QUANTUM THEORY 1

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MIDTERM EVALUATION

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(1) A particle is confined to a one dimensional infinite square well

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

(a) What are the energy eigenvalues and the corresponding time independent wave functions?

The Schrodinger equation for this potential is given by

$$-\left(\frac{\hbar^2}{2m}\right)\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi,$$

but, in the region of interest, we have that the potential is zero, then

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0,$$

thus, let's propose this expression as the solution of the previous ODE,

$$\psi = A\sin(\alpha x),$$

inserting that expression into the ODE, we have

$$-A\alpha^2\sin(\alpha x) + A\frac{2m}{\hbar^2}E\sin(\alpha x) = 0,$$

$$\iff \left(-\alpha^2 + \frac{2m}{\hbar^2}E\right)\sin(\alpha x) = 0,$$

$$\implies E = \alpha^2\frac{\hbar^2}{2m},$$

and, using the boundary conditions, we have that

$$\sin(\alpha L) = 0, \iff \alpha L = n\pi \iff \alpha = \frac{n\pi}{L},$$

thus, the energy will be given by

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL}.$$

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On the other, hand, for the constant A we need to normalize the wave function, thus

$$1 = \int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) = A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right),$$

but we know that

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) = \left[\frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L = \frac{L}{2},$$

thus

$$1 = A^2\frac{L}{2} \iff A = \sqrt{\frac{2}{L}},$$

and then, the wave function will be

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right),$$

with energies

$$E_n = \frac{\hbar^2\pi^2 n^2}{2mL}.$$

(b) What is the dispersion of position x and momentum p for an eigenstate of energy E? Do the results satisfy the Heisenberg Uncertainty Principle?
The dispersion of any observable A is given by

$$\left\langle (\Delta A)^2 \right\rangle = \left\langle A^2 \right\rangle - \langle A \rangle^2,$$

thus, the dispersion in x will be given by

$$\left\langle (\Delta x)^2 \right\rangle = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

where

$$\langle x \rangle = \int_0^L dx \left[x\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)^2\right],$$

$$\left\langle x^2 \right\rangle = \int_0^L dx \left[x^2\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)^2\right],$$

then, for $\langle x \rangle$, we have

$$\langle x \rangle = \frac{2}{L}\left(-\frac{L^2\cos\left(\frac{2\pi n x}{L}\right)}{8\pi^2 n^2} - \frac{Lx\sin\left(\frac{2\pi n x}{L}\right)}{4\pi n} + \frac{x^2}{4}\right)\Bigg|_0^L,$$

$$\implies \langle x \rangle = -\frac{L\left(-2\pi^2 n^2\right)}{4\pi^2 n^2} = \frac{L}{2},$$

$$\therefore \langle x \rangle = \frac{L}{2},$$

and following the same approach, for $\left\langle x^2 \right\rangle$ we have

$$\left\langle x^2 \right\rangle = \frac{2}{L}\left[-\frac{L^2 x\cos\left(\frac{2\pi n x}{L}\right)}{4\pi^2 n^2} - \frac{L\left(2\pi^2 n^2 x^2 - L^2\right)\sin\left(\frac{2\pi n x}{L}\right)}{8\pi^3 n^3} + \frac{x^3}{6}\right]\Bigg|_0^L,$$

$$\implies \left\langle x^2 \right\rangle = \frac{L^2\left(4\pi^3 n^3 - 6\pi n\cos(2\pi n)\right)}{12\pi^3 n^3},$$

$$\implies \left\langle x^2 \right\rangle = \frac{L^2\left(4\pi^3 n^3 - 6\pi n\right)}{12\pi^3 n^3} = L^2\left(\frac{4\pi^3 n^3}{12\pi^3 n^3} - \frac{6\pi n}{12\pi^3 n^3}\right),$$

$$\implies \left\langle x^2 \right\rangle = \frac{L^2\left(4\pi^3n^3 - 6\pi n\right)}{12\pi^3n^3} = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right),$$
$$\therefore \left\langle x^2 \right\rangle = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right),$$

and therefore, the dispersion of x is given by

$$\left\langle (\Delta x)^2 \right\rangle = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right) - \frac{L^2}{4},$$

and, in addition, we know that

$$p \to -i\hbar\partial_x,$$

thus

$$\langle p \rangle = \int_0^L dx\left[-i\hbar\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\partial_x\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\right],$$
$$\implies \langle p \rangle = -i\hbar\frac{2\pi n}{L^2}\int_0^L dx\left[\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right],$$

but because of the parity of the functions, we have that

$$\langle p \rangle = 0,$$

on the other, hand, for $\left\langle p^2 \right\rangle$ we have

$$\left\langle p^2 \right\rangle = \int_0^L dx\left[(i\hbar)^2\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\partial_x^2\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)\right],$$
$$\implies \left\langle p^2 \right\rangle = \frac{2}{L}(i\hbar)^2\left(\frac{n\pi}{L}\right)^2\int_0^L dx\left[\sin^2\left(\frac{n\pi x}{L}\right)\right],$$
$$\implies \left\langle p^2 \right\rangle = \frac{2}{L}(i\hbar)^2\left(\frac{n\pi}{L}\right)^2\left[\frac{x}{2} - \frac{L\sin\left(\frac{2\pi nx}{L}\right)}{4\pi n}\right]\Bigg|_0^L,$$
$$\implies \left\langle p^2 \right\rangle = \frac{2}{L}(i\hbar)^2\left(\frac{n\pi}{L}\right)^2\frac{L}{2} = -\hbar^2\frac{\pi^2n^2}{L^2},$$
$$\therefore \left\langle (\Delta p)^2 \right\rangle = -\hbar^2\frac{\pi^2n^2}{L^2}.$$

(c) What linear combination of Energy eigenstates at time 0, y $\psi_E(x,0)$ will form a wave packet that has a triangular shape

$$\psi(x,0) = \begin{cases} \frac{2x}{L} & 0 < x < L/2 \\ \frac{2}{L}(L-x) & L/2 < x < L \end{cases}$$

For this part, we need to express the fgiven function $\psi(x,0)$ as a sum of sine functions, which could be done by means of the Fourier series, thus we want an expansion of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right),$$

where the coeficients a_n and b_n are given by

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, n = 0,1,2,3\ldots,$$
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, n = 1,2,\ldots,$$

and even more, if the function $f(x)$ is even, then

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, n = 0,1,2,3\ldots, \quad b_n = 0, \forall n,$$

on the other hand, if the function is odd, then the coefficients are given by

$$a_n = 0, \forall n, \quad b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, n = 1,2,\ldots,$$

and now, with this information at hand, because the given function is odd, then all the coeficcients a_n are equal to zero, an then we just need to compute the b_n coefficients, which are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^{L/2}\frac{2x}{L}\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L}\frac{2}{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx,$$

$$\implies b_n = \frac{2}{L}\left(\frac{2}{L}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx\right),$$

now, if we make the change $\frac{n\pi x}{L} = t$, the previous equations transforms into

$$b_n = \frac{2}{L}\left(\frac{2L}{\pi^2}\int_0^{\pi/2} t\sin(nt)\,dt + \frac{2L}{\pi^2}\int_{\pi/2}^{\pi}(\pi - t)\sin(nt)\,dt\right),$$

then, for the first integral we have

$$\frac{2L}{\pi^2}\int_0^{\pi/2} t\sin(nt)\,dt = \frac{2L}{\pi^2}\left(-\frac{t\cos(nt)}{n}\right)\Bigg|_{t=0}^{t=\pi/2} + \frac{2L}{\pi^2 n}\int_0^{\pi/2}\cos(nt)\,dt,$$

while for the second, we have

$$\frac{2L}{\pi^2}\int_{\pi/2}^{\pi}(\pi - t)\sin(nt)\,dt = \frac{2L}{\pi^2}\left(-\frac{(\pi - t)\cos(nt)}{n}\right)\Bigg|_{t=\pi/2}^{t=\pi} - \frac{2L}{\pi^2}\int_{\pi/2}^{\pi}\cos(nt)\,dt,$$

and, as we can see, some of the terms, cancel leaving us with

$$b_n = \frac{2}{L}\left(\frac{4L}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right)\right) = \frac{8}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right),$$

then, the Fourier expansion for this function is

$$\psi(x,0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{n\pi x}{l}\right)$$

(d) How will that wave packet shape change in time for several cycles at the ground state frequency (Lowest E/h)?

The general solution, i.e, the time dependent wave function, is given by

$$\psi(x,t) = \sum_{k=1}^{\infty} c_k\psi_k(x)\exp\left[-i\frac{E_k t}{\hbar}\right],$$

where the $\psi_n(x)$ are the wave functions for the time-independent problem. Now, in this problem, we want to study the behaviour of the wave package within the first allowed energy, thus, we have

$$\psi(x,0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\exp\left[-i\frac{E_1 t}{\hbar}\right],$$

or more explicitly, using $E_1 = \frac{\hbar^2\pi^2}{2mL}$, we have

$$\psi(x,0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{\pi n}{2}\right)\sin\left(\frac{n\pi x}{l}\right)\exp\left[-i\frac{\hbar^2\pi^2}{2mL}t\right].$$

(e) If the particle is an electron, for what value of L (in metric units) will the ground state velocity equal the speed of light, c? Consider the same question for a proton.
In the gound state, i.e., $n = 1$, the energy is given by

$$E_1 = \frac{\hbar^2 \pi^2}{2mL},$$

then using $E_1 = mc^2$, we have that

$$mc^2 = \frac{\hbar^2 \pi^2}{2mL},$$

$$\implies L = \frac{\hbar^2 \pi^2}{2m^2 c^2},$$

now, in metric units, $m_e = 9.10938 \times 10^{-31}$Kg, $m_p = 1.6725 \times 10^{-27}$Kg, $c = 2.998 \times 10^8$m/s, $\hbar = 6.62607015 \times 10^{-34}$, then, for the electron we have

$$L_e = 2.904965060387787 \times 10^{-23}\text{m},$$

and for the proton

$$L_p = 8.617599344270647 \times 10^{-30}\text{m}.$$

(2) Let

$$J_\pm = \hbar a_\pm^\dagger a_\mp, \quad J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \quad N = a_+^\dagger a_+ + a_-^\dagger a_-,$$

where $a_\pm$ and $a_\mp$ are the anhilation and creation operators of two independent simple harmonic oscillators satisfying the usual simple harmonic oscillator computation relations, prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \quad \left[\mathbf{J}^2, J_z\right] = 0, \quad \mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right) N \left[\left(\frac{N}{2}\right) + 1\right]$$

(a) We know the number operator is given by

$$N_+ = a_+^\dagger a_+, \quad N_- = a_-^\dagger a_-,$$

then, we can rewrite J_z as follows

$$J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right) = \frac{\hbar}{2}\left(N_+ - N_-\right),$$

$$\implies J_z = \frac{\hbar}{2}\left(N_+ - N_-\right),$$

and we can do the same for N,

$$N = a_+^\dagger a_+ + a_-^\dagger a_- = N_+ + N_-,$$

$$\implies N = N_+ + N_-.$$

Now, with this information, let's prove

$$[J_z, J_\pm] = \pm J_\pm,$$

and let's start with J_+: by definition

$$[J_z, J_+] = \left[J_z, \hbar a_+^\dagger a_-\right] = \frac{\hbar^2}{2}\left[N_+ - N_-, a_+^\dagger a_-\right],$$

$$\implies [J_z, J_+] = \frac{\hbar^2}{2}\left[N_+, a_+^\dagger a_-\right] - \frac{\hbar^2}{2}\left[N_-, a_+^\dagger a_-\right],$$

in which I've used the linearity of the commutator. Moving on with the proof, we know that the number operator N_+ satisfies the following properties

$$[N_+, a_+] = -a_+, \quad \left[N_+, a_+^\dagger\right] = a_+^\dagger,$$

and the same for N_-

$$[N_-, a_-] = -a, \quad \left[N_-, a_-^\dagger\right] = a_-^\dagger.$$

On the other hand, for any operators X, Y, Z we have that

$$[X, YZ] = XYZ - YZX = XYZ - YXZ + YXZ - YZX = [X, Y]\, Z + Y\, [X, Z]\,,$$

$$\Longrightarrow [X, YZ] = [X, Y]\, Z + Y\, [X, Z]\,, \tag{0.1}$$

thus, if we aplly the previous relation to the commutators $\left[N_+, a_+^\dagger a_-\right]$ and $\left[N_-, a_+^\dagger a_-\right]$ we have that

$$\left[N_+, a_+^\dagger a_-\right] = \left[N_+, a_+^\dagger\right] a_- + a_+^\dagger\, [N_+, a_-] = a_+^\dagger a_-,$$

$$\Longrightarrow \left[N_+, a_+^\dagger a_-\right] = a_+^\dagger a_-,$$

and the same holds for

$$\left[N_-, a_+^\dagger a_-\right] = \left[N_-, a_+^\dagger\right] a_- + a_+^\dagger\, [N_-, a_-] = -a_+^\dagger a_-,$$

$$\Longrightarrow \left[N_-, a_+^\dagger a_-\right] = -a_+^\dagger a_-,$$

because N_+ is independent from any $a_-, a_-^\dagger$ and the same fort N_- with $a_+, a_+^\dagger$. Then, with this result, we have that the commutator $[J_z, J_+]$ can be expressed as

$$[J_z, J_+] = \frac{\hbar^2}{2} a_+^\dagger a_- - \frac{\hbar^2}{2}\left(-a_+^\dagger a_-\right) = \frac{\hbar^2}{2} a_+^\dagger a_- + \frac{\hbar^2}{2}\left(a_+^\dagger a_-\right),$$

$$\Longrightarrow [J_z, J_+] = \hbar^2 a_+^\dagger a_- = \hbar\left(\hbar a_+^\dagger a_-\right) = \hbar J_+,$$

$$\therefore [J_z, J_+] = \hbar J_+.$$

Now, let's move with $[J_z, J_+]$, and following the same procedure as before, we can write the commutator as

$$[J_z, J_+] = \left[J_z, \hbar a_-^\dagger a_+\right] = \frac{\hbar^2}{2}\left[N_+ - N_-, a_-^\dagger a_+\right],$$

$$\Longrightarrow [J_z, J_+] = \frac{\hbar^2}{2}\left[N_+, a_-^\dagger a_+\right] - \frac{\hbar^2}{2}\left[N_-, a_-^\dagger a_+\right],$$

$$\Longrightarrow [J_z, J_+] = \frac{\hbar^2}{2}\left[N_+, a_-^\dagger\right] a_+ + \frac{\hbar^2}{2} a_-^\dagger\, [N_+, a_+] - \frac{\hbar^2}{2}\left[N_-, a_-^\dagger\right] a_+ - \frac{\hbar^2}{2} a_-^\dagger\, [N_-, a_+]\,,$$

$$\Longrightarrow [J_z, J_+] = \frac{\hbar^2}{2} a_-^\dagger\, [N_+, a_+] - \frac{\hbar^2}{2}\left[N_-, a_-^\dagger\right] a_+,$$

$$\Longrightarrow [J_z, J_+] = -\frac{\hbar^2}{2} a_-^\dagger a_+ - \frac{\hbar^2}{2} a_-^\dagger a_+,$$

$$\Longrightarrow [J_z, J_+] = -\hbar^2 a_-^\dagger a_+ = -\hbar\left(\hbar a_-^\dagger a_+\right) = -\hbar J_-,$$

$$\therefore [J_z, J_+] = -\hbar J_-,$$

and finally, we can group the two previous expressions as

$$[J_z, J_\pm] = \pm\hbar J_\pm. \tag{0.2}$$

(b) Now, let's move on, in this part we're going to prove that

$$\left[\mathbf{J}^2, J_z\right] = 0,$$

where $\mathbf{J}^2$ is given by

$$\mathbf{J}^2 = J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right),$$

then let's begin. By definition

$$\left[\mathbf{J}^2, J_z\right] = \left[\left(J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right)\right), J_z\right] = \left[J_z^2, J_z\right] + \left[\frac{1}{2}\left(J_+J_- + J_-J_+\right), J_z\right],$$

$$\implies \left[\mathbf{J}^2, J_z\right] = \left[J_z^2, J_z\right] + \left[\frac{1}{2}\left(J_+J_- + J_-J_+\right), J_z\right]$$

but clearly, the commutator $\left[J_z^2, J_z\right]$ is equal to zero, then

$$\left[\mathbf{J}^2, J_z\right] = \left[\frac{1}{2}\left(J_+J_- + J_-J_+\right), J_z\right]$$

$$\implies \left[\mathbf{J}^2, J_z\right] = \frac{1}{2}\left[J_+J_-, J_z\right] + \frac{1}{2}\left[J_-J_+, J_z\right],$$

then

$$\left[\mathbf{J}^2, J_z\right] = -\frac{1}{2}\left[J_z, J_+J_-\right] - \frac{1}{2}\left[J_z, J_-J_+\right]$$

and using the property given by equation (0.1)we have

$$\left[\mathbf{J}^2, J_z\right] = -\frac{1}{2}\left[J_z, J_+\right]J_- - \frac{1}{2}J_+\left[J_z, J_-\right] - \frac{1}{2}\left[J_z, J_-\right]J_+ - \frac{1}{2}J_-\left[J_z, J_+\right],$$

and now, we can use the previous result, the one given by equation (0.2), thus

$$\left[\mathbf{J}^2, J_z\right] = -\frac{1}{2}\hbar J_+J_- - \frac{1}{2}J_+\left(-\hbar J_-\right) - \frac{1}{2}\left(-\hbar J_-\right)J_+ - \frac{1}{2}\hbar J_-J_+,$$

$$\implies \left[\mathbf{J}^2, J_z\right] = -\frac{\hbar}{2}J_+J_- + \frac{\hbar}{2}J_+\left(J_-\right) + \frac{\hbar}{2}J_-J_+ - \frac{\hbar}{2}J_-J_+ = 0,$$

$$\therefore \left[\mathbf{J}^2, J_z\right] = 0. \tag{0.3}$$

(c) Finally, let's prove

$$\mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right)N\left[\left(\frac{N}{2}\right) + 1\right].$$

By definition, $\mathbf{J}^2$ is given by

$$\mathbf{J}^2 = J_z^2 + \frac{1}{2}\left(J_+J_- + J_-J_+\right),$$

but we know that

$$J_z = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right), \quad N = a_+^\dagger a_+ + a_-^\dagger a_-,$$

thus

$$J_z^2 = \frac{\hbar^2}{4}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)^2 = \frac{\hbar^2}{4}\left(N_+^2 - N_+N_- - N_-N_+ + N_-^2\right),$$

and in addition, we know that

$$\left[a, a^\dagger\right] = 1 \implies aa^\dagger = 1 + a^\dagger a = 1 + N,$$

then

$$J_+J_- = \hbar^2 a_+^\dagger a_- a_-^\dagger a_+ = \hbar^2 a_+^\dagger a_+ a_- a_-^\dagger = \hbar^2 N_+\left(1 + N_-\right),$$

$$\implies J_+J_- = \hbar^2 N_+ + \hbar^2 N_+N_-,$$

and the same goes for

$$J_-J_+ = \hbar^2 N_- + \hbar^2 N_-N_+,$$

then

$$J_+J_- + J_-J_+ = \hbar^2 N_+ + \hbar^2 N_+N_- + \hbar^2 N_- + \hbar^2 N_-N_+,$$
$$\implies J_+J_- + J_-J_+ = \hbar^2 (N_+ + N_-) + \hbar^2 N_+N_- + \hbar^2 N_-N_+,$$

then

$$\mathbf{J}^2 = \frac{\hbar^2}{4}\left(N_+^2 - N_+N_- - N_-N_+ + N_-^2\right) + \frac{\hbar^2}{2}(N_+ + N_-) + \frac{\hbar^2}{2}N_+N_- + \frac{\hbar^2}{2}N_-N_+,$$
$$\implies \mathbf{J}^2 = \frac{\hbar^2}{4}\left(N_+^2 + N_-^2\right) + \frac{\hbar^2}{2}(N_+ + N_-) + \frac{\hbar^2}{2}N_+N_- - \frac{\hbar^2}{4}N_+N_- + \frac{\hbar^2}{2}N_-N_+ - \frac{\hbar^2}{4}N_-N_+,$$
$$\implies \mathbf{J}^2 = \frac{\hbar^2}{4}\left(N_+^2 + N_-^2\right) + \frac{\hbar^2}{2}(N_+ + N_-) + \frac{\hbar^2}{4}N_+N_- + \frac{\hbar^2}{4}N_-N_+,$$
$$\implies \mathbf{J}^2 = \frac{\hbar^2}{4}\left(N_+^2 + N_-^2 + N_+N_- + N_-N_+\right) + \frac{\hbar^2}{2}(N_+ + N_-),$$
$$\implies \mathbf{J}^2 = \frac{\hbar^2}{4}N^2 + \frac{\hbar^2}{2}N,$$

and from we have

$$\mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right) N \left[\left(\frac{N}{2}\right) + 1\right].$$

(3) Problem 1.24.

(a) Prove that $(1/\sqrt{2})(1 + i\sigma_x)$, acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)
We know that σ_x is given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then, let's operate $(1/\sqrt{2})(1 + i\sigma_x)$ on the spinor representation of $|+\rangle$

$$\frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix},$$

on the other hand, the rotation operator, for a finite rotation, is given by

$$\mathcal{D}(\hat{\mathbf{n}}, \phi) = \exp\left(\frac{-i\mathbf{S}\cdot\hat{\mathbf{n}}\phi}{\hbar}\right),$$

which can be represented in terms of the Pauli matrices as

$$\mathcal{D}(\hat{\mathbf{n}}, \phi) = \exp\left(\frac{-i\sigma\cdot\hat{\mathbf{n}}\phi}{\hbar}\right),$$

then, if we use

$$(\sigma\cdot\hat{\mathbf{n}})^n = \begin{cases} 1 & n \text{ even} \\ \sigma\cdot\hat{\mathbf{n}} & n \text{ odd} \end{cases}$$

then, we can Taylor expand the exponential, and that will give us

$$\exp\left(\frac{-i\sigma\cdot\hat{\mathbf{n}}\phi}{\hbar}\right) = \mathbf{1}\cos\left(\frac{\phi}{2}\right) - i\sigma\cdot\hat{\mathbf{n}}\sin\left(\frac{\phi}{2}\right),$$

thus

$$\mathcal{D}(\hat{\mathbf{n}}, \phi) = \mathbf{1}\cos\left(\frac{\phi}{2}\right) - i\sigma\cdot\hat{\mathbf{n}}\sin\left(\frac{\phi}{2}\right),$$

and if we use $\sigma \cdot \hat{\mathbf{n}} = \sigma_x$, with $\phi = -\pi/2$, we have that

$$\mathcal{D}\left(\hat{\mathbf{x}}, \phi\right) = \mathbf{1}\cos\left(\frac{-\pi}{4}\right) - i\sigma_x \sin\left(\frac{-\pi}{4}\right),$$

but, using the parity of the trigonometric functions, we have that

$$\mathcal{D}\left(\hat{\mathbf{x}}, \frac{\pi}{4}\right) = \mathbf{1}\cos\left(\frac{\pi}{4}\right) + i\sigma_x \sin\left(\frac{\pi}{4}\right),$$

but

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},$$

then

$$\mathcal{D}\left(\hat{\mathbf{x}}, \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\mathbf{1} + i\frac{1}{\sqrt{2}}\sigma_x,$$

$$\therefore \mathcal{D}\left(\hat{\mathbf{x}}, \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\mathbf{1} + i\sigma_x\right),$$

and therefore, we just prove that $(1/\sqrt{2})(1 + i\sigma_x)$, acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$.

(b) Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.
Now, instead of representing the represent the operators S_x, S_y and S_z in the z basis, we want to represent it in the x basis, and because we are using the x-basis then, we expect that S_x will be diagonal. On the other, hand, we know that

$$\left[S_i, S_j\right] = i\epsilon_{ijk}\hbar S_k,$$

thus, any change in the representation will also have to follow the previous equation, then if we make the permutation

$$S_x \to S_z, \quad S_z \to S_y, \quad S_y \to S_x,$$

the commutation relation will hold, then

$$S_x = \left(\frac{\hbar}{2}\right)\left(|+\rangle\langle+| - |-\rangle\langle-|\right),$$

$$S_y = \left(\frac{\hbar}{2}\right)\left(-|+\rangle\langle-| + |-\rangle\langle+|\right),$$

$$S_z = \left(\frac{\hbar}{2}\right)\left(|+\rangle\langle-| + |-\rangle\langle+|\right).$$

References

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