# EXAM REDO

EMMANUEL FLORES

Let $X$ be a Hausdorff space. Prove that $X$ is compact if and only of for any open set $O$ in $X$ and any collection of closed sets $\{C_\alpha\}_{\alpha\in\Lambda}$ such that $\cap_{\alpha\in\Lambda}C_\alpha \subset O$, then there exist a finite number of the sets $C_\alpha$ whose intersection lies in O.

*Proof.* $\implies$

Let $O$ be an open set in $X$, and $\{C_\alpha\}_{\alpha\in\Lambda}$ be a collection of closed sets in $X$ such that $\cap_{\alpha\in\Lambda}C_\alpha \subset O$. Since the intersection of $C_\alpha$'s in contained in $O$, by taking complements we have

$$X \setminus O \subset \cup_{\alpha\in\Lambda}(X \setminus C_\alpha),$$

but becasue $O$ is closed, its complement is open, and for the same reason; each one of $X \setminus C_\alpha$ is open. Now we just found an open cover of the closed set $X \setminus O$, this is $\cup_{\alpha\in\Lambda}(X \setminus C_\alpha)$. Now, since $X$ is compact, and $X \setminus O$ is closed, it follows that $X \setminus O$ is also compact. Thus there exists a finite subcover

$$\{X \setminus C_{\alpha_1}, \cdots X \setminus C_{\alpha_n}\}$$

of $X \setminus O$, this is

$$X \setminus O \subset (X \setminus C_{\alpha_1}) \cup \cdots \cup (X \setminus C_{\alpha_n}),$$

and taking complements again, we have

$$C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} \subset O.$$

$\impliedby$

Now, let's prove the other direction. Let $\{U_\alpha\}$ be an open cover of $X$. Since each one of the $U_\alpha$'s is open, their complement is closed, and even more, their intersection is empty, since the union of all of them covers the whole space $X$. This is $\cap_{\alpha\in\Lambda}(X \setminus U_\alpha) = \emptyset$. On the other hand, the empty set is subset of any set, in particular any open set; so by taking any open set $U_\beta$ from the original cover, we have $\cap_{\alpha\in\Lambda}(X \setminus U_\alpha) \subset U_\beta$, and by assumption, there exist finitely many sets $X \setminus U_{\alpha_1} \cdots X \setminus U_{\alpha_n}$ such that

$$(X \setminus U_{\alpha_1} \cap \cdots \cap (X \setminus U_{\alpha_n})) \subset U_\beta,$$

and taking the complements back again, we have

$$X \setminus U_\beta \subset U_{\alpha_1} \cup \cdots U_{\alpha_n},$$

and from this it follows that $X = U_\beta \cup (X \setminus U_\beta) \subset$, this is we have a finite subsocer of $X$, thus $X$ is compact. $\square$

*Date*: November 25, 2024.