QUANTUM THEORY II | ASSIGNMENT 2

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Problem 1. (3.34)

Solution. We want to express $|jm\rangle$, with $j = 0, 1, 2$ in terms of $|j_1 j_2; m_1 m_2\rangle$ and the notation used will be $\pm, 0$ for $m_{1,2} = \pm 1, 0$ with $j_1 = j_2 = 1$. But we know that $m = m_1 + m_2$, then we should have $|jm\rangle = |2, 2\rangle = |++\rangle$ and $|2, -2\rangle = |--\rangle$, and from this, we can build the other $j = 2$ states, but in order to do that, we must remember the following equations

$$J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle, \quad J_- = J_{1-} + J_{2-},$$

thus we have

$$J_-|2,2\rangle = \sqrt{4}|2,1\rangle = 2|2,1\rangle,$$
$$J_-|2,2\rangle = (J_{1-} + J_{2-})|++\rangle = \sqrt{2}\left(|0+\rangle + |+0\rangle\right),$$

then, we have

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|0+\rangle + |+0\rangle\right),$$
$$J_-|2,1\rangle = \sqrt{6}|2,0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|-+\rangle + \sqrt{2}|00\rangle + \sqrt{2}|00\rangle + \sqrt{2}|+-\rangle\right),$$
$$\implies |2,0\rangle = \frac{1}{\sqrt{6}}\left(|-+\rangle + 2|00\rangle + |+-\rangle\right)$$
$$J_-|2,0\rangle = \sqrt{6}|2,-1\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}|-0\rangle + 2\sqrt{2}|-0\rangle + 2|0-\rangle + \sqrt{2}|0-\rangle\right)$$
$$\implies |2,-1\rangle = \frac{1}{\sqrt{2}}\left(|-0\rangle + |0-\rangle\right)$$

and finally

$$J_-|2,-1\rangle = \sqrt{4}|2,-2\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|--\rangle + \sqrt{2}|--\rangle\right),$$
$$\implies |2,-2\rangle = |--\rangle.$$

Now, for $j = 1$, we have

$$|1,\pm 1\rangle = a|0\pm\rangle + b|\pm 0\rangle$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 = 1$, but

$$\langle 2, \pm 1|1, \pm 1\rangle = 0, \quad a + b = 0,$$

$$\implies |1,\pm 1\rangle = \frac{1}{\sqrt{2}}\left(|\pm 0\rangle - |0\pm\rangle\right),$$

Date: Feb 27, 2024.

then, we have

$$J_-|1,1\rangle = \sqrt{2}|1,0\rangle, \quad J_-|1,1\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}|+-\rangle + \sqrt{2}|00\rangle - \sqrt{2}|-+\rangle\right),$$

then

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right),$$

and finally, if we put

$$|0,0\rangle = \alpha|+-\rangle + \beta|00\rangle + \gamma|-+\rangle,$$

and using the normalization condition together with the fact that

$$\langle 2,0|0,0\rangle = 0 = \langle 1,0|0,0\rangle,$$

then

$$\alpha + 2\beta + \gamma = 0 \implies \alpha - \gamma = 0 \quad \& \quad \alpha^2 + \beta^2 + \gamma^2 = 1$$

therefore

$$\beta = -\frac{\alpha+\gamma}{2} = -\alpha = -\frac{1}{\sqrt{3}},$$

thus

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|+-\rangle - |00\rangle + |-+\rangle\right)$$

Problem 2. (3.37)

Solution. a) For this part, we want to compute the following

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}\left(\beta\right) \right| m,$$

and in order to compute that, we're going to star with

$$\mathcal{D}\left(R\right) T_q^{(k)} \mathcal{D}^{\dagger}\left(R\right) = \sum \mathcal{D}_{q'q}^{(k)}\left(R\right) T_q^{(k)},$$

which holds for the "complete" Wigner functions. Now, by definition we know that

$$\mathcal{D}_{mm'}^{(j)}\left(\alpha, \beta, \gamma\right) = \exp\left[-i\left(m\alpha + m'\gamma\right)\right] \langle jm| \exp\left(-\frac{i}{\hbar} J_y \beta\right) jm'\rangle,$$

and from this, we define

$$d_{mm'}^{(j)}\left(\beta\right) = \langle jm| \exp\left(-\frac{i}{\hbar} J_y \beta\right) jm'\rangle,$$

thus

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}\left(\beta\right) \right| m = \sum_{m=-j}^{j} \langle jm'| \exp\left(-\frac{i}{\hbar} J_y \beta\right) |jm\rangle \langle jm| \exp\left(\frac{i}{\hbar} J_y \beta\right) |jm\rangle m,$$

but $J_z|jm\rangle = m\hbar|jm\rangle$, then we have

$$\begin{aligned}\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}\left(\beta\right) \right| m &= \sum_{m=-j}^{j} \langle jm'| \exp\left(\frac{i}{\hbar} J_y \beta\right) |jm\rangle m \langle jm| \exp\left(\frac{i}{\hbar} J_y \beta\right) |jm\rangle, \\ &= \frac{1}{\hbar} \sum_{m=-j}^{j} \langle jm'| \exp\left(-\frac{i}{\hbar} J_y \beta\right) J_z |jm\rangle \langle jm| \exp\left(\frac{i}{\hbar} J_y \beta\right) |jm\rangle,\end{aligned}$$

thus we have

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}\left(\beta\right) \right| m = \frac{1}{\hbar} \langle jm'| \exp\left(-\frac{i}{\hbar} J_y \beta\right) J_z \exp\left(\frac{i}{\hbar} J_y \beta\right) |jm\rangle,$$

and in order to evaluate the RHS of the previous equation, we make use of the Baker-Haussforf Lemma, which says that

$$\exp\left(\frac{i}{\hbar} J_y \beta\right) J_z \exp\left(-\frac{i}{\hbar} J_y \beta\right) = J_z + i\left(\frac{\beta}{\hbar}\right)\left[J_y, J_z\right] + \left(\frac{i^2\beta^2}{2\hbar^2}\right)\left[J_y, \left[J_y, J_z\right]\right] + \dots$$

which the result given by

$$\exp\left(\frac{i}{\hbar} J_y \beta\right) J_z \exp\left(-\frac{i}{\hbar} J_y \beta\right) = J_z \cos\beta + J_x \sin\beta,$$

then, we have that

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}(\beta) \right| m = \frac{1}{\hbar} \langle jm' | J_z \cos\beta + J_x \sin\beta | jm \rangle$$

$$= \frac{\cos\beta}{\hbar} \langle jm' | J_z | jm \rangle + \frac{\sin\beta}{\hbar} \langle jm' | J_x | jm \rangle$$

$$= \frac{\cos\beta}{\hbar} m\hbar\delta_{mm'} + \frac{\sin\beta}{\hbar} \langle jm' | J_x | jm \rangle,$$

Now, for J_x we can make use of the ladder operators, this is

$$J_{\pm} = J_x \pm iJ_y \implies J_x = \frac{1}{2}(J_+ + J_-),$$

but as we can see, at the end, the effect of $J_{\pm}$ is to change the state $|jm\rangle$ to $|jm+1\rangle, |jm-1\rangle$, and by orthogonality, the inner product vanish, therefore, we end with

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}(\beta) \right| m = m' \cos\beta.$$

b) Now, for this part we have something similar

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}(\beta) \right| m^2 = \sum_{m=-j}^{j} \langle jm' | \exp\left(\frac{i}{\hbar} J_y \beta\right) | jm \rangle \langle jm | \exp\left(-\frac{i}{\hbar} J_y \beta\right) | jm \rangle m^2,$$

and again, using $J_z|jm\rangle = m\hbar|jm\rangle \implies J_z^2|jm\rangle = m^2\hbar^2|jm\rangle$, we have that

$$\sum_{m=-j}^{j} \left| d_{mm'}^{(j)}(\beta) \right| m^2 = \sum_{m=-j}^{j} \langle jm' | \exp\left(\frac{i}{\hbar} J_y \beta\right) | jm \rangle m^2 \langle jm | \exp\left(-\frac{i}{\hbar} J_y \beta\right) | jm \rangle,$$

$$= \frac{1}{\hbar^2} \langle jm' | \exp\left(\frac{i}{\hbar} J_y \beta\right) J_z^2 \exp\left(-\frac{i}{\hbar} J_y \beta\right) | jm \rangle,$$

now, for the last part, we're going to see the tensor properties of J_z^2, this is, we can decompose the dyadic product as

$$U_i V_j = \frac{1}{3} \mathbf{U} \cdot \mathbf{V} + \frac{(U_i V_j - U_j V_i)}{2} + \left(\frac{U_i V_j + U_j V_i}{2} - \frac{\mathbf{U} \cdot \mathbf{V}}{3} \delta_{ij} \right),$$

and in this case, we have $\mathbf{U} = \mathbf{V} = \mathbf{J}$ and in particular $U_i V_j = J_z J_z$, thus

$$J_z^2 = \frac{1}{3} \mathbf{J}^2 + \left(J_z^2 - \frac{\mathbf{J}^2}{3} \right),$$

but we can identify the part inside the parenthesis as

$$T_0^{(2)},$$

then, we have

$$J_z^2 = \frac{1}{3} \mathbf{J}^2 + T_0^{(2)},$$

now, with this we have

$$\sum_{m=-j}^{j}\left|d_{mm'}^{(j)}\left(\beta\right)\right|m^{2}=\frac{1}{\hbar^{2}}\langle jm'|\exp\left(\frac{i}{\hbar}J_{y}\beta\right)\left(\frac{1}{3}\mathbf{J}^{2}+T_{0}^{(2)}\right)\exp\left(-\frac{i}{\hbar}J_{y}\beta\right)|jm\rangle,$$

$$=\frac{1}{3\hbar^{2}}\hbar^{2}j\left(j+1\right)\delta_{mm'}+\frac{1}{\hbar^{2}}\langle jm'|\exp\left(\frac{i}{\hbar}J_{y}\beta\right)T_{0}^{(2)}\exp\left(-\frac{i}{\hbar}J_{y}\beta\right)|jm\rangle$$

which is a consequence from the eigenvalues of the $\mathbf{J}^2$, thus, we the only thing left is

$$\langle jm'|\exp\left(\frac{i}{\hbar}J_{y}\beta\right)T_{0}^{(2)}\exp\left(-\frac{i}{\hbar}J_{y}\beta\right)|jm\rangle,$$

but we can use the definition of an spherical tensor in terms of the rotation operators as follows

$$\mathcal{D}^{\dagger}\left(R\right)T_{q}^{(2)}\mathcal{D}\left(R\right)=\sum_{q'=-2}^{2}\mathcal{D}_{qq'}^{(2)^{*}}\left(R\right)T_{q}^{(2)},$$

but because we want the expectation value in the state $|jm'\rangle$, only the $T_0^{(2)}$ gives a result different that zero, thus

$$\sum_{m=-j}^{j}\left|d_{mm'}^{(j)}\left(\beta\right)\right|m^{2}=\frac{1}{3}j\left(j+1\right)\delta_{mm'}+\frac{1}{\hbar^{2}}\mathcal{D}_{00}^{(2)}\langle jm'|J_{z}^{2}-\frac{\mathbf{J}^{2}}{3}|jm\rangle,$$

$$=\frac{1}{3}j\left(j+1\right)\delta_{mm'}+\frac{1}{\hbar^{2}}\mathcal{D}_{00}^{(2)}\left(m^{2}\hbar^{2}-\frac{\hbar^{2}}{3}j\left(j+1\right)\right)\delta_{mm'},$$

but

$$\mathcal{D}_{00}^{(2)}=P_{2}\left(\cos\beta\right)=\frac{1}{2}\left(3\cos^{2}\beta-1\right),$$

thus

$$\sum_{m=-j}^{j}\left|d_{mm'}^{(j)}\left(\beta\right)\right|m^{2}=\frac{1}{3}j\left(j+1\right)+\frac{1}{2}\left(3\cos^{2}\beta-1\right)\left(m'^{2}-\frac{1}{3}j\left(j+1\right)\right),$$

$$=\frac{1}{3}\left(1-\frac{1}{2}\left(3\cos^{2}\beta-1\right)\right)j\left(j+1\right)+\frac{1}{2}\left(3\cos^{2}\beta-1\right)\left(m'^{2}\right),$$

$$\therefore\sum_{m=-j}^{j}\left|d_{mm'}^{(j)}\left(\beta\right)\right|m^{2}=\frac{1}{2}j\left(j+1\right)\sin^{2}\beta+\frac{\left(m'\right)^{2}}{2}\left(3\cos^{2}\beta-1\right).$$

Problem 3. (3.43)

Solution. Using $U_q^{(1)}$ and $V_q^{(1)}$ we know that the product of tensors is given by

$$T_q^{(k)} = \sum_{q_1=-1}^{1} \sum_{q_2=-1}^{1} \langle 11; q_1 q_2 | 11; kq \rangle U_{q_1}^{(1)} V_{q_2}^{(1)},$$

where, explicitly

$$U_0^{(1)} = U_z, U_\pm^{(1)} = \mp \frac{U_x \pm iU_y}{\sqrt{2}},$$

and

$$V_0^{(1)} = V_z, V_\pm^{(1)} = \mp \frac{V_x \pm iV_y}{\sqrt{2}},$$

then, from this we have

$$\begin{aligned}
T_{+1}^{(1)} &= \langle 11; 01 | 11; 11 \rangle U_0^{(1)} V_{+1}^{(1)} + \langle 11; 10 | 11; 11 \rangle U_{+1}^{(1)} V_0^{(1)}, \\
\implies T_{+1}^{(1)} &= \frac{1}{\sqrt{2}} \left(-U_0^{(1)} V_{+1}^{(1)} + U_{+1}^{(1)} V_0^{(1)} \right), \\
T_0^{(1)} &= \langle 11; -11 | 11; 10 \rangle U_{-1}^{(1)} V_{+1}^{(1)} + \langle 11; 00 | 11; 10 \rangle U_0^{(1)} V_0^{(1)} \langle 11; 1-1 | 11; 10 \rangle U_{+1}^{(1)} V_{-1}^{(1)} \\
\implies T_0^{(1)} &= \frac{1}{\sqrt{2}} \left(-U_{-1}^{(1)} V_{+1}^{(1)} + U_{+1}^{(1)} V_{-1}^{(1)} \right), \\
T_{-1}^{(1)} &= \langle 11; -10 | 11; 1-1 \rangle U_{-1}^{(1)} V_0^{(1)} + \langle 11; 0-1 | 11; 1-1 \rangle U_0^{(1)} V_{-1}^{(1)}, \\
\implies T_{-1}^{(1)} &= \frac{1}{\sqrt{2}} \left(-U_{-1}^{(1)} V_0^{(1)} + U_0^{(1)} V_{-1}^{(1)} \right)
\end{aligned}$$

and the same procedure follows for every one of the components of the spherical tensor, this is

$$\begin{aligned}
T_{+2}^{(2)} &= U_{+1}^{(1)} V_{+1}^{(1)} \\
T_{+1}^{(2)} &= \frac{1}{\sqrt{2}} \left(U_0^{(1)} V_{+1}^{(1)} + U_{+1}^{(1)} V_0^{(1)} \right) \\
T_0^{(2)} &= \frac{1}{\sqrt{6}} \left(U_{-1}^{(1)} V_{+1}^{(1)} + 2U_0^{(1)} V_0^{(1)} + U_{+1}^{(1)} V_{-1}^{(1)} \right) \\
T_{-1}^{(2)} &= \frac{1}{\sqrt{2}} \left(U_{-1}^{(1)} V_0^{(1)} + U_0^{(1)} V_{-1}^{(1)} \right) \\
T_{-2}^{(2)} &= U_{-1}^{(1)} V_{-1}^{(1)},
\end{aligned}$$

we can express those components in terms of (U_x, U_y, U_z) and (V_x, V_y, V_z) by using the definition of $U_0^{(1)}, U_\pm^{(1)}$ which, after a lot of algebra will have

$$T_{+1}^{(2)} = \frac{1}{2} \left(U_z V_z - U_x V_z + i \left(U_z V_y - U_y V_z \right) \right)$$

$$T_{+0}^{(2)} = \frac{i}{\sqrt{2}} \left(U_x V_y - U_y V_x \right)$$

$$T_{-1}^{(2)} = \frac{1}{2} \left(U_z V_x - U_x V_z + i \left(U_y V_z - U_z V_y \right) \right)$$

$$
\begin{aligned}
T^{(2)}_{+2} &= \frac{1}{2}\left(U_x V_x - U_y V_y + i\left(U_y V_x - U_z V_y\right)\right)\\
T^{(2)}_{+1} &= -\frac{1}{2}\left(U_z\left(V_x + iV_y\right) + \left(U_x + iU_y\right)V_z\right)\\
T^{(2)}_{0} &= \frac{1}{\sqrt{6}}\left(U_x V_x + U_y V_y + 2U_z V_z\right)\\
T^{(2)}_{-1} &= \frac{1}{2}\left(U_z V_x + U_x V_z - i\left(U_y V_z + U_z V_y\right)\right)\\
T^{(2)}_{-2} &= \frac{1}{2}\left(U_x V_x - U_y V_y - i\left(U_y V_x + U_x V_y\right)\right).
\end{aligned}
$$

Problem 4. (3.44)

Solution. a) We have a spin-less particle bound to a fixed center by a central force potential, the problem ask for the matrix elements

$$\langle n'l'm'| \mp \frac{1}{\sqrt{2}}(x \pm iy)|nlm\rangle, \quad \langle n'l'm'|z|nlm\rangle,$$

so in order to make the relation with the Wigner Eckart Theorem, we have to relate those quantities as some spherical tensors, but we now that

$$T_0^{(1)} = \sqrt{\frac{3}{4\pi}}V_z, \quad T_{\pm 1}^{(1)} = \sqrt{\frac{3}{4\pi}}\left(\mp\frac{V_x \pm iV_y}{\sqrt{2}}\right),$$

and if we make the following association

$$V_x \to x, V_y \to y, V_z \to z,$$

we see that our matrix elements can be seen as the matrix elements of some spherical tensors of rank 1. Now, the Wigner Eckart Theorem says that

$$\langle n'l'm'|T_q^{(1)}|nlm\rangle = \langle l1; mq|l1; l'm'\rangle \frac{\langle n'l'|T^{(1)}|nl\rangle}{\sqrt{2l+1}},$$

in which al the dependence on m and m' is contained int the Clebsch, and with this in mind we have

$$\frac{\langle n'l'm'|T_{\pm 1}^{(1)}|nlm\rangle}{\langle n'l'm'|T_0^{(1)}|nlm\rangle} = \frac{\langle l1; m\pm 1|l1; l'm'\rangle}{\langle l1; m0|l1; l'm'\rangle}.$$

b) Now for this part, we have the following: if we use wave functions, we know that the problem can be decomposed into a radial part, angle-dependent, and an azimuthal part, so the Wigner-Eckart Theorem express that, this is given the wave function

$$\psi(\mathbf{x}) = R_{nl}(r)\, Y_l^m(\theta,\phi),$$

we know that one property of the spherical harmonics is

$$\int d\Omega Y_l^{m^*}(\theta,\phi)\, Y_{l_1}^{m_1}(\theta,\phi)\, Y_{l_2}^{m_2}(\theta,\phi) = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}}\langle l_1 l_2; 00|l_1 l_2; l0\rangle\langle l_1 l_2; m_1 m_2|l_1 l_2; lm\rangle,$$

then if we go to the continuum using $1 = \int d^3x|x\rangle\langle x|$, we have that

$$\langle n'l'm'|T_q^{(1)}|nlm\rangle = \int_0^\infty r^3 dr R_{n'l'}(r)\, R_{nl}(r) \times \sqrt{\frac{4\pi}{3}}\int d\Omega Y_{l'}^{m^*}(\theta,\phi)\, Y_{l'}^{m}(\theta,\phi)\, Y_1^q(\theta,\phi),$$

$$\implies \langle n'l'm'|T_q^{(1)}|nlm\rangle = \int_0^\infty r^3 dr R_{n'l'}(r)\, R_{nl}(r) \times \sqrt{\frac{2l+1}{2l'+1}}\langle l1; 00|l1; l'0\rangle\langle l1; mq|l1; l'm'\rangle,$$

where the second Clebsh is the same as in the Wigner-Eckart Theorem.

Problem 5. (3.45)

Solution. a) For this part, we know that

$$Y_2^{\pm 2} = \sqrt{\frac{15}{32\pi}} \frac{(x \pm iy)^2}{r^2},$$

and we also know that

$$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta \exp(i\phi), \quad Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$$

where

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r}, \quad \cos\theta = \frac{z}{r} \quad \exp(i\phi) = \frac{x+iy}{\sqrt{x^2+y^2}},$$

and from this we have that

$$\sqrt{\frac{32\pi}{15}} r^2 Y_2^2 = x^2 - y^2 + 2ixy,$$
$$\sqrt{\frac{32\pi}{15}} r^2 Y_2^{-2} = x^2 - y^2 - 2ixy,$$

thus

$$x^2 - y^2 = \frac{1}{2}\sqrt{\frac{32\pi}{15}} r^2 \left(Y_2^2 + Y_2^{-2}\right),$$
$$xy = \frac{1}{2i}\sqrt{\frac{32\pi}{15}} r^2 \left(Y_2^2 - Y_2^{-2}\right),$$

and for the other product, we have that the $Y_2^{\pm 1}$ can be rewritten as

$$Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \left(\frac{\sqrt{x^2+y^2}}{r}\right)\left(\frac{z}{r}\right)\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right),$$

$$\implies Y_2^{\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \left(\frac{z(x+iy)}{r^2}\right),$$

then, from this, we have that

$$\sqrt{\frac{8\pi}{15}} r^2 Y_2^1 = -z(x+iy),$$
$$\sqrt{\frac{8\pi}{15}} r^2 Y_2^{-1} = z(x+iy),$$

thus

$$xz = \frac{1}{2}\sqrt{\frac{8\pi}{15}} r^2 \left(Y_2^{-1} - Y_2^1\right),$$

and finally,

$$Y_2^0 = \sqrt{\frac{5}{16\pi}} \left(3\frac{z^2}{r^2} - 1\right),$$

$$\implies 3z^2 - r^2 = \sqrt{\frac{16\pi}{5}} r^2 Y_2^0$$

b) Now, for this part need to evaluate

$$e\langle \alpha j, m = j | (x^2 - y^2) | \alpha, j, m = j \rangle$$

in terms of

$$Q = e\langle \alpha j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle,$$

and in order to do that, we're going to use the Wigner-Eckart Theorem

$$\begin{aligned} e\langle \alpha j, m = j|(x^2 - y^2)|\alpha, j, m = j\rangle &= e\langle \alpha j, m = j|\frac{1}{2}\sqrt{\frac{32\pi}{15}}r^2\left(Y_2^2 + Y_2^{-2}\right)|\alpha, j, m = j\rangle \\ &= e\frac{1}{2}\sqrt{\frac{32\pi}{15}}\frac{\langle \alpha j|Y^2|\alpha j\rangle}{\sqrt{2j+1}}\left[\langle j2j, 2|j2jm'\rangle + \langle j2j, -2|j2jm'\rangle\right], \end{aligned}$$

but $\langle j2j, 2|j2jm'\rangle = 0$ because $m' = j + 2$ is not possible. On the other hand, we have that

$$\begin{aligned} Q &= e\langle \alpha j, m = j|\sqrt{\frac{16\pi}{5}}r^2 Y_2^0|\alpha, j, m = j\rangle, \\ &= e\sqrt{\frac{16\pi}{5}}\frac{\langle \alpha j|Y^2|\alpha j\rangle}{\sqrt{2j+1}}\langle j2j, 0|j2jj\rangle, \end{aligned}$$

therefore, we have that

$$e\langle \alpha j, m = j|(x^2 - y^2)|\alpha, j, m = j\rangle = \frac{Q}{\sqrt{2}}\frac{\langle j2j, -2|j2jm'\rangle}{\langle j2j, 0|j2jj\rangle}.$$

Problem 6. (3.46)

Solution. The Hamiltonian is given by

$$H = \frac{eQ}{2s(s-1)\hbar^2}\left[\partial_{xx}\phi S_x^2 + \partial_{yy}\phi S_y^2 + \partial_{zz}\phi S_z^2\right].$$

So, let's rewrite the interaction Hamiltonian, but first let's make use of the following

$$S_\pm^2 = \left(S_x \pm iS_y\right)^2$$
$$\implies S_\pm^2 = S_x^2 - S_y^2 \pm i\left(S_xS_y + S_yS_x\right),$$

and from this, we have

$$S_x^2 - S_y^2 = \frac{S_+^2 + S_-^2}{2},$$

and on the other hand

$$\mathbf{S^2} = S_x^2 + S_y^2 + S_z^2,$$
$$\implies S_x^2 + S_y^2 = \mathbf{S^2} - S_z^2$$

and from those equations we have

$$2S_x^2 = \frac{S_+^2 + S_-^2}{2} + \mathbf{S^2} - S_z^2,$$
$$\implies S_x^2 = \frac{S_+^2 + S_-^2 + 2\left(\mathbf{S^2} - S_z^2\right)}{4}$$

and similar for S_y^2,

$$-2S_y^2 = \frac{S_+^2 + S_-^2}{2} - \left(\mathbf{S^2} - S_z^2\right)$$
$$\implies S_y^2 = -\frac{S_+^2 + S_-^2 - 2\left(\mathbf{S^2} - S_z^2\right)}{4}$$

thus, the Hamiltonian will be, taking into account that $s = 3/2$

$$H = \frac{eQ}{2\frac{3}{2}(\frac{3}{2}-1)\hbar^2}\left[\partial_{xx}\phi\left(\frac{S_+^2 + S_-^2 + 2\left(\mathbf{S^2} - S_z^2\right)}{4}\right) + \partial_{yy}\phi\left(-\frac{S_+^2 + S_-^2 - 2\left(\mathbf{S^2} - S_z^2\right)}{4}\right) + \partial_{zz}\phi\left(S_z^2\right)\right],$$
$$\implies H = \frac{2eQ}{3\hbar^2}\left[\partial_{xx}\phi\left(\frac{S_+^2 + S_-^2 + 2\left(\mathbf{S^2} - S_z^2\right)}{4}\right) + \partial_{yy}\phi\left(-\frac{S_+^2 + S_-^2 - 2\left(\mathbf{S^2} - S_z^2\right)}{4}\right) + \partial_{zz}\phi\left(S_z^2\right)\right],$$
$$\implies H = \frac{2eQ}{3\hbar^2}\left[\left(\partial_{xx}\phi + \partial_{yy}\phi\right)\left(\frac{2\left(\mathbf{S^2} - S_z^2\right)}{4}\right) + \left(\partial_{xx}\phi - \partial_{yy}\phi\right)\left(\frac{S_+^2 + S_-^2}{4}\right) + \partial_{zz}\phi\left(S_z^2\right)\right],$$
$$\implies H = \frac{2eQ}{3\hbar^2}\left[\frac{1}{4}\left(\partial_{xx}\phi + \partial_{yy}\phi\right)\left(2\mathbf{S^2} - 2S_z^2\right) + \frac{1}{\$}\left(\partial_{xx}\phi - \partial_{yy}\phi\right)\left(S_+^2 + S_-^2\right) + \partial_{zz}\phi\left(S_z^2\right)\right],$$

but because ϕ is harmonic, we know that

$$\partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0,$$
$$\partial_{zz}\phi = -\left(\partial_{xx}\phi + \partial_{yy}\phi\right),$$

then

$$H = \frac{2eQ}{3\hbar^2}\left[\frac{1}{4}\left(\partial_{xx}\phi + \partial_{yy}\phi\right)\left(2\mathbf{S^2} - 2S_z^2 - 4S_z^2\right) + \frac{1}{\$}\left(\partial_{xx}\phi - \partial_{yy}\phi\right)\left(S_+^2 + S_-^2\right)\right],$$

$$\implies H = -\frac{4eQ}{3\hbar^2}\left(\frac{1}{4}\left(\partial_{xx}\phi + \partial_{yy}\phi\right)\right)\left(-\mathbf{S^2} + 3S_z^2\right) + \frac{2eQ}{3\hbar^2}\left(\frac{1}{4}\left(\partial_{xx}\phi - \partial_{yy}\phi\right)\right)\left(S_+^2 + S_-^2\right),$$

thus, if we make

$$A = -\frac{4eQ}{3\hbar^2}\left(\frac{1}{4}\left(\partial_{xx}\phi + \partial_{yy}\phi\right)\right) \quad , B = \frac{2eQ}{3\hbar^2}\left(\frac{1}{4}\left(\partial_{xx}\phi - \partial_{yy}\phi\right)\right),$$

we have

$$H = A\left(3S_z^2 - \mathbf{S^2}\right) + B\left(S_+^2 + S_-^2\right),$$

just as we wanted. Now for this system we have the following relations

$$\left(3S_z^2 - \mathbf{S^2}\right)|m\rangle = \hbar^2\left(3m^2 - \frac{15}{4}\right),$$

$$S_\pm^2|m\rangle = \hbar^2\sqrt{\left(s \mp m - 1\right)\left(s \pm m + 2\right)\left(s \mp m\right)\left(s \pm m + 1\right)}|m \pm 2\rangle,$$

and now, after a little bit of algebra, we can obtain the matrix representation of the Hamiltonian Operator, which is

$$H = \begin{pmatrix} 3A & 2B\sqrt{3} & 0 & 0 \\ 2B\sqrt{3} & -3A & 0 & 0 \\ 0 & 0 & 3A & 2B\sqrt{3} \\ 0 & 0 & 2B\sqrt{3} & -3A \end{pmatrix},$$

and now, for the eigenvector y eigenvalues we have to solve the problem

$$Hx = \lambda x.$$

Then the eigenvalues are given by

$$\left\{-\sqrt{3}\sqrt{3A^2 + 4B^2}, -\sqrt{3}\sqrt{3A^2 + 4B^2}, \sqrt{3}\sqrt{3A^2 + 4B^2}, \sqrt{3}\sqrt{3A^2 + 4B^2}\right\},$$

whereas the eigenvectors associated with each eigenvalue are given by

$$\left(0, 0, -\frac{\sqrt{3A^2 + 4B^2} - \sqrt{3}A}{2B}, 1\right), \left(-\frac{\sqrt{3A^2 + 4B^2} - \sqrt{3}A}{2B}, 1, 0, 0\right),$$

$$\left(0, 0, -\frac{-\sqrt{3A^2 + 4B^2} - \sqrt{3}A}{2B}, 1\right), \left(-\frac{-\sqrt{3A^2 + 4B^2} - \sqrt{3}A}{2B}, 1, 0, 0\right).$$

Problem 7. Problem 3.42 (Mann-1)

Solution. Instead of just solving for one component of the problem, I decided to do the problem with all the components, so let's begin. We need to evaluate the following sum

$$\sum_{q'} d^{(j)}_{qq'}(\beta)\, V^{(1)}_{q'},$$

where d is given by the following matrix representation

$$d^{(1)} = \begin{pmatrix} \frac{1}{2}(\cos(\beta)+1) & -\frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}(1-\cos(\beta)) \\ \frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{\sin(\beta)}{\sqrt{2}} \\ \frac{1}{2}(1-\cos(\beta)) & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}(\cos(\beta)+1) \end{pmatrix},$$

and in this case, the vector is given by the following expression

$$V^{(1)} = \begin{pmatrix} V_1 \\ V_0 \\ V_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{V_x+iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x-iV_y}{\sqrt{2}} \end{pmatrix}.$$

Now, with this information at hand we proceed with the following products

$$\sum_{q'} d^{(j)}_{1q'}(\beta)\, V^{(1)}_{q'}, \quad \sum_{q'} d^{(j)}_{0q'}(\beta)\, V^{(1)}_{q'}, \quad \sum_{q'} d^{(j)}_{-1q'}(\beta)\, V^{(1)}_{q'},$$

thus, we have for $q=1$

$$\sum_{q'} d^{(j)}_{1q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{1}{2}(\cos(\beta)+1), -\frac{\sin(\beta)}{\sqrt{2}}, \frac{1}{2}(1-\cos(\beta))\right)\cdot \begin{pmatrix} -\frac{V_x+iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x-iV_y}{\sqrt{2}} \end{pmatrix},$$

$$\implies \sum_{q'} d^{(j)}_{1q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{1}{2}(\cos(\beta)+1)\right)\left(-\frac{V_x+iV_y}{\sqrt{2}}\right) + \left(-\frac{\sin(\beta)}{\sqrt{2}}\right)(V_z) + \left(\frac{1}{2}(1-\cos(\beta))\right)\left(\frac{V_x-iV_y}{\sqrt{2}}\right).$$

$$\implies \sum_{q'} d^{(j)}_{1q'}(\beta)\, V^{(1)}_{q'} = \frac{-V_x\cos(\beta) - iV_y - V_z\sin(\beta)}{\sqrt{2}},$$

for $q=0$

$$\sum_{q'} d^{(j)}_{0q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{\sin(\beta)}{\sqrt{2}}, \cos(\beta), -\frac{\sin(\beta)}{\sqrt{2}}\right)\cdot \begin{pmatrix} -\frac{V_x+iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x-iV_y}{\sqrt{2}} \end{pmatrix},$$

$$\implies \sum_{q'} d^{(j)}_{0q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{\sin(\beta)}{\sqrt{2}}\right)\left(-\frac{V_x+iV_y}{\sqrt{2}}\right) + V_z\cos(\beta) + \left(-\frac{\sin(\beta)}{\sqrt{2}}\right)\left(\frac{V_x-iV_y}{\sqrt{2}}\right)$$

$$\implies \sum_{q'} d^{(j)}_{0q'}(\beta)\, V^{(1)}_{q'} = V_z\cos(\beta) - V_x\sin(\beta),$$

and finally, for $q=-1$

$$\sum_{q'} d^{(j)}_{-1q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{1}{2}(1-\cos(\beta)), \frac{\sin(\beta)}{\sqrt{2}}, \frac{1}{2}(\cos(\beta)+1)\right)\cdot\begin{pmatrix} -\frac{V_x+iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x-iV_y}{\sqrt{2}} \end{pmatrix},$$

$$\implies \sum_{q'} d^{(j)}_{-1q'}(\beta)\, V^{(1)}_{q'} = \left(\frac{1}{2}(1-\cos(\beta))\right)\left(-\frac{V_x+iV_y}{\sqrt{2}}\right) + V_z\frac{\sin(\beta)}{\sqrt{2}} + \left(\frac{1}{2}(\cos(\beta)+1)\right)\left(\frac{V_x-iV_y}{\sqrt{2}}\right),$$

$$\implies \sum_{q'} d^{(j)}_{-1q'}(\beta)\, V^{(1)}_{q'} = \frac{V_x\cos(\beta)-iV_y+V_z\sin(\beta)}{\sqrt{2}}.$$

On the other hand, a rotation about the y-axis by an angle β is given by the matrix

$$R_y = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix},$$

thus

$$\left(V^{(1)}\right)' = R_y V^{(1)} = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}\cdot\begin{pmatrix} -\frac{V_x+iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x-iV_y}{\sqrt{2}} \end{pmatrix},$$

$$\implies \left(V^{(1)}\right)' = \begin{pmatrix} V_x\cos(\beta)+V_z\sin(\beta) \\ V_Y \\ V_z\cos(\beta)-V_x\sin(\beta) \end{pmatrix},$$

and now, in order to make the comparison, using our rotated vector, we have to construct the following "tensor"

$$\left(V^{(1)}\right)'' = \begin{pmatrix} (V_1)'' \\ (V_0)'' \\ (V_{-1})'' \end{pmatrix} = \begin{pmatrix} -\frac{(V_x)'+i\left(V_y\right)'}{\sqrt{2}} \\ (V_z)' \\ \frac{(V_x)'-i\left(V_y\right)'}{\sqrt{2}} \end{pmatrix},$$

$$\implies (V_1)'' = -\frac{V_x\cos(\beta)+iV_y+V_z\sin(\beta)}{\sqrt{2}},$$

$$\implies (V_0)'' = V_z\cos(\beta)-V_x\sin(\beta),$$

$$\implies (V_{-1})'' = \frac{V_x\cos(\beta)-iV_y+V_z\sin(\beta)}{\sqrt{2}}.$$

And as we can see, we have the following result

$$\sum_{q'} d^{(j)}_{1q'}(\beta)\, V^{(1)}_{q'} = (V_1)'',$$

$$\sum_{q'} d^{(j)}_{0q'}(\beta)\, V^{(1)}_{q'} = (V_0)'',$$

$$\sum_{q'} d^{(j)}_{-1q'}(\beta)\, V^{(1)}_{q'} = (V_{-1})'',$$

therefore, this concludes the proof.