Supersymmetry

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Motivation

Combining Bosonic and Fermionics Systems

By taking tensor products ($\otimes$) of bosonic and fermionic systems, the operators (bosonic of fermionic) will continue to act on the combined systems.

And the idea is to consider some very special operators:

- They appear to mix bosonic and fermionic systems.
- They commute with the hamiltonian H.

The supersymmetric oscillator

Bosonic Harmonic Oscillator

Definition
The Hamiltonian of a bosonic harmonic oscillator in d-dimensions, with state space $\mathcal{F}_d$ is given by:

$$H = \frac{1}{2}\hbar\omega\sum_{j=1}^{d}(a^\dagger_{B_j}a_{B_j} + a_{B_j}a^\dagger_{B_j}),$$

or

$$H = \sum_{j=1}^{d}(N_{B_j} + \frac{1}{2})\hbar\omega$$

where N_{B_j} has eigenvalues $n_{B_j} = 0, 1, 2, \ldots$

Fermionic Harmonic Oscillator

Definition
The Hamiltonian of a fermionic harmonic oscillator in d-dimensions, with state space $\mathcal{F}_d^\dagger$ is given by:

$$H = \frac{1}{2}\hbar\omega \sum_{j=1}^{d}(a_{F_j}^\dagger a_{F_j} - a_{B_j} a_{B_j}^\dagger),$$

or

$$H = \sum_{j=1}^{d}(N_{F_j} - \frac{1}{2})\hbar\omega$$

where N_{B_j} has eigenvalues $n_{B_j} = 0, 1$

Definition of the Full System

By taking the state spaces $\mathcal{F}_d$ and $\mathcal{F}_d^\dagger$ we can make the new state

$$\mathcal{H} = \mathcal{F}_d \otimes \mathcal{F}_d^\dagger$$

with a corresponding Hamiltonian given by

$$H = \sum_{j=1}^{d} (N_{B_j} + N_{F_j})\hbar\omega,$$

and the lowest energy state has energy $|0\rangle$ has energy 0.

Supersymmetric quantum mechanics with a superpotential

Supersymmetric quantum mechanics and differential forms