MATH 171: HOMEWORK 1

J. EMMANUEL FLORES

Problem 1. Let $f : X \to Y$ be a function and $A, B \subset Y$.

(1) Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
(2) If f is injective and $y \in Y$, then $f^{-1}(y) = f^{-1}(\{y\})$ contains at most one point.
(3) If $f$is surjective and $y \in Y$, then $f^{-1}(y) = f^{-1}(\{y\})$ contains at least one point.

Solution. Let X and Y be sets, $f : X \to Y$ a function and $A, B \subset X$.

(1) By definition, the inverse image is given by

$$f^{-1}(B) = \{x, f(x) \in B\},$$

thus, let

$$x \in f^{-1}(A \cup B) \iff x \in X, f(x) \in A \cup B,$$

it follows that

$$\begin{aligned} & x \in X, f(x) \in A, \text{ or } f(x) \in B, \\ \iff & x \in X, f(x) \in A \text{ or } x \in X, f(x) \in B, \end{aligned}$$

thus, we have the

$$f^{-1}(A \cup B) \subset f^{-1}(A) \cup f^{-1}(B),$$

and

$$f^{-1}(A) \cup f^{-1}(B) \subset f^{-1}(A \cup B),$$

leaving us with

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

On the other hand, let

$$\begin{aligned} x \in f^{-1}(A \cap B) &\iff x \in X, f(x) \in A \cap B \\ &\iff x \in X, f(x) \in A \text{ and } f(x) \in B, \end{aligned}$$

thus

$$x \in f^{-1}(A) \cap f^{-1}(B).$$

(2) Let's assume that$f$ is injective, $y \in Y$, and we consider $x_1, x_2 \in X$ such that

$$x_1, x_2 \in f^{-1}(\{y\}) \implies f(x_1) = y \quad \& \quad f(x_2) = y,$$

$$\implies f(x_1) = y = f(x_2) \implies f(x_1) = f(x_2),$$

but f is injective, thus it follows that

$$x_1 = x_2,$$

therefore, $f^{-1}(\{y\})$ contains at most one point.

(3) Let f be a surjective function, this is

$$\forall y \in Y, \exists x \in X \quad \text{s.t} \quad y = f(x),$$

thus if $y \in Y$, and we consider $f^{-1}(\{y\})$, there exists, at least one $x \in X$ such that

$$y = f(x),$$

thus $f^{-1}(\{y\})$ contains at least one point.

Problem 2. Prove that the union of countably many countable sets is countable.

Solution. Let's consider the family $\{E_n\}$ where E_n is countable, for each $n \in \mathbb{N}$. Thus we want to prove that

$$\cup_{n\in\mathbb{N}} E_n,$$

is also countable. Now, because each E_n is countable, then, there exist a bijection between E_n and $\mathbb{N}$, for each $n \in \mathbb{N}$,this is, we can enumerate all the elements of E_n, in particular we can make that enumeration as follows

$$E_n = \{x_{nk} : k = 1, 2, \cdots\},$$

for all $n \in \mathbb{N}$, and now, let's make an enumeration of $\cup_{n\in\mathbb{N}} E_n$, as follows

$$\begin{array}{cccccc}
x_{11} & x_{12} & x_{13} & x_{14} & \cdots & E_1 \\
x_{21} & x_{22} & x_{23} & x_{24} & \cdots & E_2 \\
x_{31} & x_{32} & x_{33} & x_{34} & \cdots & E_3 \\
x_{41} & x_{42} & x_{43} & x_{44} & \cdots & E_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array},$$

this is, we have an enumeration

$$x_{11}; x_{21}, x_{21}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; \cdots,$$

in which all the elements of $\cup_{n\in\mathbb{N}} E_n$ have a tag, thus we have constructed a bijection between $\cup_{n\in\mathbb{N}} E_n$ and $\mathbb{N}$, and therefore the union of countably many countable sets is countable.

Problem 3. For a subset A of $\mathbb{R}$, let $\delta(A) = \sup\{|x-y|, x, y \in A\}$ be the diameter of A. Prove that $\delta(A) < \infty$ if and only if A is bounded.

Solution. Let $A \subset \mathbb{R}$, such that $A \neq \emptyset$, and let's consider

$$\delta(A) = \sup\{|x-y|, x, y \in A\}.$$

($\Longleftarrow$)Let's suppose that A is bounded, and in particular that implies that A is bounded from above, that is,

$$\exists M/2 \in \mathbb{R} \quad \text{s.t} \quad x \leq M/2, \forall x \in A,$$

on the other hand, we have that

$$\begin{aligned} |x-y| = |x-0+0-y| &\leq |x-0| + |0-y|, \\ &\leq |x| + |-y|, \\ &\leq |x| + |y| \leq M/2 + M/2, \\ &\leq M, \end{aligned}$$

thus, we have that

$$|x-y| \leq M, \quad \forall x, y \in A,$$

in other words the set

$$\{|x-y|, x, y \in A\},$$

is bounded from above, and is not empty, which implies that the sup $\{|x-y|, x, y \in A\}$ exists, and from that it follows that

$$\delta(A) < \infty.$$

($\Longrightarrow$)Now, let's suppose that $\delta(A) < \infty$, this is

$$\delta(A) = \sup\{|x-y|, x, y \in A\} < \infty,$$

thus, there exist $M \in \mathbb{R}$ such that

$$\begin{aligned} |x-y| &\leq M, \\ |x-0+0-y| &\leq M, \\ |x-0| + |0-y| &\leq M, \\ |x| + |-y| &\leq M, \end{aligned}$$

and from this it follows that

$$|x| \leq M - |y|,$$

if $M - |y| < M$, then $M - |y|$ is not an upper bound of A, thus

$$|x| \leq M,$$

and therefore A is bounded.

Problem 4. Let $\{x_n\}$ be a Cauchy sequence of real numbers. In this problem we prove that this sequence converges. For each $n \geq 1$, let $A_n = \{x_k : k \geq n\}$.

(1) Prove that the sequence of diameters $\{\delta(A_n)\}$ is decreasing and tends to 0.
(2) Justify why $\alpha_n = \inf A_n$ and $\beta_n = \sup A_n$ exist, and that $\{\alpha_n\}$ is increasing while $\{\beta_n\}$ is decreasing with $\alpha_n \leq \beta_n$.

(3) Conclude that $\alpha = \lim \alpha_n$ and $\beta = \lim \beta_n$ both exist, and that $\alpha, \beta \in [\alpha_n, \beta_n]$ for all $n \geq 1$.
(4) Conclude that $\alpha = \beta = \lim x_n$.

Solution. Let's consider $\{x_n\}$ be a Cauchy sequence of real numbers and

$$A_n = \{x_k : k \geq n\} \quad \forall n \geq 1.$$

(1) Now, for each A_n we have that

$$A_{n+1} \subset A_n, \quad \forall n \in \mathbb{N},$$

thus we have that the sequence is decreasing, this is

$$\delta(A_{n+1}) \leq \delta(A_n).$$

On the other hand, using the condition of Cauchy sequence, $\{x_n\}$, there exist $\epsilon > 0$, and $N \in \mathbb{N}$, such that for all $m, n > N$, we have

$$|x_n - x_m| < \epsilon,$$

and we and from this it follows that the diameters

$$\delta(A_n) = \sup\left\{\left|x_{k_i} - x_{k_j}\right|; k_i, k_j > n\right\},$$

tend to zero.

(2) Now, because of the Cauchy sequence property it follows that A_n is a non-empty bounded set of real numbers which implies that the infimum α_n and supremum β_n exist. We also have that

$$A_{n+1} \subset A_n, \quad \forall n \in \mathbb{N},$$

which implies that

$$\alpha_n \leq \alpha_{n+1} \quad \& \quad \beta_{n+1} \leq \beta_n,$$

from this we have that

$$\{\alpha_n\} \text{ is increasing,}$$

whereas

$$\{\beta_n\} \text{ is decreasing,}$$

is decreasing, but we due to the fact that α_n and β_n are the infimum and supremum of A_n, respectively, it follows that

$$\alpha_n \leq x_k \leq \beta_n, \quad \forall k \geq n.$$

(3) Because $\{\alpha_n\}$ is increasing and bounded from above by any β_n, it follows that it has a limit, and let's call it α. On the other hand, because $\{\beta_n\}$ is decreasing and bounded from below by any α_n, it also follows that it has a limit, and let's call it β. Thus we have

$$\forall n \in \mathbb{N}, \quad \alpha_n \leq \alpha \leq \beta \leq \beta_n,$$

which implies that

$$\alpha \in [\alpha_n, \beta_n] \quad \& \quad \beta \in [\alpha_n, \beta_n] \quad \forall n.$$

(4) Now, we have

$$\alpha_n \leq x_n \leq \beta_n, \quad \forall n \in \mathbb{N},$$

if we take the limit as $n \to \infty$, we get

$$\alpha \leq \lim_{n\to\infty} x_n \leq \beta,$$

but α and β are both limits of the sequence $\{x_n\}$,and the limit is unique, thus it follows that

$$\alpha = \beta = \lim_{n\to\infty} x_n,$$

which completes the proof.

Problem 5. Show why the intersection of an infinite number of open sets in a topological space $(X, \mathcal{T})$ is not necessarily open.

Solution. Let's consider the topological space $(\mathbb{R}, \tau)$, where τ is the usual topology in the real line, in this case the open sets are open intervals, that is, let $x, y \in \mathbb{R}$,

$$(x, y) \quad \text{is open},$$

and now, let's consider the collection of open sets; let $n \in \mathbb{N}$

$$\mathcal{U}_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \text{is open } \forall n \in \mathbb{N}.$$

If we allow that the arbitrary intersection of open sets is also an open set, we can make

$$\cap_{n\in\mathbb{N}}\mathcal{U}_n = \{0\},$$

which is not open.

Problem 6. Recall that the Euclidean distance of $\mathbb{R}^n$ is given by

$$d(x, y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2\right)^{1/2}, \quad \text{for } x, y \in \mathbb{R}^n.$$

The standard topology $\mathcal{T}_{std}$ of $\mathbb{R}^n$ is defined as follows $\mathcal{U} \subset \mathbb{R}^n$ is open if and only if for each $p \in \mathcal{U}$ there exist $\epsilon_p > 0$ such that

$$B(p, \epsilon_p) = \{x \in \mathbb{R}^n : d(x, p) < \epsilon_p\} \subset \mathcal{U}.$$

Prove that $(\mathbb{R}^n, \mathcal{T}_{std})$ is a topological space.

Solution. In order to prove that $(\mathbb{R}^n, \mathcal{T}_{std})$ is a topological space we need to prove that $\emptyset, \mathbb{R}^n \in \mathcal{T}_{std}$, the arbitrary union of open sets is open and the finite intersection of open sets is open.

(1) Let's suppose that $\emptyset \notin \mathcal{T}_{std}$, which implies that $\exists x \in \emptyset$ with some property, but this is a contradiction, therefore
$$\emptyset \in \mathcal{T}_{std},$$
on the other hand, let $x \in \mathbb{R}^n$, we can always find an $\epsilon > 0$, such that
$$B(p, \epsilon) \subset \mathbb{R}^n,$$
therefore
$$\mathbb{R}^n \in \mathcal{T}_{std}.$$
(2) Let $\mathcal{U}_\alpha$ be a family of open sets, indexed by $\alpha \in \lambda$ and let's consider
$$\cup_{\alpha\in\lambda}\mathcal{U}_\alpha,$$
if $\cup_{\alpha\in\lambda}\mathcal{U}_\alpha = \emptyset$,then, there's nothing to prove, because $\emptyset \in \mathcal{T}$, thus, let's assume that $\cup_{\alpha\in\lambda}\mathcal{U}_\alpha = \emptyset$, and let
$$x \in \cup_{\alpha\in\lambda}\mathcal{U}_\alpha,$$
it follows that there exist an $i \in \lambda$ such that
$$x \in \mathcal{U}_i \implies \exists\epsilon_i > 0 \quad \text{s.t} \quad x \in B(x, \epsilon_i) \subset \mathcal{U}_i,$$
and from this it follows that
$$x \in B(x, \epsilon_i) \subset \mathcal{U}_i \subset \cup_{\alpha\in\lambda}\mathcal{U}_\alpha,$$
thus
$$B(x, \epsilon_i) \subset \cup_{\alpha\in\lambda}\mathcal{U}_\alpha,$$
therefore, the arbitrary union of open sets is open.

(3) Now, let's consider the following finite collection of sets
$$\{\mathcal{U}_i : i \in \{1, 2, \cdots, n\}, \quad U_i \in \mathcal{T}_{std}\},$$
and let's consider the intersection
$$\cap_{i=1}^n \mathcal{U}_i,$$
if
$$\cap_{i=1}^n \mathcal{U}_i = \emptyset,$$
then, there's nothing to prove, because $\emptyset$ is open. Then, let's consider that $\cap_{i=1}^n \mathcal{U}_i \neq \emptyset$, and let
$$x \in \cap_{i=1}^n \mathcal{U}_i,$$
thus, it follows that for each $\mathcal{U}_i \in \mathcal{T}$, there is an epsilon ball $B(x, \epsilon_i)$, such that
$$x \in B(x, \epsilon_i) \subset \mathcal{U}_i,$$
now, let be $\epsilon > 0$, such that
$$\epsilon = \min\{\epsilon_i : i \in \{1, 2, \cdots, n\}\},$$

and let's consider $B(x,\epsilon)$, thus, it follows that $x \in B(x,\epsilon)$, and

$$\begin{gathered} B(x,\epsilon) \subset B(x,\epsilon_i) \subset \mathcal{U}_i, \\ \implies B(x,\epsilon) \subset \mathcal{U}_i, \quad \forall i \in \{1,2,\cdots,n\}, \\ \implies B(x,\epsilon) \subset \cap_{i=1}^{n} \mathcal{U}_i, \end{gathered}$$

therefore $\cap_{i=1}^{n} \mathcal{U}_i$ is open.