Cosmology, Problem Set 5: Neutron Abundance and Hydrogen Recombination

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Problem 1a: The Boltzmann equation reads

$$\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = -\Gamma_1\left(n_1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}\frac{n_3 n_4}{n_2}\right),$$

and by making $n_1 = n_n$, $n_3 = n_p$, with n_2, n_4 to be leptons n_l, we have

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n\left(n_n - \left(\frac{n_n n_l}{n_p n_l}\right)_{eq}\frac{n_p n_l}{n_l}\right),$$

which will lead us to

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n\left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

just as we wanted.

Problem 1b: The equilibrium distribution of number density of species i is given by

$$(n_i)_{eq} = g_i\left(\frac{m_i T}{2\pi}\right)^{3/2}\exp\left[-\frac{m_i - \mu_i}{T}\right],$$

then by neglecting the chemical potential, we will have

$$(n_n)_{eq} = g_n\left(\frac{m_n T}{2\pi}\right)^{3/2}\exp\left[-\frac{m_n}{T}\right],$$

and also

$$(n_p)_{eq} = g_p\left(\frac{m_p T}{2\pi}\right)^{3/2}\exp\left[-\frac{m_p}{T}\right],$$

from this we can make the ratio

$$\left(\frac{n_i}{n_p}\right)_{eq} = \frac{g_n\left(\frac{m_n T}{2\pi}\right)^{3/2}\exp\left[-\frac{m_n}{T}\right]}{g_p\left(\frac{m_p T}{2\pi}\right)^{3/2}\exp\left[-\frac{m_p}{T}\right]}$$

but $g_n = g_p$, thus

$$\left(\frac{n_i}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{3/2}\exp\left[\frac{-m_n + m_p}{T}\right],$$

and if we define $Q = m_n - m_p$, we have $-Q = -m_n + m_p$, thus

$$\left(\frac{n_i}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[\frac{-Q}{T}\right],$$

just as we wanted.

Problem 1c: We know that

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

on the other hand, if we define

$$X_n = \frac{n_n}{n_n + n_p},$$

we can write

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3 \frac{n_n + n_p}{n_n + n_p}) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

which implies that

$$\frac{1}{a^3}\frac{d}{dt}(X_n a^3 (n_n + n_p)) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

and if we take the time derivative we have

$$\frac{1}{a^3}\left[a^3(n_n + n_p)\frac{d}{dt}(X_n) + X_n \frac{d}{dt}(a^3(n_n + n_p))\right] = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right).$$

On the other hand, if we assume that the baryon number is conserved, we have the following condition

$$\frac{d}{dt}(a^3(n_n + n_p)) = 0,$$

which implies that

$$\frac{1}{a^3}\left[a^3(n_n + n_p)\frac{d}{dt}(X_n)\right] = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

thus

$$\frac{d}{dt}X_n = -\Gamma_n \left(\frac{n_n}{(n_n + n_p)} - \left(\frac{n_n}{n_p}\right)_{eq} \frac{n_p}{(n_n + n_p)}\right),$$

on the other hand we know that

$$\left(\frac{n_n}{n_p}\right)_{eq} \approx \exp(-Q/T),$$

thus, we have

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - \exp(-Q/T)\frac{n_p}{(n_n + n_p)}\right).$$

On the other hand, we have the following identity

$$\frac{n_p}{n_n + n_p} = 1 - \frac{n_n}{n_n + n_p} = 1 - X_n,$$

which leads us to

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - \exp(-Q/T)(1 - X_n) \right),$$

therefore, we finally have

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - (1 - X_n) \exp(-Q/T) \right),$$

as desired.

Problem 1d: For this part we have the following derivative

$$\frac{dX_n}{dt} = \frac{dx}{dt}\frac{dX_n}{dx},$$

but $x = x(T)$ and $T = T(a)$, thus by the chain rule we have

$$\frac{dX_n}{dt} = \frac{dx}{dT}\frac{dT}{da}\frac{da}{dt}\frac{dX_n}{dx},$$

and more explicitly $x = Q/T$, whereas by assuming $T \propto a^{-1}$, we have

$$\frac{dx}{dT} = -\frac{Q}{T^2}, \frac{dT}{da} = -\frac{1}{a^2},$$

thus we have

$$\frac{dX_n}{dt} = \left(-\frac{Q}{T^2}\right)\left(-\frac{1}{a^2}\right)\frac{da}{dt}\frac{dX_n}{dx},$$

$$\implies \frac{dX_n}{dt} = \left(\frac{Q}{T}\right)\left(\frac{1}{T}\right)\left(\frac{1}{a}\right)\left(\frac{1}{a}\frac{da}{dt}\right)\frac{dX_n}{dx},$$

and again, using the fact that $T \propto a^{-1}$ we have $T/a = 1$, thus

$$\implies \frac{dX_n}{dt} = xH\frac{dX_n}{dx},$$

just as we wanted.

Problem 1e: For this problem, I append the solution of the ode as a Mathematica notebook. On the other hand, the limi of neutron fraction for longer times is given by

$$X_e \to 0.149533$$

Problem 2a: Starting with

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left[-\frac{(m_i - \mu_i)}{T}\right],$$

we have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{\left(\frac{m_H T}{2\pi}\right)^{3/2}}{\left(\frac{m_e T}{2\pi}\right)^{3/2}\left(\frac{m_p T}{2\pi}\right)^{3/2}}\right) \exp\left[-\frac{(m_H - \mu_H)}{T} + \frac{(m_e - \mu_e)}{T} + \frac{(m_p - \mu_p)}{T}\right],$$

and from this we have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{m_e + m_p - m_H}{T} + \frac{\mu_H - \mu_e - \mu_p}{T}\right],$$

and by using the condition of equilibrium given, we have

$$\mu_H - \mu_e - \mu_p = 0,$$

thus

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{m_e + m_p - m_H}{T}\right],$$

and by making

$$E_I = m_e + m_p - m_H,$$

we finally have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

just as we wanted.

Problem 2b: The mass of the Hydrogen comes mostly from the proton, thus to a good approximation we have

$$\frac{m_H}{m_p} \approx 1,$$

thus this implies that

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

on the other hand, the degrees of freedom are $g_p = 2$, $g_e = 2$, whereas for the hydrogen, the spins of the electron and proton can be either aligned or anti aligned, which will give one singlet state and one triplet state, therefore $g_H = 1 + 3 = 4$, and with this

$$\frac{n_H}{n_e n_p} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

moving on, the universe isn't electrically charged, so $n_e = e_p$, thus

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

as desired.
Problem 2c: If we define

$$X_e = \frac{n_e}{n_e + n_H},$$

then we have

$$\frac{1 - X_e}{X_e^2} = \frac{1 - \frac{n_e}{n_e + n_H}}{\frac{n_e^2}{(n_e + n_H)^2}} = \frac{\frac{n_e + n_H - n_e}{n_e + n_H}}{\frac{n_e^2}{(n_e + n_H)^2}},$$

thus, we can write

$$\frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2}(n_e + n_H) \implies \frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2} n_b,$$

where $n_b = n_e + n_H$.
Problem 2d: In the previous homework we found the following result for bosons

$$n = \frac{\zeta(3)}{\pi^2} g T^3,$$

and in particular, for photons we have $g = 2$, thus

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3,$$

and if we use $n_b = \eta n_\gamma$ we have

$$\frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2} \frac{2\eta\zeta(3)}{\pi^2} T^3,$$

but we also know that

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

thus

$$\frac{1 - X_e}{X_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right] \times \frac{2\eta\zeta(3)}{\pi^2} T^3,$$

but $T^{-3/2} T^3 = T^{3/2}$, thus

$$\frac{1 - X_e}{X_e^2} = \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right] \times \frac{2\eta\zeta(3)}{\pi^2},$$

and we arrive at the desired result.
Problem 2e: I used Mathematica and I append the notebook. But here I'll do some algebra to find a nicer expression. The solution is given by

$$X_e(T) = -\frac{e^{-\frac{E_I}{T}}\left(1 - \sqrt{8\sqrt{2}\pi^{3/2}\eta X_0 e^{E_I/T}\left(\frac{T}{m_e}\right)^{3/2} + 1}\right)}{4\sqrt{2}\pi^{3/2}\eta X_0\left(\frac{T}{m_e}\right)^{3/2}},$$

where I defined

$$X_0 = \frac{2\zeta(3)}{\pi^2},$$

but $8 \times 2^{1/2} = 4 \times 2^{3/2}$, and $4 \times 2^{1/2} = 2 \times 2^{3/2}$, and with that we have

$$X_e(T) = -\frac{\left(1 - \sqrt{4 \times 2^{3/2} \pi^{3/2} \eta X_0 \exp[E_I/T] \left(\frac{T}{m_e}\right)^{3/2} + 1}\right)}{2 \times 2^{3/2} \pi^{3/2} \eta X_0 \left(\frac{T}{m_e}\right)^{3/2} \exp[E_I/T]},$$

which can be simplified to

$$X_e(T) = \frac{-1 + \sqrt{1 + 4X_0 \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right]}}{2X_0 \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right]},$$

and if we define

$$f = X_0 \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

the final solution take the form

$$X_e(T) = \frac{-1 + \sqrt{1 + 4f}}{2f},$$

which is much ”prettier”.
Problem 2f: I append a Mathematica Notebook with the plot.
Problem 2g: The temperature of recombination can be calculated by

$$X_e(T) = 0.5,$$

which I solved numerically, with the result given by

$$T_{rec} = 0.323887 \text{eV}.$$

Problem 2h: For a_{rec} we have

$$T = \frac{T_0}{a} \implies a_{rec} = \frac{T_0}{T_{rec}},$$

and if we use $T_0 = 2.3 \times 10^{-4}$eV we have

$$a_{rec} = 0.000710124.$$

Problem 2i: Finally, we need to compute t_{rec}, where t is given by

$$t(a) = \frac{1}{H_0} \int_0^a \frac{dx}{x\sqrt{\Omega_{M,0}/x^3 + \Omega_{R,0}/x^4}},$$

and if we use $a = a_{rec}$ together with $\Omega_{M,0} = 0.3$, $\Omega_{R,0} = 8.6 \times 10^-5$ and $H_0 = 0.7 \times 10^{-10}$/years, we have

$$t_{rec} = 243884 \text{ years}.$$

Cosmology: Problem Set 5

In[1]:=
```
<< NumericalCalculus`;(*Load Package for Finding Limits Numerically*)
```

In[2]:=
```
vMap = ResourceFunction["ViridisColor"];
```

Problem 1

In[3]:=
```
params = {τ → 1.4*^18, Q → 1.3*^6, G → 6.7087*^-57, g → 10};
```

In[4]:= $H = \mathrm{Sqrt}\left[\frac{8\,\pi\,G}{3} \times \frac{\pi\,{}^\wedge 2}{30}\ g\right] \frac{Q^2}{x^2};$

In[5]:= $\Gamma = \frac{3060}{\tau}\ \frac{1}{x^5} \left(1 + \frac{x}{2} + \frac{x^2}{12}\right);$

In[6]:= $\text{ode} = \left(D[f[x], x] == -\frac{\Gamma}{H\,x}\ (f[x] - (1 - f[x])\ \mathrm{Exp}[-x])\right)\ /.\ \text{params}$

Out[6]= $f'[x] == -\frac{3.00773 \left(1 + \frac{x}{2} + \frac{x^2}{12}\right) \left(-e^{-x}\ (1 - f[x]) + f[x]\right)}{x^4}$

In[7]:=
```
x0 = 0.01;
xF = 100.0;
sol = NDSolveValue[{ode, f[x0] == 0.5}, f, {x, x0, xF}]
```

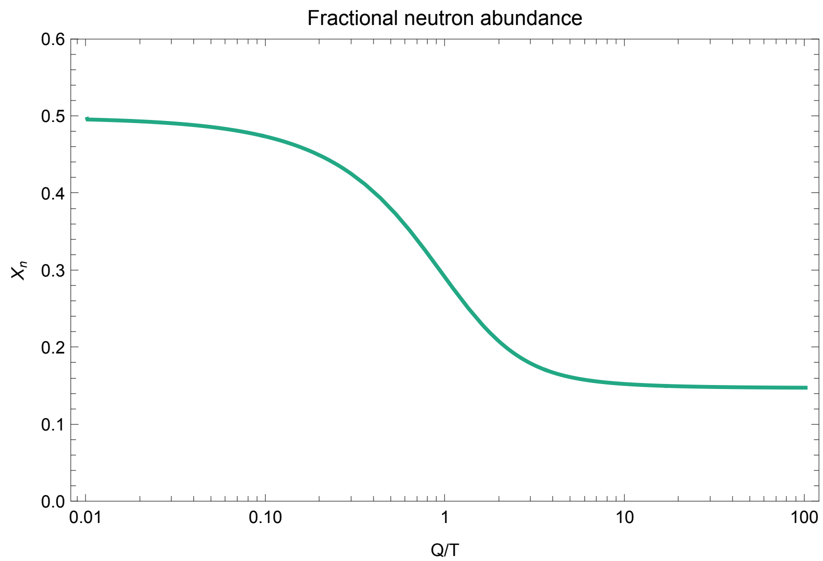
Out[9]= InterpolatingFunction[Domain: {{0.01, 100.}} Output: scalar]

In[10]:=
```
ϵ = 1*^-6;
solLim = NLimit[sol[x], x → (xF - ϵ), Direction → xF]
```

Out[11]=
```
0.149533
```

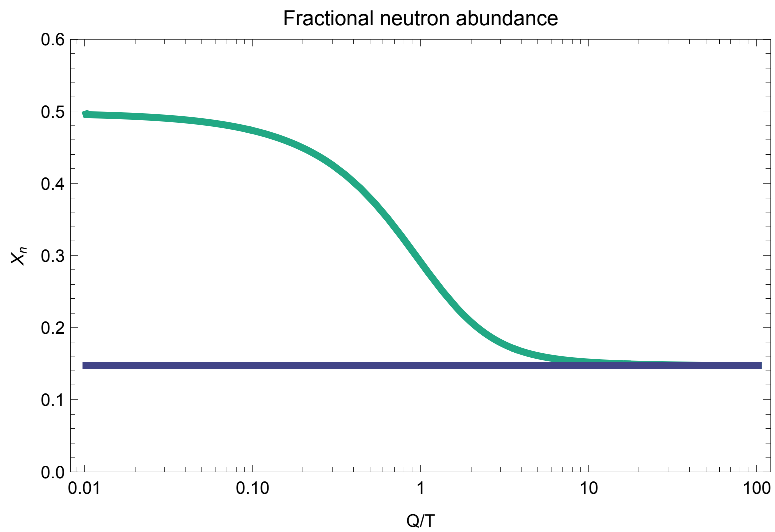
In[12]:=
```
Plot[sol[x], {x, x0, xF},
 ScalingFunctions → {"Log"},
 PlotLabel → "Fractional neutron abundance",
 PlotStyle → vMap[0.6],
 Frame → True,
 FrameLabel → {"Q/T", "Xn"},
 PlotRange → {{0, 0.6}}]
```

Out[12]=



In[13]:=
```
Plot[{sol[x], solLim}, {x, x0, xF},
 ScalingFunctions → {"Log"},
 PlotLabel → "Fractional neutron abundance",
 PlotStyle → {{Thickness[0.01], vMap[0.6]}, {Thickness[0.01], vMap[0.2]}},
 Frame → True,
 FrameLabel → {"Q/T", "Xn"},
 PlotRange → {{0, 0.6}}]
```

Out[13]=



Problem 2:

In[14]:=

$$\text{solX} = \text{DSolve}\left[\frac{1-\chi[T]}{\chi[T]^2} == \chi0\,\eta\left(\frac{2\pi T}{m_e}\right)^{3/2}\text{Exp}[EI/T],\ \chi[T],\ T,\ \text{Assumptions} \to \{m_e \to \text{PositiveReals}\}\right]$$

Out[14]=

$$\left\{\left\{\chi[T] \to -\frac{e^{-\frac{EI}{T}}\left(1-\sqrt{1+8\sqrt{2}\,e^{EI/T}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}\right)}{4\sqrt{2}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}\right\},\right.$$
$$\left.\left\{\chi[T] \to -\frac{e^{-\frac{EI}{T}}\left(1+\sqrt{1+8\sqrt{2}\,e^{EI/T}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}\right)}{4\sqrt{2}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}\right\}\right\}$$

In[15]:=
```
χSol1 = χ[T] /. solX[[1]]
```

Out[15]=

$$-\frac{e^{-\frac{EI}{T}}\left(1-\sqrt{1+8\sqrt{2}\,e^{EI/T}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}\right)}{4\sqrt{2}\,\pi^{3/2}\,\eta\,\chi0\left(\frac{T}{m_e}\right)^{3/2}}$$

In[16]:=
```
χSol1 // TeXForm (*Get String for LaTeX document*)
```

Out[16]//TeXForm=
```
-\frac{e^{-\frac{\text{EI}}{T}} \left(1-\sqrt{8 \sqrt{2} \pi
   ^{3/2} \eta  \text{$\chi $0} e^{\text{EI}/T}
   \left(\frac{T}{m_e}\right){}^{3/2}+1}\right)}{4 \sqrt{2} \pi
   ^{3/2} \eta  \text{$\chi $0}
   \left(\frac{T}{m_e}\right){}^{3/2}}
```

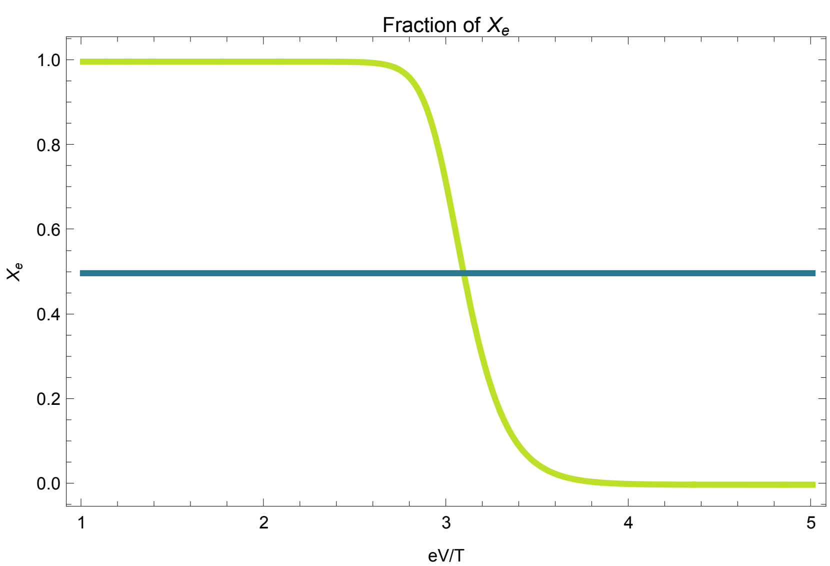
```
paramsP2 = {EI → 13.6, me → 0.511*^6, η → 6*^-10}; (*New list of parameters*)
```

In[23]:= $X = \dfrac{-1 + \text{Sqrt}[1 + 4\,\text{f2}]}{2\,\text{f2}};$

In[43]:= $\text{f2} = \dfrac{2\,\text{Zeta}[3]}{\pi\char`^2}\,\eta\,\left(\dfrac{2\,\pi}{\text{me x}}\right)^{3/2}\,\text{Exp}[\text{EI x}];$

In[62]:=
```
Plot[{(X /. paramsP2), 0.5}, {x, 1, 5},
 PlotLabel → "Fraction of Xe",
 PlotStyle → {{Thickness[0.008], vMap[0.9]}, {Thickness[0.008], vMap[0.4]}},
 Frame → True,
 FrameLabel → {"eV/T", "Xe"}]
```

Out[62]=



In[21]:=
```
XEvaluated = X /. paramsP2;
```

In[36]:=
```
xRec = NSolve[XEvaluated == 0.5, x, Reals]
```

Out[36]=

$\{\{x \to 2.1492 \times 10^{-12}\},\ \{x \to 3.08749\}\}$

In[42]:= $\text{TRec} = \left(\dfrac{1}{x}\right)$ /. xRec[[2]][[1]]

Out[42]=

0.323887

In[51]:=
```
aRec = (T0 / TRec) /. (T0 → 2.3*^-4)
```
Out[51]=
```
0.000710124
```

```
params3 = {H0 → 0.7*^-10, ΩM0 → 0.3, ΩR0 → 8.6*^-5 };(*Third List of Parameters*)
```

In[60]:=

$$t = \left(\frac{1}{H0} \text{ /. params3}\right) \text{Integrate}\left[\left(\frac{1}{x\,\text{Sqrt}\left[\frac{\Omega M0}{x^3} + \frac{\Omega R0}{x^4}\right]}\right) \text{ /. params3, \{x, 0, aRec\}}\right]$$

Out[60]=
```
243 884.
```