Electricity and Magnetism

Tufts University

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Short Assignment 2

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October 2, 2023

1. Q1: The Electric field in Coulomb's and Gauss' Law

   (a) Why does Gauss's law constitute a generalization of Coulomb's law? What are the elements of Coulomb's law, both fundamental and restrictive, that are still contained in the concept of electric field as defined by Gauss' law

Answer.

Coulomb's Law states the force between two point particles. With this at hand, you can define a new quantity, a new field called the electric field. In math, we have

$$\overrightarrow{F} = \frac{1}{4\pi\epsilon_0}\frac{q_0 q}{r^2}\overrightarrow{r} \implies \exists\, \overrightarrow{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\overrightarrow{r},$$

such that

$$\overrightarrow{F} = q_0\overrightarrow{E}$$

However, from the beginning, you state the law for point charges. With this at hand, you can generalize to a continuum distribution of charges using a Dirac delta function for each particle and performing a sum over the whole number of particles, i.e. you can state that

$$\rho = \sum_k q_k \delta\left(\overrightarrow{r} - \overrightarrow{r}_k\right).$$

On the other hand, when you state Gauss's Law, in its differential or integral form, you are allowing the possibility of a continuum distribution of the charges:

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \iff \oint d\overrightarrow{S} \cdot \overrightarrow{E} = \frac{Q}{\epsilon_0}$$

When you make the change to a point charge distribution, you end up with the electric field for a point charge, the same as you would obtain using Coulomb's law. But in this case, it was a particular case for Gauss's Law. Therefore, from a mathematical perspective, this law corresponds to a generalization of Coulomb's law.

On the other hand, as discussed previously, with Coulomb's Law, from the beginning, you're assuming point charges. In some sense, this is fundamental because we know matter and the charge is quantized.

However, on the other hand, when we're dealing with the macroscopic world, this becomes restrictive given the order of magnitude of the particles involved in such charge distributions. By the same reasoning, when you are dealing with Gauss's Law, you are considering the whole amount of charge enclosed in a given closed surface, which obviously can be seen as the sum of the charge of each particle

2. Q2: Applications of Gauss' law: Earnshaw's theorem

   (a) Use Gauss' law to demonstrate that a positive charge placed at the center of an equilateral triangle with equal negative charges –q at each summit of the triangle will not be in a stable equilibrium.

   (b) Use vector calculus to demonstrate that a generic constant electric dipole p cannot be in a stable equilibrium in a generic electrostatic field E.

   [Hint: Stable equilibrium required $\nabla \cdot \mathbf{F} < 0$. This can also be expressed in terms of the potential energy U. For an electric dipole, $U = \mathbf{p} \cdot \mathbf{E}$]

Solution.

a) From the theoryt of differential calculus, we all know that if we want to find the extremal points of a function we should compute the derivate and the derivative at the point in question shhould vanish. And even more, if we perform its second derivative we can obtain information about the point, i.e., if it's a maximum, minimum or an inflection point. Now, that's the case for functions of one variable, and for functions of several variables, the line of tought it's the same but we have to generalize the previous ideas. Now, let's write in mathematical terms, the things that I've just explained.

On the other hand, we can use the Gauss function to prove that the centroid of the equilateral triangle is an unstable point, which in fact is a consequence of the Earnshaw theorem which states that a collection of point charges (or magnets) cannot be maintained in a stable equilibrium solely by the electrostatic (or magnetic) interactions, i.e., in order to matain such a equilibrium configuration another forces must enter into the game.

For the problem at hand, we are goint to have that the total potential will be the artimetic sum of each individual potential, i.e.,

$$V_T = V_1 + V_2 + V_3, \tag{0.1}$$

and now, for the centroid to be placed at an equilibrium point, it must satisfy the following condition

$$\nabla \cdot \overrightarrow{F} < 0,$$

but, we know that the electric force is just the negative of the gradient of the potential, thus we can rewrite the previous condition in the following way:

$$\nabla^2 V \geq 0. \tag{0.2}$$

Now, because of the symmetry of the system, we're going to utilize speherical coordinates, and moreover, we also know that the potential for a single particle is given by the expression

$$V(r) = \frac{q}{4\pi\epsilon_0 r},$$

in which, we've choosen the refernce point at infinity. Therefore, for this problem, we have that the total potential, as stated in (0.1)is given by

$$V_T(r) = \frac{3q}{4\pi\epsilon_0 r}.$$

Now, we also know that the laplacian in polar coordinates is given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

but it's clear that in this case we only have a dependence in r, so we can drop the derivative with respect to θ and focus on the left part. Thus, we have that

$$\nabla^2 V_T(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{3q}{4\pi\epsilon_0 r}\right)\right),$$

$$\implies \nabla^2 V_T(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{3q}{4\pi\epsilon_0 r}\right) = \frac{3q}{4\pi\epsilon_0 r^3},$$

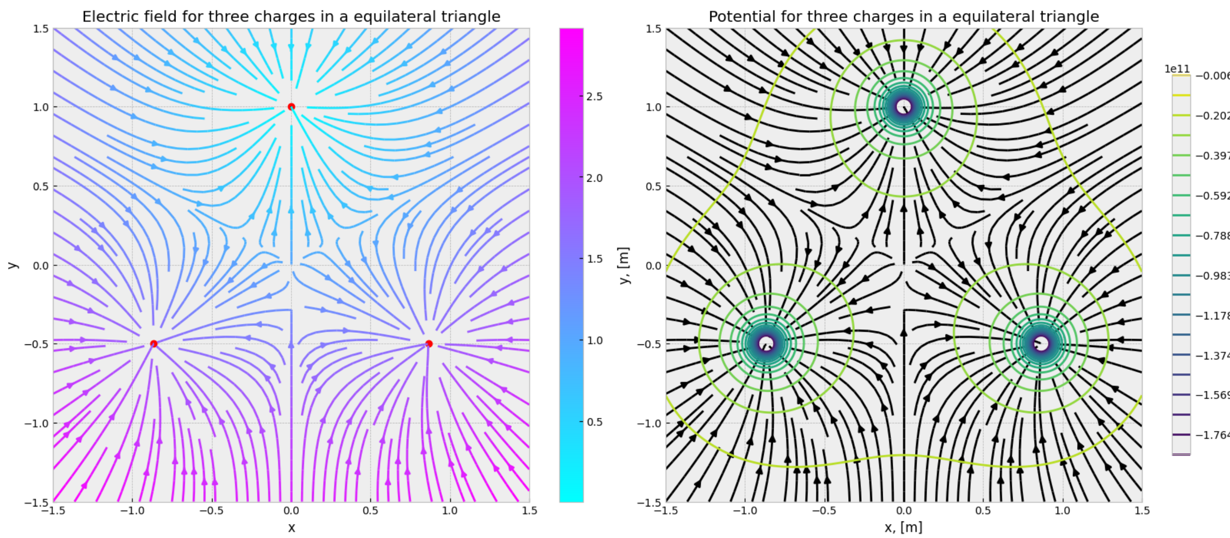

Figure 0.1: Potential and Electric field for a configuration of three charges placed in an equilateral triangle, with centroid placed in the origin of coordinates.

$$\implies \nabla^2 V_T(r) = \frac{3q}{4\pi\epsilon_0 r^3} > 0,$$

$$\therefore \nabla^2 V_T(r) > 0$$

and from the above expression we have that the potential is greater that zero, which implies that the centroid it's an unstable equilibrium point.

I could've started by plotting the electric and potential field, but I choose to present it until now, just to check that our result makes sense.

Now, from the figure above, we can see, or interpret, that if we perturbe any particle placed at the centroid, from the surface levels, it will "fall" to a lower potential value, therefore, the point it's unstable.

b) Now let's move into the next part, for this we have an electric dipole. We know that in the condition for a point to be in equilibrium is $\nabla \cdot \mathbf{F} < 0$, and moreover, in this case we have

$$V = \overrightarrow{p} \cdot \overrightarrow{E},$$

therefore, we have that the equilibrium is ustable if

$$\nabla^2 V = \nabla^2 \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) = \nabla \cdot \nabla \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) \geq 0.$$

But, we can use one of the many vector identities for the previous derivative, in particular, we can use

$$\nabla(a \cdot b) = (a \cdot \nabla) b + (b \cdot \nabla) a + a \times (\nabla \times b) + b \times (\nabla \times a),$$

and if we make the identification of a with p and b with E. Thus, we have

$$\nabla \cdot \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) = (\overrightarrow{p} \cdot \nabla) \overrightarrow{E} + (\overrightarrow{E} \cdot \nabla) \overrightarrow{p} + \overrightarrow{p} \times (\nabla \times \overrightarrow{E}) + \overrightarrow{E} \times (\nabla \times \overrightarrow{p}),$$

but, because we are dealing with phenomena in static conditions, we have, for one of the Maxwell's equations $\nabla \times \overrightarrow{E} = 0$, and because $\overrightarrow{p}$ is constant, any derivative of it is zero, thus $\nabla \times \overrightarrow{p} = 0$ and $(\overrightarrow{E} \cdot \nabla) \overrightarrow{p} = 0$. Therefore, we have that

$$\nabla \cdot \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) = (\overrightarrow{p} \cdot \nabla) \overrightarrow{E},$$

then we have that

$$\nabla^2 \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) = \left(\overrightarrow{p} \cdot \nabla^2 \right) \overrightarrow{E},$$

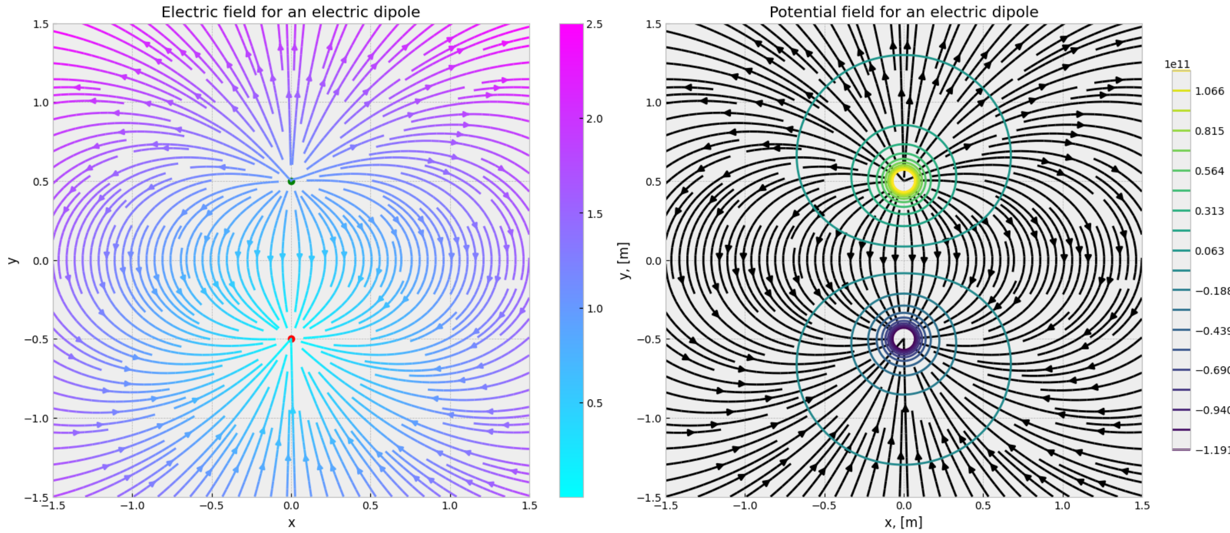

Figure 0.2: Electric field and Potential field for a dipole.

bur we also know that for statis situations, we have $\nabla^2 \overrightarrow{E} = 0$, thus we end with

$$\nabla^2 \left(\overrightarrow{p} \cdot \overrightarrow{E} \right) = 0,$$

just as we wanted, and therefore, the arbitrary point in question is unstable. Just for completness, here I show the potential and electric field for such configuration, i.e., for a dipole.