GR-HW-03

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Problem 1 (A Causal 1+1 dimensional theory)

i) By considering

$$\frac{dp_1^\mu}{d\tau} = B\varepsilon^\mu_\nu \frac{dx_1^\nu}{d\tau}, \tag{1}$$

where B is a Lorentz scalar and ε^μ_ν is given by the matrix

$$\varepsilon^\mu_\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This epsilon tensor is an antisymmetric rank-2 tensor, it's invariant under proper Lorentz transformations, and flips sign under improper ones. On the other hand, we know thath in 1+1D, Lorentz transformations preserve the antisymmetry and normalization; thus, ε^{mu}_ν transforms appropriately as a (pseudo-)tensor. Its invariance under Lorentz boosts makes it a good Lorentz tensor.

ii) On the other hand, the given equation is fully Lorentz covariant:

- In the LHS, the 4-momentum is a 4-vector and τ is the proper time, which is a Lorentz scalar; thus the derivative is a 4-vector.
- On the RHS, B is given and it's a Lorentz scalar, as I comment previously, ε is a rank-2 tensor, a good Lorentz tensor, and the four velocity is a 4-vector, therefore the whole expression it's also a good four vector.
- And then, the whole equation is fully covariant.

iii) Let's consider $mu = 1$, in the given equation. It follows that

$$\frac{dp_1^1}{d\tau} = B\epsilon^1_\nu \frac{dx_1^\nu}{d\tau}, \tag{2}$$

but $\epsilon^1_0 = 1$ and $\epsilon^1_1 = 0$, we have

$$\frac{dp_1^1}{d\tau} = B(1)\frac{dx_1^0}{d\tau} + B(0)\frac{dx_1^1}{d\tau} = B\frac{dx_1^0}{d\tau}, \tag{3}$$

by the chain rule we have

$$\frac{dp_1^1}{dt}\frac{dt}{d\tau} = B\frac{dt}{dt}\frac{dt}{d\tau}, \tag{4}$$

then we have

$$\frac{dp_1^1}{dt} = B, \tag{5}$$

and from this it follows that

$$B = -C_{12}\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}. \tag{6}$$

Problem 2 (Nonlinear Field Theory)

i) The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{4}\phi^4, \tag{7}$$

and we know the Euler-Lagrange equation is given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{8}$$

Second term term in the previous equation is easy, and it's given by

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m_\phi^2\phi - \lambda\phi^3 \implies \frac{\partial\mathcal{L}}{\partial\phi} = -\left(m_\phi^2 + \lambda\phi^2\right)\phi. \tag{9}$$

And for the first term we have to work a little more;

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}\eta^{\mu\nu}\left[\partial_\nu\phi + \partial_\mu\phi\delta_\mu^\nu\right] \implies \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \eta^{\mu\nu}\partial_\nu\phi, \tag{10}$$

and from this we have

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi, \tag{11}$$

therefore, for the Euler-Lagrange equation we have

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \left(m_\phi^2 + \lambda\phi^2\right)\phi = 0, \tag{12}$$

or more explicitly

$$\partial_t^2\phi - \nabla^2\phi + \left(m_\phi^2 + \lambda\phi^2\right)\phi = 0, \tag{13}$$

or

$$\Box\phi + \left(m_\phi^2 + \lambda\phi^2\right)\phi = 0, \tag{14}$$

and as we can see, in the case $\lambda = 0$, we recover the Klein-Gordon equation.

ii) The energy momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}, \tag{15}$$

thus, we have

$$T^{\mu\nu} = \eta^{\mu\nu}\partial_\nu\phi\partial^\nu\phi - \eta^{\mu\nu}\left[\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{4}\phi^4\right] \tag{16}$$

iii) Let's compute the divergence of the energy-momentum tensor, this is

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\partial^\mu\phi\partial^\nu\phi) - \partial_m u(\eta^{\mu\nu}\mathcal{L}), \tag{17}$$

then, we have

$$\partial_\mu T^{\mu\nu} = (\partial_\mu\partial^\mu\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \partial^\nu\mathcal{L}, \tag{18}$$

and using the chain rule for the last term, we have

$$\partial^\nu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\partial^\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\partial^\nu(\partial_\alpha\phi) = \frac{\partial\mathcal{L}}{\partial\phi}\partial^\nu\phi + (\partial^\alpha\phi)\partial_\alpha\partial^\nu\phi, \tag{19}$$

then we have

$$\partial_\mu T^{\mu\nu} = (\Box\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \left(\frac{\partial\mathcal{L}}{\partial\phi}\partial^\nu\phi + \partial^\alpha\phi(\partial_\alpha\partial^\nu\phi)\right), \tag{20}$$

and as we can see, the second and last term cancel between them, thus

$$\partial_\mu T^{\mu\nu} = \Box\phi\partial^\nu\phi - \frac{\partial\mathcal{L}}{\partial\phi}\partial^\nu\phi = \left(\Box\phi - \frac{\partial\mathcal{L}}{\partial\phi}\right)\partial^\nu\phi, \tag{21}$$

but the term in the parenthesis vanishes due to the Euler-Lagrange equation of motion, therefore, we have that

$$\partial_\mu T^{\mu\nu} = 0, \tag{22}$$

just as we wanted to prove.

Problem 3 (Scalar Gravity with Non-Universal Couplings) *The action is given by*

$$S = S_{kin} + S_{int}, \tag{23}$$

where

$$S_{kin} = \int d^4x \left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_\phi^2\phi^2\right] - \sum_i m_i \int d\tau_i, \tag{24}$$

and

$$S_{int} = -\sum_i g_i \int d^4x\phi T_i, \tag{25}$$

where T_i is the trace of the energy-momentum tensor associated with the ith particle.

i) Let's derive the equation of motion for ϕ. And for this first we need to derive the Euler-Lagrange equation which, again, is given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0, \tag{26}$$

and from this we have

$$\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m_\phi^2\phi^2 = -\sum_i g_i T_i, \tag{27}$$

which can be writen as

$$\Box\phi + m_\phi^2\phi^2 = -\sum_i g_i T_i, \tag{28}$$

which is the equation of motion for ϕ.

ii) In the static non-relativistic limit we have the following condition $\partial_t\phi = 0$, which implies that $\Box \to -\nabla^2$, and if the particles are non-relativistic, we have

$$\sqrt{1-v_i^2} \approx 1, \tag{29}$$

then the trace of the energy-momentum tensor becomes

$$T_i \approx m_i\delta^3(\mathbf{x}-\mathbf{x}_i), \tag{30}$$

and then the equation of motion simplifies to

$$\left(\nabla^2 - m_\phi^2\right)\phi = \sum_i g_i m_i \delta^3(\mathbf{x}-\mathbf{x}_i), \tag{31}$$

and by following the same procedure derived in class we have the solution

$$\phi(\mathbf{x}) = \int d^3x' G(\mathbf{x}-\mathbf{x}_i)(\sum_i g_i m_i \delta^3(\mathbf{x}-\mathbf{x}_i)), \tag{32}$$

where

$$G(\mathbf{r}) = -\frac{e^{-m_\phi|\mathbf{r}|}}{4\pi|\mathbf{r}|}, \tag{33}$$

which implies that

$$\phi(\mathbf{x}) = -\sum_i g_i m_i \frac{e^{-m_\phi|\mathbf{x}-\mathbf{x}_i|}}{4\pi|(\mathbf{x}-\mathbf{x}_i|}, \tag{34}$$

and if we define

$$\rho_g = \sum_i g_i m_i \delta(\mathbf{x}-\mathbf{x}'), \tag{35}$$

then the solution is given by

$$\phi(\mathbf{x}) = -\int d^3x' \frac{\rho_g(\mathbf{x}')e^{-m_\phi|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{36}$$

iii) Now, let's compute the acceleration of a particle: the action for a single particle i interacting with the field ϕ is given by

$$S_i = -m_i \int d\tau_i - g_i \int d^4\phi(x)T_i(x) = -m_i \int (1 + g_i\phi(x_i(\tau_i)))d\tau_i, \tag{37}$$

and in the non-relativistic limit we have

$$d\tau_i = \sqrt{1-\dot{\mathbf{x}}^2}dt \approx (1 - \frac{1}{2}\dot{\mathbf{x}}^2)dt, \tag{38}$$

adn thus, the Lagrangian becomes

$$L_i \approx -m_i(1 + g_i\phi(\mathbf{x}_i))(1 - \frac{1}{2}\dot{\mathbf{x}}^2) \approx \frac{1}{2}m_i\dot{\mathbf{x}}_i^2 - m_ig_i(\mathbf{x}_i) - m_i. \tag{39}$$

The potential energy is $V_i = m_ig_i\phi(\mathbf{x}_i)$, which implies that the force on the particle is

$$\mathbf{F}_i = -\nabla_iV_i = -m_ig_i\nabla_i\phi(\mathbf{x}_i), \tag{40}$$

then, we have

$$\frac{d^2\mathbf{x}_i}{dt^2} = -g_i\phi_i(\mathbf{x_i}), \tag{41}$$

and since the field ϕ at the location of particle i is generated by all the other particles $j \neq i$ we have that

$$\phi(\mathbf{x}_i) = -\sum_{j\neq i} g_jM_j \frac{e^{-m_\phi|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}, \tag{42}$$

and by taking the gradient we have

$$\nabla_i\phi(\mathbf{x}_i) = \sum_{j\neq i} \frac{g_jM_j}{4\pi} \frac{\mathbf{x_i}-\mathbf{x_j}}{|\mathbf{x_i}-\mathbf{x_j}|} e^{-m_\phi|\mathbf{x_i}-\mathbf{x_j}|}(1 + m_\phi|\mathbf{x_i}-\mathbf{x_j}|), \tag{43}$$

and then, the accelation of the particle is given by

$$\frac{d^2\mathbf{x}_i}{dt^2} = -\sum_{j\neq i} \frac{g_ig_jM_j}{4\pi} \frac{\mathbf{x_i}-\mathbf{x_j}}{|\mathbf{x_i}-\mathbf{x_j}|} e^{-m_\phi|\mathbf{x_i}-\mathbf{x_j}|}(1 + m_\phi|\mathbf{x_i}-\mathbf{x_j}|), \tag{44}$$

iv) The charactesitic length scale is giveb by the Compton wavelength of the scalaron, λ_ϕ, this is

$$\lambda_\phi = \frac{\hbar}{m_\phi c}, \tag{45}$$

and since $m_\phi = 10^{-20}eVc^{-2}$, we have that

$$\lambda_\phi = \frac{\hbar c}{m_\phi c^2} = \frac{1.97\times 10^-7eVm}{10^-20eV} \approx 1.97 \times^{13} m, \tag{46}$$

which is a really big number.

v) By setting $m_\phi = 0$, the exponential term in the acceleration formula from part (iii) vanishes. The acceleration of a particle i in the gravitational field of a large source S (with mass M_S and coupling g_S) simplifies to:

$$\mathbf{a}_i = -\frac{g_ig_SM_S}{4\pi}\frac{\mathbf{r}}{r^3} \tag{47}$$

where as usual $\mathbf{r}$ is the position vector from the source S to the particle i. And we know that the acceleration $\mathbf{a}_i$ is directly proportional to the particle's own coupling, g_i. Now, if we consider two different test bodies, 1 and 2, with couplings g_1 and g_2 falling toward the source. Since they are at the same position, their acceleration magnitudes are given by:

$$a_1 = \frac{g_1 g_S M_S}{4\pi r^2}$$
$$a_2 = \frac{g_2 g_S M_S}{4\pi r^2}$$

From these we have

$$\left|\frac{a_1 - a_2}{a_1 + a_2}\right| \lesssim 10^{-13}$$
$$\left|\frac{(g_1 g_S M_S/4\pi r^2) - (g_2 g_S M_S/4\pi r^2)}{(g_1 g_S M_S/4\pi r^2) + (g_2 g_S M_S/4\pi r^2)}\right| \lesssim 10^{-13}$$
$$\left|\frac{g_1 - g_2}{g_1 + g_2}\right| \lesssim 10^{-13}$$

Which implies that the couplings g_1 and g_2 must be almost identical. If we assume $g_2 \approx g_1$, then the denominator $g_1 + g_2 \approx 2g_1$. The constraint on the relative difference is thus given by:

$$\left|\frac{g_1 - g_2}{g_1}\right| \lesssim 2 \times 10^{-13}. \tag{48}$$

Problem 4 (Gravitational Redshift) *By considering*

$$\omega = \omega_0 \left(1 + \frac{G_N M}{Rc^2}\right), \tag{49}$$

derive the fractional change in frequency.

The fractional change in frequency is given by

$$\frac{\Delta\omega}{\omega_0} = \frac{\omega - \omega_0}{\omega_0} \approx \frac{\Delta\Phi}{c^2}, \tag{50}$$

where $\Delta\Phi$ is the change in gravitational potential. Now, with this at hand, the change in potential is given by

$$\Delta\Phi = gh, \tag{51}$$

where g is the acceleration due to gravity. Since the light is moving away from the mass, its frequency is decreasing (by redshift), thus the change is negative, this is

$$\frac{\Delta\omega}{\omega_0} = \frac{-gh}{c^2}, \tag{52}$$

and now, if we use the values $g = 9.8ms^-2$, $h = 22.6m$ and $c = 3 \times 10^8 ms^-2$, we have

$$\frac{\Delta\omega}{\omega_0} = -\frac{221.706}{9 \times 10^-16} \approx -2.46 \times 10^{-15}. \tag{53}$$

According to Wikipedia, the experimental value measured was

$$\left(\frac{\Delta\omega}{\omega_0}\right)_{exp} = -2.1 \pm 0.25 \times 10^{-15}, \tag{54}$$

which is not bad, it's actually pretty close.