GR-HW-01

J Emmanuel Flores

September 16, 2025

Problem 1 (Two-Body Problem) *By performing a Lorentz transformation, show that $T_{\mu\nu}B^{\mu}A^{\nu}$ is a Lorentz scalar*

Proof 1 *By definition a Lorentz scalar is an entity for which the following relationship holds*

$$\bar{T}_{\mu\nu}\bar{B}^{\mu}\bar{A}^{\nu} = T_{\mu\nu}B^{\mu}A^{\nu}, \tag{1}$$

this is, it has the same form under a Lorentz transformation. So let's proceed with the proof. By definition, we have the following

$$\bar{T}_{\mu\nu} = \Lambda_{\mu}^{\alpha_1}\Lambda_{\nu}^{\alpha_2}T_{\alpha_1\alpha_2} \tag{2}$$

and as well for the other two tensors of order one

$$\bar{B}^{\mu} = \Lambda_{\alpha_3}^{\mu}B^{\alpha_3}, \;\; \bar{A}^{\nu} = \Lambda_{\alpha_4}^{\nu}A^{\alpha_4}, \tag{3}$$

and from this we have that

$$\bar{T}_{\mu\nu}\bar{B}^{\mu}\bar{A}^{\nu} = \Lambda_{\mu}^{\alpha_1}\Lambda_{\nu}^{\alpha_2}\Lambda_{\alpha_3}^{\mu}\Lambda_{\alpha_4}^{\nu}T_{\alpha_1\alpha_2}B^{\alpha_3}A^{\alpha_4}. \tag{4}$$

After rearangin the terms and using the fact that

$$\Lambda_{\mu}^{\alpha_1}\Lambda_{\alpha_3}^{\mu} = \delta_{\alpha_3}^{\alpha_1}, \;\; \Lambda_{\nu}^{\alpha_2}\Lambda_{\alpha_4}^{\nu} = \delta_{\alpha_4}^{\alpha_2}, \tag{5}$$

then we have

$$\bar{T}_{\mu\nu}\bar{B}^{\mu}\bar{A}^{\nu} = \delta_{\alpha_3}^{\alpha_1}\delta_{\alpha_4}^{\alpha_2}T_{\alpha_1\alpha_2}B^{\alpha_3}A^{\alpha_4}, \tag{6}$$

and then the RHS of the previous equation takes the form

$$T_{\alpha_1\alpha_2}B^{\alpha_1}A^{\alpha_2}, \tag{7}$$

but since those are dummy indices, we can change them to make the equation prettier, leaving us with

$$\bar{T}_{\mu\nu}\bar{B}^{\mu}\bar{A}^{\nu} = T_{\mu\nu}B^{\mu}A^{\nu}, \tag{8}$$

just as we wanted.