QUANTUM THEORY II | ASSIGNMENT 1

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Problem 1. (Problem 3.2)

Solution. Let's compute the eigenvalues and eigenvector of the Pauli matrix σ_y, which we know is defined by

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The eigenvalue problem, for this matrix is thus

$$\sigma_y x = \lambda x,$$

which is equivalent to

$$\det(\sigma_y - \lambda \mathrm{I}) = 0,$$

this is

$$\det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0, \iff \lambda^2 + i^2 = 0,$$
$$\iff \lambda^2 = 1 \iff \lambda_\pm = \pm 1.$$

Now, with these, we can find the correspondent eigenvectors, let's begin with $\lambda_+ = 1$,thus

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \iff -iy = x \iff y = ix,$$

now, if we choose $x = 1$, then we have the following normalized vector

$$\lambda_+ \to x_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},$$

on the other hand, for $\lambda_- = -1$, we have

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} x \\ y \end{pmatrix} \iff -iy = -x \iff y = -ix,$$

thus, if we choose $x = 1$, the normalized eigenvector will be

$$\lambda_- \to x_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

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Now, let's suppose that an electron is in the spin state $|\phi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, and we measure s_y we're interested in the probability of the measure result $\hbar/2$. Now, to answer this question we're going to make use of the previous eigenvectors, in this case x_+, that's the case because we want to compute $|\langle x_+|\phi\rangle|^2$, thus

$$\langle x_+|\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\alpha - i\beta\right),$$

$$\implies |\langle x_+|\phi\rangle|^2 = \left(\frac{1}{\sqrt{2}} \left(\alpha - i\beta\right)\right)^2.$$

Problem 2. (Problem 3.5)

Solution. a) Before anything, it should be clear from the start that we're dealing with a product of spaces, and following Sakurai's notation, I will call this space as the direct product of the spaces for the electron and the positron. Now, the full Hamiltonian for this problem is given by

$$H = A\mathbf{S}^{(e^-)} \cdot \mathbf{S}^{(e^+)} + \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right),$$

now, we're interested in the limit $A \to 0$, the Hamiltonian becomes

$$H = \left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right),$$

we're also assuming that the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$, and finally, that we're working with the z-basis, thus using the definition of the S_z operator in the z-basis, in general we have

$$S_z = \frac{\hbar}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right],$$

it's important to notice that we have the same operator for each one of the spaces but it acts in different kets, thus we have

$$S_z^{(e^-)}\chi_+^{(e^-)} = \frac{\hbar}{2}\chi_+^{(e^-)} \;\&\; S_z^{(e^+)}\chi_-^{(e^-)} = -\frac{\hbar}{2}\chi_-^{(e^-)},$$

then

$$\left(S_z^{(e^-)} - S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)} = S_z^{(e^-)}\chi_+^{(e^-)}\chi_-^{(e^+)} - S_z^{(e^+)}\chi_+^{(e^-)}\chi_-^{(e^+)},$$

$$\implies \left(S_z^{(e^-)} - S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)} = \frac{\hbar}{2}\chi_+^{(e^-)}\chi_-^{(e^+)} - \left(-\frac{\hbar}{2}\right)\chi_+^{(e^-)}\chi_-^{(e^+)},$$

$$\implies \left(S_z^{(e^-)} - S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)} = \hbar\chi_+^{(e^-)}\chi_-^{(e^+)},$$

thus

$$\left(\frac{eB}{mc}\right)\left(S_z^{(e^-)} - S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)} = \hbar\left(\frac{eB}{mc}\right)\chi_+^{(e^-)}\chi_-^{(e^+)},$$

which is equivalent to

$$H\chi_+^{(e^-)}\chi_-^{(e^+)} = \hbar\left(\frac{eB}{mc}\right)\chi_+^{(e^-)}\chi_-^{(e^+)},$$

therefore $\chi_+^{(e^-)}\chi_-^{(e^+)}$ it is en eigenfunction.

b) In this case we're interested in the following limit $\frac{eB}{mc} \to 0$, thus, the Hamiltonian becomes

$$H = A\mathbf{S}^{(e^-)} \cdot \mathbf{S}^{(e^+)},$$

which can be written as

$$H = A\left(S_x^{(e^-)} \cdot S_x^{(e^+)} + S_y^{(e^-)} \cdot S_y^{(e^+)} + S_z^{(e^-)} \cdot S_z^{(e^+)}\right)$$

and for this we need the operators S_x and S_y expressed in this basis, which are

$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right],$$

$$S_y = \frac{\hbar}{2}\left[-i|+\rangle\langle-| + i|-\rangle\langle+|\right],$$

now, if we apply those operators to $\chi_+^{(e^-)}$ we have

$$S_x^{(e^-)}\chi_+^{(e^-)} = \frac{\hbar}{2}\chi_-^{(e^-)},$$

$$S_y^{(e^-)}\chi_+^{(e^-)} = \frac{i\hbar}{2}\chi_-^{(e^-)},$$

whereas for $\chi_-^{(e^+)}$ we have

$$S_x^{(e^+)}\chi_-^{(e^+)} = \frac{\hbar}{2}\chi_+^{(e^+)},$$

$$S_y^{(e^+)}\chi_-^{(e^+)} = -\frac{i\hbar}{2}\chi_+^{(e^+)},$$

therefore, we have

$$\left(S_x^{(e^-)}\cdot S_x^{(e^+)} + S_y^{(e^-)}\cdot S_y^{(e^+)} + S_z^{(e^-)}\cdot S_z^{(e^+)}\right)\chi_+^{(e^-)}\chi_-^{(e^+)} =$$

$$\left(\frac{\hbar}{2}\right)^2\chi_-^{(e^-)}\chi_+^{(e^+)} - \left(\frac{i\hbar}{2}\right)^2\chi_-^{(e^-)}\chi_+^{(e^+)} + \left(\frac{\hbar}{2}\right)^2\chi_+^{(e^-)}\chi_-^{(e^+)},$$

which clearly is not an eigenfunction, thus in the limit $\frac{eB}{mc}\to 0$ is not an eigenfunction.

Problem 3. (Problem 3.6)

Solution. For spin 1, if we use the z-basis, we have the following states $|1\rangle, |0\rangle, |-1\rangle$ with eigenvalues $\hbar, 0, -\hbar$ respectively, then by construction the S_z operator will be diagonal, and it'll be given by

$$S_z = \hbar \left(|1\rangle\langle 1| + 0|0\rangle\langle 0| - |-1\rangle\langle -1| \right),$$

which in matrix notation is represented by

$$S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Now, this is one way of getting the matrix representation of the operator, on the other hand, we know that, in the general case, the matrix elements of the angular momentum are obtained via

$$\langle j'm'|J_z|jm\rangle = m\hbar\delta_{j'j}\delta_{m'm},$$

$$\langle j'm'|J_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar\delta_{j'j}\delta_{m'm\pm 1}.$$

And in order to make the notation more clear, I'm going to change a little the notation: in this case $j = j' = 1$, therefore we're going to drop that term from the ket-bra notation, and I'm going to make the following change $J \to S$, then, for the matrix elements of S_z we have

$$\langle m'|S_z|m\rangle = m\hbar\delta_{m'm},$$

and, as we can see, only the diagonal terms survive, each one of them contributes with $m = 1, m = 0, m = -1$, thus, the matrix representation is

$$S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

just as we previously found. On the other hand, for the S_x operator, we know that

$$S_x = \frac{S_+ + S_-}{2},$$

thus, if we find the matrix representation for $S_\pm$ we find the matrix representation for S_x. Thus, we have

$$\langle m'|S_+|m\rangle = \sqrt{(1-m)(1+m+1)}\hbar\delta_{(m',m+1)},$$

but, from the previous equation, only the terms $m' = m + 1$ survive, which implies that the only surviving terms are $\{1, 0\}, \{0, -1\}$, thus

$$\langle 1|S_+|0\rangle = \sqrt{(1-0)(1+0+1)}\hbar\delta_{(1,0+1)} = \sqrt{2}\hbar,$$

$$\langle 0|S_+|-1\rangle = \sqrt{(1+1)(1-1+1)}\hbar\delta_{(0,-1+1)} = \sqrt{2}\hbar,$$

now, for the other operator, we have

$$\langle m'|S_-|m\rangle = \sqrt{(1+m)(1-m+1)}\hbar\delta_{(m',m-1)},$$

and again, almost all of the coefficients are zero, except the ones for which $m' = m-1$ holds, which implies that the only surviving terms are $\{0,1\}, \{-1,0\}$, thus

$$\langle 0|S_-|1\rangle = \sqrt{(1+1)(1-1+1)}\hbar\delta_{(0,1-1)} = \sqrt{2}\hbar,$$

$$\langle -1|S_-|0\rangle = \sqrt{(1+0)(1-0+1)}\hbar\delta_{(-1,0-1)} = \sqrt{2}\hbar,$$

therefore, the matrix representation for S_x is given by

$$S_x = \frac{1}{2}\left(S_+ + S_-\right),$$

$$\implies S_x \doteq \frac{\sqrt{2}}{2}\hbar\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\therefore S_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, with this at hand, we can perform the matrix multiplication given by

$$S_z\left(S_z + \hbar\mathbb{I}\right)\left(S_z - \hbar\mathbb{I}\right) \text{ and } S_x\left(S_x + \hbar\mathbb{I}\right)\left(S_x - \hbar\mathbb{I}\right),$$

for the first one, we have

$$S_z\left(S_z + \hbar\mathbb{I}\right)\left(S_z - \hbar\mathbb{I}\right) \doteq \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\left\{\left[\hbar\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\left[\hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right]\right\},$$

thus, we have

$$\implies S_z\left(S_z + \hbar\mathbb{I}\right)\left(S_z - \hbar\mathbb{I}\right) \doteq \hbar^3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

doing one of the matrix products

$$\implies S_z\left(S_z + \hbar\mathbb{I}\right)\left(S_z - \hbar\mathbb{I}\right) \doteq \hbar^3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and from that, we can immediately see that the matrix is zero in all its entries

$$\therefore S_z\left(S_z + \hbar\mathbb{I}\right)\left(S_z - \hbar\mathbb{I}\right) \doteq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

now, for the other operator, we have

$$S_x\left(S_x+\hbar\mathbb{I}\right)\left(S_x-\hbar\mathbb{I}\right)\doteq\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\times$$

$$\left\{\left[\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}+\hbar\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\left[\left[\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}-\hbar\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\right]\right\},$$

which after a little algebra can be proof that it also gives us a zero matrix

$$\therefore S_x\left(S_x+\hbar\mathbb{I}\right)\left(S_x-\hbar\mathbb{I}\right)\doteq\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Problem. (Problem 3.7)

Solution. In general, for an observable A, the Heisenberg equation is given by

$$\frac{dA}{dt} = \frac{1}{i\hbar}\left[A, H\right],$$

but using the fact that $[A, B] = -[B, A]$ and $\frac{1}{i} = -i$, then we can express the previous equation as

$$\frac{dA}{dt} = \left(-\frac{i}{\hbar}\right)\left(-\left[H, A\right]\right) = \frac{i}{\hbar}\left[H, A\right],$$

thus

$$\frac{dA}{dt} = \frac{i}{\hbar}\left[H, A\right].$$

Now, in this case, we have the following Hamiltonian

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_2^2}{I_2}\right),$$

where $\mathbf{K}$ is the angular momentum in the body frame, and we want to obtain the equation of motion for this $\mathbf{K}$ operator, which is

$$\frac{d\mathbf{K}}{dt} = \frac{i}{\hbar}\left[H, \mathbf{K}\right],$$

$$\implies \frac{d\mathbf{K}}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_2^2}{I_2}, \mathbf{K}\right],$$

now following the notation of H we can express $\mathbf{K}$ as follows

$$\mathbf{K} = K_1\mathbf{e}_1 + K_2\mathbf{e}_2 + K_3\mathbf{e}_3,$$

thus

$$\frac{d\mathbf{K}}{dt} = \frac{dK_1}{dt}\mathbf{e}_1 + \frac{dK_2}{dt}\mathbf{e}_2 + \frac{dK_3}{dt}\mathbf{e}_3,$$

and from that if follows that

$$\frac{dK_1}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}, K_1\right],$$

$$\frac{dK_2}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}, K_2\right],$$

$$\frac{dK_3}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}, K_3\right].$$

So we have to calculate the previous commutators, but before doing any algebra, let's remember the following property of the commutator

$$\left[A^2, B\right] = A\left[A, B\right] + \left[A, B\right]A,$$

and from that it's easy to see that any commutator $\left[K_i^2, K_i\right] = 0$, thus, we just have to calculate

$$\frac{dK_1}{dt} = \frac{i}{2\hbar}\left[\frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}, K_1\right],$$

$$\frac{dK_2}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1}+\frac{K_3^2}{I_3},K_2\right],$$
$$\frac{dK_3}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1}+\frac{K_2^2}{I_2},K_3\right].$$

Then, for K_1, we have

$$\frac{dK_1}{dt} = \frac{i}{2\hbar}\left[\frac{K_2^2}{I_2}+\frac{K_3^2}{I_2},K_1\right] = \frac{i}{2\hbar}\left\{\frac{1}{I_2}\left[K_2^2,K_1\right]+\frac{1}{I_3}\left[K_3^2,K_1\right]\right\},$$

and using the previous property we have

$$\frac{dK_1}{dt} = \frac{i}{2\hbar}\left\{\frac{1}{I_2}K_2\left[K_2,K_1\right]+\frac{1}{I_2}\left[K_2,K_1\right]K_2+\frac{1}{I_3}K_3\left[K_3,K_1\right]+\frac{1}{I_3}\left[K_3,K_1\right]K_3\right\},$$

now, because this is an angular momentum, we expect that it follows the commutation relations for angular momentum, however, this system is rotating, so we have to change a little bit the relations that we have, this is for the following reason: in the development of all the relations we assume a coordinate system that remains unchanged, but that differs from the problem that we have at hand, which is just a difference between passive versus active rotations. If we go back to the matrix rotations around some axis, we must change the sign in one of the entries, which overall adds a minus sign to the relations, therefore, for this point of view, we have

$$[K_1,K_2] = -i\hbar K_3,$$
$$[K_2,K_3] = -i\hbar K_1,$$
$$[K_3,K_1] = -i\hbar K_2,$$

it follows that the time equation for K_1 can be write as

$$\frac{dK_1}{dt} = \frac{i}{2\hbar}\left\{\frac{1}{I_2}K_2\left(i\hbar K_3\right)+\frac{1}{I_2}\left(i\hbar K_3\right)K_2+\frac{1}{I_3}K_3\left(-i\hbar K_2\right)+\frac{1}{I_3}\left(-i\hbar K_2\right)K_3\right\},$$
$$\implies \frac{dK_1}{dt} = \frac{i}{2\hbar}\left\{\frac{i\hbar}{I_2}\left(K_2K_3+K_3K_2\right)+\frac{i\hbar}{I_3}\left(-K_3K_2-K_2K_3\right)\right\},$$
$$\implies \frac{dK_1}{dt} = \frac{i}{2\hbar}\left\{\left(\frac{i\hbar}{I_2}-\frac{i\hbar}{I_3}\right)K_2K_3+\left(\frac{i\hbar}{I_2}-\frac{i\hbar}{I_3}\right)K_3K_2\right\},$$
$$\implies \frac{dK_1}{dt} = \frac{i}{2\hbar}\left(\frac{i\hbar}{I_2}-\frac{i\hbar}{I_3}\right)\left(K_2K_3+K_3K_2\right),$$

but we can write the last term as the anti commutator, thus

$$\implies \frac{dK_1}{dt} = -\frac{1}{2}\left(\frac{1}{I_2}-\frac{1}{I_3}\right)\left\{K_2,K_3\right\},$$
$$\therefore \frac{dK_1}{dt} = \frac{I_2-I_3}{2I_2I_3}\left\{K_2,K_3\right\}.$$

Now, for K_2 we have to do the same calculation, this is

$$\frac{dK_2}{dt} = \frac{i}{2\hbar}\left[\frac{K_1^2}{I_1}+\frac{K_3^2}{I_3},K_2\right] = \frac{i}{2\hbar}\left(\left[\frac{K_1^2}{I_1},K_2\right]+\left[\frac{K_3^2}{I_3},K_2\right]\right),$$

$$\implies \frac{dK_2}{dt} = \frac{i}{2\hbar}\left(\frac{1}{I_1}K_1\left[K_1, K_2\right] + \frac{1}{I_1}\left[K_1, K_2\right]K_1 + \frac{1}{I_3}K_3\left[K_3, K_2\right] + \frac{1}{I_3}\left[K_3, K_2\right]K_3\right),$$

$$\implies \frac{dK_2}{dt} = \frac{i}{2\hbar}\left(\frac{1}{I_1}K_1\left(-i\hbar K_3\right) + \frac{1}{I_1}\left(-i\hbar K_3\right)K_1 + \frac{1}{I_3}K_3\left(i\hbar K_1\right) + \frac{1}{I_3}\left(i\hbar K_1\right)K_3\right),$$

$$\implies \frac{dK_2}{dt} = \frac{i}{2\hbar}\left(\left(-\frac{1}{I_1} + \frac{1}{I_3}\right)i\hbar K_1 K_3 + \left(-\frac{1}{I_1} + \frac{1}{I_3}\right)i\hbar K_3 K_1\right),$$

$$\implies \frac{dK_2}{dt} = \frac{i}{2\hbar}\left(\left(-\frac{1}{I_1} + \frac{1}{I_3}\right)i\hbar\left\{K_1, K_3\right\}\right) = -\frac{1}{2}\left(-\frac{1}{I_1} + \frac{1}{I_3}\right)\left\{K_1, K_3\right\},$$

$$\therefore \frac{dK_2}{dt} = \frac{I_3 - I_1}{2 I_3 I_1}\left\{K_3, K_1\right\},$$

and from this we can infer the last one of the equations

$$\frac{dK_3}{dt} = \frac{I_2 - I_1}{2 I_2 I_1}\left\{K_2, K_1\right\}.$$

And finally, in the classical limit we have

$$\frac{dK_1}{dt} = \frac{I_2 - I_3}{2 I_2 I_3}\left\{K_2, K_3\right\} \to \frac{dK_1}{dt} = \frac{I_2 - I_3}{2 I_2 I_3}\left(2 K_2 K_3\right) = \frac{I_2 - I_3}{I_2 I_3}\left(K_2 K_3\right),$$

and we have to change from $K \to \omega$, thus

$$\frac{d\omega_1}{dt} = \frac{I_2 - I_3}{I_2 I_3}\left(\omega_2 \omega_3\right),$$

and the same for the other two components

$$\frac{d\omega_2}{dt} = \frac{I_3 - I_1}{I_3 I_1}\omega_3 \omega_1,$$

$$\frac{d\omega_3}{dt} = \frac{I_2 - I_1}{I_2 I_1}\omega_2 \omega_1,$$

and it's no surprise that this correspond to the Euler's equations of motion for rigid-body rotation.

Problem 4. (Problem 3.10)

Solution. The given sequence of Euler Rotations is given by

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right),$$

$$\implies \mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha,\beta,\gamma) = \begin{pmatrix} \exp\left[-i\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2} & -\exp\left[-i\left(\alpha-\gamma\right)/2\right]\cos\frac{\beta}{2} \\ \exp\left[i\left(\alpha-\gamma\right)/2\right]\cos\frac{\beta}{2} & \exp\left[i\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2} \end{pmatrix},$$

now, we know that these operator inherit the group properties of the rotations matrices, so, in general there should exist some rotation around some axis in such a way that corresponds to the combination of the given rotations. In math terms, we expect a rotation given by

$$\mathcal{D}^{\left(\frac{1}{2}\right)}(\mathbf{n},\phi) \doteq \exp\left(\frac{-i\sigma\cdot\mathbf{n}}{2}\phi\right) \doteq \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & \left(-in_x - n_y\right)\sin\left(\frac{\phi}{2}\right) \\ \left(-in_x + n_y\right)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) \end{pmatrix},$$

now, first let's notice that if we compute the trace of both matrices, we find a functional dependence on the angles, this is

$$\operatorname{tr}\left[\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha,\beta,\gamma)\right] = \exp\left[-i\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2} + \exp\left[i\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2},$$

$$\implies \operatorname{tr}\left[\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha,\beta,\gamma)\right] = \left(\cos\left[\left(\alpha+\gamma\right)/2\right] - i\sin\left[\left(\alpha+\gamma\right)/2\right]\right)\cos\frac{\beta}{2} +$$

$$\left(\cos\left[\left(\alpha+\gamma\right)/2\right] + i\sin\left[\left(\alpha+\gamma\right)/2\right]\right)\cos\frac{\beta}{2},$$

$$\implies \operatorname{tr}\left[\mathcal{D}^{\left(\frac{1}{2}\right)}(\alpha,\beta,\gamma)\right] = 2\cos\left[\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2},$$

and for $\mathcal{D}^{\left(\frac{1}{2}\right)}(\mathbf{n},\phi)$ we have something similar

$$\operatorname{tr}\left(\mathcal{D}^{\left(\frac{1}{2}\right)}(\mathbf{n},\phi)\right) = \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) + \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) = 2\cos\left(\frac{\phi}{2}\right),$$

$$\implies \operatorname{tr}\left(\mathcal{D}^{\left(\frac{1}{2}\right)}(\mathbf{n},\phi)\right) = 2\cos\left(\frac{\phi}{2}\right),$$

therefore, in order to find the angle, we have

$$\cos\left(\frac{\phi}{2}\right) = \cos\left[\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2},$$

$$\implies \phi = 2\cos^{-1}\left[\cos\left[\left(\alpha+\gamma\right)/2\right]\cos\frac{\beta}{2}\right],$$

while the previous expression is not very aesthetic, it makes clear, at least in theory, that we can always find this angle of rotation.

Problem 5. (Problem 3.19)

Solution. By definition, we have that

$$\mathbf{J}^2 = J_x^2 + J_y^2 + J_z^2,$$

and for now, let's focus on the first two terms of the RHS of the previous equation. So, we can write

$$J_+J_- = (J_x + iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 - i(J_xJ_y - J_yJ_x),$$

but, we also know, in virtue of the commutation relationships between the components of the angular momentum, that

$$[J_x, J_y] = i\hbar J_z,$$

thus, we have

$$-i(J_xJ_y - J_yJ_x) = -i(i\hbar J_z) = \hbar J_z,$$

and this implies that

$$J_x^2 + J_y^2 = J_+J_- - \hbar J_z,$$

therefore, we end up with

$$J^2 = J_+J_- - \hbar J_z + J_z^2,$$

thus, ordering the equation, we have

$$\mathbf{J}^2 = J_z^2 + J_+J_- - \hbar J_z,$$

just as we wanted.

Now, in order to determine the constant c_- we must do the following calculation

$$\langle j, m|J_-^\dagger J_-|j, m\rangle,$$

but, because of the definition of the ladder operators, we know that

$$J_-^\dagger J_- = J_+J_-,$$

thus, we can use use previous result as follows:

$$\langle j, m|J_-^\dagger J_-|j, m\rangle = \langle j, m|\mathbf{J}^2 - J_z^2 + \hbar J_z|j, m\rangle,$$

but, we know the eigenvalues of $\mathbf{J}$ and J_z are $j(j+1)\hbar^2$ and $m\hbar$ respectively, thus, for the previous expression we have

$$\langle j, m|J_-^\dagger J_-|j, m\rangle = j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2,$$

$$\implies \langle j, m|J_-^\dagger J_-|j, m\rangle = \hbar^2 \left[j(j+1) - m^2 + m\right],$$

and now, let's play a little with the algebra

$$\implies \langle j, m|J_-^\dagger J_-|j, m\rangle = \hbar^2 \left[j(j+1) - m^2 + m - mj + mj\right],$$

$$\implies \langle j, m|J_-^\dagger J_-|j, m\rangle = \hbar^2 \left[(j+m)(j-m+1)\right],$$

and therefore, we have

$$c_{-} = \hbar\sqrt{(j+m)(j-m+1)}.$$