Numerical Techniques in Cosmology

Emmanuel Flores

April 28, 2025

Cosmology Final Presentation,
Tufts University

Outline

Motivation

The more tricks you have up your sleeve, the easier it is to handle stuff.

The more tricks you have up your sleeve, the easier it is to handle stuff.

Why should we care about numerical techniques?

The more tricks you have up your sleeve, the easier it is to handle stuff.

Why should we care about numerical techniques?

- Sometimes it's not possible to find analytical solutions, but an approximate solution is better than no solution at all

The more tricks you have up your sleeve, the easier it is to handle stuff.

Why should we care about numerical techniques?

- Sometimes it's not possible to find analytical solutions, but an approximate solution is better than no solution at all
- Highly complex systems often requiere some numerical treatment(nonlinearities, lots DOF, etc.)

The more tricks you have up your sleeve, the easier it is to handle stuff.

Why should we care about numerical techniques?

- Sometimes it's not possible to find analytical solutions, but an approximate solution is better than no solution at all
- Highly complex systems often requiere some numerical treatment(nonlinearities, lots DOF, etc.)

The formal way to proceed:

The more tricks you have up your sleeve, the easier it is to handle stuff.

Why should we care about numerical techniques?

- Sometimes it's not possible to find analytical solutions, but an approximate solution is better than no solution at all
- Highly complex systems often requiere some numerical treatment(nonlinearities, lots DOF, etc.)

The formal way to proceed:

- Is to find criteria under which the solution exists (well posed problem), and prove that under some discretization (on desired function spaces) the approximate solution is bounded from below and it converges

Some Numerical Techniques in Cosmology

We can separate observables into: deterministic and stochastic

In general we can separate

$$g_{\mu\nu} = g^0_{\mu\nu}$$

arXiv:1908.00116v1(Selected Topics in Numerical Methods for Cosmology)

We can separate observables into: deterministic and stochastic

In general we can separate

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

- General Relativity $\rightarrow$ homogeneous and isotropic background

arXiv:1908.00116v1(Selected Topics in Numerical Methods for Cosmology)

We can separate observables into: deterministic and stochastic

In general we can separate

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

- General Relativity $\rightarrow$ homogeneous and isotropic background
- Perturbation theory $\rightarrow$ inhomogeneous description

arXiv:1908.00116v1(Selected Topics in Numerical Methods for Cosmology)

We can separate observables into: deterministic and stochastic

In general we can separate

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}$$

- General Relativity $\rightarrow$ homogeneous and isotropic background
- Perturbation theory $\rightarrow$ inhomogeneous description

Initial conditions in the universe are given in terms of initial perturbations

arXiv:1908.00116v1(Selected Topics in Numerical Methods for Cosmology)

We can also separate numerical cosmology into two groups

1. Deterministic: background observables $\rightarrow$ ODE theory
    - H_0 determination for example

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

We can also separate numerical cosmology into two groups

1. Deterministic: background observables $\rightarrow$ ODE theory
   - H_0 determination for example
2. Stochastic: inhomogeneous part $\rightarrow$ linear perturbation theory and beyond
   - CMB radiation
   - Large Scale Structure Observables: galaxy spatial correlations, galaxy cluster count, gravitational lensing, etc.
   - We want to describe the evolution of the universe from initial primordial fluctuations to the structure formation $\rightarrow$ observables we can measure

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

We take the FLRW metric with small perturbations and do GR

Start with FLRW metricand assume that all deviations are described by small perturbations. Then

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

We take the FLRW metric with small perturbations and do GR

Start with FLRW metricand assume that all deviations are described by small perturbations. Then

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

We take the FLRW metric with small perturbations and do GR

Start with FLRW metricand assume that all deviations are described by small perturbations. Then

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

which lead us to

$$h_{00} = 2\phi, h_{0i} = -aD_iB, h_{ij} = 2a^2(\psi\gamma_{ij} - D_iD_jE)$$

where D_i is the covariant derivative, (ψ, ϕ, E, B) are scalar field and γ is the spatial projection of the FLRW metric.

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

We take the FLRW metric with small perturbations and do GR

Start with FLRW metricand assume that all deviations are described by small perturbations. Then

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu},$$

which lead us to

$$h_{00} = 2\phi, h_{0i} = -aD_iB, h_{ij} = 2a^2(\psi\gamma_{ij} - D_iD_jE)$$

where D_i is the covariant derivative, (ψ, ϕ, E, B) are scalar field and γ is the spatial projection of the FLRW metric. And from here, the idea is to obtain Einstein's equations and solve them...

arXiv:1908.00116v1 (Selected Topics in Numerical Methods for Cosmology)

Background Dynamics and Machine Learning

For some function spaces, a neural network is an universal approximator

One of very interesting results from ML is the following:

For some function spaces, a neural network is an universal approximator

One of very interesting results from ML is the following:

Universal Approximation Theorem

Given a family of neural networks $\forall f \in \mathcal{F}$ where $\mathcal{F}$ is some function space, there exist a family of functions $\{\phi_n\}$, such that $\phi_n \to f$. We can also say that $\{\phi_n\}$ is dense in $\mathcal{F}$.

For some function spaces, a neural network is an universal approximator

One of very interesting results from ML is the following:

Universal Approximation Theorem

Given a family of neural networks $\forall f \in \mathcal{F}$ where $\mathcal{F}$ is some function space, there exist a family of functions $\{\phi_n\}$, such that $\phi_n \to f$. We can also say that $\{\phi_n\}$ is dense in $\mathcal{F}$.

Thus it makes sense to try to use ML with ODE's: Physics Informed Neural Networks (PINN's)

Cosmology-Informed Neural Networks

Starting with the FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + +sin^2\theta d\phi^2) \right],$$

and assuming the universe is a perfect fluid, we have

$$\dot{\rho} + 3H(\rho + p) = 0$$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

ΛCMD: the background cosmological evolution considering an T^{μ}_{ν} with only nonrelativistic matter

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

ΛCMD: the background cosmological evolution considering an T^{μ}_{ν} with only nonrelativistic matter

$$\frac{dx}{dz} = \frac{3x}{1+z}, \quad x(z)|_{z=0} = \frac{\kappa \rho_{m,0}}{3H_0^2} = \Omega_{m,0}$$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

ΛCMD: the background cosmological evolution considering an T^{μ}_{ν} with only nonrelativistic matter

$$\frac{dx}{dz} = \frac{3x}{1+z}, \quad x(z)|_{z=0} = \frac{\kappa \rho_{m,0}}{3H_0^2} = \Omega_{m,0}$$

Parametric Dark Matter: incorporation of new component of T^{μ}_{ν}, whose equation of state is that of a fluid and a function of redshift (DM)

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

ΛCMD: the background cosmological evolution considering an T^{μ}_{ν} with only nonrelativistic matter

$$\frac{dx}{dz} = \frac{3x}{1+z}, \quad x(z)|_{z=0} = \frac{\kappa\rho_{m,0}}{3H_0^2} = \Omega_{m,0}$$

Parametric Dark Matter: incorporation of new component of T^{μ}_{ν}, whose equation of state is that of a fluid and a function of redshift (DM)

$$\frac{dx}{dz} = \frac{3x}{1+z}\left(1 + \omega_0 + \frac{\omega_1 z}{1+z}\right), \quad x(z)|_{z=0} = \frac{\kappa\rho_{DE,0}}{3H_0^2} = 1 - \Omega_{m,0}$$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

Quintessence: alternative proposal for the expansion of the universe via a scalar field ϕ minimally coupled to gravity via $V(\phi)$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

Quintessence: alternative proposal for the expansion of the universe via a scalar field ϕ minimally coupled to gravity via $V(\phi)$

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x\left(1 + x^2 - y^2\right),$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}xy\lambda + \frac{3}{2}y\left(1 + x^2 - y^2\right)$$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

$f(R)$ gravity: GR modifications

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Models: ΛCDM, parametric dark energy, quintessence and $f(R)$ gravity

$f(R)$ gravity: GR modifications

$$\frac{dx}{dz} = \frac{1}{1+z}\left(-\Omega + 2v + x + 4y + xv + x^2\right),$$

$$\frac{dy}{dz} = -\frac{1}{1+z}\left(vx\Gamma - xy + 4y - 2vy\right),$$

$$\frac{dv}{dz} = -\frac{v}{1+z}\left(x\Gamma + 4 - 2v\right), \quad \frac{d\Omega}{dz} = \frac{\Omega}{1+z}\left(-1 + 2v + x\right),$$

$$\frac{dr}{dz} = -\frac{r\Gamma x}{1+z}$$

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Core Methodology and Trainig Details

- Unsupervised learning: the training process does not rely on pre-computed analytical or numerical solutions of the differential equations

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Core Methodology and Trainig Details

- Unsupervised learning: the training process does not rely on pre-computed analytical or numerical solutions of the differential equations
- Optimization problem: the network's internal parameters (weights and biases) are adjusted during training to minimize a loss function

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Core Methodology and Trainig Details

- Unsupervised learning: the training process does not rely on pre-computed analytical or numerical solutions of the differential equations
- Optimization problem: the network's internal parameters (weights and biases) are adjusted during training to minimize a loss function
- 1 NN is assigned per dependent variable in the system, and each network have two hidden layers of 32 units each, with *tanh* as activation function

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Core Methodology and Trainig Details

- Unsupervised learning: the training process does not rely on pre-computed analytical or numerical solutions of the differential equations
- Optimization problem: the network's internal parameters (weights and biases) are adjusted during training to minimize a loss function
- 1 NN is assigned per dependent variable in the system, and each network have two hidden layers of 32 units each, with *tanh* as activation function
- Use of ADAM optimizer for the gradient descent

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Core Methodology and Trainig Details

- Unsupervised learning: the training process does not rely on pre-computed analytical or numerical solutions of the differential equations
- Optimization problem: the network's internal parameters (weights and biases) are adjusted during training to minimize a loss function
- 1 NN is assigned per dependent variable in the system, and each network have two hidden layers of 32 units each, with *tanh* as activation function
- Use of ADAM optimizer for the gradient descent
- Minimize loss on batches of points until convergence

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)
- Type Ia Supernovae (SNIa)

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)
- Type Ia Supernovae (SNIa)
- Baryon Acoustic Oscillations (BAO)

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)
- Type Ia Supernovae (SNIa)
- Baryon Acoustic Oscillations (BAO)

Statistical Analysis (MCMC)

- Trained models give $H(z)$ as output

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)
- Type Ia Supernovae (SNIa)
- Baryon Acoustic Oscillations (BAO)

Statistical Analysis (MCMC)

- Trained models give $H(z)$ as output
- Standard likelihood constructed on the dataset

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Validation Methodology

Having trained their models, they validate with

- Cosmic Chronometers (CC)
- Type Ia Supernovae (SNIa)
- Baryon Acoustic Oscillations (BAO)

Statistical Analysis (MCMC)

- Trained models give $H(z)$ as output
- Standard likelihood constructed on the dataset
- MCMC to explore the parameter space of each cosmological model

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Main Punch Lines

Key takeways

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Main Punch Lines

Key takeaways

1. They can solve the equations

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Main Punch Lines

Key takeaways

1. They can solve the equations
2. Successful implementation of bundle solution: NN can output solutions across a continuous landscape of parameters.

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Main Punch Lines

Key takeways

1. They can solve the equations
2. Successful implementation of bundle solution: NN can output solutions across a continuous landscape of parameters.
3. Parameter constraints were found to be consistent with those obtained in previous studies that used numerical solvers

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Main Punch Lines

Key takeways

1. They can solve the equations
2. Successful implementation of bundle solution: NN can output solutions across a continuous landscape of parameters.
3. Parameter constraints were found to be consistent with those obtained in previous studies that used numerical solvers
4. In some cases can be more efficient than traditional numerical solvers after the initial training phase, especially with the $f(R)$ model

Phys. Rev. D 107, 063523 (Cosmology-informed neural networks to solve the background dynamics of the Universe)

Thanks!