QUANTUM THEORY 1
TUFTS UNIVERSITY
GRADUATE SCHOOL OF ARTS AND SCIENCES
FINAL EVALUATION

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(1) The fine structure constant (for QM and E&M) is written as $\alpha = 1/137$. It was asked in class "what are the units?" Answer this.

Solution. The fine structure constant is a dimensionless quantity, and it plays a fundamental role in the interpretation of electromagnetism in the quantum realm, because it quentifies the strenght of the electromagnetic interaction between elementary charged particles. It's also important to say that this is constant independent of the system of units, and, even more interesting, is defined in terms of fundamental constants as follows. Now, if we use CGS units, then we can express α as follows,

$$\alpha = \frac{e}{\hbar c},$$

where e is the charge of the electron, $\hbar$ is the Plank constant divided by two π and c is the speed of light

(2) Suppose the Stern-Gerlach experiment is repeated with a beam of spin 1 particles. What would the pattern on the detector screen look like? On what physical property of the particles does the scale of the pattern depend?

In the Stern-Gerlach experiment with spin-1 particles, the pattern on the detector screen would exhibit three discrete spots. This is because spin-1 particles have three possible projection values: $+1, 0$, and -1.

The Stern-Gerlach experiment is designed to measure the component of a particle's angular momentum along a particular axis. For spin-1 particles, the angular momentum projection can take one of the three values mentioned above. The scale of the pattern on the detector screen depends on the physical property of the particles related to their spin, specifically the quantum number associated with their spin angular momentum.

In summary, the pattern on the detector screen for spin-1 particles in the Stern-Gerlach experiment would consist of three discrete spots, corresponding to the three possible projection values of the particles' spin along the chosen axis. The scale of the pattern depends on the quantum nature of the particles' spin.

(3) Feynman path integral: After eqn. 2.6.42 the text says "The reader may work out a similar comparison for the simple harmonic oscillator." Do that.

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The Lagrangian for this system is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2,$$

then the classical action is given by

$$S(n, n-1) = \int_{t_{n-1}}^{t_n} dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right),$$

thus, if we assume that the time interval is so small, we can make a stright-line approximation to the path joining (x_{n-1}, t_{n-1}) and (x_n, t_n), and with that, we can approximate the classical action as follows

$$S(n, n-1) \approx \Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{x_n - x_{n-1}}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(\frac{x_n + x_{n-1}}{2}\right)^2 \right\},$$

but we can write

$$x_n + x_{n-1} = 2x_n - (x_n - x_{n-1}) = 2x_n - \Delta x,$$

thus the action becomes

$$\implies S(n, n-1) \approx \Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{\Delta x}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(\frac{2x_n - \Delta x}{2}\right)^2 \right\},$$

$$\implies S(n, n-1) \approx \Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{\Delta x}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(x_n - \frac{\Delta x}{2}\right)^2 \right\},$$

$$\implies S(n, n-1) \approx \Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{\Delta x}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(x_n^2 - x_n\frac{\Delta x}{2} + \frac{(\Delta x)^2}{4}\right) \right\},$$

and if we keep only the lowest terms, we have that

$$S(n, n-1) \approx \Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{\Delta x}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(x_n^2\right) \right\}.$$

Now, with this information, we know that $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ is given by

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left[\frac{i}{\hbar} S(n, n-1)\right],$$

therefore, we have

$$\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \exp\left[\frac{i}{\hbar}\Delta t \left\{ \left(\frac{m}{2}\right) \left[\frac{\Delta x}{\Delta t}\right]^2 - \frac{1}{2}m\omega^2 \left(x_n^2\right) \right\}\right].$$

(4) ψ_1 and ψ_2 are two linearly independent and normalized (but not orthogonal) eigenfunctions of H belonging to the same energy eigenvalue E . They are degenerate states.

(a) Show that $c_1\psi_1 + c_2\psi_2$ is also an eigenfunction of H belonging to the same eigenvalue as ψ_1 and ψ_2.
(b) Construct two linear combinations of ψ_1 and ψ_2 that are orthogonal to each other.
(c) Suppose that at $t = 0$ this system is purely ψ_1 . What is the probability that at a later time the system is still in ψ_1? How does that probability very with time?

Solution.

(a) Because ψ_1 and ψ_2 are eigenfunctions of H, we have that the following relationships holds

$$\psi_1 H = E\psi_1, \quad \psi_2 H = E\psi_2,$$

then if we define

$$\psi_3 = c_1\psi_1 + c_2\psi_2,$$

we're going to prove that this linear combination is also an eigenfunction. So, using the fact that ψ_1 and ψ_2 are eigenfunctions of H, then, the following is true

$$c_1\psi_1 H = c_1 E\psi_1, \quad c_2\psi_2 H = c_2 E\psi_2,$$
$$\implies \psi_3 H = (c_1\psi_1 + c_2\psi_2)\, H,$$
$$\psi_3 H = c_1\psi_1 H + c_2\psi_2 H = c_1 E\psi_1 + c_2 E\psi_2,$$
$$\psi_3 H = c_1 E\psi_1 + c_2 E\psi_2 = E\,(c_1\psi_1 + c_2\psi_2) = E\psi_3,$$
$$\therefore \psi_3 H = E\psi_3,$$

thus, indeed, ψ_3 is also an eigenfunction of the Hamiltonian, as stated before.

(b) For this part of the problem, we're going to use the Gram-Schmith procedure, which given a set of linearly independent vectors permit us to construc an orthogonoal (orthonormal) set of vectors. Let's begin by chosing $u_1 = \psi_1$, and because ψ_1 is normalized, then u_1 will be normalized as well, then, let's construct another vector as follows

$$u_2 = \psi_2 - \text{proj}_{\psi_1}(\psi_2)\,,$$

where $\text{proj}_{\psi_1}(\psi_2)$ is defined as follows

$$\text{proj}_{\psi_1}(\psi_2) = \frac{\langle\psi_2|\psi_1\rangle}{\langle\psi_1|\psi_1\rangle}\psi_1,$$

but, because ψ_1 is normalized, we have

$$\text{proj}_{\psi_1}(\psi_2) = \langle\psi_2|\psi_1\rangle\,\psi_1,$$

thus

$$u_2 = \psi_2 - \langle\psi_2|\psi_1\rangle\,\psi_1,$$

then, the two orthogonal linear combinations are given by

$$u_1 = \psi_1, \quad u_2 = \psi_2 - \langle\psi_2|\psi_1\rangle\,\psi_1.$$

(c) For the final part we have the following: They evolve exactly the same way and their evolution is determined by the Schroedinger equation. For two eigenstates with the same eigenvalue the evolution is therefore the same. Another argument is that the superposition of two degnerate eigenstate is also an eigenstate and is therefore stationary, from which we can conclude that their time evolution must be the same.

(5) Write the Hamiltonian for a 2 dimensional simple harmonic oscillator in terms of x_1 and x_2. Assume the 2 directions have the same classical frequency ω.

(a) What are the energy eigenvalues of the eigenstates?
(b) Is there a conserved angular momentum (like $r \times p$)? If there is, prove it and determine the eigenvalues?

Solution. For a SHO in one dimension, we know that the Hamiltonian is given by

$$H = \frac{p_x^2}{2m} + \frac{m\omega_x^2 x^2}{2},$$

where ω_x is the frequency of the oscillator. Thus for a 2D SHO, if we treat x and y as independent coordinates, we have that the Hamiltonian reads

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_x^2 x^2}{2} + \frac{m\omega_y^2 y^2}{2},$$

now, if we change the variables as follows $x \to x_1$ and $y \to x_2$, we have

$$H = \frac{p_{x_1}^2}{2m} + \frac{p_{x_2}^2}{2m} + \frac{m\omega_1^2 x_1^2}{2} + \frac{m\omega_2^2 x_2^2}{2},$$

now, if we assume that $\omega_1 = \omega_2 = \omega$, then the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_{x_1}^2 + p_{x_2}^2\right) + \frac{m\omega^2}{2}\left(x_1^2 + x_2^2\right).$$

Now, the system can be decomposed into two subsystems, that is

$$H = H_{x_1} + H_{x_2},$$

where

$$H_{x_1} = \frac{1}{2m}p_{x_1}^2 + \frac{m\omega^2}{2}x_1^2, \quad H_{x_2} = \frac{1}{2m}p_{x_2}^2 + \frac{m\omega^2}{2}x_2^2,$$

and then, we can form the state kets for this system as follows

$$|\psi\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle,$$

this is, as the outer product of the two SHO.

Now, we also know that for the SHO in one dimension, the energy eigenkets are given by

$$E = \left(n + \frac{1}{2}\right)\hbar\omega,$$

then, for the two subsystems that we have, the energies are given by

$$E_{x_2} = \left(n_{x_2} + \frac{1}{2}\right)\hbar\omega, \quad E_{x_2} = \left(n_{x_2} + \frac{1}{2}\right)\hbar\omega,$$

thus, for the whole systems, we have

$$E = E_{x_1} + E_{x_2} = \left(n_{x_2} + \frac{1}{2}\right)\hbar\omega + \left(n_{x_2} + \frac{1}{2}\right)\hbar\omega,$$

$$\implies E = (n_{x_2} + n_{x_2} + 1)\,\hbar\omega.$$

Now for the eigenfuncuions we can develop the same idea of the creation and anhilation operators, so, let's define

$$a_{x_1} = \frac{1}{\sqrt{2}}\left(\beta x_1 + i\frac{p_{x_1}}{\beta\hbar}\right), \quad a_{x_2} = \frac{1}{\sqrt{2}}\left(\beta x_2 + i\frac{p_{x_2}}{\beta\hbar}\right),$$

where $\beta = \sqrt{m\omega/\hbar}$. But because those operators act on different spaces, we have that

$$\left[a_{x_1}, a_{x_1}^\dagger\right] = \left[a_{x_2}, a_{x_2}^\dagger\right] = 1,$$

and even more, we can define the number operator for each one of the spaces, thus

$$N_{x_1} = a_{x_1}^\dagger a_{x_1}, \quad N_{x_2} = a_{x_2}^\dagger a_{x_2},$$

which enables us to rewrite the Hamiltonian as follows

$$H = (N_{x_1} + N_{x_2} + 1)\,\hbar\omega,$$

thus, any state $|\psi\rangle$ can be constructed by successive aplications of the creation operators for each space, this is

$$|\psi\rangle = \frac{1}{\sqrt{n_{x_1}!n_{x_2}!}} \left(a^\dagger_{x_1} a^\dagger_{x_2}\right) |\psi_{0,0}\rangle \,,$$

and the corresponding wave function is the product of the wave function of each subspace, but we know that the solution of this is in terms of the Hermite polynomials, that is

$$u_{n_{x_1}} = \frac{1}{\sqrt{\pi^{1/2} 2^{n_{x_1}} n_{x_1}!}} H_{n_{x_1}} \left(\beta x_1\right) \exp\left[-\beta^2 x_1^2\right] ,$$

then, the solution will be

$$|\psi\rangle = \frac{1}{\sqrt{\pi 2^{\left(n_{x_1}+n_{x_2}\right)} n_{x_1}! n_{x_2}!}} \exp\left[-\beta^2 \left(x_1^2 + x_2^2\right)\right] H_{n_{x_1}} \left(\beta x_1\right) H_{n_{x_2}} \left(\beta x_2\right) .$$

Finally, if we define the angular momentum operator as

$$L_z = x p_y - y p_x ,$$

in which I've decided to make the change of variables $x_1 \to x$ and $x_2 \to y$. But we can rewrite that operator in terms of the creation and anhilation operators as follows

$$L_z = i\hbar \left(a_x a^\dagger_y - a^\dagger_x a_y\right) ,$$

and we can also write the Hamiltonian in terms of these operators, that is

$$H = \left(a^\dagger_x a_x + a^\dagger_y a_y + 1\right) \hbar\omega ,$$

and now let's calculate the commutator,

$$\left[a_x a^\dagger_y , a^\dagger_x a_x + a^\dagger_y a_y\right] = a_x a^\dagger_y - a_x a^\dagger_y = 0,$$

and

$$\left[a^\dagger_x a_y , a^\dagger_x a_x + a^\dagger_y a_y\right] = -a^\dagger_x a_y + a^\dagger_x a_y = 0,$$

therefore, we have

$$[H, L_z] = 0,$$

which means that L_z is conserved quantity.

References

[1] Sakurai, J. J. "Jim Napolitano, Modern Quantum Mechanics." (2017).