Central Potentials and the Hydrogen Atom

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Contents

Quantum particle in a central potential

Symmetries of the Hamiltonian

- If H is invariant under rotations, then their corresponding eigenspaces carry representations of $SO(3)$
- This eigenspaces, break up into irreducible representations, which are labeled by l and have dimension $2l+1$.

Central Potential

The hamiltonian function in phase space reads

$$h = \frac{p^2}{2m} + V(q),$$

and in 3D

$$h = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$

Schrodinger Representation

The Hamiltonian in the Schrodinger representation reads,

$$H = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}) + V(q_1, q_2, q_3)$$

and the focus wil be in potentials that are only functions of

$$q_1^2 + q_2^2 + q_3^2,$$

this is, functions that depend just on the radial distance to the origin.

Casimir Operator

Reminder: The casimir operator is defined as

$$L^2 = L_1^2 + L_2^2 + L_3^2.$$

And one can write the Laplacian in 3D spherical coordinates in terms of L^2, as

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}L^2$$

Casimir Operator

Reminder: The eigenvalues fo the Casimir Operator

$$l(l+1)$$

and thus, the Laplacian reads

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}$$

Eigenfunctions

The space of eigenfunctions of energy E will be a sum of irreducible representations of $SO(3)$, with the $SO(3)$ acting on the angular coordinates of th wavefunctions, leaving the radial coordinate invariant. And we will seek for functions that only have raidial dependence; $g_{lE}(r)$

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}+2\frac{2}{dr}-\frac{l(l+1)}{r^2}\right)+V(r)\right)g_{lE}(r)=Eg_{lE}(r)$$

Spherical Harmonics

Representationd of $SO(3)$ on functions of angular coordinates can be explicitly expressed in ters of spherical harmonics, $Y_l^m(\theta, \phi)$. For each solution, $g_{lE}(r)$ will have the eigenvalue equation

$$Hg_{lE}(r)Y_l^m(\theta, \phi) = Eg_{lE}(r)Y_l^m(\theta, \phi).$$

however, for a general potential function $V(r)$, exact solutions for the eigenvalues E and corresponding functions g_{lE} cannot be found in closed form.

Coulomb Potential

This potential describes a light charged particle moving in the potential due to the electric field of a much heavier charged particle. The potential reads

$$V = -\frac{e^2}{r^2},$$

and we're looking for solutions to

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + 2\frac{2}{dr} - \frac{l(l+1)}{r^2}\right) - \frac{e^2}{r^2}\right) g_{lE}(r) = E g_{lE}(r).$$

Coulomb Potential

By a change of coordinates

$$\left(-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)-\frac{e^2}{r^2}\right)rg_{lE}(r)=Erg_{lE}(r),$$

and the solution of this equation is genellary done via the Frobenius method, which is a power series solution.

- For $E \geq 0$; non-normalizable solutions that descrive scattering phenomena.
- For $E < 0$ solutions correspond to an integer $n = 1, 2, 3, . \cdots$, and describe bound states.

Coulomb Potential

- Bound state: A bound state occurs when a particle is trapped or localized within a specific region of space due to the potential acting upon it.
- Scattering state: A scattering state describes a particle that is not confined and can move freely to infinity.

$\mathfrak{so}(4)$ symmetry and the Coulomb potential

The hydrogen atom