Tensor Model for Particle Transport

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We have the following dynamical model for the system

$$\beta \frac{\partial S}{\partial t} = \nabla \cdot \frac{\partial F}{\partial \nabla S} - \frac{\partial F}{\partial S}, \tag{1}$$

with the free energy given by

$$F = \frac{1}{2}\alpha \left(\nabla S\right)^2 + \frac{3}{4}a_0 \Delta T S^2 - \frac{1}{4}bS^3 + \frac{9}{16}cS^4, \tag{2}$$

where β it's a trasnsport coefficient. Now, what we want is to generalize the previous model in the case of 2D, and in order to achieve that, we need to make the switch from the scalar order parameter to the tensor order parameter. With the previous assumptions, we can parametrize the tensor order parameter as

$$\mathbf{Q} = S\left(\mathbf{nn} - \frac{\mathbf{I}}{2}\right), \tag{3}$$

where $\mathbf{n}$ it's a unit vector in 2D and $\mathbf{I}$ is the identity matrix, and for this case, we can consider the free energy as

$$F = \frac{1}{2}\alpha \left(\nabla \mathbf{Q}\right)^2 + \frac{3}{4}a_0 \Delta T \mathrm{tr}\mathbf{Q}^2 - \frac{1}{4}b\mathrm{tr}\mathbf{Q}^3 + \frac{9}{16}c\mathrm{tr}\mathbf{Q}^4, \tag{4}$$

when tr means the trace of the tensor $\mathbf{Q}$, but, we can express that in terms of the scalar order parameter in the following way

$$\mathrm{tr}\left(\mathbf{Q}^2\right) = \frac{S^2}{2}, \tag{5}$$

$$\mathrm{tr}\left(\mathbf{Q}^4\right) = \frac{1}{8}S^4, \tag{6}$$

and the odd terms vanish. Therefore, we have that we can write the equation (4)as

$$F = \frac{1}{2}\alpha \left(\nabla \mathbf{Q}\right)^2 + \frac{3}{8}a_0 \Delta T S^2 + \frac{9}{128}cS^4,$$

Question, what kind of combination of the derivatives of the free energy should we consider?