Legendre Transformations

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This kind of transfiormations appear in the are of thermodynamics and, in fact we can see the Hamilatonian as a Legendre tranformation for the Lagrangian.

Let's begin with an arbitrary function of three variables (the generalization for n coordinates is immediate) $f = f(x, y, z)$, we can calculate its differential, which its expressed in the following way

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

Now, the goal is to perform a Legendre transformation for the variable x, then, for the ease of the lecture, let's rename the partial derivatives in the following way

$$\frac{\partial f}{\partial x} = X, \frac{\partial f}{\partial y} = Y, \frac{\partial f}{\partial z} = Z,$$

then we have the following expression

$$df = X_1 dx_1 + Y dy + Z dz,$$

and with this, we can say, that

$$X = X(x, y, z),$$

i.e., the function X is a function of the variables $\{x, y, z\}$, but, if we make the following change of variables

$$g = f - xX,$$

then we have that the differential of g is given by the

$$dg = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz - d(xX) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz - (xdX + Xdx),$$

but

$$Xdx = \frac{\partial f}{\partial x}dx,$$

then we have that the differential of g is given by

$$dg = -xdX + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$

but
$$-x = \frac{\partial g}{\partial X},$$
therefore, we have that the differential of g becomes
$$dg = \frac{\partial g}{\partial X}dX + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz,$$
and moreover, by the way in which we construct the function g we have that for the variables y and z
$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y}, \frac{\partial g}{\partial z} = \frac{\partial f}{\partial z},$$
therefore, we have
$$dg = \frac{\partial g}{\partial X}dX + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz,$$
which is equivalent to say that
$$g = g(X, y, z).$$

And just to add some language , in the first set of coordinates, for the function f we say that x is natural, and with the transformation we can say the same for $g$but now in terms of X.

It's easy to see that if we perform another Legendre Transformation for the new "changed" variable, we end with the original function, therefore, we can say that the Legendre Transformation can be seen as an idempotent operator.

References

[1] Classical Mechanics, Dieter Strauch.