# Problem 1: Evolution of Universe with general Energy of Particles

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The Friedmann equation is given by

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

where

$$\rho = \frac{\rho_0 R_0^3}{R^3}.$$

Problem 1. Rewrite the Friedmann equation by replacing derivatives with respect to the time variable $t$ by derivatives with respect to the conformal time variable $\eta$ defined by

$$dt = Rd\eta$$

Solution 1. By the chain rule, we have

$$\frac{dR}{dt} = \frac{dR}{d\eta}\frac{d\eta}{dt} \implies \frac{dR}{dt} = \frac{dR}{d\eta}\frac{1}{R},$$

and from this, the Friedmann equation is rewriten as

$$\left(\frac{dR}{d\eta}\frac{1}{R^2}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mR^2},$$

thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi GR^4}{3}\rho + \frac{2ER^4}{mR^2},$$

but $\rho$ is a function of $R$, thus

$$\left(\frac{dR}{d\eta}\right)^2 = \frac{8\pi\rho_0 R_0^3 G}{3}R + \frac{2E}{m}R^2.$$

and by making

$$\alpha = \frac{8\pi\rho_0 R_0^3 G}{3}, \beta = \frac{2E}{m},$$

we have

$$\left(\frac{dR}{d\eta}\right)^2 = \alpha R + \beta R^2.$$

Problem 2. Use separation of variables to re-write the Friedmann equation as an expression for $\eta$ in terms of an integral of $R$.

Solution 2. By separation of variables we have

$$\frac{dR}{\sqrt{\alpha R + \beta R^2}} = d\eta,$$

thus, the problem now resides on solving the preovious integral.

Problem 3. Carry out the integral and invert to obtain $R(\eta)$; simplify for $E > 0$, $E = 0$, and $E < 0$.
Solution 3. In the variables that I'm using $E = 0 \implies \beta = 0$, $E > 0 \implies \beta > 0$, and $E < 0 \implies \beta < 0$, So let's proceed accordingly.

**Case 1**, $\beta = 0$: in this case, the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha}{2}\eta^2$$

**Case 2**, $\beta < 0$: Here, I'm going to add the sign of $\beta$ by hand (not elegant at all, but just to be more explicit),

$$\int \frac{dR}{\sqrt{\alpha R - \beta R^2}} = \int d\eta,$$

and here the idea is to make a change of variables (I attach the notes with the algebra at the end of the document). Having done that, the solution reads

$$R(\eta) = \frac{\alpha}{\beta}\cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right)$$

**Case 3**, $\beta > 0$: Here the integral becomes

$$\int \frac{dR}{\sqrt{\alpha R + \beta R^2}} = \int d\eta,$$

with solution given by

$$R(\eta) = \frac{\alpha e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}},$$

and again, details about the algebra are attached at the end of the document.

Problem 4. Determine $t(\eta)$; again simplify for $E > 0$, $E = 0$, and $E < 0$.
Solution 4. We know that

$$dt = Rd\eta,$$

and in the previous bullet we found exactly $R(\eta)$, thus we only need to integrate, this is

$$t(\eta) = \int d\eta R(\eta).$$

**Case 1**, $\beta = 0$: in this case we have

$$t(\eta) = \frac{\alpha}{2} \int d\eta \left(\eta^2\right),$$

with solution given by

$$t = \frac{\alpha}{6}\eta^3.$$

**Case 2**, $\beta < 0$: here we have

$$t(\eta) = \frac{\alpha}{\beta} \int d\eta \cos^2\left(\frac{\sqrt{\beta}}{2}\eta\right),$$

with solution

$$t = \frac{\alpha}{\beta} \frac{2\sin\left(\frac{\sqrt{\beta}}{2}\eta\right)}{\sqrt{\beta}}$$

**Case 3**, $\beta > 0$: and finally, we have

$$t(\eta) = \alpha \int d\eta \frac{e^{\alpha\eta}}{1 - \beta e^{\alpha\eta}}$$

with solution given by

$$t(\eta) = -\frac{\log\left(1 - \beta e^{\alpha x}\right)}{\beta}.$$

Problem 5. Using the above pair of parametric equations, plot $R(t)$ (use some plotting software, such as Mathematica) for 3 different choices of $E$, namely $E > 0$, $E = 0$, and $E < 0$.

Solution 5. By combining the two previous parts we can write a parametric equation

$$\gamma(\eta) = (R(\eta), t(\eta)),$$

therefore.

**Case 1**, $\beta = 0$: in this case we have

$$\gamma(\eta) = \frac{\alpha}{2}\left(\eta^2, \eta^3\right)$$

**Case 2**, $\beta < 0$: again, we have

$$\gamma(\eta) = \frac{\alpha}{\beta}\left(\cos^2\left(\frac{\sqrt{\beta}}{2}t\right), \frac{2}{\sqrt{\beta}}\sin\left(\frac{\sqrt{\beta}}{2}t\right)\right)$$

**Case 3**, $\beta > 0$: and finally

$$\gamma(\eta) = \left(\frac{\alpha e^{\alpha\eta}}{1-\beta e^{\alpha\eta}}, -\frac{\log\left(1-\beta e^{\alpha x}\right)}{\beta}\right)$$

The plots are shown in the next page.