QUANTUM THEORY 1

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HOMEWORK 1

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Problem. Prove

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB.$$

Solution. By definition, we have that

$$[AB, CD] = ABCD - CDAB$$

But $\{B, C\} = BC + CB \implies BC = \{B, C\} - CB$, and we also know that $\{B, C\} = \{C, B\}$, therefore, we have

$$ABCD = A(\{C, B\} - CB)D = A\{C, B\}D - ACBD$$

Putting into the main equation, we have

$$[AB, CD] = A\{C, B\}D - ACBD - CDAB$$

Now, let's work with the $ACBD$ term

$$BD = \{B, D\} - DB$$

Therefore, we have

$$ACBD = AC(\{B, D\} - DB) = AC\{B, D\} - ACDB$$

Again, inserting this into the main equation, we have

$$[AB, CD] = A\{C, B\}D - AC\{B, D\} + ACDB - CDAB$$

Now, for $ACDB$ we have

$$AC = \{C, A\} - CA$$

Then, we have

$$ACDB = (\{C, A\} - CA)DB = \{C, A\}DB - CADB$$

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And again, inserting into the main equation, we have

$$[AB, CD] = A\{C, B\}D - AC\{B, D\} + \{C, A\}DB - CADB - CDAB$$

But in the last expression, the last two therms on the rhs, could be written as

$$CADB + ADAB = C\{D, A\}B$$

Therefore, we have

$$[AB, CD] = A\{C, B\}D - AC\{B, D\} + \{C, A\}DB - C\{D, A\}B$$

Just as we wanted to prove.

Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

Problem. Suppose a 2x2 matrix $\hat{X}$ (not necessarily Hermitian, nor unitary) is written as

$$\hat{X} = a_0\mathbb{I} + \hat{\sigma} \cdot \mathbf{a}$$

where a_0 and $a_{1,2,3}$ are just numbers.

(1) How are a_0 and a_k $(k = 1, 2, 3)$ related to $tr(\hat{X})$ and $tr(\sigma_k\hat{X})$?
(2) Obtain a_0 and a_k in terms of the matrix elements X_{ij}.

Solution. Let's work with the first part of the problem

We know that the σ_i for $i = \{x, y, z\}$ represent the Pauli matrices, and those are represented in the eigenbase for S_z as the following

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{0.1}$$

And we also know that for those matrices the following relationship holds:

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k. \tag{0.2}$$

Thus, for part **1)**, we have the following

$$\hat{X} = a_0\delta_{ij} + \sigma_k a_k, \text{ with } i, j = 1, 2,\ k = 1, 2, 3,$$

$$\implies \hat{X} = a_0\delta_{ij} + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3,$$

Now, if we do the following identification, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$ y $\sigma_3 = \sigma_z$, we have

$$\hat{X} = a_0\delta_{ij} + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z, \tag{0.3}$$

or in matrix notation

$$\hat{X} = a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\implies \hat{X} = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}. \tag{0.4}$$

Now, for the tr($\hat{X}$), let's take a look at the equations given in (0.1) and note that all the Pauli matrices are traceless, i.e.:

$$tr(\sigma_i) = 0\ \forall i = x, y, z\ ,$$

then, we have

$$tr(\hat{X}) = tr\left(a_0\delta_{ij} + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z\right) = a_0 tr(\delta_{ij}) + a_1 tr(\sigma_x) + a_2 tr(\sigma_y) + a_3 tr(\sigma_z),$$

$$\implies tr(\hat{X}) = a_0 tr(\delta_{ij}) = a_0\delta_{ii} = 2a_0,$$

$$\therefore\ tr(\hat{X}) = 2a_0$$

Now, for the $tr(\sigma_k\hat{X})$, we have that

$$\sigma_k\hat{X} = a_0\sigma_k\delta_{ij} + a_1\sigma_k\sigma_1 + a_2\sigma_k\sigma_2 + a_3\sigma_k\sigma_3, \text{ with } k = 1, 2, 3,$$

Or, in component form, writing explicitly the products, we have

$$\sigma_1\hat{X} = a_0\sigma_1\delta_{ij} + a_1\sigma_1\sigma_1 + a_2\sigma_1\sigma_2 + a_3\sigma_1\sigma_3, \tag{0.5}$$

$$\sigma_2\hat{X} = a_0\sigma_2\delta_{ij} + a_1\sigma_2\sigma_1 + a_2\sigma_2\sigma_2 + a_3\sigma_2\sigma_3 \text{ and} \tag{0.6}$$

$$\sigma_3\hat{X} = a_0\sigma_3\delta_{ij} + a_1\sigma_3\sigma_1 + a_2\sigma_3\sigma_2 + a_3\sigma_3\sigma_3, \tag{0.7}$$

Now, if we write the product $\sigma_k\delta_{ij}$ in matrix form, we have

$$\sigma_1\delta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_2\delta_{ij} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \sigma_3\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies

$$\sigma_1\delta_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ \sigma_2\delta_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \sigma_3\delta_{ij} = \sigma_3. \tag{0.8}$$

Instead of doing the calculation for (0.5), (0.6) and (0.7), using matrices, let's use the property shown in the equation (0.2), and for a better reading, let's write it down again:

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

In general ϵ_{ijk} it's a 3-rank tensor totally asntisymetric, but for the purposes of the following calculations, we are going to consider it just a constant which depends on the order of the indices i, j, k, and its value could be $0, 1$ o -1. Now, having said that, for our purposes, let's make $\alpha = i\epsilon_{ijk}$.

On the other hand, in general, the product $\epsilon_{ijk}\sigma_k$ is just another Pauli matrix, and let's call it σ_α, where the index α it's different from the actual values of i and j (and we do that for the properties of the tensor ϵ_{ijk}).

Therefore, with the previous statements, and with the changes of variables, we have the following

$$\sigma_i\sigma_j = \delta_{ij} + \alpha\sigma_\alpha.$$

Now, if we take the trace of the previous expression, we have

$$tr(\sigma_i\sigma_j) = tr\left(\delta_{ij} + \alpha\sigma_\alpha\right) = tr(\delta_{ij}) + \alpha tr(\sigma_\alpha) = tr(\delta_{ij}).$$

In other words, the trace $\sigma_i\sigma_j$ is not zero if and only if $i = j$.

Now, with the previous result, and with the equations given in (0.8) and applying this to the equations (0.5), (0.6) y (0.7), we got

$$tr(\sigma_1\hat{X}) = tr\left(a_0\sigma_1\delta_{ij} + a_1\sigma_1\sigma_1 + a_2\sigma_1\sigma_2 + a_3\sigma_1\sigma_3\right) = tr\left(a_0\sigma_1\delta_{ij} + a_1\sigma_1\sigma_1\right),$$
$$\implies tr(\sigma_1\hat{X}) = a_1 tr\left(\sigma_1\sigma_1\right) = a_1 tr(\delta_{11}) = 2a_1$$
$$\therefore\ tr(\sigma_1\hat{X}) = 2a_1.$$

Noe, for $\sigma_2\hat{X}$, we got

$$tr(\sigma_2\hat{X}) = tr\left(a_0\sigma_2\delta_{ij} + a_2\sigma_2\sigma_2\right) = a_2 tr(\sigma_2\sigma_2) = a_2 tr(\delta_{22})$$
$$\implies tr(\sigma_2\hat{X}) = 2a_2.$$

And finally, for $\sigma_3\hat{X}$, we got

$$tr(\sigma_3\hat{X}) = tr\left(a_0\sigma_3\delta_{ij} + a_3\sigma_3\sigma_3\right) = a_0 tr(\sigma_3) + a_3 tr(\sigma_{33}) = 2a_3 tr(\delta_{33}),$$

$$\implies tr(\sigma_3 \hat{X}) = 2a_3 tr(\delta_{33}).$$

Therefore, we end with the following result if $k = 1, 2, 3$, then

$$tr(\sigma_k \hat{X}) = 2a_k.$$

For **2)**, let's focus on the equation (0.4).

It's clear that

$$\hat{X}_{11} + \hat{X}_{22} = a_0 + a_3 + a_0 - a_3 = 2a_0,$$

$$\therefore\ a_0 = \frac{1}{2}\left(\hat{X}_{11} + \hat{X}_{22}\right).$$

For a_1, we have

$$\hat{X}_{12} + \hat{X}_{21} = a_1 - ia_2 + a_1 + ia_2 = 2a_1,$$
$$\therefore\ a_1 = \frac{1}{2}\left(\hat{X}_{12} + \hat{X}_{21}\right).$$

On the other hand, for a_2, we have

$$-\hat{X}_{12} + \hat{X}_{21} = -a_1 + ia_2 - a_1 + ia_2 = 2ia_2,$$

$$\therefore\ a_2 = \frac{1}{2i}\left(\hat{X}_{21} - \hat{X}_{12}\right).$$

And finally, for a_3, we have

$$\hat{X}_{11} - \hat{X}_{22} = a_0 + a_3 - a_0 + a_3 = 2a_3,$$
$$\therefore\ a_3 = \frac{1}{2}\left(\hat{X}_{11} - \hat{X}_{22}\right).$$

Problem. Using the rules of bra-ket algebra, prove or evaluate the following:

(1) $tr(XY) = tr(YX)$, where X and Y are operators;
(2) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;
(3) $exp[if(A)] =?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;
(4) $\sum_{a'} \psi'^*(x')\psi'(x'')$, where $\psi'(x') = \langle x'|a'\rangle$.

Solution. For 1), we have that in general, any operator can be expressed in a matrix form in the following way

$$A \to A_{ij} = \langle i|A|j\rangle,$$

and moreover, by definition, the trace of the operator is given by the following expression

$$tr(A) = \sum_i A_{ii} = \sum_i \langle i|A|i\rangle$$

on the other hand, we have that XY in matrix notation can be represented as

$$XY \to (XY)_{ij} = X_{ik}Y_{kj},$$

therefore, the trace of the previous operator will be

$$\sum_i (XY)_{ii} = \sum_i X_{ik}Y_{ki} = \sum_i \sum_k \langle i|X|k\rangle\langle k|Y|i\rangle = \sum_k \sum_i \langle k|Y|i\rangle\langle i|X|k\rangle,$$

where we've used that

$$\sum_k |k\rangle\langle k| = \sum_k |i\rangle\langle i| = 1,$$

thus, we have

$$\sum_i (XY)_{ii} = \sum_k \sum_i \langle k|Y|i\rangle\langle i|X|k\rangle = \sum_k \langle k|YX|k\rangle = \sum_i (YX)_{ii},$$

therefore, we end up with the following expression

$$tr(XY) = tr(YX).$$

Now, for 2) we have that in general, the dual correspondence between an operator can be expressed in the following way

$$X|\alpha\rangle \Longleftrightarrow_{DC} \langle\alpha|X^{\dagger},$$

and in order to make realy explicit, for two operators we'll use the following notation

$$X\left(Y|\alpha\rangle\right) = X|\beta\rangle,$$

and by the dual correspondence, we have

$$X|\beta\rangle \Longleftrightarrow_{DC} \langle\beta|X^{\dagger},$$

but the bra $\langle\beta|$ is given by

$$\langle\beta| = \langle\alpha|Y^{\dagger},$$

by simply using the dual correspondence again. Therefore, we have the following result

$$X\left(Y|\alpha\rangle\right) \Longleftrightarrow_{DC} \langle\alpha|Y^{\dagger}X^{\dagger},$$

on in terms of the notation used in the statement of the problem,

$$(XY)^{\dagger} = Y^{\dagger}X^{\dagger}.$$

For 3) we know that, we can expand the exponential function as an infinite sum in the following way

$$\exp(x) = \sum_{n}^{\infty} \frac{1}{n!} x^{n},$$

and using the same line of thought, for the expression at hand, we have

$$\exp\left(if\left(A\right)\right) = \sum_{n}^{\infty} \frac{1}{n!} \left(if\left(A\right)\right)^{n}.$$

Now, if we use a matrix representation of the above operator, we have that

$$\langle\alpha_i| \exp\left(if\left(A\right)\right)|\alpha_j\rangle = \langle\alpha_i| \sum_{n}^{\infty} \frac{1}{n!} \left(if\left(A\right)\right)^{n} |\alpha_j\rangle.$$

On the other hand, we know that $f\left(A\right)$ can be expanded as a Taylor serie, in the following way

$$f\left(A\right) = \sum_{n} \frac{f^{n}}{n!} A^{n},$$

and if we follow the same procedure and use a matrix representation, we have that

$$\langle\alpha_i|f\left(A\right)|\alpha_j\rangle = \langle\alpha_i| \sum_{n} \frac{f^{n}}{n!} A^{n}|\alpha_j\rangle,$$

but by assumption A is Hermitian and its eigenvalues are known, therefore, we can say that

$$A|\alpha_j\rangle = a_j|\alpha_j\rangle,$$

and with that, we have

$$\langle\alpha_i|f\left(A\right)|\alpha_j\rangle = \langle\alpha_i| \sum_{n} \frac{f^{n}}{n!} A^{n}|\alpha_j\rangle = \langle\alpha_i| \sum_{n} \frac{f^{n}}{n!} \alpha_j^{n}|\alpha_j\rangle = \langle\alpha_i|f\left(\alpha_j\right)|\alpha_j\rangle,$$

$$\Longrightarrow \langle\alpha_i|f\left(A\right)|\alpha_j\rangle = \langle\alpha_i|f\left(\alpha_j\right)|\alpha_j\rangle.$$

And using that previous results, we have

$$\langle\alpha_i|\exp\left(if\left(A\right)\right)|\alpha_j\rangle = \langle\alpha_i|\sum_n^{\infty}\frac{1}{n!}\left(if\left(\alpha_j\right)\right)^n|\alpha_j\rangle,$$

$$\implies \langle\alpha_i|\exp\left(if\left(A\right)\right)|\alpha_j\rangle = \sum_n^{\infty}\frac{1}{n!}i^n f^n\left(\alpha_j\right)\langle\alpha_i|\alpha_j\rangle = \sum_n^{\infty}\frac{1}{n!}i^n f^n\left(\alpha_j\right)\delta_{ij},$$

but, in the rhd of the last expression, we do the following identification

$$\sum_n^{\infty}\frac{1}{n!}i^n f^n\left(\alpha_j\right) = \exp\left(if\left(\alpha_j\right)\right),$$

therefore, we have

$$\langle\alpha_i|\exp\left(if\left(A\right)\right)|\alpha_j\rangle = \exp\left(if\left(\alpha_j\right)\right)\delta_{ij},$$

and finally we know that the δ_{ij} can be expressed as an outer product of the elements of the basis, thus, we end with

$$exp[if(A)] = \sum_{\alpha_j}\exp\left(if\left(\alpha_j\right)\right)|\alpha_j\rangle\langle\alpha_j|.$$

It's important to notice that in all the previous calculations, the matrix notation was performed in terms of the eigenbase of the A operator.

Finally, for part 4), we have that

$$\sum_{a'}\psi'^*(x')\psi'(x'') = \sum_{a'}\langle x'|a'\rangle\langle a'|x''\rangle,$$

but, using the completeness relationship, which states that $\sum_{a'}|a'\rangle\langle a'| = \mathbb{I}$, we have

$$\sum_{a'}\psi'^*(x')\psi'(x'') = \langle x'|x''\rangle = \delta(x'-x''),$$

$$\sum_{a'}\psi'^*(x')\psi'(x'') = \delta(x'-x'').$$

Problem. Construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that

$$\mathbf{S}\cdot\hat{\mathbf{n}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \frac{\hbar}{2}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$$

where $\hat{n}$ is characterized by the angles shown in the figure. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$.

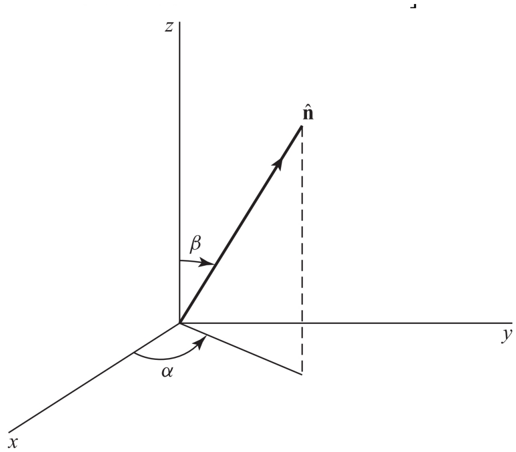

FIGURE 0.1. Figure of problem 1.11

Solution. Let's write the normal vector $\hat{\mathbf{n}}$ as function of the angles given, which is clear it's going to be the usual expression for a vector in terms of spherical coordinates,

$$\hat{\mathbf{n}} = \sin(\alpha)\cos(\alpha)\hat{\mathbf{x}} + \sin(\alpha)\sin(\beta)\hat{\mathbf{y}} + \cos(\beta)\hat{\mathbf{z}},$$

then, the inner product $\mathbf{S}\cdot\hat{\mathbf{n}}$ is given by

$$\mathbf{S}\cdot\hat{\mathbf{n}} = S_x\sin(\alpha)\cos(\alpha) + S_y\sin(\alpha)\sin(\beta) + S_z\cos(\beta).$$

On the other hand, we know that in matrix notation the operators S_x, S_y, S_z, in the S_z basis, are given by

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ y } S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

thus, the product $\mathbf{S}\cdot\hat{\mathbf{n}}$, in matrix notation is given by

$$\mathbf{S}\cdot\hat{\mathbf{n}} = \frac{\hbar}{2}\sin(\beta)\cos(\alpha)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\sin(\beta)\sin(\alpha)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{\hbar}{2}\cos(\beta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\implies \mathbf{S}\cdot\hat{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos(\beta) & \sin(\beta)\cos(\alpha) - i\sin(\beta)\sin(\alpha) \\ \sin(\beta)\cos(\alpha) + i\sin(\beta)\sin(\alpha) & -\cos(\beta) \end{pmatrix},$$

or

$$\mathbf{S}\cdot\hat{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos(\beta) & \sin(\beta)\exp(-i\alpha) \\ \sin(\beta)\exp(i\alpha) & -\cos(\beta) \end{pmatrix}.$$

Now, we must construct $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ such that the following equation holds

$$\frac{\hbar}{2}\begin{pmatrix} \cos(\beta) & \sin(\beta)\exp(-i\alpha) \\ \sin(\beta)\exp(i\alpha) & -\cos(\beta) \end{pmatrix}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \frac{\hbar}{2}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle, \tag{0.9}$$

but we can express the ket $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ as a linear combination of the basis kets $|+\rangle$ y $|-\rangle$. Therefore, let's we want to find $\mathcal{C}_+,\mathcal{C}_- \in \mathbb{C}$, such that

$$|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \mathcal{C}_+|+\rangle + \mathcal{C}_-|-\rangle,$$

or, in matrix notation as $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \begin{pmatrix} \mathcal{C}_+ \\ \mathcal{C}_- \end{pmatrix}$, and with this, the equation given in (0.9) reads

$$\frac{\hbar}{2}\begin{pmatrix} \cos(\beta) & \sin(\beta)\exp(-i\alpha) \\ \sin(\beta)\exp(i\alpha) & -\cos(\beta) \end{pmatrix}\begin{pmatrix} \mathcal{C}_+ \\ \mathcal{C}_- \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \mathcal{C}_+ \\ \mathcal{C}_- \end{pmatrix},$$

or

$$\begin{pmatrix} \mathcal{C}_+\cos(\beta) + \mathcal{C}_-\sin(\beta)\exp(-i\alpha) \\ \mathcal{C}_+\sin(\beta)\exp(i\alpha) - \mathcal{C}_-\cos(\beta) \end{pmatrix} = \begin{pmatrix} \mathcal{C}_+ \\ \mathcal{C}_- \end{pmatrix},$$

then, we have the following equations

$$\mathcal{C}_+\cos(\beta) + \mathcal{C}_-\sin(\beta)\exp(-i\alpha) = \mathcal{C}_+, \tag{0.10}$$

$$\mathcal{C}_+\sin(\beta)\exp(i\alpha) - \mathcal{C}_-\cos(\beta) = \mathcal{C}_-. \tag{0.11}$$

Before moving on with the calculations, lets remember that the ket $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ should be normalized, and this condition, translates into

$$\|\langle +;\mathbf{S}\cdot\hat{\mathbf{n}}|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle\| = \mathcal{C}_+^2 + \mathcal{C}_-^2 = 1. \tag{0.12}$$

Now, let's play a little with the equation (0.10)

$$\mathcal{C}_+\cos(\beta) + \mathcal{C}_-\sin(\beta)\exp(-i\alpha) = \mathcal{C}_+ \implies \mathcal{C}_+(1-\cos(\beta)) = \mathcal{C}_-\sin(\beta)\exp(-i\alpha),$$

and, remember, in general $\mathcal{C}_+,\mathcal{C}_- \in \mathbb{C}$, therefor, if we multiply by its complex conjugate, we have

$$\mathcal{C}_+^2\,(1-\cos(\beta))^2 = \mathcal{C}_-^2\,\sin^2(\beta).$$

Now, if we use the following trigonometric identity, $\sin^2(\beta/2) = 1/2\,(1-\cos(\beta))$, and the relation $\mathcal{C}_-^2 = 1-\mathcal{C}_+^2$, we have

$$4\mathcal{C}_+^2\,\sin^4(\beta/2) = \left(1-\mathcal{C}_+^2\right)\sin^2(\beta). \tag{0.13}$$

Now, let's use the identity $\sin(\theta)\cos(\phi) = 1/2(\sin(\theta+\phi)+\sin(\theta-\phi))$ with the changes given by $\theta = \beta/2$ and $\phi = \beta/2$, which gives

$$\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right) = \frac{1}{2}\left(\sin\left(\beta\right)+\sin(0)\right) \implies \sin\left(\beta\right) = 2\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right),$$
$$\implies \sin^2\left(\beta\right) = 4\sin^2\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right),$$

if we subsitute the previous result in the equation (0.13), we have

$$4\mathcal{C}_+^2\,\sin^4(\beta/2) = 4\left(1-\mathcal{C}_+^2\right)\sin^2\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right),$$
$$\implies \mathcal{C}_+^2\sin^2\left(\frac{\beta}{2}\right) = \left(1-\mathcal{C}_+^2\right)\cos\left(\frac{\beta}{2}\right) \implies \mathcal{C}_+^2\sin^2\left(\frac{\beta}{2}\right)+\mathcal{C}_+^2\cos^2\left(\frac{\beta}{2}\right) = \cos^2\left(\frac{\beta}{2}\right),$$
$$\implies \mathcal{C}_+^2\left(\sin^2\left(\frac{\beta}{2}\right)+\cos^2\left(\frac{\beta}{2}\right)\right) = \cos^2\left(\frac{\beta}{2}\right) \implies \mathcal{C}_+^2 = \cos^2\left(\frac{\beta}{2}\right),$$
$$\therefore\ \mathcal{C}_+ = \cos\left(\frac{\beta}{2}\right)$$

On the other hand, for the constant $\mathcal{C}_-$, as in the previous case, let's look at the equation (0.10), from which we have

$$\mathcal{C}_-\,\sin(\beta)\exp(-i\alpha) = \mathcal{C}_+\left(1-\cos\left(\beta\right)\right).$$

And using the relationship before shown $\sin^2\left(\beta/2\right) = 1/2\,(1-\cos(\beta))$ y $\sin\left(\beta\right) = 2\sin(\beta/2)\cos(\beta/2)$, we have

$$2\mathcal{C}_-\,\sin(\beta/2)\cos(\beta/2)\exp(-i\alpha) = 2\mathcal{C}_+\sin^2\left(\beta/2\right),$$
$$\implies \mathcal{C}_-\,\cos(\beta/2)\exp(-i\alpha) = \mathcal{C}_+\sin\left(\beta/2\right),$$

but, $\mathcal{C}_+ = \cos\left(\frac{\beta}{2}\right)$, then

$$\mathcal{C}_-\,\cos(\beta/2)\exp(-i\alpha) = \cos\left(\frac{\beta}{2}\right)\sin\left(\beta/2\right), \implies \mathcal{C}_-\,\exp(-i\alpha) = \sin\left(\beta/2\right),$$

$$\therefore\ \mathcal{C}_- = \sin\left(\frac{\beta}{2}\right)\exp(i\alpha) \tag{0.14}$$

Therefore, the ket $|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle$ written in terms if the basis kets $\{|+\rangle,|-\rangle\}$ its given by

$$|\mathbf{S}\cdot\hat{\mathbf{n}};+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle+\sin\left(\frac{\beta}{2}\right)\exp(i\alpha)|-\rangle.$$

Problem. The Hamiltonian operator for a two-state system is given by

$$H = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

where a is a number with the dimensions of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$).

Solution. Let's fix a basis, and let be $|1\rangle = (1,0)^T$ with $|2\rangle = (0,1)^T$, therefore, the Hamiltonian, expressed in terms of this basis is given by the following matrix

$$H = a \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and for ehe eigenvalues and eigenvectors, we have to solve the following eigenvalue problem

$$H|\alpha\rangle = \lambda|\alpha\rangle \tag{0.15}$$

, for all $|\alpha\rangle$, which can also be stated in the following way $(H - \lambda\mathbb{I})\,|\alpha\rangle = 0$, and the problem can be translated into the following

$$\det\left(H - \lambda\mathbb{I}\right) = 0,$$

therefore, we have that

$$\det\left(H - \lambda\mathbb{I}\right) = 0 \iff \det\begin{pmatrix} a-\lambda & a \\ a & -a-\lambda \end{pmatrix} = 0 \iff -\left(a-\lambda\right)\left(a+\lambda\right) - a^2 = 0,$$

$$\iff a^2 - \lambda^2 + a^2 = 0 \iff \lambda^2 = 2a^2$$

and then, we have that the eigenvalues $\lambda_\pm$ are given by

$$\lambda_\pm = \pm\sqrt{2}a.$$

Now, for the eigenvectors we have to solve the eigenvalue problem given in the equation 0.15 for any of the eigenvalues given in the previous equation. If we plug $\lambda_\pm = \pm\sqrt{2}a$, and take $|\alpha\rangle = (x_1, x_2)^T$, and then we end with the following system of equations

$$\begin{aligned} \left(a \pm \sqrt{2}a\right) x_1 + a x_2 &= \sqrt{2}a x_1 \\ a x_1 + \left(-a \pm \sqrt{2}a\right) x_2 &= \sqrt{2}a x_2 \end{aligned}$$

which can be solved by the conventional methods at hand. Giving the following eigenkets, for λ_+, we have

$$|\alpha_+\rangle = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix},$$

and, for λ_-

$$|\alpha_-\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}.$$

And finally, if we normalize the previous kets, we have that

$$|\hat{\alpha}_+\rangle = \frac{1}{2\sqrt{1-\frac{1}{\sqrt{2}}}} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix},$$

$$|\hat{\alpha}_-\rangle = \frac{1}{2\sqrt{1+\frac{1}{\sqrt{2}}}} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}.$$