Problem 1

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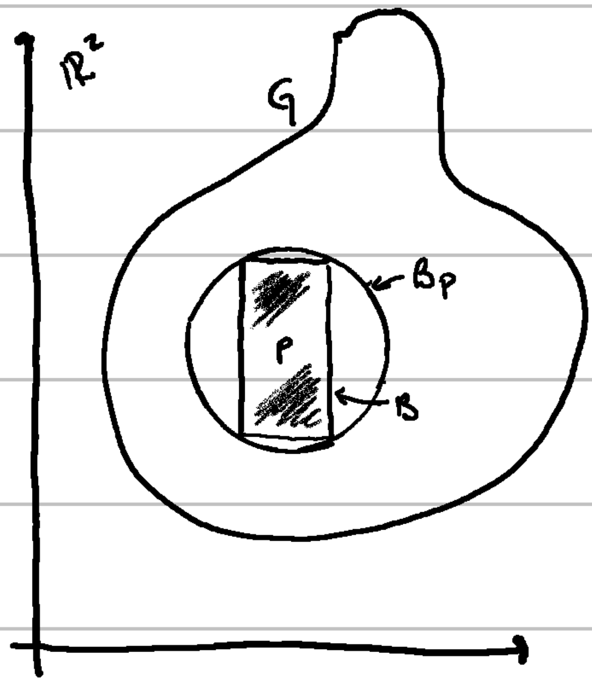
Using the standard topology on $\mathbb{R}$, is the product topology on $\mathbb{R}\times\mathbb{R}$ the same as the standard topology on $\mathbb{R}^2$?

Yes. Using $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$ considered as the product topology of $\mathbb{R}$ with open intervals corresponds to open rectangles, $\mathcal{B}$.



Let's consider $\mathbb{R}^2$ with the standard topology, let $G\subset\mathbb{R}^2$ be an open set, and let $p\in G$, thus it follows that there's an open disc centered at p, $B_p(r)$, $r>0$;

$$B_p(r) = \{q\in\mathbb{R}^2 \mid d(p,q)<r\},$$

such that $p\in B_p(r)\subset G$. But if we consider open rectangles, we have that any open rectangle $B\in\mathcal{B}$ whose vertices lie on the boundary of $B_p(r)$ satisfies:

$$p\in B\subset B_r(p)\subset G \implies p\in B\subset G,$$

therefore it generates the same topology.

Problem 2.

Prove that the topological space X is Hausdorff space only if $\Delta = \{(x,x), x \in X\}$ is closed in the product space $X \times X$.

Proof

($\Rightarrow$) let's suppose that (X, τ) is a Hausdorff space, this means that given $x, y \in X$, $x \neq y$, we can find neighborhoods of x and y such that their intersection is empty.

Now, with this in mind let $x, y \in X$, $x \neq y$, this is equivalent to: $(x,y) \in X \times X \setminus \Delta$,

but because X is Hausdorff, it follows that there exists opensets $U \ni x$ and $V \ni y$ such that $U \cap V = \varphi$,

but this implies that $U \times V$ is an open set in $X \times X$ that contains (x,y) and it does not intersect Δ,

$\Rightarrow$ $(X \times X) \setminus \Delta$ is open $\therefore$ Δ is closed.

($\Leftarrow$) let's suppose that Δ is closed in $X \times X$, and let $x, y \in X$ such that $x \neq y$, thus it follows that $(X \times X) \setminus \Delta \ni (x,y)$ and it's also open thus, let $\mathcal{U} \times V$ be an open set in $\underline{X} \times \underline{X}$ containing (x,y), but again; $\mathcal{U}$ and V are open sets such that $x \in \mathcal{U}$, $y \in V$ and $\mathcal{U} \cap V = \phi$,

therefore $\underline{X}$ is Hausdorff.

$\blacksquare$

Problem 3.

Let (X, τ) be a topological space and $Y \subset X$ equipped with the subspace topology. Prove that in order that every openset in Y to be an openset on X, its necessary and sufficient that Y be open in X.

Proof.

Let (X, τ_x) be a topological space and $Y \subset X$, we define the subspace topology $X \cap Y$ as:

$$\tau_Y = \{ V \cap Y \mid V \in \tau_x \}.$$

We want to prove that $U \in \tau_Y$ is open in τ_x if and only if Y is open in X.

$(\Leftarrow)$ Let $U \in \tau_Y$ and $Y \in \tau_x$, then because U is open in τ_Y, then exists $V \in \tau_x$ such that $U = V \cap Y$, and the finite intersection of open sets is open

$$\Rightarrow U \in \tau_x \text{ (} U \text{ is open in } X \text{)}.$$

$(\Rightarrow)$

Problem 4

Prove that a set A is dense in a topological space (X, T) if and only if every non-empty set in X contains a point of A.

Proof

$(\Rightarrow)$

Let (X, τ) be a topological space and $A \subset X$, A dense means that $\bar{A} = X$. But we know that the closure of A is the collection of all points $p \in A$ such that every neighborhood $\mathcal{U}$ of p satisfy $\mathcal{U} \cap A \neq \emptyset$, but $\mathcal{U}$ being a neighborhood of p means that exist $O \in \tau$ such that $p \in O \subset \mathcal{U}$,

therefore, if $A \subset X$ is dense in X every nonempty open set of X contains a point of A.

$(\Leftarrow)$

Let $A \subset X$ be such that every nonempty set of X contains a point of A, then $\bar{A} = X$. Indeed lets suppose that is not the case, and let's consider

$$\mathcal{U} = X / \bar{A} \quad \leftarrow \text{open},$$

therefore $\mathcal{U}$ is an open set that does not intersect A, which is a contradiction with our first assumption, therefore $\bar{A} = X$.

$\blacksquare$

Problem 5

5.1 Find a topology on $\mathbb{R}$ that is not separable.

5.2. Find a separable space that contains a subspace that is not separable in the subspace topology.

<u>Sol</u>.

1. $\mathbb{R}$ with the discrete topology is not separable.

By definition, a topological space is separable if it contains a countable dense subset. Now, we know $\mathbb{R}$ is uncountable, and in the discrete topology every subset is closed and open, from this it follows that the closure is the space itself, therefore if we have a dense subset, it must be the whole space, which in this case is uncountable ∎

2. Let's consider the following topology defined on $\mathbb{R}$,

$\mathcal{U} \subseteq \mathbb{R}$ is open if and only if $0 \in \mathcal{U}$ or $\mathcal{U} = \emptyset$.

We claim that $\{0\}$ is dense in this space:

indeed every non-empty set contains 0 or is the empty set, thus it follows that:

$$\{0\} \cap V = \{0\} \text{ for all non-empty sets } V,$$

which is the same as saying that $\{0\}$ intersects all open sets. Because the only non-empty open sets in this topology must contain 0 and the set $\{0\}$ intersects all of them, it follows that $\{0\}$ is dense in $\mathbb{R}$ with this topology.

But on the other hand $\mathbb{R}\setminus\{0\}$ is discrete, because given $x \in \mathbb{R}\setminus\{0\} \Rightarrow \{x, 0\}$ is open, which implies that $\{x\}$ is relatively open, and uncountable discrete spaces cannot be separable because in order to have separability we need to have the existence of a countable dense subset, and in a discrete space, no proper subset can be dense. $\blacksquare$

Problem 6.

Suppose that $(\mathbb{X}, \tau)$ is a topological space that has a countable basis. Prove that X is separable.

Proof.

We know that a topological space $(\mathbb{X}, \tau)$ is separable if there exists a countable subset $A \subset \mathbb{X}$ such that $\bar{A} = \mathbb{X}$.

Let $(\mathbb{X}, \tau)$ be a topological space, and let $\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$ be a countable base for $\mathbb{X}$.

Now, let's consider the set $A = \{a_n \mid n \in \mathbb{N} \text{ and } a_n \in B_n\}$, we claim, that this is a countable dense subset of $\mathbb{X}$.

Clearly A is countable, this is by construction.

Now, let's prove that this set is dense, that is $\bar{A} = X$, but we know that if $x \in \bar{A} \Rightarrow \forall\, \mathcal{U} \in \tau \;\; \mathcal{U} \cap A \neq \phi$.

So, let $\mathcal{U}$ be an open set in $\mathbb{X}$, because $\mathcal{B}$ is basis, it follows that there exist $k \in \mathbb{N}$ such that $B_k \subseteq \mathcal{U}$, but by the definition of A, there exist $x_k \in B_k$, thus $x \in \mathcal{U} \cap A$, $\Rightarrow \mathcal{U} \cap A \neq \phi$ and therefore A is dense in $\mathbb{X}$. ∎