Problem 1 – Spin 2 Polarization:

Let's define $\hbar = 1$, and the J1 operator

In[26]:=
```
h = 1;
J1 = {{0, -I h, 0}, {I h, 0, 0}, {0, 0, 0}};
```

Now, let' s define the five epsilon matrices (these are given):

In[28]:=
```
epsA = 1 / 2 {{1, I, 0}, {I, -1, 0}, {0, 0, 0}};
epsB = 1 / 2 {{1, -I, 0}, {-I, -1, 0}, {0, 0, 0}};
epsC = 1 / 2 {{0, 0, 1}, {0, 0, I}, {1, I, 0}};
epsD = 1 / 2 {{0, 0, 1}, {0, 0, -I}, {1, -I, 0}};
epsE = -1 / Sqrt[6] {{1, 0, 0}, {0, 1, 0}, {0, 0, -2}};
```

And let's also store them in a single list:

In[33]:=
```
epsilons = {epsA, epsB, epsC, epsD, epsE};
```

Show the epsilon matrices

In[34]:=
```
Table[epsilons[[i]] // MatrixForm, {i, Length[epsilons]}]
```

Out[34]=

$$\left\{ \begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \right\}$$

(i) Let's verify that the ϵ matrices are indeed eigenvectors of J^2:

In[35]:=
```
J2Action[eps_] := J1.eps - eps.J1;
```

Compute J2

In[36]:=
```
results = Map[J2Action, epsilons]
```

Out[36]=

$$\left\{\{\{1, i, 0\}, \{i, -1, 0\}, \{0, 0, 0\}\}, \{\{-1, i, 0\}, \{i, 1, 0\}, \{0, 0, 0\}\}, \left\{\left\{0, 0, \frac{1}{2}\right\}, \left\{0, 0, \frac{i}{2}\right\}, \left\{\frac{1}{2}, \frac{i}{2}, 0\right\}\right\}, \left\{\left\{0, 0, -\frac{1}{2}\right\}, \left\{0, 0, \frac{i}{2}\right\}, \left\{-\frac{1}{2}, \frac{i}{2}, 0\right\}\right\}, \{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}\right\}$$

Show both matrices

In[37]:=
```
Table[results[[i]] // MatrixForm, {i, Length[results]}]
```
Out[37]=

$$\left\{\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ -\frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

In[38]:=
```
Table[epsilons[[i]] // MatrixForm, {i, Length[epsilons]}]
```
Out[38]=

$$\left\{\begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\right\}$$

From this we can see that their eigenvalues are as follows

- ϵ_A has eigenvalue $\lambda = 2\hbar$
- ϵ_A has eigenvalue $\lambda = -2\hbar$
- ϵ_A has eigenvalue $\lambda = \hbar$
- ϵ_A has eigenvalue $\lambda = -\hbar$
- ϵ_A has eigenvalue $\lambda = 0$

(ii) Let’s lift the 3×3 ϵ matrices to 4×4 tensors and let’s boost them:

In[39]:=
```
ϵTensors = Table[ArrayPad[epsilons[[i]], {{1, 0}, {1, 0}}, 0], {i, Length[epsilons]}]
```
Out[39]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\},\left\{0,\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\},\left\{0,-\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,\frac{i}{2}\right\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,-\frac{i}{2}\right\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\}\right\},$$
$$\left.\left\{\{0,0,0,0\},\left\{0,-\frac{1}{\sqrt{6}},0,0\right\},\left\{0,0,-\frac{1}{\sqrt{6}},0\right\},\left\{0,0,0,\sqrt{\frac{2}{3}}\right\}\right\}\right\}$$

In[40]:=
```
Table[ϵTensors[[i]] // MatrixForm, {i, Length[ϵTensors]}]
```
Out[40]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\right\}$$

In[41]:=
```
Λ = {{γ, 0, 0, γ v}, {0, 1, 0, 0}, {0, 0, 1, 0}, {γv, 0, 0, γ}};
```

In[42]:=
```
Λ // MatrixForm
```

Out[42]//MatrixForm=

$$\begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix}$$

In[43]:=
```
ϵTBoosted = Table[Λ.ϵTensors⟦i⟧, {i, Length[ϵTensors]}]
```

Out[43]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\},\left\{0,\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\},\left\{0,-\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\left\{0,\frac{v\gamma}{2},\frac{i v\gamma}{2},0\right\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,\frac{i}{2}\right\},\left\{0,\frac{\gamma}{2},\frac{i\gamma}{2},0\right\}\right\},$$
$$\left\{\left\{0,\frac{v\gamma}{2},-\frac{1}{2}i v\gamma,0\right\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,-\frac{i}{2}\right\},\left\{0,\frac{\gamma}{2},-\frac{i\gamma}{2},0\right\}\right\},$$
$$\left.\left\{\left\{0,0,0,\sqrt{\frac{2}{3}}\,v\gamma\right\},\left\{0,-\frac{1}{\sqrt{6}},0,0\right\},\left\{0,0,-\frac{1}{\sqrt{6}},0\right\},\left\{0,0,0,\sqrt{\frac{2}{3}}\,\gamma\right\}\right\}\right\}$$

In[44]:=
```
Table[ϵTBoosted⟦i⟧ // MatrixForm, {i, Length[ϵTBoosted]}]
```

Out[44]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{v\gamma}{2} & \frac{i v\gamma}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & \frac{\gamma}{2} & \frac{i\gamma}{2} & 0 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0 & \frac{v\gamma}{2} & -\frac{1}{2} i v\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & \frac{\gamma}{2} & -\frac{i\gamma}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{2}{3}}\,v\gamma \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}}\,\gamma \end{pmatrix}\right\}$$

(iii) Taking massless limit:

In the massless limit, we can see that the only tensors that are well behaved are the ϵ_A and ϵ_B.

(iv) Let’s compute $\epsilon^{ij} p_j$:

Since the boost was along the z-direction, we’re going to consider a particle moving along the z-axis. In this case, the contravariant 4-momentum is given by $p^\mu = (E, 0, 0, E)$, and the covariant form reads $p_\mu = (-E, 0, 0, E)$, then we can define p as follows

In[45]:=
```
p = {-1, 0, 0, 1};
```

And now let’s perform the contraction

```
resultVector = Table[ϵTBoosted[[i]].p, {i, Length[ϵTBoosted]}]
```

Out[47]=

$$\left\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \left\{0, \frac{1}{2}, \frac{i}{2}, 0\right\}, \left\{0, \frac{1}{2}, -\frac{i}{2}, 0\right\}, \left\{\sqrt{\frac{2}{3}}\, v\, \gamma, 0, 0, \sqrt{\frac{2}{3}}\, \gamma\right\}\right\}$$

And from this we can see that the only ones that satisfy the condition are the 4×4 tensors ϵ_A and ϵ_B.