Chapter 7: From Particles to Fields

Key Concepts

- There are no Lorentz invariant and causal theories between **classical particles** in three spatial dimensions.
- Therefore, we turn to quantum mechanics, this is: **quantum fields** and **force carriers**.
- And the **starting point** is this: force carriers must be **bosons** (for rotation invariance).
- As a first go, we are going to study **spinless particles** (scalars).

Creation and Annihilation Operators

Since the number of particles is not always the same (emission or absorption processes), we use **creation** ($\hat{a}^\dagger$) and **annihilation** ($\hat{a}$) operators with their usual commutation rules.

The Hamiltonian

For a collection of free particles, the total energy is:

$$E = \sum_i E_i, \quad \text{where} \quad E_i = \sqrt{p_i^2 c^2 + m^2 c^4} \tag{1}$$

And iff we set $\hbar = c = 1$, we have $E = \sqrt{p^2 + m^2}$. Alternatively, the total energy can be written in terms of the number of particles as follows:

$$E = \sum_p E_p N_p \tag{2}$$

where N_p is the **number operator**. From this, we can construct a Hamiltonian operator as:

$$H = \sum_{\vec{p}} E_{\vec{p}} \hat{N}_{\vec{p}} = \sum_{\vec{p}} E_{\vec{p}} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \tag{3}$$

In the continuous limit ($\sum_{\vec{p}} \to V \int \frac{d^3p}{(2\pi)^3}$), the Hamiltonian becomes:

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \tag{4}$$

Similarly, the total momentum operator becomes:

$$\vec{P}_{\text{tot}} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \hat{N}_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \hat{a}^\dagger_{\vec{p}} \hat{a}_{\vec{p}} \tag{5}$$

Physics Turn: Position Space Fields

We need a **causal** theory, which implies the theory must be **LOCAL** in position space. We use a Fourier transform to go to position space.

Scalar Field Definition

We define the **scalar field** operator:

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(\hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \tag{6}$$

- This is known as a **Lorentz scalar field.**
- The factor $\frac{1}{\sqrt{2E_{\vec{p}}}}$ ensures $\hat{\phi}$ transforms as a Lorentz scalar.
- The creation operator ($\hat{a}^\dagger_{\vec{p}}$) is included to make $\hat{\phi}$ **real** (Hermitian).

Conjugate Gradient Field (Conjugate Momentum)

We define the conjugate gradient field (or conjugate momentum operator):

$$\hat{\pi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(\hat{a}_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - \hat{a}^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \tag{7}$$

It's important to notice that $\hat{\pi}$ is **NOT Lorentz invariant.** The canonical commutation relation is:

$$[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{8}$$

Hamiltonian Density

Using the inverse Fourier theorem, the Hamiltonian can be written as an integral over the **Hamiltonian density** $\mathcal{H}(\vec{x})$:

$$\hat{H} = \int d^3x \, \hat{\mathcal{H}}(\vec{x}), \quad \text{where} \tag{9}$$

$$\hat{\mathcal{H}}(\vec{x}) = \frac{1}{2}\hat{\pi}(\vec{x})^2 + \frac{1}{2}\vec{\nabla}\hat{\phi}(\vec{x}) \cdot \vec{\nabla}\hat{\phi}(\vec{x}) + \frac{1}{2}m^2\hat{\phi}(\vec{x})^2 \tag{10}$$

The Lagrangian Formalism

The Lagrangian formulation is ideal because the **action** (S) must be a **Lorentz invariant**. We decompose the Lagrangian into an integral over a **density** $\mathcal{L}$:

$$L = \int d^3x \, \mathcal{L}, \quad \text{with } S = \int d^4x \, \mathcal{L} \tag{11}$$

The **Lagrangian density** arises from the Hamiltonian density as follows:

$$\mathcal{L} = \dot{\phi}\pi - \mathcal{H}, \quad \text{where } \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{12}$$

Using the previous Hamiltonian density, we get the **Lagrangian density** for the free scalar field:

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{13}$$

This is **manifestly Lorentz invariant**. In covariant notation, this is:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi \, \partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{14}$$

and even more, this is the **only Lorentz invariant quadratic Lagrangian density**.

Takeaways

- Field formalism is great to later include **interactions** and ensure they are **local**.
- Fields are useful mathematical tools that help us build **local theories** in position space.