Structure Formation, Statistics, and Scalar Field

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Problem 1. Dark Matter and Baryon Density Growth.
a. Let's start by defining

$$\epsilon = \delta_b(t) - \delta_d(t),$$

and we're given

$$\ddot{\delta}_d(t) + 2H\dot{\delta}_d(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

$$\ddot{\delta}_b(t) + 2H\dot{\delta}_b(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

by taking the difference of the two previous equations, we have the following

$$\ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H\dot{\delta}_b(t) - 2H\dot{\delta}_d(t) = 0 \implies \ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H(\dot{\delta}_b(t) - \dot{\delta}_d(t)) = 0$$

but we know that $H = 12/3t$, thus

$$\ddot{\epsilon} + \frac{4}{3t}\dot{\epsilon} = 0,$$

b. The solution of this equation is given by
c. I append the Mathematica notebook with the solution and the corresponding plots.
d. From the numerical plot we can see that a late times δ_b becomes almost equal to δ_d

Problem 2. Matter Growth with Dark Energy

Problem 3. Power Spectrum

Problem 4. Scalar Field (Inflaton) in Expanding Universe
a. Let's begin with the following action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],$$

where

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2),$$

and with this metric, we have

$$\sqrt{-g} = a^3,$$

whereas

$$g^{\mu\nu} = \mathrm{diag}(1, -1/a^2, -1/a^2, -1/a^2),$$

therefore, the action takes the form

$$S = \int dt d^3x a^3 \left[\frac{1}{2} (\dot{\phi})^2 - \frac{1}{2a^2} (\nabla \phi)^2 - V(\phi) \right]$$

which can be simplified to

$$S = \int dt d^3x \left[\frac{1}{2} a^3 (\dot{\phi})^2 - \frac{1}{2} a (\nabla \phi)^2 - a^3 V(\phi) \right].$$

And if we perform the variation, we have

$$\delta S = \int dt d^3x \left[a^3 \dot{\phi} \delta \dot{\phi} - a \nabla \phi \cdot \nabla \delta \phi - a^3 \frac{V(\phi)}{d\phi} \delta \phi \right],$$

for the first two terms, we can perform integration by parts as follows

$$\int dt (a^3 \dot{\phi}) \delta \dot{\phi} = - \int \frac{d}{dt} \left(a^3 \dot{\phi} \right) \delta \phi + \text{boundary terms},$$

and

$$\int d^3x a \nabla \phi \cdot \nabla \delta \phi = - \int d^3x (a \nabla^2 \phi) \delta \phi + \text{boundary terms},$$

thus, if we neglect the boundary terms, the variation becomes

$$\delta S = \int dt d^3x \left[- \frac{d}{dt} \left(a^3 \dot{\phi} \right) \delta \phi + (a \nabla^2 \phi) \delta \phi - a^3 \frac{V(\phi)}{d\phi} \delta \phi \right],$$

and impossing the condition $\delta S = 0$ we have

$$- \frac{d}{dt} \left(a^3 \dot{\phi} \right) \delta \phi + (a \nabla^2 \phi) \delta \phi - a^3 \frac{V(\phi)}{d\phi} \delta \phi = 0,$$

which can be simplified to

$$-3a^2 \dot{a} \dot{\phi} - a^3 \ddot{\phi} + a \nabla^2 \phi - a^3 \frac{V(\phi)}{d\phi} = 0,$$

since $a \neq 0$ we have

$$-3 \frac{\dot{a}}{a} \dot{\phi} - \ddot{\phi} + \frac{\nabla^2 \phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

and by using the fact $H = \dot{a}/a$ we have

$$-3H\dot{\phi} - \ddot{\phi} + \frac{\nabla^2\phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

which can also be writen as

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{V(\phi)}{d\phi} = 0,$$

just as we wanted.
b. By assuming ϕ is homogeneous in space and that $V = \frac{1}{2}m^2\phi^2$ we have

$$\frac{dV}{d\phi} = m^2\phi, \nabla^2\phi = 0,$$

thus the previous equation becomes:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$$

just as we wanted.
c. Finally, by assuming H a constant we seek solutions of the form

$$\phi = \phi_0 \exp(rt),$$

we have the following equation

$$\phi_0 \exp(rt)\left(r^2 + 3Hr + m^2\right) = 0$$

and solving the characteristic equation lead us to the following expression

$$r = \frac{-3H \pm \sqrt{1 - \frac{2m^2}{9H^2}}}{2},$$

which can be simplified to

$$r = \frac{3H}{2}\left(1 \pm \sqrt{1 - \left(\frac{2m}{3H}\right)^2}\right)$$

or we can also make

$$r = \frac{m}{\alpha}(1 \pm \sqrt{1 - \alpha^2}),$$

where $\alpha = \frac{2m}{3H}$. Therefore, the general solution takes the form

$$\phi(t) = \phi_+ \exp\left[\frac{m}{\alpha}(1 + \sqrt{1-\alpha^2})t\right] + \phi_- \exp\left[\frac{m}{\alpha}(1 - \sqrt{1-\alpha^2})t\right].$$

Now, the solution will oscillate when the following condition hold

$$1 - \left(\frac{2m}{3H}\right)^2 < 0 \iff 2m > 3H.$$

On the other hand, the solution will not oscillate when

$$1 - \left(\frac{2m}{3H}\right)^2 \geq 0 \iff 3H \geq 2m.$$

And finally, if $m \ll H$, this implies that $\frac{m}{H} \ll 1$, thus the solution will take the form

$$r \approx 3H \implies \phi \propto e^{3Ht}$$

therefore, ϕ will evolve quickly.