MATH 171: HOMEWORK 2

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Problem 1. Given a set X, consider the poset P of all subset of X partially ordered by inclusion. Show that X is the unique maximum element of P , and show that the empty set is the unique least element of P.

Solution. By definition, a partially ordered set (poset) is a set equipped with a binary relation that is reflexive, antisymmetric and transitive. Now, with this in mind, let X be a non empty set and let's consider the poset $(P(X), \subset)$. It's clear that $\emptyset, X \in P$.

Now, let's suppose that there's an $O \neq \emptyset$ such that O is the "least element", then it follows that $O \subset \emptyset$, which is clearly a contradiction, therefore the empty set is the least element and it's unique.

On the other hand, it's clear that X is the greatest element. Now let's suppose that there's an X', with the following property, $A \subset X'$ for each $A \in P$, thus it follows that $X \subset X'$, but X is the greatest element, thus $X' \subset X$, therefore

$$X = X',$$

it follows that the maximum element of P is unique.

Problem 2. Given examples of sets A in various topological spaces $(X, \mathcal{T})$ with:

2.1 a limit point of A that is an element of A;

2.2 a limit point of A that is not an element of A;

2.3 an isolated point of A;

2.4 a point not in A that is not a limit point of A.

Solution. Let $(X, \mathcal{T})$ be a topological space.

(1) Let $(X, \mathcal{T})$ be $\mathbb{R}$ with $\mathcal{T}$ be the usual topology.
   - (a) Let $A = [0, 1]$, thus 0 and 1 are limit points of A and they belong to A.
   - (b) Let $A = (0, 1)$, in this case 0 and 1 are limit points of A and they do not belong to A.
   - (c) Let $A = \{0\} \cup (1, 2)$, in this case 0 is an isolated point because we can find a neighborhood that does not intersect any other element of the set. Indeed any interval with $0 < \epsilon < 1$ does not intersect the set A other than in 0.
   - (d) Let $A = (0, 1)$, and $p = 5$, thus $p \in \mathbb{R}$ is a point not in A that is not a limit point of A.

Problem 3. Let $(X, \mathcal{T})$ be a topological space:

3.1 Which sets are closed in a set X with the discrete topology?

3.2 Which sets are closed in a set X with the indiscrete topology?

3.3 Which set are closed in a set X with the finite complement topology?

3.4 Which set are closed in a set X with the countable complement topology?

Solution. Let $(X, \mathcal{T})$ be a topological space.

3.1 In the case in which $(X, \mathcal{T})$ is the discrete topology, we have $\mathcal{T} = \mathcal{P}(X)$, this is every possible subset of X is an element of the topology $\mathcal{T}$. Moreover, we know that a set is closed if its complement is open, but the complement of any subset of X is also a subset of X thus it follows that every subset of X is also closed.

3.2 In the case in which $(X, \mathcal{T})$ is the indiscrete topology, we have $\mathcal{T} = \{\emptyset, X\}$. Moreover, the complement of an open set is, by definition, a closed set. Therefore , the closed sets in the indiscrete topology are $\{\emptyset, X\}$.

3.3 Let X be a set, the finite complement (or cofinite) topology on X is defined as: $\mathcal{U} \subset X$ is open if and only if $X/\mathcal{U}$ is finite or $\mathcal{U} = \varnothing$. And again, using the definition that a set if closed when its complement is open we have that closed sets in the cofinite topology are finite sets, X or the empty set $\emptyset$.

3.4 Let X be a set, the countable complement (or cocountable) topology on X is defined as: $\mathcal{U} \subset X$ is open if and only if $X/\mathcal{U}$ is countable or $\mathcal{U} = \varnothing$. As previously a set if closed when its complement is open, but the open sets in this topology are defined as the sets whose complements are countable, thus the closed sets are countable sets, X or the empty set $\emptyset$.

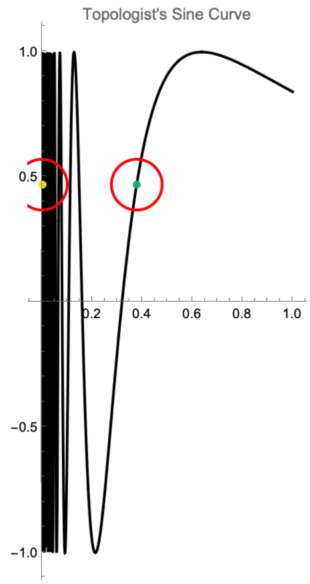
Problem 4. In $\mathbb{R}^2$ with the standard topology, describe the limit points and the closure of the set

$$S = \{(x, \sin(1/x)) | x \in (0,1)\}.$$

Solution. Clearly, in for for each $p \in (0,1)$ we can always find a neighborhood U_p of x such that

$$(U_x / \{p\}) \cap S \neq \emptyset.$$

On the other hand the when the function approaches 0 from the left, the function $\sin(1/x)$ oscillates abruptly between the interval $[-1, 1]$ without settling at any particular value, and that behavior is shown in the next figure.



Moreover, for each $y \in [-1, 1]$ we can always solve the equation

$$y = \sin\left(\frac{1}{x}\right),$$

because the sine function is dense in $[-1, 1]$, for each $y \in [-1, 1]$ and for each $\epsilon > 0$ there is an x such that

$$\left|\sin\left(\frac{1}{x}\right) - y\right| < \epsilon.$$

Thus, we have that the limit points of point in S, and also points in A, where A is given by

$$A = \left\{(0, y) \subset \mathbb{R}^2; y \in [-1, 1]\right\}.$$

On the other hand, by definition, the closure of S is the union of the set with its accumulation points, therefore, we have

$$\overline{S} = S \cup \left\{(0, y) \subset \mathbb{R}^2; y \in [-1, 1]\right\}.$$

Problem 5. Prove or disprove each of the following statements.

5.1 If $A \subset B$ then $\overline{A} \subset \overline{B}$,

5.2 $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

5.3 $\overline{\cup_\alpha A_\alpha} = \cup_\alpha \overline{A_\alpha}$,

5.4 $\overline{\overline{A}} = \overline{A}$,

5.5 $\text{Int}\,(\text{Int}\,(A)) = \text{Int}\,(A)$,

5.6 $\text{Int}\,(A \cap B) = \text{Int}\,(A) \cap \text{Int}\,(B)$,

Solution. Let $(X, \mathcal{T})$ be a topological space.

5.1 Let's suppose that $A \subset B$, by definition

$$\overline{A} = \cup \{\mathcal{V} : \mathcal{V} \text{ is closed and } \quad A \subset \mathcal{V}\},$$

it follows that $A \subset \overline{A}$, but $A \subset B$ which implies that

$$A \subset B \subset \overline{B} \implies A \subset \overline{B},$$

but $\overline{A}$ is the smallest closed set that contains A, thus we have

$$\overline{A} \subset \overline{B}.$$

5.2 Let's prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. By definition

$$\overline{A \cup B} = \cap \{\mathcal{V} : \mathcal{V} \text{ is closed and } \quad A \cup B \subset \mathcal{V}\},$$

but clearly $\overline{A} \subset \overline{A}$ and $B \subset \overline{B}$, which implies that

$$A \cup B \subset \overline{A} \cup \overline{B},$$

but $\overline{A \cup B}$ is the smallest closed set that contains $A \cup B$, thus any other set containing $A \cup B$, should be greater than $\overline{A \cup B}$, this is

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}.$$

On the other hand, $A \subset A \cup B$ and $B \subset A \cup B$, and again, by 5.1, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$, it follows that

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B},$$

therefore, we have

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

5.3 Let $\{A_\alpha\}_{\alpha \in \lambda}$ a collection of sets in a topological space $(X, \mathcal{T})$ but we know that the arbitrary union of closed sets in general is not a closed set. Therefore, the statement

$$\overline{\cup_\alpha A_\alpha} = \cup_\alpha \overline{A_\alpha},$$

in general is not true.

5.4 Let $A \subset X$, thus by definition

$$\overline{A} = \cap \{\mathcal{V} : \mathcal{V} \text{ is closed and } \quad A \subset \mathcal{V}\},$$

it follows that $A \subset \overline{A}$, and by 5.1, we have

$$\overline{A} \subset \overline{\overline{A}},$$

on the other hand

$$\overline{\overline{A}} = \cap \{\mathcal{V} : \mathcal{V} \text{ is closed and } \quad \overline{A} \subset \mathcal{V}\},$$

but $\overline{A}$ is closed by definition, thus

$$\overline{\overline{A}} \subset \overline{A},$$

therefore

$$\overline{\overline{A}} = \overline{A}.$$

5.5 Let $A \subset X$. By definition

$$\text{Int}\,(A) = \cup \{\mathcal{U} : \mathcal{U} \in \mathcal{T} \quad \& \quad \mathcal{U} \subset A\},$$

and because the interior is an union of open sets it follows that is an open set. Now if we consider the interior of the interior of a set $A \subset X$, because the interior is open, and even more, it's the greatest open contained in the given set, it follows that

$$\text{Int}\,(\text{Int}\,(A)) = \text{Int}\,(A).$$

5.6 Let $A, B \subset X$. As an auxiliary step, let's prove that if $C \subset D$, then $\text{Int}\,(C) \subset \text{Int}\,(D)$; indeed

$$\text{Int}\,(C) \subset C \subset D,$$

but the interior of D is the greatest open set contained in D, thus

$$\text{Int}\,(C) \subset \text{Int}\,(D),$$

just as we wanted. Moreover, we have that

$$A \cap B \subset A \quad \& \quad A \cap B \subset B,$$

thus, using the previous result we have that

$$\text{Int}\,(A \cap B) \subset \text{Int}\,(A) \quad \& \quad \text{Int}\,(A \cap B) \subset \text{Int}\,(B),$$

and from this it follows that

$$\text{Int}\,(A \cap B) \subset \text{Int}\,(A) \cap \text{Int}\,(B).$$

On the other hand, because the interior is the greatest open contained in the given set, we have that

$$\text{Int}\,(A) \subset A \quad \& \quad \text{Int}\,(B) \subset B,$$

it follows that

$$\text{Int}\,(A) \cap \text{Int}\,(B) \subset A \cap B,$$

but the interior of $A \cap B$ is the greatest open contained in $A \cap B$, thus

$$\text{Int}\,(A) \cap \text{Int}\,(B) \subset \text{Int}\,(A \cap B),$$

therefore, we have that

$$\text{Int}\,(A) \cap \text{Int}\,(B) = \text{Int}\,(A \cap B).$$

Problem 6. Let A be a subset of X where $(X, \mathcal{T})$ is a topological space. Prove that $\text{Int}\,(A)$, $\partial\,(A)$, and $\text{Int}\,(X/A)$ form a partition of X, that is, the sets are pairwise disjoint and their union is X.

Solution. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$, the boundary of A is by definition

$$\partial (A) = \overline{A} \cap \overline{(X/A)}.$$

Clearly

$$\operatorname{Int}(A) \cap \operatorname{Int}(X/A) = \emptyset.$$