$, and from this, it follows that S is closed. Let's prove that it's also closed, and let's proceed by contradiction, this is, suppose that S is not bounded, it follows that there exists x_n such that $x_n > n$ for each n. And even more, let's focus on the convergent sequence $\{x_n\}$ because S is not bounded, it follows that every subsequence of $\{x_n\}$ is also unbounded which implies that it does not converge, and we've reached a contradiction. Therefore, it follows that S is also bounded. On the other hand, suppose that S is closed and bounded to prove that S is sequentially compact. Indeed, let $\{x_n\}$ be a bounded sequence, but every bounded sequence has a convergent subsequence. Thus, it follows that $x_n \to x$ with $x \in S$, and thus, S is sequentially compact.*

2. *Let's prove that compactness implies boundedness. Indeed, let's suppose S is compact, and by contradiction, let's assume that S is unbounded, which means that for every number $M > 0$, there is an element $x \in S$ such that $x > M$. Now let's consider the family of open intervals $(-n, n)$ for all natural numbers n. This is a collection of intervals that covers the entire real line, in particular S. But because S is unbounded, it follows that for any finite family of these intervals, there will always be elements of S outside the largest interval, but this is a contradiction because we assume that S is compact. Therefore, S is bounded. Now, let's prove that compactness implies closedness, and as before, let's assume that S is compact, and let's proceed by contradiction that S is not closed, which implies that there exists a limit point x of S that is not in S. Now, for each point $s \in S$, let's consider the open interval $(s - d(s,x)/2, s + d(s,x)/2)$, where $d(s,x)$ is the distance between s and x. This collection of intervals covers S. And because x is a limit point, it follows that every neighborhood of x contains infinitely many points of S. Thus, any finite subcollection of these intervals will miss some points of S close to x, which, again, contradicts our assumption of S being compact. Therefore, it follows that S is closed.*

3. *Let's prove that being sequentially compact implies compactness. And let's proceed by contradiction, let's assume that S is not compact, then there exists an open cover of S that has no finite subcover, and let's call it $\{U_\alpha\}$. From this, let's select a sequence as follows for each n let's choose $x_n \in S \backslash \cup_{i=1}^{n} U_i$ where U_i are finitely many open sets in the cover; we can do this because no finite subcover exists. Thus, the sequence $(x_n) \in S$ by construction, S is sequentially compact by assumption. Thus, this sequence must have a convergent sequence (x_{n_k}) with limit $x \in S$. But because $\{U_\alpha\}$ is an*

open cover of S, it follows that x must belong to some open set in $\{U_\alpha\}$, but this is a contradiction because each term in the sequence was chosen to lie outside the finite subcover. Therefore, sequential compactness implies compactness.

Problem 2 *For two points $x = (x_k)_{k=1}^n$, $y = (y_k)_{k=1}^n \in \mathbb{R}$, consider the following three functions:*

$$d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k|$$

$$d_2(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

$$d_\infty(x, y) = max\{|x_k - y_k|: k = 1, \cdots, n.\}$$

1. *Verify that each of these functions defines a metric on $\mathbb{R}^n$.*
2. *Prove that the three distances generate the same topology on $\mathbb{R}^n$.*

Proof 2 1. *Let's verify that each one is a metric. Indeed*

$$d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k| \geq 0,$$

because each one of the terms in the sum is greater or equal to zero. On the other hand

$$d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k| = 0 \iff x_k = y_k \forall k \implies, d_1(x, y) = 0 \iff x = y.$$

The metric is also symmetric, this is

$$d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k| = \sum_{k=1}^{n} |-1||y_k - x_k| = \sum_{k=1}^{n} |y_k - x_k| = d_1(y, x)$$

Now, let's prove the triangle inequality, let $x, y, z \in \mathbb{R}^n$,

$$d_1(x, y) = \sum_{k=1}^{n} |x_k - y_k| = \sum_{k=1}^{n} |x_k - z_k + z_k - y_k| \leq \sum_{k=1}^{n} |x_k - z_k| + |z_k - y_k| = d_1(x, z) + d_1(z, y)$$

Let's move to the metric d_2, and again

$$d_2(x, y) \geq 0 \text{ and } , d_2(x, y) = 0 \iff x = y.$$

d_2 is also symmetric

$$d_2(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} = \sqrt{\sum_{k=1}^{n} (y_k - x_k)^2} = d_2(y, x).$$

Finally, for the triangle inequality, we're going to make use of the Minkowsky inequality, which is a consequence of the Cauchy-Schwarz inequality, which in this case reads

$$\sqrt{\sum (x_k + y_k)^2} \leq \sqrt{\sum x_k^2} + \sqrt{\sum y_k^2},$$

from this it follows that

$$\sum (x_k + y_k)^2 = \sum (x_k - z_k + z_k + y_k)^2 \leq (\sum (x_k - z_k)^2)^{1/2} + (\sum (-z_k + y_i^2)^{1/2},$$

it follows that $d_2(x, y) \leq d_2(x, z) + d_2(z, y)$.

Finally, $d_\infty(x,y) = max\{|x_k - y_k|\} \geq 0$*,and also* $d_\infty(x,y) = max\{|x_k - y_k|\} = 0 \iff x = y$*. On the other hand* $d_\infty(x,y) = max\{|x_k - y_k|\} = max\{|y_k - x_k|\}$*, which implies that*

$$d_\infty(x,y) = d_\infty(y,x).$$

And finally, let $x, y, z \in \mathbb{R}^n$*, thus*

$$d_\infty(x,y) = max\{|x_k - y_k|\} = max\{|x_k - z_k + z_k - y_k|\} \leq max\{|x_k - z_k|\} + max\{|z_k - y_k|\},$$

therefore

$$d_\infty(x,y) \leq d_\infty(x,z) + d_\infty(z,y)$$

2. *In order to prove that they generate the same topology, we need to prove that the basic open sets generated by these topologies are the same.*

Problem 3 *A topological space $(E, \mathcal{T})$ is said to be locally compact provided that it is Hausdorff and every point in E has a least one compact neighborhood.*

1. *Prove that every compact space is locally compact.*
2. *Prove that E equipped with the discrete topology is locally compact.*
3. *Every closed subspace of a locally compact space is locally compact.*

Proof 3

1. *Let's suppose that $(E, \mathcal{T})$ is compact. By definition, is Hausdorff, and even more, $\forall x \in X$ follows that E is a neighborhood of x, therefore $(E, \mathcal{T})$ is locally compact.*
2. *Let's consider the topological space $(E, \mathcal{T})$ equipped with the discrete topology, to prove that it is compact. Let $x \in E$, and let's consider the singleton $\{x\}$, which is open. Now, in this topology every set is also closed, thus $\{x\}$ is both open and closed, which means that the $\{x\}$ equals to its closure. Finally, we know that any finite set in a topological space is compact, and since $\{x\}$ is finite, it's compact.*
3. *Let E be a locally compact space and let F be a closed subspace of E. To prove; F is locally compact. Indeed, let $x \in F$, since E is locally compact, there exists a compact neighborhood K of x in E. This implies that there's an open set U in $E \ni x$, such that $U \subset K$. From this, let's consider the set $V = U \cap F$, which is an open set in the subspace topology on F, and it contains x. K is compact in E, and F is closed in E, because the intersection of a compact set and a closed set is always compact. Therefore, $V = U \cap F$ is compact in E. And since it's a subset of F, it's also compact in F.*

Problem 4 *Let d, d' be two metrics on a set E, and let $\psi : [0,\infty][0,\infty]$ be an increasing function whose derivative $\psi : [0,\infty) \to [0,\infty]$ is also increasing with $\psi(0) = \psi'(0) = 0$. Suppose that for all $x, y \in E$*

$$d'(x,y) \leq \varphi(d(x,y)) \text{ and } d(x,y) \leq \varphi'(d'(x,y))$$

Prove that these two distances generate the same topology on E.

Proof 4 *Again, the idea is to show that for any open ball around a point x with respect to the metric d, we can find an open ball around the same point x with respect to the metric d' that is contained within the first ball, and vice-versa. Let $B_d(x,r)$ be the open ball centered at x with radius r with respect to the metric d, and similarly let denote $B_{d'}(x,r)$ the open ball with respect to the other metric.*

Let $x \in E$ and $r > 0$. We want to prove that $B_{d'}(x,r) \subset B_d(x,r)$. Indeed, for any $y \in B_{d'}(x,r)$ we have $d'(x,y) < \epsilon$, ans using the given inequality we have that

$$d(x,y) \leq \psi'(d'(x,y)) < \psi'(\epsilon),$$

but because ψ' is an increasing function and $\psi'(0) = 0$ we can choose ϵ small enough such that $\psi'(\epsilon) < r$, which ensures that $d(x,y) < r$, and from this we have $y \in B_d(x,r)$, therefore $B_{d'}(x,r) \subset B_d(x,r)$.

Now, let's prove the other contention. Let $x \in E$ and $r > 0$. We want to find a $\delta > 0$ such that $B_d(x,\delta) \subset B_{d'}(x,r)$. Indeed for any $y \in B_d(x,\delta)$ we have $d(x,y) < \delta$, and again, using the given inequality we have

$$d'(x,y) \leq \psi(d(x,y)) \implies d'(x,y) \leq \psi(d(x,y)) < \psi(\delta),$$

and as before, because ψ is increasing and $\psi(0) = 0$ we can choose δ small enough such that $psi(\delta) < r$, which ensures that $d'(x,y) < r$, and hence $y \in B_{d'}(x,r)$, and therefore $B_d(x,\delta) \subset B_{d'}(x,r)$.

Problem 5 *Let (A_n)be a decreasing sequence of subsets of R, each of which is a finite union of pairwise disjoint closed intervals. We also assume that each of the intervals making up A_n contains exactly two of the intervals which make up A_{n+1}, and that the diameter of these intervals tends to 0 with $1/n$. Show that the set $A = \cap_n A_n$ is a compact set without any isolated points.*

Proof 5 *Let's prove that A is compact. Indeed, each A_n is a finite union of closed intervals, thus is closed, and even more, the intersection of closed sets is closed, so $A = \cap_n A_n$ is closed. On the other hand, since (A_n) is decreasing, and because A_1 is a finite union of pairwise disjoint closed intervals, it follows that it is bounded, and from this, we have that all A_n and their intersection are bounded. Using Heine-Borel Theorem it follows that the $A = \cap_n A_n$ is compact. Now, let's prove that is has no isolated points. Indeed, let $x \in A$, and we need to show that this is not an isolated point, which means that $\forall \epsilon > 0$ there exists $y \in A$ with $y \neq x$ such that $|x-y| < \epsilon$. Now, since the diameter of A_n tends to 0 as n increases, we can find an n big enough such that the diameter of A_n is less than $\epsilon/2$, and even more, $x \in A$, and hence to A_n, thus there exist some closed interval $I_n \subset A$ such that $x \in I_n$. Now using the condition given I_n contains two disjoint closed intervals, let's call it I'_{n+1} and I''_{n+1} that make up A_{n+1}. Since $x \in I_n$, it follows that it must belong to either $I^{'}_{n+1}$ or I''_{n+1}, let's choose $x \in I'_{n+1}$ and let $y \in I''_{n+1}$. Since both I'_{n+1} and I''_{n+1} are contained in I_n and the diameter of I_n is less than $\epsilon/2$, and from this we have that $y \in A_n + 1 \subset A_n$, $y \neq x$, and even more $|x - y| \leq \delta(I_n) < \epsilon/2$, where δ stands for diameter, and from this we have that x is not an isolated point, and since x was arbitrary it follows that A has no isolated points.*