Problem 1: Evolution of Universe with general Energy of Particles

Emmanuel Flores

January 31, 2025

The Friedmann equation is given by

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

where

$$\rho = \frac{\rho_0 R_0^3}{R^3}.$$

Problem 1. Rewrite the Friedmann equation by replacing derivatives with respect to the time variable t by derivatives with respect to the conformal time variable η defined by

$$dt = R d\eta$$

Solution 1. By the chain rule, we have

$$\frac{dR}{dt} = \frac{dR}{d\eta}\frac{d\eta}{dt} \implies \frac{dR}{dt} = \frac{dR}{d\eta}\frac{1}{R},$$

and from this, the Friedmann equation is rewriten as

$$\left(\frac{dR}{d\eta}\frac{1}{R^2}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi G R^4}{3}\rho + \frac{2ER^4}{mR^2},$$

but ρ is a function of R, thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi \rho_0 R_0^3 G}{3}R + \frac{2E}{m}R^2.$$

and by making

$$\alpha = \frac{8\pi \rho_0 R_0^3 G}{3}, \beta = \frac{2E}{m},$$

we have

$$\left(\frac{dR}{d\eta}\right)^2 = \alpha R + \beta R^2.$$

Problem 2. Use separation of variables to re-write the Friedmann equation as an expression for η in terms of an integral of R.

Solution 2. By separation of variables we have

$$\frac{dR}{\sqrt{\alpha R + \beta R^2}} = d\eta,$$

thus, the problem now resides on solving the preovious integral.

Problem 3. Carry out the integral and invert to obtain $R(\eta)$; simplify for $E > 0$, $E = 0$, and $E < 0$.

Solution 3. In the variables that I'm using $E = 0 \implies \beta = 0$, $E > 0 \implies \beta > 0$, and $E < 0 \implies \beta < 0$, So let's proceed accordingly.

Case 1, $\beta = 0$: in this case, the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha}{2}\eta^2$$

Case 2, $\beta < 0$: Here, I'm going to add the sign of β by hand (not elegant at all, but just to be more explicit),

$$\int \frac{dR}{\sqrt{\alpha R - \beta R^2}} = \int d\eta,$$

and here the idea is to make a change of variables (I attach the notes with the algebra at the end of the document). Having done that, the solution reads

$$R(\eta) = \frac{\alpha}{\beta}\cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)$$

Case 3, $\beta > 0$: Here the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R + \beta R^2}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}},$$

and again, details about the algebra are attached at the end of the document.

Problem 4. Determine $t(\eta)$; again simplify for $E > 0$, $E = 0$, and $E < 0$.
Solution 4. We know that

$$dt = R d\eta,$$

and in the previous bullet we found exactly $R(\eta)$, thus we only need to integrate, this is

$$t(\eta) = \int d\eta R(\eta).$$

Case 1, $\beta = 0$: in this case we have

$$t(\eta) = \frac{\alpha}{2} \int d\eta \left(\eta^2\right),$$

with solution given by

$$t = \frac{\alpha}{6}\eta^3.$$

Case 2, $\beta < 0$: here we have

$$t(\eta) = \frac{\alpha}{\beta} \int d\eta \cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right),$$

with solution

$$t = \frac{\alpha}{\beta} \frac{2\sin\left(\frac{\sqrt{\beta}}{2}\eta\right)}{\sqrt{\beta}}$$

Case 3, $\beta > 0$: and finally, we have

$$t(\eta) = \alpha \int d\eta \frac{e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}}$$

with solution given by

$$t(\eta) = -\frac{\log\left(1 - \beta e^{\alpha x}\right)}{\beta}.$$

Problem 5. Using the above pair of parametric equations, plot $R(t)$ (use some plotting software, such as Mathematica) for 3 different choices of E, namely $E > 0$, $E = 0$, and $E < 0$.

Solution 5. By combining the two previous parts we can write a parametric equation

$$\gamma(\eta) = (t(\eta), R(\eta)),$$

therefore.

Case 1, $\beta = 0$: in this case we have

$$\gamma(\eta) = \frac{\alpha}{2}\left(\eta^3, \eta^2\right)$$

Case 2, $\beta < 0$: again, we have

$$\gamma(\eta) = \frac{\alpha}{\beta}\left(\frac{2}{\sqrt{\beta}} \sin\left(\frac{\sqrt{\beta}}{2}t\right), \cos^2\left(\frac{\sqrt{\beta}}{2}t\right)\right)$$

Case 3, $\beta > 0$: and finally

$$\gamma(\eta) = \left(-\frac{\log\left(1 - \beta e^{\alpha x}\right)}{\beta}, \frac{\alpha e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}}\right)$$

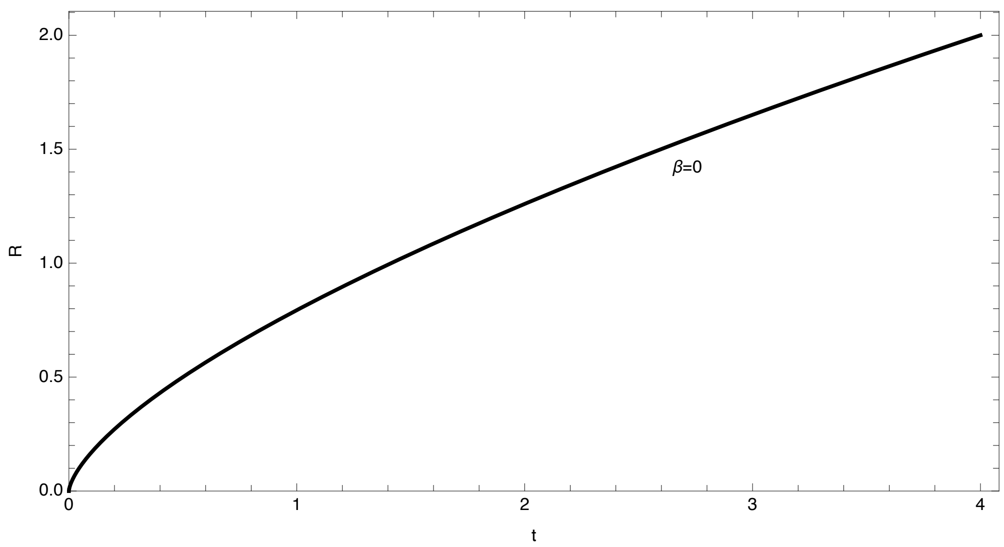
The plots are shown in the next page.

Cosmology: Homework 1

Problem 4. Parametric plot: $\gamma(\eta) = (R(\eta), t(\eta))$.
Case $\beta = 0$ $(E = 0)$

In[1]:=
```
params = {α → 1, β → 1};
```

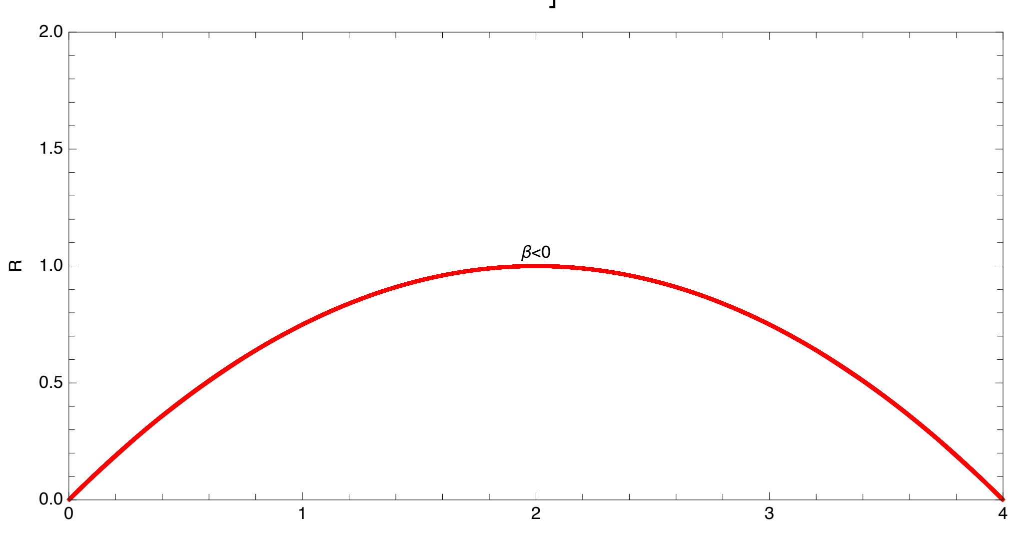
In[2]:=
```
figPlotE0 = ParametricPlot[(α/2 {η^3, η^2}) /. params, {η, 0, 2}, PlotRange → {{0, 4}, {0, 2}},
  PlotStyle → {Black, Thick},
  Frame → True, (*PlotLabel→"Evolution of R(t)(E=0)",*)
  FrameLabel → {"t", "R"}, LabelStyle → (FontFamily → "Helvetica"),
  PlotLabels → Placed[{"β=0"}, Scaled[0.7]]]
```

Out[2]=


In[3]:= Case $\beta < 0$ $(E < 0)$

Out[3]= Case $\beta < 0$

In[4]:=
```
figPlotEN = ParametricPlot[
  (α/β {2 + 2/Sqrt[β] Sin[Sqrt[β]/2 η], Cos[Sqrt[β]/2 η]^2}) /. params, {η, -100, 100}, PlotRange → {{0, 4}, {0, 2}},
  PlotStyle → {Red, Thick},
  Frame → True, (*PlotLabel→"Evolution of R(t)(E<0)",*)
  FrameLabel → {"t", "R"}, LabelStyle → (FontFamily → "Helvetica"),
  PlotLabels → Placed[{"β<0"}, Above]]
```

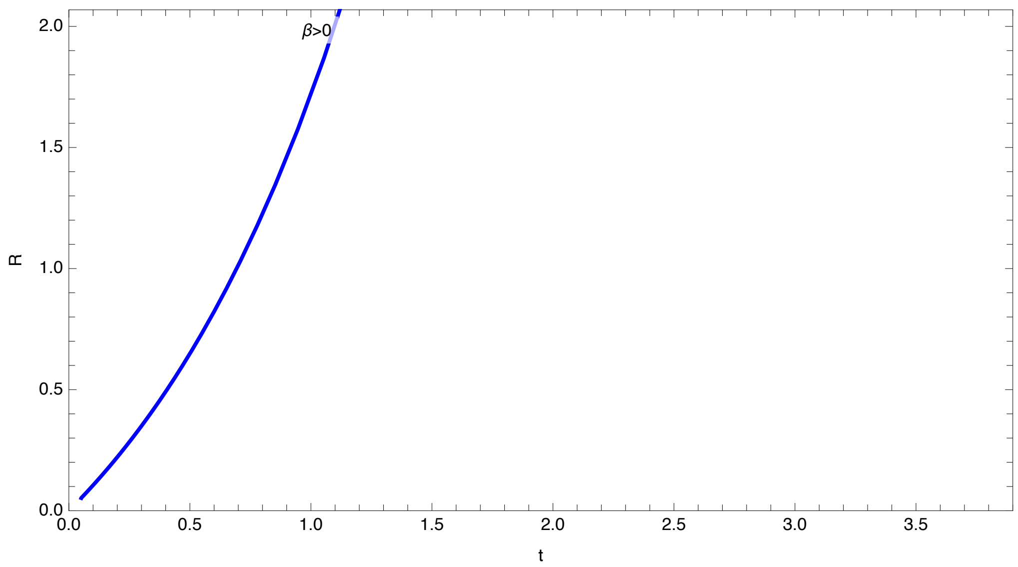
Out[4]=


In[5]:= Case $\beta > 0$ $(E > 0)$

Out[5]= Case $\beta > 0$

In[6]:=
```
figPlotEP = ParametricPlot[({-Log[1 - β Exp[α η]]/β, α Exp[α η]/(1 - β Exp[α η])}) /. params, {η, -3, 0}, PlotRange → {{0, 3.9}, {0, 2}},
  PlotStyle → {Blue, Thick},
  Frame → True, (*PlotLabel→"Evolution of R(t)(E>0)"*)
  FrameLabel → {"t", "R"}, LabelStyle → (FontFamily → "Helvetica"),
  PlotLabels → Placed[{"β>0"}, Above]]
```

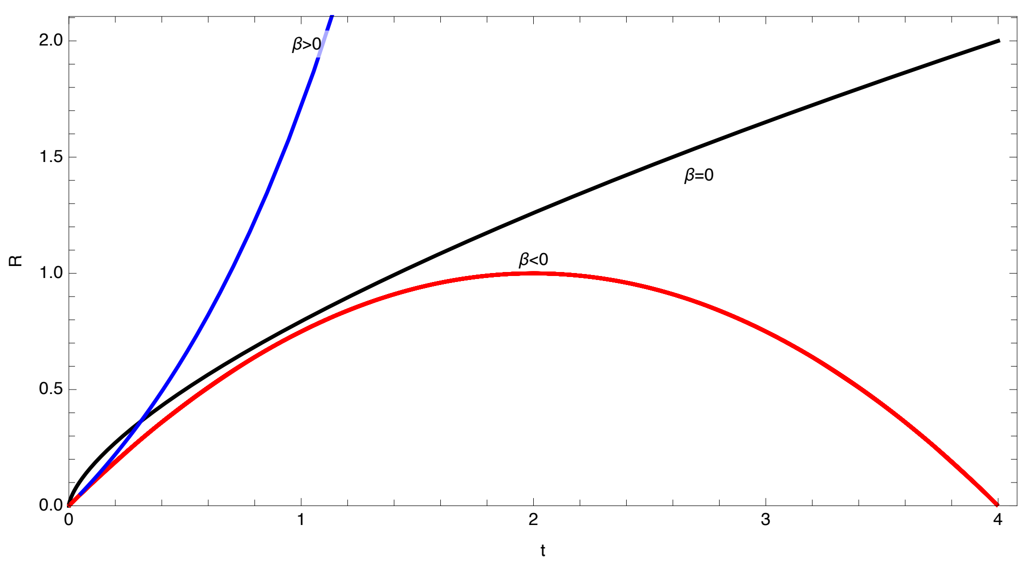
Out[6]=



Putting everything together

In[7]:=
```
Show[figPlotE0, figPlotEN, figPlotEP]
```

Out[7]=



Details about the algebra.

By separation of variables:

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi G R^4}{3}\rho + \frac{2ER^4}{mR^2} \qquad \rho = \frac{\rho_0 R_0^3}{R^3}$$

$$\Rightarrow \left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi G \rho_0 R_0^3}{3} R + \frac{2E}{m} R^2$$

$$\Rightarrow \left(\frac{dR}{d\eta}\right)^2 = \alpha R + \beta R^2 \Rightarrow \boxed{\frac{dR}{d\eta} = \sqrt{\alpha R + \beta R^2}}$$

Performing the integral by cases

$\Rightarrow E = 0 \Rightarrow \beta = 0$

$$\Rightarrow \frac{dR}{d\eta} = \sqrt{\alpha}\, R^{1/2} \Rightarrow \frac{dR}{\sqrt{\alpha}\, R^{1/2}} = d\eta$$

$$\Rightarrow \frac{2\sqrt{R}}{\sqrt{\alpha}} = \eta \Rightarrow \boxed{R(\eta) = \frac{\alpha}{2}\eta^2}$$

On the other hand;

$$dt = R\,d\eta \;,\quad R = \frac{\alpha}{2}\eta^2$$

$$\Rightarrow \quad t = \int \frac{\alpha}{2}\eta^2\,d\eta = \frac{\alpha}{6}\eta^3$$

$$\Rightarrow \boxed{\gamma(\eta) = \frac{\alpha}{2}\left(\eta^2 \,,\, \frac{\eta^3}{3}\right)}$$

Let $E<0 \Rightarrow \beta<0$

$$\Rightarrow \quad \frac{dR}{d\eta} = \sqrt{\alpha R - \beta R^2} = \sqrt{\beta\left(\frac{\alpha}{\beta}R - R^2\right)}$$

$$= \sqrt{\beta}\left(\frac{\alpha}{\beta}R - R^2\right)^{1/2}$$

Let $a = \frac{\alpha}{\beta}$, then;

$$\frac{dR}{\sqrt{\beta}\,(aR + R^2)^{1/2}} = d\eta \;,\quad \text{by making}$$

$R = a\cos^2(u)$, we have

$$\Rightarrow \frac{1}{\sqrt{\beta}\,(a\ell+\ell^2)^{1/2}} = \frac{1}{\sqrt{\beta}}\,\frac{}{\left(a^2\cos^2(u)\left(-1+\cos^2(u)\right)\right)^{1/2}}$$

$$-\sin^2(u) - \cos^2(u) = 1$$

$$\Rightarrow \frac{1}{\sqrt{\beta}}\,\frac{1}{\left(a^2\cos^2(u)\left(\cos^2(u)-1\right)\right)^{1/2}}$$

$$= \frac{1}{\sqrt{\beta}}\,\frac{1}{\left(a^2\cos^2(u)\sin^2 u\right)^{1/2}} = \frac{}{\sqrt{\beta}\,a\cos(u)\sin(u)}$$

and

$$d\ell = -2a\cos(u)\sin(u)\,du$$

$$\Rightarrow \frac{d\ell}{\sqrt{\beta}\,(a\ell+\ell^2)^{1/2}} = \frac{-2\cancel{a}\cancel{\cos(u)}\cancel{\sin(u)}\,du}{\sqrt{\beta}\,\cancel{a}\cancel{\cos(u)}\cancel{\sin(u)}}$$

$$\Rightarrow -\int\frac{2}{\sqrt{\beta}}\,du = \frac{-2}{\sqrt{\beta}}\,u = -\frac{2}{\sqrt{\beta}}\arccos\left[\left(\frac{\ell}{a}\right)^{1/2}\right]$$

$$\Rightarrow -\frac{2}{\sqrt{\beta}}\cos^{-1}\left(\left(\frac{\ell}{a}\right)^{1/2}\right) = \eta$$

$$-\frac{2}{\sqrt{\beta}} \cos^{-1}\left(\left(\frac{R}{a}\right)^{1/2}\right) = \eta$$

$$\Rightarrow \quad R = a\left[-\cos\left(+\frac{\sqrt{\beta}}{2}\eta\right)\right]^2$$

$$\Rightarrow \quad R = a\cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)$$

On the other hand:

$$dt = R\,d\eta \quad \Rightarrow \quad t = \int a\cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)d\eta$$

$$\Rightarrow \quad t = a\int \cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)d\eta$$

$$\Rightarrow \quad t = \frac{2a\sin\left(\frac{\sqrt{\beta}}{2}\eta\right)}{\sqrt{\beta}}, \quad \text{but} \quad a = \frac{\alpha}{\beta}$$

$$\boxed{\therefore \quad r(\eta) = a\left(\cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right), \frac{2}{\sqrt{\beta}}\sin\left(\frac{\sqrt{\beta}}{2}\eta\right)\right)}$$

Case 3: $E > 0 \Rightarrow \beta > 0$

$$\therefore \quad \frac{dR}{d\eta} = \sqrt{\alpha R + \beta R^2} \Rightarrow \frac{dR}{\sqrt{\alpha R + \beta R^2}} = d\eta$$

$$\frac{1}{a}\left(\ln(R) - \ln(\alpha + \beta R)\right) = \eta$$

$$\Rightarrow \frac{1}{a} \ln\left(\frac{R}{\alpha + \beta R}\right) = \eta \Rightarrow \frac{\ln(R)}{\ln(\alpha + \beta R)} = a\eta$$

$$\Rightarrow \exp\left(\ln\left(\frac{R}{\alpha + \beta R}\right)\right) = \exp(\alpha \eta)$$

$$\Rightarrow \frac{R}{\alpha + \beta R} = \exp(\alpha\eta) \Rightarrow R = (\alpha + \beta R) e^{\alpha\eta}$$

$$\Rightarrow R(1 - \beta \exp(\alpha\eta)) = \alpha \exp(\alpha\eta)$$

$$\Rightarrow \quad R = \frac{\alpha \exp(\alpha\eta)}{1 - \beta \exp(\alpha\eta)}$$

and, on the other hand;

$$dt = R\,d\eta \Rightarrow \quad t = \int \frac{\alpha \, d\eta}{1 - \beta \exp(\alpha\eta)}$$

Done with Mathematica