Structure Formation, Statistics, and Scalar Field

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April 18, 2025

Problem 1. Dark Matter and Baryon Density Growth.
a. Let's start by defining

$$\epsilon = \delta_b(t) - \delta_d(t),$$

and we're given

$$\ddot{\delta}_d(t) + 2H\dot{\delta}_d(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

$$\ddot{\delta}_b(t) + 2H\dot{\delta}_b(t) - \frac{3}{2}H^2(\Omega_d\delta_d + \Omega_b\delta_b) = 0,$$

by taking the difference of the two previous equations, we have the following

$$\ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H\dot{\delta}_b(t) - 2H\dot{\delta}_d(t) = 0 \implies \ddot{\delta}_b(t) - \ddot{\delta}_d(t) + 2H(\dot{\delta}_b(t) - \dot{\delta}_d(t)) = 0$$

but we know that $H = 12/3t$, thus

$$\ddot{\epsilon} + \frac{4}{3t}\dot{\epsilon} = 0,$$

b. Let's seek solutions of the previous equation with the following form

$$\epsilon = t^n,$$

then, we have

$$t^{n-2}\left(n(n-1) + \frac{4}{3}n\right) = 0 \iff n\left(n + \frac{1}{3}\right) = 0$$

which implies that

$$n \in \{0, -\frac{1}{3}\},$$

and with this, the general solution will be

$$\epsilon(t) = \epsilon_0 + \frac{\epsilon_1}{t^{1/3}},$$

and from the form of the solution, we can see that a late times $\epsilon \to \epsilon_0$, which implies that at late times

$$\delta_b - \delta_d \to \epsilon_0 \implies \frac{\delta_b - \delta_d}{\delta_b} \to \frac{\epsilon_0}{\delta_b} \implies 1 - \frac{\delta_d}{\delta_b} \to \frac{\epsilon_0}{\delta_b},$$

which is valid for all ϵ_0, and in particular is valid for $\epsilon_0 = 0$, thus we have

$$\frac{\delta_d}{\delta_b} \to 1.$$

c. I append the Mathematica notebook with the solution and the corresponding plots.
d. From the numerical plot we can see that a late times δ_b becomes almost equal to δ_d which is in agreement with that I found in b. And finally,

$$t_{dec} = 3.17432 \times 10^7 \text{ years.}$$

Problem 2. Matter Growth with Dark Energy

a. Let's take derivatives
$$\frac{d\delta}{dt} = \frac{da}{dt}\frac{d\delta}{da} \implies \dot{\delta} = \delta'\dot{a},$$
where $'$ means derivative with respect to a. On the other hand, we also have
$$\ddot{\delta} = \frac{d}{dt}(\dot{\delta}) = \frac{d}{dt}(\delta'\dot{a}) = \dot{a}\frac{d\delta'}{dt} + \delta'\frac{d}{dt}(\dot{a}) \implies \ddot{\delta} = \dot{a}^2\delta'' + \delta'\ddot{a},$$
just as we wanted.

b. The first Friedmann equation with just matter and dark matter is given by
$$\frac{H^2}{H_0^2} = \Omega_{M0}a^{-3} + \Omega_{V0}$$
and from this we have
$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right),$$
which implies that
$$\dot{a}^2 = a^2H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right).$$
Now, if we take the time derivative of the previous expression we have
$$2\dot{a}\ddot{a} = 2a\dot{a}H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) + a^2H_0^2(-3\Omega_{M0}a^{-2}\dot{a}),$$
which implies that
$$2\dot{a}\ddot{a} = 2a\dot{a}H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) - 3\dot{a}a^{-2}H_0^2\Omega_{M0},$$
and by cancelind the $\dot{a}$ we have
$$2\ddot{a} = 2aH_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right) - 3a^{-2}H_0^2\Omega_{M0},$$
and from this we have
$$2\ddot{a} = -a^{-2}H_0^2\Omega_{M0} + 2aH_0^2\Omega_{V0},$$
and by factorizing some terms
$$2\ddot{a} = -aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0}),$$
which leads us to
$$\ddot{a} = -aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0})/2$$
just as we wanted.

c. Starting with
$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\Omega_{M0}\delta = 0,$$
and using the previous results we have that
$$a^2H_0^2\left(\Omega_{M0}a^{-3} + \Omega_{V0}\right)\delta'' + \delta'(-aH_0^2(\Omega_{M0}a^{-3} - 2\Omega_{V0})/2) + 2H\delta'\dot{a} - \frac{3}{2}H^2\Omega_{M0}\delta = 0$$

d. Let's look at both limit cases

- By considering a small and dark matter dominating, we have that Ω_{M0}/a^3 dominates, and therefore, the equation becomes

$$a^2(\Omega_{M0}/a^3)\delta'' + \frac{3}{2}(\Omega_{M0}/a^3)\delta' - \frac{3}{2}\Omega_{M0}/a^3\delta = 0,$$

  which implies that

$$a^2\delta'' + \frac{3}{2}\delta' - \frac{3}{2}\delta = 0,$$

  and if we seek solutions of the form $\delta = a^n$, we'll have the following equation

$$a^n(n^2 + \frac{1}{2}n - \frac{3}{2}) = 0,$$

  with solutions $n = -3/2$ and $n = 1$, therefore $\delta \propto a$ is a valis solution.

- On the other hand a is large and dark energy dominates, we have that $\Omega_{M0}/a^3 \ll 1$, which implies that terms without $\Omega_{M0}/a^3 \ll 1$ will dominate, and in limit the equation will take the form

$$a^2(1-\Omega_{M0})\delta'' + 3a(1-\Omega_{M0})\delta' = 0,$$

  and from here we can see that δ proportional to a constant is a valid solution.

Therefore such possible scenarios have either $\delta \propto a$ or $\delta \propto$ a constant.
e. I append a Mathematica notebook with the graph.
f. In the plot I made I include a fit taken into account the firts points, wich indeed resemble a linear function in a, whereas for higer values of a one can see that the solution reached a plateau, which is consistent witht the previous analysis in both scenarios/limits.

Problem 3. Power Spectrum
By definition $\langle\delta(\mathbf{k})\delta^*(\mathbf{k})'\rangle$ is given by

$$\langle\delta(\mathbf{k})\delta^*(\mathbf{k})'\rangle = \int d^3x d^3x' \exp(-i\mathbf{k}\cdot\mathbf{x})\exp(-i\mathbf{k}\cdot\mathbf{x}')\langle\delta(\mathbf{x})\delta(\mathbf{x}')\rangle,$$

and if we make $\mathbf{r} = |\mathbf{x}-\mathbf{x}'|$ we have

$$\exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}\cdot\mathbf{x}' - i\mathbf{k}\cdot\mathbf{x}' + i\mathbf{k}'\cdot\mathbf{x}'),$$

then we have

$$\exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')],$$

$$\implies \exp(-i\mathbf{k}\cdot\mathbf{x} + i\mathbf{k}'\cdot\mathbf{x}') = \exp(-i\mathbf{k}\cdot\mathbf{r})\exp(-i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')).$$

With this information, we can arrive at the following expression

$$\langle\delta(\mathbf{k})\delta^*(\mathbf{k})'\rangle = \int d^3r d^3x' e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{x}\cdot(\mathbf{k}-\mathbf{k}')}\xi(r),$$

where

$$\xi(r) = \langle\delta(\mathbf{x})\delta(\mathbf{x}')\rangle.$$

And if now we integrate over the x' variables we have

$$\langle\delta(\mathbf{k})\delta^*(\mathbf{k})'\rangle = (2\pi)^3\delta^3(\mathbf{x}-\mathbf{x}')\int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}}\xi(r),$$

and by making

$$\mathcal{P}(k) = \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}}\xi(r),$$

we arrive at the following result

$$\langle\delta(\mathbf{k})\delta^*(\mathbf{k})'\rangle = (2\pi)^3\delta^3(\mathbf{x}-\mathbf{x}')\mathcal{P}(k)$$

just as we wanted.

Problem 4. Scalar Field (Inflaton) in Expanding Universe

a. Let's begin with the following action

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right],$$

where

$$g_{\mu\nu} = \text{diag}(1, -a^2, -a^2, -a^2),$$

and with this metric, we have

$$\sqrt{-g} = a^3,$$

whereas

$$g^{\mu\nu} = \text{diag}(1, -1/a^2, -1/a^2, -1/a^2),$$

therefore, the action takes the form

$$S = \int dtd^3xa^3\left[\frac{1}{2}(\dot{\phi})^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi)\right]$$

which can be simplified to

$$S = \int dtd^3x\left[\frac{1}{2}a^3(\dot{\phi})^2 - \frac{1}{2}a(\nabla\phi)^2 - a^3V(\phi)\right].$$

And if we perform the variation, we have

$$\delta S = \int dtd^3x\left[a^3\dot{\phi}\delta\dot{\phi} - a\nabla\phi\cdot\nabla\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi\right],$$

for the first two terms, we can perform integration by parts as follows

$$\int dt(a^3\dot{\phi})\delta\dot{\phi} = -\int\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + \text{boundary terms},$$

and

$$\int d^3xa\nabla\phi\cdot\nabla\delta\phi = -\int d^3x(a\nabla^2\phi)\delta\phi + \text{boundary terms},$$

thus, if we neglect the boundary terms, the variation becomes

$$\delta S = \int dtd^3x\left[-\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + (a\nabla^2\phi)\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi\right],$$

and impossing the condition $\delta S = 0$ we have

$$-\frac{d}{dt}\left(a^3\dot{\phi}\right)\delta\phi + (a\nabla^2\phi)\delta\phi - a^3\frac{V(\phi)}{d\phi}\delta\phi = 0,$$

which can be simplified to

$$-3a^2\dot{a}\dot{\phi} - a^3\ddot{\phi} + a\nabla^2\phi - a^3\frac{V(\phi)}{d\phi} = 0,$$

since $a \neq 0$ we have

$$-3\frac{\dot{a}}{a}\dot{\phi} - \ddot{\phi} + \frac{\nabla^2\phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

and by using the fact $H = \dot{a}/a$ we have

$$-3H\dot{\phi} - \ddot{\phi} + \frac{\nabla^2\phi}{a^2} - \frac{V(\phi)}{d\phi} = 0,$$

which can also be writen as

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} + \frac{V(\phi)}{d\phi} = 0,$$

just as we wanted.

b. By assuming ϕ is homogeneous in space and that $V = \frac{1}{2}m^2\phi^2$ we have

$$\frac{dV}{d\phi} = m^2\phi, \nabla^2\phi = 0,$$

thus the previous equation becomes:

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0,$$

just as we wanted.

c. Finally, by assuming H a constant we seek solutions of the form

$$\phi = \phi_0 \exp(rt),$$

we have the following equation

$$\phi_0 \exp(rt) \left(r^2 + 3Hr + m^2\right) = 0$$

and solving the characteristic equation lead us to the following expression

$$r = \frac{-3H \pm \sqrt{1 - \frac{2m^2}{9H^2}}}{2},$$

which can be simplified to

$$r = \frac{3H}{2}\left(1 \pm \sqrt{1 - \left(\frac{2m}{3H}\right)^2}\right)$$

or we can also make

$$r = \frac{m}{\alpha}(1 \pm \sqrt{1 - \alpha^2}),$$

where $\alpha = \frac{2m}{3H}$. Therefore, the general solution takes the form

$$\phi(t) = \phi_+ \exp\left[\frac{m}{\alpha}(1 + \sqrt{1 - \alpha^2})t\right] + \phi_- \exp\left[\frac{m}{\alpha}(1 - \sqrt{1 - \alpha^2})t\right].$$

Now, the solution will oscillate when the following condition hold

$$1 - \left(\frac{2m}{3H}\right)^2 < 0 \iff 2m > 3H.$$

On the other hand, the solution will not oscillate when

$$1 - \left(\frac{2m}{3H}\right)^2 \geq 0 \iff 3H \geq 2m.$$

And finally, if $m \ll H$, this implies that $\frac{m}{H} \ll 1$, thus the solution will take the form

$$r \approx 3H \implies \phi(t) \propto e^{3Ht}$$

therefore, ϕ will evolve quickly.