Groupwork 5

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Problem 1 *Let X be a topological space, D a dense subset of X, and Y be a Hausdorff topological space. Suppose f and g are two continuous functions from X to Y such that $f(x) = g(x)$ for all $x \in D$. Prove that $f(x) = g(x)$ for all $x \in X$.*

Proof 1 *Let $f : X \to Y$ and $g : X \to Y$ be two continuous functions, where X is a topological space and Y is a Hausdorff topological space, and let's suppose that $f(x) = g(x)$ for all $x \in D$, where $D \subset X$ is dense.*

Let's prove that $f(x) = g(x)$ for all $x \in X$, and let's proceed by contradiction, let's assume that there exists $x \in X$ such that $f(x) \neq g(x)$.

Because Y is Hausdorff, it follows that there exist open sets $U \ni f(x)$ and $V \ni g(x)$ subsets of Y such that

$$U \cap V = \emptyset.$$

Now, on the other hand, we know that both f and g are continuous, which implies that $f^{-1}(U)$ is open in X, and $g^{-1}(V)$ is open also in X, and even more,

$$x \in f^{-1}(U) \text{ and } x \in g^{-1}(V).$$

Now, let's consider the open set $W = f^{-1}(U) \cap g^{-1}(V)$, it's clear that $x \in W$, thus W is an open set that contains x, i.e., is a neighborhood of x. But D is dense in X which implies that W must contain some point $d \in D$, thus,

$$d \in W \implies d \in f^{-1}(U) \cap g^{-1}(V),$$

thus

$$f(d) \in U \text{ and } f(d) \in U,$$

but because $d \in D$ it follows that $f(d) = g(d)$, but this contradicts the fact that $U \cap V = \emptyset$, therefore $f(x) = g(x)$ for all $x \in X$