Rotations and the Spin $1/2$ Particle in a Magnetic Field

Emmanuel Flores

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Advanced Mathematical Methods, Tufts University

Spinor Representation

Some Motivation 1

- The existence of spin 1/2 particles shows that is $Spin\left(3\right)$ rather than $SO\left(3\right)$ that is the symmetry group of corresponding of rotations of fundamental quantum systems.
- The idea is to study $\mathcal{H} = \mathbb{C}^2$ with the group action given by rotations in $3D$.

Definition
The spinor representation of $Spin\left(3\right) = SU\left(2\right)$ is the representation on $\mathbb{C}^2$ given by

$$g \in SU\left(2\right) \to \pi_{spinor}\left(g\right) = g,$$

and elements of the representation space $\mathbb{C}^2$ are called spinors.

Some Motivation 2

- States of elementary particles such as the electron are described by a state space $\mathcal{H} = \mathbb{C}^2$.
- We want to study rotations acting on this space, and this is done via: the two dimensional irreducible representation of $SU(2) = Spin(3)$

Spin $\frac{1}{2}$ in a Magnetic Field

Elements of the Lie algebra

- We will consider only the $SU(2)$ subgroup of $U(2)$.
- "When it occurs in its role as double cover of the rotational group, the quantum system is said to carry "spin", in particular "spin 1/2" for the two dimensional irreducible representation."
- Elements of the Lie algebra

$$X_j = -i\frac{\sigma_j}{2},$$

with commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, [X_3, X_1] = X_2.$$

Physics Connection

- Making contact with physics

$$S_j = i\hbar X_j,$$

we like this as observables because the eigenvalues are real $\pm 1/2$ (experimental measures).

- Elements of the group are given by

$$\Omega\left(\theta, \mathbf{w}\right) = \exp\left(-\frac{i}{\hbar}\mathbf{w}\cdot\mathbf{S}\right) \in SU\left(2\right).$$

- States in $\mathcal{H}$ that have a well-defined value of the observable S_j will be eigenvectors of S_j with eigenvalues $\pm 1/2$.

Step Back: Axioms of QM

- A1: The state of a QMS is given by a non-zero vector in a complex vector space $\mathcal{H}$ with Hermitian inner product.
- A2: The observables of a QMS are given by self-adjoint linear operators on $\mathcal{H}$
- A3: There is a distinguished quantum observable, the Hamiltonian H. Time evolution of states $|\psi(t)\rangle \in \mathcal{H}$ is given by the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Action of Ω

- Let $|\psi\rangle \in \mathcal{H}$, thus we have

$$|\psi\rangle \to \Omega |\psi\rangle .$$

- Baker-Hausdorff lemma

$$\begin{aligned}\exp(iG\lambda) A \exp(-iG\lambda) &= A + i\lambda [G, A] \\ &\quad + \left(\frac{i^2\lambda^2}{2!}\right) [G, [G, A]] + \dots\end{aligned}$$

The Hamiltonian

- The spin degree of freedom that we are describing by $\mathcal{H}$ has a dynamics given by

$$\mathsf{H} = -\mu \cdot \mathsf{B},$$

where

$$\mu = -\frac{ge}{2mc}\mathsf{S},$$

is the magnetic moment operator.

With Schrödinger Equation

The Schrödinger equation is given by

$$\frac{d}{dt}\left|\psi\right\rangle = -i\left(-\mu \cdot B\right)\left|\psi\right\rangle$$

and solution

$$\left|\psi\left(t\right)\right\rangle = U\left(t\right)\left|\psi\left(0\right)\right\rangle,$$

where

$$U\left(t\right) = \exp\left(it\mu \cdot B\right).$$

Explicitly

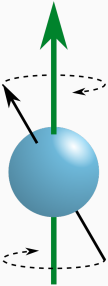
- Assuming **B** with just a component in the *z*-direction, we have

$$\mathsf{H} = \omega S_z,$$

thus

$$U(t) = \exp\left(-\frac{iS_z\omega t}{\hbar}\right),$$

we see that this Hamiltonian causes spin precession.



Physical Systems

- The Zeeman effect,
- Stern Gerlach experiment,
- Nuclear magnetic resonance spectroscopy,
- Quantum computing.

Heisenberg Picture

Heisenberg Picture

- Time evolution operator can be used to make a unitary transformation that puts the time-dependence on the observables. This is:

$$|\psi(t)\rangle \to |\psi(t)\rangle_H = U^{-1}(t)|\psi(t)\rangle,$$

$$\mathcal{O} \to \mathcal{O}_H(t) = U^{-1}(t)\mathcal{O}U(t)$$

- The dynamics is given by a differential equation on the operators

$$\frac{d}{dt}\mathcal{O}_H(t) = i[H, \mathcal{O}_H(t)]$$

Rotation of the Spin Vector

- Putting all together:

$$\frac{d}{dt}\mathsf{S}_H(t) = i[H, \mathsf{S}_H(t)]$$

  the solution will be

$$\mathsf{S}_H(t) = U(t)\mathsf{S}_H(0)U(t)^{-1}.$$

- The spin vector observable evolves in the Heisenberg picture by rotating about the magnetic field vector B with angular velocity $(ge/2mc)|\mathsf{B}|$

Schrödinger and Heisenberg Pictures

	Heisenberg Picture	Schrödinger Picture
State Ket	No Change	Evolution Given by H
Observable	Evolution Given by H	No Change

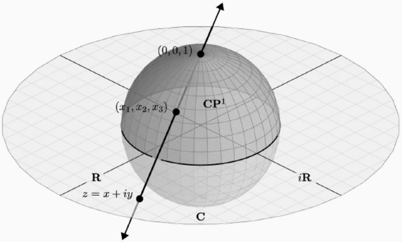
Complex Projective Space

Trying Another Characterization

- Multiplication on $\mathcal{H}$ by non-zero complex number do not change eigenvectors $\implies$ no physical effect.
- The relevant part is the quotient space $\left(\mathbb{C}^2 - \{0\}\right)/\mathbb{C}^*$, and constructed by: taking all non-zero elements of $\mathbb{C}^2$ and identifying those related by multiplication by a non-zero complex number.
- In some sense the space CP^1 is the complex plane, but with a "point at infinity" added.

Riemann Sphere

- CP^1: "Riemann sphere" with the relation to the plane and the point at infinity given by stereographic projection.



Coordinates relationship

- Relation between coordinates on the sphere (x_1, x_2, x_3) and complex coordinates $z_1/z_2 = z = x + iy$ is given by

$$x = \frac{x_1}{1 - x_3}, y = \frac{x_2}{1 - x_3},$$

  and

$$x_1 = \frac{2x}{x^2 + y^2 + 1}, x_2 = \frac{2y}{x^2 + y^2 + 1}, x_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

The Bloch Sphere

Another Characterization

- The unit sphere $S^2 \subset \mathbb{R}^3$ can be mapped to operators by

$$\mathsf{x} \to \sigma \cdot \mathsf{x},$$

and for each point $\mathsf{x} \in S^2$, $\sigma \cdot \mathsf{x}$ has eigenvalues ± 1. Eigenvectors with eigenvalue $+1$ are solutions to

$$\sigma \cdot \mathsf{x} \left|\psi\right\rangle = \left|\psi\right\rangle .$$

Interpretation in terms of spin operators

- One can characterize the $\mathbb{C} \subset \mathcal{H}$ corresponding to $\mathsf{x} \in S^2$ as the solutions to

$$\mathsf{S} \cdot \mathsf{x} \, |\psi\rangle = \frac{1}{2} \, |\psi\rangle \, ,$$

  thus, the North pole of the sphere is a "spin-up" state and the South pole is a "spin down" state.
- Along the equator one finds two points corresponding to states with definite values for S_1, as well as two for states that have definite values for S_2.