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Let X be a nonempty set and $\mathcal{T}$ be the set of all subset E of T such that X/E is countable or $E = \emptyset$.

(1) Prove that $(X, \mathcal{T})$ is a topological space. ($\mathcal{T}$ is called the countable complement topology on X.)

1. Let's prove that $\emptyset, X \in \mathcal{T}$. Indeed, $\emptyset \in \mathcal{T}$ by definition. On the other hand, $X \in \mathcal{T}$ because $X/X = \emptyset$ which is a member of $\mathcal{T}$.

2. Let's prove that the finite intersection of open sets is an open set. Indeed, let $U_1, U_2 \in \mathcal{T}$, then, by the definition of $\mathcal{T}$, we have that U_1 is empty or X/U_1 countable, and the same as well for X/U_2. So let's assume that both U_1 and U_2 are non-empty, thus, it follows that

$$X/U_1 \quad \& \quad X/U_2,$$

are both countable, now let's take $U_1 \cap U_2$, we want to prove that $U_1 \cap U_2 \in \mathcal{T}$. Indeed, if $U_1 \cap U_2 = \emptyset$, then $U_1 \cap U_2 \in \mathcal{T}$, now, let's assume that the intersection is non-empty, thus using the Morgan's Laws, we have that

$$X/\left(U_1 \cap U_2\right) = \left(X/U_1\right) \cup \left(X/U_2\right),$$

but we know that the union of countable sets is countable, and by assumption X/U_1 and X/U_2 are countable, thus we have that

$$U_1 \cap U_2 \in \mathcal{T},$$

just as we wanted.

3. Now, we want to prove that the arbitrary union of open sets is open. Indeed let U_α be an indexed family of open sets, with $\alpha \in \lambda$. Then, let's consider $\cup_{\alpha \in \lambda} U_\alpha$, and again, we have two options, for each $\alpha \in \lambda$ either U_α is the empty set or the complement is countable, so let's assume that the complement is countable. Again, using the Morgan's Laws, we have that

$$X/\left(\cup_{\alpha \in \lambda} U_\alpha\right) = \cap_{\alpha \in \lambda}\left(X/U_\alpha\right),$$

but each one of X/U_α are countable, thus the intersection is at most countable, with means that

$$\cup_{\alpha \in \lambda} U_\alpha \in \mathcal{T},$$

just as we wanted.

(2) For $X = \mathbb{R}$ given an example of a set that is open in both the standard and countable complement topologies.

The standard topology of $\mathbb{R}$ is given by the open intervals, thus if we consider

$$\mathcal{U} = (-\infty, 0) \cup (0, \infty),$$

then, using the definition of countable topology we have that

$$\mathbb{R}/\mathcal{U} = \{0\},$$

which is finite, and therefore, countable.

(3) For $X = \mathbb{R}$ given an example of a set that is open in the standard topology but not open in the countable complement topology.

Again, the standard topology of $\mathbb{R}$ is given by the open intervals, thus if we consider the open interval $(0, 1)$ and taking complement, we have

$$X/\,(0,1) = (-\infty, 0] \cup [1, \infty),$$

which is clearly non countable, thus we found an open set which is open in the standard topology but not in the countable complement topology.

(4) For $X = \mathbb{R}$ given an example of a set that is closed in both the standard and countable complement topologies.

The complement of a closed set is an open set, thus 0 is an closed set in the standard topology because the complement is the union of two open sets, this is

$$X/\,\{0\} = (-\infty, 0) \cup (0, \infty),$$

on the other hand in the countable complement topology the complements of open sets are the empty set of countable, thus $\{0\}$ is closed too in the countable complement topology.