Position and the Free Particle

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Contents

Canonical Quantization

Poisson Bracket and Commutator

Quantization: transition from a classical description to a quantum version, "procedure for constructing quantum mechanics from classical mechanics."

Dirac notice the following connection

$$\{q, p\} = 1, -\frac{i}{\hbar}[Q, P] = 1,$$

together with

$$\frac{df}{dt} = \{f, h\}, \frac{d\mathcal{O}}{dt} = -\frac{i}{\hbar}[\mathcal{O}, H]$$

Procedure

Given any classical system, we can quantize it by finding a rule as follows: for each function f defined in the phase space, we associate a self-adjoint operator $\mathcal{O}_f$, acting on a state space $\mathcal{H}$, such that

$$O_{\{f,g\}} = -\frac{i}{\hbar}\left[\mathcal{O}_f, \mathcal{O}_g\right]$$

Linear Functions and Schrodinger Representation

"The Heisenberg Lie algebra is isomorphic to the threedimensional subalgebra of functions on phase space given by linear combinations of the constant function, the function q and the function p." We have

$$O_1 = \mathbf{1}, O_q = Q, O_p = P,$$

with

$$\Gamma'_S(1) = -i\mathbf{1},$$

$$\Gamma'_S(q) = -iQ = -iq,$$

$$\Gamma'_S(p) = -P = \frac{d}{dq}.$$

Quadratic Polynomials

Quadratic polynomials can be quantized as follows

$$O_{p^2/2} = \frac{P^2}{2}, O_{q^2/2} = \frac{Q^2}{2}$$

but, we need to work a more for pq since the order here matters. It turns out that

$$O_{pq} = \frac{1}{2}\left(PQ + QP\right).$$

And the issue here is: "Γ'_S has the same sort of problem as the spinor representation of $su(2) = so(3)$, which was not a representation of $SO(3)$, but only of its double cover $SU(2) = Spin(3)$"

The Groenwold-van Hove no-go theorem

Main challenges

The issue lies here: how can we quantize polynomial functions on phase space with a degree greater than two?

- Operator Ordering Ambiguities: Ordering matters.
- Inconsistency with Poisson Bracket Relation.
- Lowest Order Approximation.
- Limited Lie Algebra Representation.

And from the physics point of view, different ways of ordering the P and Q operators will lead to different operators O_f for the same function f, with physically different observables.

Main challenges

For polynomials of degree greater than two there is no possible way to do this consistent with the following relation:

$$Q_{\{f,g\}} = -\frac{i}{\hbar}\left[O_f, O_g\right].$$

"Whatever method one devises for quantizing higher-degree polynomials, it can only satisfy that relation to lowest order in $\hbar$, and there will be higher-order corrections, which depend upon one's choice of quantization scheme."

Groenwold-van Hove no-go theorem

Theorem
There is no map $f \to O_f$ from polynomials on $\mathbf{R}^2$ to self-adjoint operators on $L^2(\mathbf{R})$ satisfying

$$O_{\{f,g\}} = -\frac{i}{\hbar}[O_f, O_g]$$

and

$$O_p = P, O_q = Q,$$

or any Lie subalgebra of the functions on $\mathbf{R}^2$ for which the subalgebra of polynomials of degree less than or equal to two is a proper subalgebra.

Canonical quantization in d dimensions

Generalization

Moving on to d dimensions, we have

$$\Gamma'_S(q_j) = -iQ_j, \Gamma'_S(p_j) = -iP_j,$$

wich satisfy the Heisenberg relations

$$[Q_j, P_k] = i\delta_{jk}$$

And for quadratic polynomials

$$\Gamma'_S(q_j q_k) = -iQ_j Q_k, \Gamma'_S(p_j p_k) = -iP_j P_k$$

$$\Gamma'_S(q_j p_k) = -i\frac{i}{2}\left(Q_j P_k + P_k Q_j\right)$$

Quantization and Symmetries

Example: Angular Momentum

"The observables that commute with the Hamiltonian operator H will make up a Lie algebra of symmetries of the quantum system and will take energy eigenstates to energy eigenstates of the same energy."

Example: The group SO(3).

The following operators provide a basis for the Lie algebra representation

$$-i\left(Q_2P_3 - Q_3P_2\right), -i\left(Q_3P_1 - Q_1P_3\right), -i\left(Q_1P_2 - Q_2P_1\right)$$

General Ways of Quantization

Feynman Path Integral

The key ideas are:

- The quantum amplitude is calculated by summing over all possible paths a system can take between two states.
- It naturally incorporates the principle of least action from classical mechanics.
- The method is particularly useful in quantum field theory and for systems with many degrees of freedom.