GR-HW-06

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Problem 1 (Transformation Rule for Inverse Metric)

Let's prove that the transformed $g^{\mu\nu}$ is the inverse of the transformed $g_{\mu\nu}$. Indeed, the transformation rule for a covariant tensor is given by

$$\bar{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \bar{x}^\mu}\frac{\partial x^\sigma}{\partial \bar{x}^\mu} g_{\rho\sigma} \tag{1}$$

Whereas for a contravariant one, we have

$$\bar{g}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} g^{\alpha\beta}, \tag{2}$$

then, we have

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial \bar{x}^\alpha}{\partial x^\sigma} g^{\rho\sigma}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\alpha} g_{\delta\gamma}, \tag{3}$$

and from this we can see that, by moving the partials, we have a δ^γ_σ in the expression, thus

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\delta^\gamma_\sigma g^{\rho\sigma} g_{\delta\gamma}, \tag{4}$$

but we know that

$$\delta^\gamma_\sigma g^{\rho\sigma} g_{\delta\gamma} = g^{\rho\sigma} g_{\delta\rho} = \delta^\rho_\delta, \tag{5}$$

which implies

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{\partial \bar{x}^\mu}{\partial x^\rho}\frac{\partial x^\delta}{\partial \bar{x}^\nu}\delta^\rho_\delta = \frac{\partial \bar{x}^\mu}{\partial \bar{x}^\nu}, \tag{6}$$

therefore, we have

$$\bar{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \delta^\mu_\nu, \tag{7}$$

which proves that indeed, the transformed $g^{\mu\nu}$ is the inverse of the transformed $g_{\mu\nu}$.

Problem 2 (Covariant Derivative of Vector)

1. Let's prove that $\partial_\mu V_\nu$ does not transform as a tensor. Indeed, from the transformation definition, we have that

$$\bar{\partial}_\mu \bar{V}_\nu = \bar{\partial}_\mu \left(\frac{\partial x^\beta}{\partial \bar{x}^\nu} V_\beta \right) = \frac{\partial x^\beta}{\partial \bar{x}^\nu} \bar{\partial}_\mu V_\beta + \frac{\partial^2 x^\beta}{\partial x^\mu \partial x^\nu} V_\beta, \tag{8}$$

and by the chain rule, we have

$$\bar{\partial}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta, \tag{9}$$

and as we can see, we have an extra term, therefore, $\partial_\mu V_\nu$ does not transform as 2-rank tensor.

2. We need to prove that the covariant derivative transforms as a tensor, this is

$$\nabla_\mu V_\nu \to \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \bar{\nabla}_\alpha \bar{V}_\beta. \tag{10}$$

Indeed, by definition, we have that

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda, \tag{11}$$

thus $\bar{\nabla}_\mu \bar{V} \nu$ is given by

$$\bar{\nabla}_\mu \bar{V}_\nu = \bar{\partial}_\mu \bar{V}_\nu - \bar{\Gamma}^\lambda_{\mu\nu} \bar{V}_\lambda. \tag{12}$$

The only term that we need to be more careful is with the Christoffel symbol, since this does not transform like a tensor, in fact, the transformation rule for this object is given by

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\alpha_{\beta\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} + \frac{\partial \bar{x}^\lambda}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu}, \tag{13}$$

from this we have

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \left(\Gamma^\alpha_{\beta\gamma} \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} + \frac{\partial \bar{x}^\lambda}{\partial x^\gamma} \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \right) \frac{\partial x^\delta}{\partial \bar{x}^\lambda} V_\delta, \tag{14}$$

from this we can see that inside the third term we have two Kronecker deltas, δ^δ_α and δ^δ_γ, respectively, thus

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \Gamma^\alpha_{\beta\gamma} \delta^\delta_\alpha \frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} V_\delta - \delta^\delta_\gamma \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\delta, \tag{15}$$

thus we have

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta + \frac{\partial^2 x^\beta}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\beta - \frac{\partial x^\beta}{\partial \bar{x}^\mu} \frac{\partial x^\gamma}{\partial \bar{x}^\nu} \Gamma^\alpha_{\beta\gamma} V_\alpha - \frac{\partial^2 x^\gamma}{\partial \bar{x}^\mu \partial \bar{x}^\nu} V_\gamma, \tag{16}$$

and from this we can see that the second term cancels with the fourth one, whereas for the third, we can relabel the indices as follows

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \partial_\alpha V_\beta - \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \Gamma^\lambda_{\alpha\beta} V_\lambda = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \left(\nabla_\mu V_\nu \right), \tag{17}$$

this is

$$\bar{\nabla}_\mu \bar{V}_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \left(\nabla_\mu V_\nu \right), \tag{18}$$

therefore, the covariant derivative is indeed a good 2-rank tensor.

Riemann Curvature:

The Riemann tensor is given by

$$R^{\rho}_{\sigma\mu\nu} := \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

where the $\Gamma^{i}{}_{jk}$ is the Christoffel symbol of second kind, which is given by

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\delta} (-\partial_{\delta} g_{\mu\nu} + \partial_{\mu} g_{\delta\nu} + \partial_{\nu} g_{\delta\nu})$$

On the other hand, the Ricci tensor is defined as the contraction with the Riemann tensor, this is

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}$$

and the Ricci scalar is defined in terms of the Ricci tensor as follows

$$R = g^{\mu\nu} R_{\mu\nu}$$

The first step is to define the metric tensor its inverse, and the set of coordinates that we will work on.

In[1]:=
```
g = {{1 + x^2/(L^2 - x^2 - y^2), (x y/(L^2 - x^2 - y^2))}, {(x y/(L^2 - x^2 - y^2)), 1 + y^2/(L^2 - x^2 - y^2)}};
ginv = Inverse[g];
dim = Dimensions[g][[1]];
coords = {x, y};
```

Let's display the metric and it's inverse:

In[5]:=
```
g // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 1 + \frac{x^2}{L^2-x^2-y^2} & \frac{x\,y}{L^2-x^2-y^2} \\ \frac{x\,y}{L^2-x^2-y^2} & 1 + \frac{y^2}{L^2-x^2-y^2} \end{pmatrix}$$

In[6]:=
```
Simplify[ginv] // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 1 - \frac{x^2}{L^2} & -\frac{x\,y}{L^2} \\ -\frac{x\,y}{L^2} & 1 - \frac{y^2}{L^2} \end{pmatrix}$$

Now, let's compute the Christoffel symbol of the second kind:

In[7]:=
```
Γ = Table[Simplify[1/2 Sum[
      ginv⟦α, δ⟧ × (
        -D[g⟦μ, ν⟧, coords⟦δ⟧] + D[g⟦δ, ν⟧, coords⟦μ⟧] + D[g⟦δ, μ⟧, coords⟦ν⟧]),
      {δ, 1, dim}]], {α, 1, dim}, {μ, 1, dim}, {ν, 1, dim}]
```

Out[7]=
$$\left\{\left\{\left\{\frac{x\left(L^2-y^2\right)}{L^2\left(L^2-x^2-y^2\right)},\frac{x^2\,y}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{x^2\,y}{L^2\left(L^2-x^2-y^2\right)},\frac{x\left(L^2-x^2\right)}{L^2\left(L^2-x^2-y^2\right)}\right\}\right\},\left\{\left\{\frac{y\left(L^2-y^2\right)}{L^2\left(L^2-x^2-y^2\right)},\frac{x\,y^2}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{x\,y^2}{L^2\left(L^2-x^2-y^2\right)},\frac{\left(L^2-x^2\right)y}{L^2\left(L^2-x^2-y^2\right)}\right\}\right\}\right\}$$

Since this object has 3 indices, and the number of dimensions is 2, the shape of this object is expected to be {3,3,3}, let's check that:

In[8]:=
```
Dimensions[Γ] == {4, 4, 4}
```

Out[8]=
```
False
```

Great, now let's move on let's compute the Riemann tensor:

In[9]:=
```
rTensor = Table[
  Simplify[
   D[Γ⟦ρ, ν, σ⟧, coords⟦μ⟧] - D[Γ⟦ρ, μ, σ⟧, coords⟦ν⟧] +
    Sum[Γ⟦ρ, μ, λ⟧ × Γ⟦λ, ν, σ⟧, {λ, 1, dim}] - Sum[Γ⟦ρ, ν, λ⟧ × Γ⟦λ, μ, σ⟧, {λ, 1, dim}]
  ], {ρ, 1, dim}, {σ, 1, dim}, {μ, 1, dim}, {ν, 1, dim}]
```

Out[9]=
$$\left\{\left\{\left\{\left\{0,\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{-\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)},0\right\}\right\},\left\{\left\{0,\frac{L^2-x^2}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{-L^2+x^2}{L^2\left(L^2-x^2-y^2\right)},0\right\}\right\}\right\},\left\{\left\{\left\{0,\frac{-L^2+y^2}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{L^2-y^2}{L^2\left(L^2-x^2-y^2\right)},0\right\}\right\},\left\{\left\{0,-\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)},0\right\}\right\}\right\}\right\}$$

Since this tensor does not vanish for all its components, we can conclude that the space is not flat. Even more, this object is a good tensor, and the shape of this object is expected to be {2,2,2,2}, so let's check again for this

In[10]:=
```
Dimensions[rTensor] == {4, 4, 4, 4}
```

Out[10]=
```
False
```

Now let's compute the version of the Riemann tensor with all the indices down:

In[11]:=
```
rDown = Table[Simplify[Sum[g[[i, m]] × rTensor[[m, j, k, l]], {m, 1, dim}]],
   {i, 1, dim}, {j, 1, dim}, {k, 1, dim}, {l, 1, dim}]
```

Out[11]=

$$\left\{\left\{\{\{0,0\},\{0,0\}\},\left\{\left\{0,\frac{1}{L^2-x^2-y^2}\right\},\left\{\frac{1}{-L^2+x^2+y^2},0\right\}\right\}\right\},\left\{\left\{\left\{0,\frac{1}{-L^2+x^2+y^2}\right\},\left\{\frac{1}{L^2-x^2-y^2},0\right\}\right\},\{\{0,0\},\{0,0\}\}\right\}\right\}$$

And as we can see, the shape remains the same, as expected:

In[12]:=
```
Dimensions[rDown]
```

Out[12]=

{2, 2, 2, 2}

And we can also compute the Ricci tensor:

In[13]:=
```
ricci = Table[Sum[rTensor[[ρ, μ, ρ, ν]], {ρ, 1, dim}], {μ, 1, dim}, {ν, 1, dim}]
```

Out[13]=

$$\left\{\left\{\frac{L^2-y^2}{L^2\left(L^2-x^2-y^2\right)},\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)}\right\},\left\{\frac{x\,y}{L^2\left(L^2-x^2-y^2\right)},\frac{L^2-x^2}{L^2\left(L^2-x^2-y^2\right)}\right\}\right\}$$

And check that indeed it has the right shape:

In[14]:=
```
Dimensions[ricci] == {4, 4}
```

Out[14]=

False

Finally, we can compute the Ricci scalar:

In[15]:=
```
ricciScalar = Simplify[Sum[ginv[[μ, ν]] × ricci[[μ, ν]], {μ, 1, dim}, {ν, 1, dim}]]
```

Out[15]=

$$\frac{2}{L^2}$$