GR-HW-04: Spin and Electromagnetism and Linear Gravitational Field Equation

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Problem 1 (Spin 2 Polarization)

I did the whole problem in Mathematica and I append the notebook.

Problem 2 (Electromagnetism with Point Charges)

(i) The expression given by

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)), \tag{1}$$

is indeed a good 4-Lorentz vector, and let me explain why.

- First, q_i is a scalar, so it's fine.
- Proper time τ_i is a good Lorentz invariant.
- The 4-velocity, $dx_i^\mu/d\tau_i$ is a good Lorentz vector.
- And finally, the 4-dimensional delta is also a Lorentz invariant.

Therefore, the whole combination in the previous expression is a good 4-Lorentz vector, since is the combination of good Lorentz quantities.

(ii) Let's express J^μ in terms of a 3-dimensional delta function. Let's start with

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)), \tag{2}$$

then, by the chain rule, we have

$$J^\mu(x) = \sum_i^N q_i \int d\tau_i \frac{dt}{d\tau_i} \frac{dx_i^\mu}{d\tau_i} \delta(t - t_i(\tau_i)) \delta^3(x^\alpha - x_i^\alpha(\tau_i)), \tag{3}$$

which implies that

$$J^\mu(x) = \sum_i^N q_i \int dt \frac{dx_i^\mu}{dt} \delta(t - t_i(\tau_i)) \delta^3(x^\alpha - x_i^\alpha(\tau_i)), \tag{4}$$

and from this we can express perform the time integral. We have

$$J^\mu(x) = \sum_i^N q_i \frac{dx_i^\mu}{dt} \delta^3(x^\alpha - x_i^\alpha(t)), \tag{5}$$

(iii) And now let's compute

$$\int d^3xJ^0(x), \tag{6}$$

from which the previous expression we have

$$\int d^3xJ^0(x) = \int d^3x \left(\sum_i^N q_i \frac{dx_i^0}{dt} \delta^3(x^\alpha - x_i^\alpha(t)) \right), \tag{7}$$

and since everything under the integral sign is well behaved we can commute the summation and the integral

$$\int d^3xJ^0(x) = \sum_i^N \int d^3x \left(q_i \frac{dx_i^0}{dt} \delta^3(x^\alpha - x_i^\alpha(t)) \right), \tag{8}$$

but $dx^0 = cdt$, thus the previous expression reduces to

$$\int d^3xJ^0(x) = \sum_i^N cq_i, \tag{9}$$

or

$$\int d^3xJ^0(x) = cQ, \text{ where } \quad Q = \sum_i^N q_i. \tag{10}$$

(iii) The interaction term is given by

$$S_{int} = -\int d^4xA_\mu J^\mu \tag{11}$$

let's massage this term a little bit. So let's begin

$$S_{int} = -\int d^4xA_\mu J^\mu = -\int d^4xA_\mu \left[\sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)) \right], \tag{12}$$

and assuming the functions inside the integrand are well behaved, we can commute the sum and the interal operations, this yields

$$S_{int} = -\sum_i^N q_i \int d^4xA_\mu \left[\int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta^4(x^\alpha - x_i^\alpha(\tau_i)) \right], \tag{13}$$

and if we unfold the 4-dimensional delta, we have

$$S_{int} = -\sum_i^N q_i \int d^4xA_\mu \left[\int d\tau_i \frac{dx_i^\mu}{d\tau_i} \delta(t - t_i(\tau_i))\delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \tag{14}$$

and we can write the inner integral just as we did before, this is

$$S_{int} = -\sum_i^N q_i \int d^4xA_\mu \left[\int dt \frac{d\tau_i}{dt} \frac{dx_i^\mu}{d\tau_i} \delta(t - t_i(\tau_i))\delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \tag{15}$$

it follows that

$$S_{int} = -\sum_i^N q_i \int d^4xA_\mu \left[\frac{dx_i^\mu}{dt} \delta^3(\mathbf{x} - \mathbf{x}_i(\tau_i)) \right], \tag{16}$$

and from this, the 3-dimensional delta collapses the 4-dimensional integral to a 1-dimensional one as follows

$$S_{int} = -\sum_i^N q_i \int dt \frac{dx_i^\mu}{dt} A_\mu(x_i), \tag{17}$$

and again, by the chain rule, we have

$$S_{int} = -\sum_i^N q_i \int d\tau_i \frac{dt}{d\tau_i} \frac{dx_i^\mu}{dt} A_\mu(x_i), \tag{18}$$

thus, the final result is given by

$$S_{int} = -\sum_i^N q_i \int d\tau_i \frac{dx_i^\mu}{d\tau_i} A_\mu(x_i), \tag{19}$$

just as we wanted.

(iv) Now, let's compute the equations of motion from the Euler-Lagrange equations. In this part, the Lagrangian is given by

$$\mathcal{L}_i = -m_i \sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu} - q_i A_\mu \dot{x}_i^\mu, \tag{20}$$

where

$$\dot{x}_i^\mu = \frac{dx_i^\mu}{d\tau} \tag{21}$$

and we know the Euler-Lagrange equation is given by

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x_i^\alpha} = 0. \tag{22}$$

For the second term, we have

$$\frac{\partial \mathcal{L}}{\partial x_i^\alpha} = -q_i \dot{x}_i^\mu \partial_\alpha A_\mu, \tag{23}$$

now, for the first term we need to use the chain rule, which gives us

$$\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \frac{\eta_{\alpha\nu} \dot{x}_i^\nu}{\sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu}} - q_i A_\alpha, \tag{24}$$

but we know that

$$\sqrt{\eta_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu} = 1, \tag{25}$$

thus, we have

$$\frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \dot{x}_i^\nu - q_i A_\alpha \implies \frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \ddot{x}_i^\nu - q_i \frac{dA_\alpha}{d\tau}, \tag{26}$$

but

$$\frac{dA_\alpha}{d\tau} = \frac{\partial A_\alpha}{\partial x_i^\mu} \frac{dx_i^\mu}{d\tau} = \partial_\mu A_\alpha \dot{x}_i^\mu, \tag{27}$$

which implies that

$$\frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \eta_{\alpha\nu} \ddot{x}_i^\nu - q_i \partial_\mu A_\alpha \dot{x}_i^\mu \implies \frac{d}{d\tau} \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i^\alpha} = -m_i \ddot{x}_{i\alpha} - q_i \partial_\mu A_\alpha \dot{x}_i^\mu \tag{28}$$

then, if we combine both terms in the Euler-Lagrange equations, we have

$$-m_i \ddot{x}_{i\alpha} - q_i \partial_\mu A_\alpha \dot{x}_i^\mu + q_i \dot{x}_i^\mu \partial_\alpha A_\mu = 0, \tag{29}$$

which implies that

$$m_i \ddot{x}_{i\alpha} = q_i \dot{x}_i^\mu (\partial_\alpha A_\mu - \partial_\mu A_\alpha), \tag{30}$$

and from this we have

$$m_i \ddot{x}_{i\alpha} = q_i \dot{x}_i^\mu F_{\alpha\mu}, \tag{31}$$

which can be written as

$$m_i \eta^{\sigma\alpha} \ddot{x}_{i\alpha} = q_i \eta^{\sigma\alpha} \dot{x}_i^\mu F_{\alpha_\mu} \implies m_i \ddot{x}_i^\sigma = q_i F^\sigma_\mu \dot{x}_i^\mu, \tag{32}$$

and if we change $\sigma \to \alpha$, we have

$$m_i \ddot{x}_i^\alpha = q_i F^\alpha_\mu \dot{x}_i^\mu, \tag{33}$$

just as we wanted.

(v) Finally, if we set $\alpha = j$ in the previous expression, by the chain rule, we have

$$m_i \frac{d\tau}{dt} \frac{d}{d\tau} (\frac{dx_i^j}{d\tau}) = q_i F^j_\mu \frac{d\tau}{dt} \frac{dx_i^\mu}{d\tau}, \tag{34}$$

which implies that

$$\frac{d}{dt} \left(m_i \frac{dx_i^j}{d\tau} \right) = q_i F^j_\mu \frac{dx_i^\mu}{dt} \quad \implies \frac{d}{dt} (\mathbf{p}_i) = q_i F^j_\mu \frac{dx_i^\mu}{dt}, \tag{35}$$

on the other hand, we know the matrix representation of the Faraday tensor, which is

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}, \tag{36}$$

and from this we know

$$F^\alpha_\mu = \eta_{\mu\beta} F^{\alpha\beta}, \tag{37}$$

or in matrix notation

$$F^\alpha_\mu = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{38}$$

but since we're interested in the spatial index in α we have

$$F^i_\mu \frac{dx_i^\mu}{dt} = \begin{bmatrix} -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \cdot \begin{bmatrix} cdt/dt \\ -dx_i/dt \\ -dy_i/dt \\ -dz+i/dt \end{bmatrix} \tag{39}$$

then we have

$$\frac{d\mathbf{p}_i}{dt} = q_i \begin{bmatrix} E_x - B_z dy_i/dt + B_y dz_i/dt \\ E_y + B_z dx_i/dt - B_x dz_i/dt \\ E_z - B_y dx_i/dt + B_x dy_i/dt \end{bmatrix}, \tag{40}$$

which can also be writen as

$$\frac{d\mathbf{p}_i}{dt} = q_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \tag{41}$$

which is the equation of motion for particle i in the presence of an electric and magnetic field, the RHS of the previous equation is the Lorentz force.

Problem 3 (Gravitational Field Equations)

The lagrangian is given by

$$\mathcal{L} = \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu} + \frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h\partial_\nu h^{\mu\nu} - h_{\mu\nu}\tau^{\mu\nu} \tag{42}$$

Let's compute the Euler-Lagrange equations of motion, which are given by

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha h^{\mu\nu})}\right) - \frac{\partial\mathcal{L}}{\partial h^{\mu\nu}} = 0, \tag{43}$$

and from here, probably the easiest term is the derivative with respect to $h^{\mu\nu}$, for that we have

$$\frac{\partial\mathcal{L}}{\partial h^{\mu\nu}} = \frac{\partial}{\partial h^{\mu\nu}}\left(h_{\mu\nu}\tau^{\mu\nu}\right) = \tau^{\mu\nu}. \tag{44}$$

Now, the hard work comes with the remainder terms, so I'll proceed one by one, but before that; in the derivation I'll be using heavily the following identity

$$\frac{\partial(\partial_k h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\delta_k^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu\right], \tag{45}$$

and with this in mind, let's begin. For the first term, we have

$$\mathcal{L}_1 = \frac{1}{4}\partial_\lambda h_{\rho\sigma}\partial^\lambda h^{\rho\sigma}, \tag{46}$$

and from there we have

$$\frac{\partial(\mathcal{L}_1)}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4}\left[\frac{\partial(\partial_\lambda h_{\rho_\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}\partial^\lambda h^{\rho\sigma} + \partial_\lambda h_{\rho\sigma}\frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}\right], \tag{47}$$

the first term in the previous expression is easy, as all the indices are lowered, and by using equation (45), we have that

$$\frac{1}{8}\delta_\lambda^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu\right]\partial^\lambda h^{\rho\sigma}, \tag{48}$$

by expanding the product we have

$$\frac{1}{8}\delta_\lambda^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu\right]\partial^\lambda h^{\rho\sigma} = \frac{1}{8}\delta_\lambda^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu\partial^\lambda h^{\rho\sigma} + \delta_\rho^\nu\delta_\sigma^\mu\partial^\lambda h^{\rho\sigma}\right] \tag{49}$$

and from this we can see that $\lambda = \alpha$, and in the first term we have $\rho = \mu$ and $\sigma = \nu$, whereas for the second one, we have $\rho = \nu$ and $\sigma = \mu$, thus

$$\frac{1}{8}\delta_\lambda^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu\right]\partial^\lambda h^{\rho\sigma} = \frac{1}{8}\left[\partial^\alpha h^{\mu\nu} + \partial^\alpha h^{\nu\mu}\right], \tag{50}$$

but $h_{\mu\nu}$ is symmetric, thus

$$\frac{1}{8}\delta_\lambda^\alpha\left[\delta_\rho^\mu\delta_\sigma^\nu + \delta_\rho^\nu\delta_\sigma^\mu\right]\partial^\lambda h^{\rho\sigma} = \frac{1}{4}h^\alpha h^{\mu\nu}, \tag{51}$$

and now, for the second term in equation (47), we need to work a little more since the indices are up, but we know the way to lower them is via the metric, thus

$$\partial^\lambda h^{\rho\sigma} = \eta^{\lambda\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}\partial_\beta h_{\gamma\delta}, \tag{52}$$

and from this we have

$$\frac{1}{4}\partial_\lambda h_{\rho_\sigma}\frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{4}\eta^{\lambda\beta}\eta^{\rho\gamma}\eta^{\sigma\delta}\partial_\lambda h_{\rho_\sigma}\frac{\partial(\partial_\beta h_{\gamma\delta})}{\partial(\partial_\lambda h_{\rho_\sigma})}, \tag{53}$$

ana again, we can use the identity given in equation (45), which in this case reads

$$\frac{\partial(\partial_\beta h_{\gamma\delta})}{\partial(\partial_\lambda h_{\rho_\sigma})}=\frac{1}{2}\delta^\alpha_\beta\left[\delta^\mu_\delta\delta^\nu_\gamma+\delta^\nu_\gamma\delta^\mu_\delta\right], \tag{54}$$

and from this, we can see that the first delta implies $\beta=\alpha$, the deltas in the first term also implies that $\delta=\mu$, $\gamma=\nu$ whereas for the second term we have $\gamma=\nu$ and $\delta=\mu$, then if we expand the product we have

$$\frac{1}{4}\partial_\lambda h_{\rho_\sigma}\frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}=\frac{1}{8}\eta^{\lambda\alpha}\eta^{\rho\nu}\eta^{\sigma\mu}\partial_\lambda h_{\rho_\sigma}+\frac{1}{8}\eta^{\lambda\alpha}\eta^{\rho\nu}\eta^{\sigma\mu}\partial_\lambda h_{\rho_\mu}, \tag{55}$$

which can also be writen as

$$\frac{1}{4}\partial_\lambda h_{\rho_\sigma}\frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}=\frac{1}{8}\partial^\alpha h^{\mu\nu}+\frac{1}{8}\partial^\alpha h^{\nu\mu}, \tag{56}$$

but again, by symmetry, we have

$$\frac{1}{4}\partial_\lambda h_{\rho_\sigma}\frac{\partial(\partial^\lambda h^{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}=\frac{1}{4}\partial^\alpha h^{\mu\nu} \tag{57}$$

therefore,

$$\frac{\partial(\mathcal{L}_1)}{\partial(\partial_\alpha h_{\mu\nu})}=\frac{1}{4}\partial^\alpha h^{\mu\nu}+\frac{1}{4}\partial^\alpha h^{\mu\nu} \tag{58}$$

thus

$$\frac{\partial\mathcal{L}_1}{\partial(\partial_\alpha h_{\mu\nu})}=\frac{1}{2}\partial^\alpha h^{\mu\nu}, \tag{59}$$

and from this

$$\partial_\alpha\left(\frac{\partial\mathcal{L}_1}{\partial(\partial_\alpha h_{\mu\nu})}\right)=\frac{1}{2}\partial_\alpha\partial^\alpha h^{\mu\nu}=\frac{1}{2}\Box h^{\mu\nu}. \tag{60}$$

Now, let's move to the second term

$$\mathcal{L}_2=-\frac{1}{2}\partial^\mu h_{\mu\nu}\partial_\alpha h^{\alpha\nu}, \tag{61}$$

then we have

$$\frac{\partial\mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu}))}=-\frac{1}{2}\left[\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}+\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h^{\sigma\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\right] \tag{62}$$

but we know that

$$\partial^\lambda=\eta^{\lambda\beta}\partial_\beta, \tag{63}$$

thus, for the first term we have

$$\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}=\eta^{\lambda\beta}\frac{\partial(\partial_\beta h_{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}, \tag{64}$$

and by making use of the equation (45), we have

$$\frac{\partial(\partial_\beta h_{\lambda\rho})}{\partial(\partial h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}=\eta^{\lambda\beta}\left[\frac{1}{2}\delta^\alpha_\beta(\delta^\mu_\lambda\delta^\nu_\rho+\delta^\nu_\lambda\delta^\mu_\rho)\right]\partial_\sigma h^{\sigma\rho}, \tag{65}$$

which implies that

$$\frac{\partial(\partial^\lambda h_{\lambda\rho})}{\partial(\partial h_{\mu\nu})}\partial_\sigma h^{\sigma\rho}=\frac{1}{2}\eta^{\alpha\mu}\partial_\sigma h^{\sigma\nu}+\frac{1}{2}\eta^{\alpha\nu}\partial_\sigma h^{\sigma\mu}, \tag{66}$$

and for the second term is pretty much the same procedure, but we need to lower the indices of the $h^{\rho\sigma}$, this is

$$h^{\sigma\rho}=\eta^{\sigma\beta}\eta^{\rho\epsilon}h_{\beta\epsilon}, \tag{67}$$

then the second term becomes

$$\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h^{\sigma\rho})}{\partial(\partial_\alpha h_{\mu\nu})} = \eta^{\sigma\beta}\eta^{\rho\epsilon}\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} \tag{68}$$

and again, by making use of the equation (45), we have

$$\eta^{\sigma\beta}\eta^{\rho\epsilon}\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\sigma\beta}\eta^{\rho\epsilon}\partial^\lambda h_{\lambda\rho}\left[\delta^\alpha_\sigma(\delta^\mu_\beta\delta^\nu_\epsilon + \delta^\nu_\beta\delta^\mu_\epsilon)\right] \tag{69}$$

the previous expression implies that $\sigma = \alpha$, and for the first deltas we have $\beta = \mu$, $\epsilon = \nu$ whereas for the second deltas we have $\beta = \nu$, $\epsilon = \mu$, thus

$$\eta^{\sigma\beta}\eta^{\rho\epsilon}\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\alpha\mu}\eta^{\rho\nu}\partial^\lambda h_{\lambda\rho} + \frac{1}{2}\eta^{\alpha\nu}\eta^{\rho\mu}\partial^\lambda h_{\lambda\rho}, \tag{70}$$

now, from that expression we can see that each one of the η's is going to mix the indices, this is; raise one index of h, and since we want the index in the derivative down (so we can match the other computed term), we can lower this and raise the other index in the field h, this results in

$$\eta^{\sigma\beta}\eta^{\rho\epsilon}\partial^\lambda h_{\lambda\rho}\frac{\partial(\partial_\sigma h_{\beta\epsilon})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\alpha\mu}\partial_\lambda h^{\lambda\nu} + \frac{1}{2}\eta^{\alpha\nu}\partial_\lambda h^{\lambda\mu}, \tag{71}$$

and from this we have

$$\frac{\partial\mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu}))} = -\frac{1}{2}\left[\eta^{\alpha\mu}\partial_\lambda h^{\lambda\nu} + \eta^{\alpha\nu}\partial_\lambda h^{\lambda\mu}\right] \tag{72}$$

which is equivalent to

$$\frac{\partial\mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2}\left[\eta^{\alpha\mu}\partial_\sigma h^{\sigma\nu} + \eta^{\alpha\nu}\partial_\sigma h^{\sigma\mu}\right], \tag{73}$$

and if we take ∂_α of the previous expression we can see that the η's are going to raise the index in this partial, thus

$$\partial_\alpha\left(\frac{\partial\mathcal{L}_2}{\partial(\partial_\alpha h_{\mu\nu})}\right) = -\frac{1}{2}\left[\partial^\mu\partial_\sigma h^{\sigma\nu} + \partial^\nu\partial_\sigma h^{\sigma\mu}\right]. \tag{74}$$

Now, for the third term

$$\mathcal{L}_3 = -\frac{1}{4}\partial_\lambda h\partial^\lambda h, \tag{75}$$

we follow the same procedure as before, but before proceding any further, we need to remember that

$$h = \eta^{\rho\sigma}h_{\rho\sigma}, \tag{76}$$

with this in mind we have

$$\frac{\partial\mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{4}\left[\eta^{\rho\sigma}\frac{\partial(\partial_\lambda h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})} + \eta^{\rho\sigma}\eta^{\lambda\delta}\partial_\lambda\frac{\partial(\partial_\delta h_{\rho\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}\right], \tag{77}$$

and again, using the identity in equation (45), we have

$$\frac{\partial\mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{4}\left[\frac{1}{2}\eta^{\sigma\rho}\delta^\alpha_\lambda(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\nu_\rho\delta^\mu_\sigma)\partial^\lambda h + \frac{1}{2}\eta^{\rho\sigma}\eta^{\lambda\delta}\partial_\lambda h\delta^\alpha_\delta(\delta^\mu_\rho\delta^\nu_\sigma + \delta^\nu_\rho\delta^\mu_\sigma)\right], \tag{78}$$

and from this expression we have that $\lambda = \alpha$, and even more, the deltas in the first term impose $\rho = \nu$, $\sigma = \nu$ and $\rho = \nu$, $\sigma = \mu$, and for the second term the deltas impose the same relationship as well, which implies that

$$\frac{\partial\mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{8}\eta^{\mu\nu}\partial^\alpha h - \frac{1}{8}\eta^{\nu\mu}\partial^\alpha h - \frac{1}{8}\eta^{\mu\nu}\partial^\alpha h - \frac{1}{8}\eta^{\nu\mu}\partial^\alpha h, \tag{79}$$

but η is symmetric, thus, the previous expression reduces to

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})} = -\frac{1}{2}\eta^{\mu\nu}\partial^\alpha h, \tag{80}$$

and if we take ∂_α, we get the box operator, this is

$$\partial_\alpha\left(\frac{\partial \mathcal{L}_3}{\partial(\partial_\alpha h_{\mu\nu})}\right) = -\frac{1}{2}\eta^{\mu\nu}\Box h. \tag{81}$$

Finally, let's move to the last term

$$\mathcal{L}_4 = \frac{1}{2}\partial_\lambda h\partial_\rho h^{\lambda\rho}, \tag{82}$$

then we have

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\left[\frac{\partial(\partial_\lambda h)}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\rho h^{\lambda\rho} + \partial_\lambda h\frac{\partial(\partial_\rho h^{\lambda\rho})}{\partial(\partial_\alpha h_{\mu\nu})}\right], \tag{83}$$

and for the first term we follow the same procedure as with $\mathcal{L}_3$, whereas for the second term, we use two η's to lower the indices in the field $h^{\lambda\rho}$, with this in mind we have

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\left[\eta^{\delta\sigma}\frac{\partial(\partial_\lambda h_{\delta\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\rho h^{\lambda\rho} + \eta^{\lambda\delta}\eta^{\rho\gamma}\partial_\lambda h\frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})}\right], \tag{84}$$

and again, we use the identity given in equation (45), which again, is making $\lambda = \alpha$ in the first term and $\rho = \alpha$ in the second, for the first one we have

$$\eta^{\delta\sigma}\frac{\partial(\partial_\lambda h_{\delta\sigma})}{\partial(\partial_\alpha h_{\mu\nu})}\partial_\rho h^{\lambda\rho} = \frac{1}{2}\eta^{\mu\nu}\partial_\rho h^{\alpha\rho} + \frac{1}{2}\eta^{\nu\mu}\partial_\rho h^{\alpha\rho} = \eta^{\mu\nu}\partial_\rho h^{\alpha\rho}, \tag{85}$$

whereas for the second term we have

$$\eta^{\lambda\delta}\eta^{\rho\gamma}\partial_\lambda h\frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\lambda\delta}\eta^{\rho\gamma}\partial_\lambda h(\delta^\mu_\delta\delta^\nu_\gamma + \delta^\nu_\delta\delta^\mu_\gamma), \tag{86}$$

which implies that

$$\eta^{\lambda\delta}\eta^{\rho\gamma}\partial_\lambda h\frac{\partial(\partial_\rho h_{\delta\gamma})}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\lambda\mu}\eta^{\alpha\nu}\partial_\lambda h + \frac{1}{2}\eta^{\alpha\mu}\partial^\nu h = \frac{1}{2}(\eta^{\alpha\nu}\partial^\mu h + \eta^{\alpha\mu}\partial^\nu h), \tag{87}$$

then, the whole expression reduces to

$$\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})} = \frac{1}{2}\eta^{\mu\nu}\partial_\rho h^{\alpha\rho} + \frac{1}{4}(\eta^{\alpha\nu}\partial^\mu h + \eta^{\alpha\mu}\partial^\nu h), \tag{88}$$

and if we take ∂_α, we have

$$\partial_\alpha\left(\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})}\right) = \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\partial_\rho h^{\alpha\rho} + \frac{1}{4}\partial_\alpha(\eta^{\alpha\nu}\partial^\mu h + \eta^{\alpha\mu}\partial^\nu h), \tag{89}$$

now, in the second term, the η's are going to raise the indec of ∂_α and since derivatives commute we have

$$\partial_\alpha\left(\frac{\partial \mathcal{L}_4}{\partial(\partial_\alpha h_{\mu\nu})}\right) = \frac{1}{2}\eta^{\mu\nu}\partial_\alpha\partial_\rho h^{\alpha\rho} + \frac{1}{2}\partial^\mu\partial^\nu h, \tag{90}$$

then, if we put everything together, and define

$$K^{\mu\nu} = \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha h_{\mu\nu})}\right), \tag{91}$$

we have

$$K^{\mu\nu} = \frac{1}{2}\Box h^{\mu\nu} - \frac{1}{2}\left[\partial^\mu \partial_\sigma h^{\sigma\nu} + \partial^\nu \partial_\sigma h^{\sigma\mu}\right] - \frac{1}{2}\eta^{\mu\nu}\Box h + \frac{1}{2}\eta^{\mu\nu}\partial_\alpha \partial_\rho h^{\alpha\rho} + \frac{1}{2}\partial^\mu \partial^\nu h, \tag{92}$$

now, the expression we got from G has its indices lowered, and the version with the indices up is given by

$$G^{\mu\nu} = \frac{1}{2}\left(\partial_\alpha \partial^\nu h^{\alpha\mu} + \partial_\alpha \partial^\mu h^{\alpha\nu} - \partial^\mu \partial^\nu h - \Box h^{\mu\nu} - \eta^{\mu\nu}\partial^\alpha \partial^\beta h_{\alpha\beta} + \eta^{\mu\nu}\Box h\right), \tag{93}$$

and we can rearrange the terms in $K^{\mu\nu}$ as follows

$$K^{\mu\nu} = \frac{1}{2}(-\partial^\mu \partial_\sigma h^{\sigma\nu} - \partial^\nu \partial_\sigma h^{\sigma\mu} + \partial^\mu \partial^\nu h + \Box h^{\mu\nu} + \eta^{\mu\nu}\partial_\alpha \partial_\rho h^{\alpha\rho} - \eta^{\mu\nu}\Box h), \tag{94}$$

and from this it follows

$$K^{\mu\nu} = -G^{\mu\nu}, \tag{95}$$

going back to the original Euler-Lagrange equation, we have

$$-G^{\mu\nu} + \tau^{\mu\nu} = 0 \quad \Longrightarrow \; G^{\mu\nu} = \tau^{\mu\nu}, \tag{96}$$

just as we wanted.

Problem 1 – Spin 2 Polarization:

Let's define $\hbar = 1$, and the J1 operator

In[26]:=
```
h = 1;
J1 = {{0, -I h, 0}, {I h, 0, 0}, {0, 0, 0}};
```

Now, let' s define the five epsilon matrices (these are given):

In[28]:=
```
epsA = 1 / 2 {{1, I, 0}, {I, -1, 0}, {0, 0, 0}};
epsB = 1 / 2 {{1, -I, 0}, {-I, -1, 0}, {0, 0, 0}};
epsC = 1 / 2 {{0, 0, 1}, {0, 0, I}, {1, I, 0}};
epsD = 1 / 2 {{0, 0, 1}, {0, 0, -I}, {1, -I, 0}};
epsE = -1 / Sqrt[6] {{1, 0, 0}, {0, 1, 0}, {0, 0, -2}};
```

And let's also store them in a single list:

In[33]:=
```
epsilons = {epsA, epsB, epsC, epsD, epsE};
```

Show the epsilon matrices

In[34]:=
```
Table[epsilons[[i]] // MatrixForm, {i, Length[epsilons]}]
```

Out[34]=

$$\left\{\begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\right\}$$

(i) Let's verify that the ϵ matrices are indeed eigenvectors of J^2:

In[35]:=
```
J2Action[eps_] := J1.eps - eps.J1;
```

Compute J2

In[36]:=
```
results = Map[J2Action, epsilons]
```

Out[36]=

$$\left\{\{\{1, i, 0\}, \{i, -1, 0\}, \{0, 0, 0\}\}, \{\{-1, i, 0\}, \{i, 1, 0\}, \{0, 0, 0\}\}, \left\{\left\{0, 0, \frac{1}{2}\right\}, \left\{0, 0, \frac{i}{2}\right\}, \left\{\frac{1}{2}, \frac{i}{2}, 0\right\}\right\}, \left\{\left\{0, 0, -\frac{1}{2}\right\}, \left\{0, 0, \frac{i}{2}\right\}, \left\{-\frac{1}{2}, \frac{i}{2}, 0\right\}\right\}, \{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}\right\}$$

Show both matrices

```
In[37]:= Table[results[[i]] // MatrixForm, {i, Length[results]}]
```

Out[37]=

$$\left\{\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ -\frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\}$$

```
In[38]:= Table[epsilons[[i]] // MatrixForm, {i, Length[epsilons]}]
```

Out[38]=

$$\left\{\begin{pmatrix} \frac{1}{2} & \frac{i}{2} & 0 \\ \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} & 0 \\ -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\right\}$$

From this we can see that their eigenvalues are as follows

- ϵ_A has eigenvalue $\lambda = 2\hbar$
- ϵ_A has eigenvalue $\lambda = -2\hbar$
- ϵ_A has eigenvalue $\lambda = \hbar$
- ϵ_A has eigenvalue $\lambda = -\hbar$
- ϵ_A has eigenvalue $\lambda = 0$

(ii) Let’s lift the 3×3 ϵ matrices to 4×4 tensors and let’s boost them:

```
In[39]:= ϵTensors = Table[ArrayPad[epsilons[[i]], {{1, 0}, {1, 0}}, 0], {i, Length[epsilons]}]
```

Out[39]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\},\left\{0,\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\},\left\{0,-\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,\frac{i}{2}\right\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,-\frac{i}{2}\right\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\}\right\},$$
$$\left.\left\{\{0,0,0,0\},\left\{0,-\frac{1}{\sqrt{6}},0,0\right\},\left\{0,0,-\frac{1}{\sqrt{6}},0\right\},\left\{0,0,0,\sqrt{\frac{2}{3}}\right\}\right\}\right\}$$

```
In[40]:= Table[ϵTensors[[i]] // MatrixForm, {i, Length[ϵTensors]}]
```

Out[40]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}\right\}$$

In[41]:=
```
Λ = {{γ, 0, 0, γ v}, {0, 1, 0, 0}, {0, 0, 1, 0}, {γv, 0, 0, γ}};
```

In[42]:=
```
Λ // MatrixForm
```

Out[42]//MatrixForm=

$$\begin{pmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix}$$

In[43]:=
```
ϵTBoosted = Table[Λ.ϵTensors⟦i⟧, {i, Length[ϵTensors]}]
```

Out[43]=

$$\left\{\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},\frac{i}{2},0\right\},\left\{0,\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,\frac{1}{2},-\frac{i}{2},0\right\},\left\{0,-\frac{i}{2},-\frac{1}{2},0\right\},\{0,0,0,0\}\right\},$$
$$\left\{\left\{0,\frac{v\gamma}{2},\frac{i v\gamma}{2},0\right\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,\frac{i}{2}\right\},\left\{0,\frac{\gamma}{2},\frac{i\gamma}{2},0\right\}\right\},$$
$$\left\{\left\{0,\frac{v\gamma}{2},-\frac{1}{2}i v\gamma,0\right\},\left\{0,0,0,\frac{1}{2}\right\},\left\{0,0,0,-\frac{i}{2}\right\},\left\{0,\frac{\gamma}{2},-\frac{i\gamma}{2},0\right\}\right\},$$
$$\left.\left\{\left\{0,0,0,\sqrt{\frac{2}{3}}v\gamma\right\},\left\{0,-\frac{1}{\sqrt{6}},0,0\right\},\left\{0,0,-\frac{1}{\sqrt{6}},0\right\},\left\{0,0,0,\sqrt{\frac{2}{3}}\gamma\right\}\right\}\right\}$$

In[44]:=
```
Table[ϵTBoosted⟦i⟧ // MatrixForm, {i, Length[ϵTBoosted]}]
```

Out[44]=

$$\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 \\ 0 & -\frac{i}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{v\gamma}{2} & \frac{i v\gamma}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{i}{2} \\ 0 & \frac{\gamma}{2} & \frac{i\gamma}{2} & 0 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 0 & \frac{v\gamma}{2} & -\frac{1}{2}i v\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{i}{2} \\ 0 & \frac{\gamma}{2} & -\frac{i\gamma}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{\frac{2}{3}}v\gamma \\ 0 & -\frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}}\gamma \end{pmatrix}\right\}$$

(iii) Taking massless limit:

In the massless limit, we can see that the only tensors that are well behaved are the ϵ_A and ϵ_B.

(iv) Let’s compute $\epsilon^{ij}p_j$:

Since the boost was along the z-direction, we’re going to consider a particle moving along the z-axis. In this case, the contravariant 4-momentum is given by $p^\mu = (E, 0, 0, E)$, and the covariant form reads $p_\mu = (-E, 0, 0, E)$, then we can define p as follows

In[45]:=
```
p = {-1, 0, 0, 1};
```

And now let's perform the contraction

```
resultVector = Table[ϵTBoosted[[i]].p, {i, Length[ϵTBoosted]}]
```

Out[47]=

$$\left\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \left\{0, \frac{1}{2}, \frac{i}{2}, 0\right\}, \left\{0, \frac{1}{2}, -\frac{i}{2}, 0\right\}, \left\{\sqrt{\frac{2}{3}}\, v\, \gamma, 0, 0, \sqrt{\frac{2}{3}}\, \gamma\right\}\right\}$$

And from this we can see that the only ones that satisfy the condition are the 4×4 tensors ϵ_A and ϵ_B.