Cosmology, Problem Set 5: Neutron Abundance and Hydrogen Recombination

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Problem 1a: The Boltzmann equation reads

$$\frac{1}{a^3}\frac{d}{dt}(n_1 a^3) = -\Gamma_1 \left(n_1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq} \frac{n_3 n_4}{n_2} \right),$$

and by making $n_1 = n_n$, $n_3 = n_p$, with n_2, n_4 to be leptons n_l, we have

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n \left(n_n - \left(\frac{n_n n_l}{n_p n_l}\right)_{eq} \frac{n_p n_l}{n_l} \right),$$

which will lead us to

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p \right),$$

just as we wanted.

Problem 1b: The equilibrium distribution of number density of species i is given by

$$(n_i)_{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_i - \mu_i}{T}\right],$$

then by neglecting the chemical potential, we will have

$$(n_n)_{eq} = g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_n}{T}\right],$$

and also

$$(n_p)_{eq} = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_p}{T}\right],$$

from this we can make the ratio

$$\left(\frac{n_i}{n_p}\right)_{eq} = \frac{g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_n}{T}\right]}{g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} \exp\left[-\frac{m_p}{T}\right]}$$

but $g_n = g_p$, thus

$$\left(\frac{n_i}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[\frac{-m_n + m_p}{T}\right],$$

and if we define $Q = m_n - m_p$, we have $-Q = -m_n + m_p$, thus

$$\left(\frac{n_i}{n_p}\right)_{eq} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left[\frac{-Q}{T}\right],$$

just as we wanted.

Problem 1c: We know that

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

on the other hand, if we define

$$X_n = \frac{n_n}{n_n + n_p},$$

we can write

$$\frac{1}{a^3}\frac{d}{dt}(n_n a^3 \frac{n_n + n_p}{n_n + n_p}) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

which implies that

$$\frac{1}{a^3}\frac{d}{dt}(X_n a^3 (n_n + n_p)) = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

and if we take the time derivative we have

$$\frac{1}{a^3}\left[a^3(n_n + n_p)\frac{d}{dt}(X_n) + X_n \frac{d}{dt}(a^3(n_n + n_p))\right] = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right).$$

On the other hand, if we assume that the baryon number is conserved, we have the following condition

$$\frac{d}{dt}(a^3(n_n + n_p)) = 0,$$

which implies that

$$\frac{1}{a^3}\left[a^3(n_n + n_p)\frac{d}{dt}(X_n)\right] = -\Gamma_n \left(n_n - \left(\frac{n_n}{n_p}\right)_{eq} n_p\right),$$

thus

$$\frac{d}{dt}X_n = -\Gamma_n \left(\frac{n_n}{(n_n + n_p)} - \left(\frac{n_n}{n_p}\right)_{eq} \frac{n_p}{(n_n + n_p)}\right),$$

on the other hand we know that

$$\left(\frac{n_n}{n_p}\right)_{eq} \approx \exp(-Q/T),$$

thus, we have

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - \exp(-Q/T)\frac{n_p}{(n_n + n_p)}\right).$$

On the other hand, we have the following identity

$$\frac{n_p}{n_n + n_p} = 1 - \frac{n_n}{n_n + n_p} = 1 - X_n,$$

which leads us to

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - \exp(-Q/T)(1 - X_n)\right),$$

therefore, we finally have

$$\frac{d}{dt}X_n = -\Gamma_n \left(X_n - (1 - X_n)\exp(-Q/T)\right),$$

as desired.

Problem 1d: For this part we have the following derivative

$$\frac{dX_n}{dt} = \frac{dx}{dt}\frac{dX_n}{dx},$$

but $x = x(T)$ and $T = T(a)$, thus by the chain rule we have

$$\frac{dX_n}{dt} = \frac{dx}{dT}\frac{dT}{da}\frac{da}{dt}\frac{dX_n}{dx},$$

and more explicitly $x = Q/T$, whereas by assuming $T \propto a^{-1}$, we have

$$\frac{dx}{dT} = -\frac{Q}{T^2}, \frac{dT}{da} = -\frac{1}{a^2},$$

thus we have

$$\frac{dX_n}{dt} = \left(-\frac{Q}{T^2}\right)\left(-\frac{1}{a^2}\right)\frac{da}{dt}\frac{dX_n}{dx},$$

$$\implies \frac{dX_n}{dt} = \left(\frac{Q}{T}\right)\left(\frac{1}{T}\right)\left(\frac{1}{a}\right)\left(\frac{1}{a}\frac{da}{dt}\right)\frac{dX_n}{dx},$$

and again, using the fact that $T \propto a^{-1}$ we have $T/a = 1$, thus

$$\implies \frac{dX_n}{dt} = xH\frac{dX_n}{dx},$$

just as we wanted.

Problem 1e: For this problem, I append the solution of the ode as a Mathematica notebook. On the other hand, the limi of neutron fraction for longer times is given by

$$X_e \to 0.149533$$

Problem 2a: Starting with

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left[-\frac{(m_i - \mu_i)}{T}\right],$$

we have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{\left(\frac{m_H T}{2\pi}\right)^{3/2}}{\left(\frac{m_e T}{2\pi}\right)^{3/2}\left(\frac{m_p T}{2\pi}\right)^{3/2}}\right) \exp\left[-\frac{(m_H - \mu_H)}{T} + \frac{(m_e - \mu_e)}{T} + \frac{(m_p - \mu_p)}{T}\right],$$

and from this we have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{m_e + m_p - m_H}{T} + \frac{\mu_H - \mu_e - \mu_p}{T}\right],$$

and by using the condition of equilibrium given, we have

$$\mu_H - \mu_e - \mu_p = 0,$$

thus

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{m_e + m_p - m_H}{T}\right],$$

and by making

$$E_I = m_e + m_p - m_H,$$

we finally have

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

just as we wanted.

Problem 2b: The mass of the Hydrogen comes mostly from the proton, thus to a good approximation we have

$$\frac{m_H}{m_p} \approx 1,$$

thus this implies that

$$\frac{n_H}{n_e n_p} = \frac{g_H}{g_e g_p} \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

on the other hand, the degrees of freedom are $g_p = 2$, $g_e = 2$, whereas for the hydrogen, the spins of the electron and proton can be either aligned or anti aligned, which will give one singlet state and one triplet state, therefore $g_H = 1 + 3 = 4$, and with this

$$\frac{n_H}{n_e n_p} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

moving on, the universe isn't electrically charged, so $n_e = e_p$, thus

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_I}{T}\right],$$

as desired.
Problem 2c: If we define

$$X_e = \frac{n_e}{n_e + n_H},$$

then we have

$$\frac{1 - X_e}{X_e^2} = \frac{1 - \frac{n_e}{n_e+n_H}}{\frac{n_e^2}{(n_e+n_H)^2}} = \frac{\frac{n_e+n_H-n_e}{n_e+n_H}}{\frac{n_e^2}{(n_e+n_H)^2}},$$

thus, we can write

$$\frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2}\left(n_e + n_H\right) \implies \frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2} n_b,$$

where $n_b = n_e + n_H$.
Problem 2d: In the previous homework we found the following result for bosons

$$n = \frac{\zeta(3)}{\pi^2} g T^3,$$

and in particular, for photons we have $g = 2$, thus

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3,$$

and if we use $n_b = \eta n_\gamma$ we have

$$\frac{1 - X_e}{X_e^2} = \frac{n_H}{n_e^2} \frac{2\eta\zeta(3)}{\pi^2} T^3,$$

but we also know that

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

thus

$$\frac{1 - X_e}{X_e^2} = \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp\left[\frac{E_i}{T}\right] \times \frac{2\eta\zeta(3)}{\pi^2} T^3,$$

but $T^{-3/2} T^3 = T^{3/2}$, thus

$$\frac{1 - X_e}{X_e^2} = \left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right] \times \frac{2\eta\zeta(3)}{\pi^2},$$

and we arrive at the desired result.
Problem 2e: I used Mathematica and I append the notebook. But here I'll do some algebra to find a nicer expression. The solution is given by

$$X_e(T) = -\frac{e^{-\frac{E_I}{T}}\left(1 - \sqrt{8\sqrt{2}\pi^{3/2}\eta X_0 e^{E_I/T}\left(\frac{T}{m_e}\right)^{3/2} + 1}\right)}{4\sqrt{2}\pi^{3/2}\eta X_0 \left(\frac{T}{m_e}\right)^{3/2}},$$

where I defined

$$X_0 = \frac{2\zeta(3)}{\pi^2},$$

but $8 \times 2^{1/2} = 4 \times 2^{3/2}$, and $4 \times 2^{1/2} = 2 \times 2^{3/2}$, and with that we have

$$X_e(T) = -\frac{\left(1 - \sqrt{4 \times 2^{3/2}\pi^{3/2}\eta X_0 \exp[E_I/T]\left(\frac{T}{m_e}\right)^{3/2} + 1}\right)}{2 \times 2^{3/2}\pi^{3/2}\eta X_0 \left(\frac{T}{m_e}\right)^{3/2} \exp[E_I/T]},$$

which can be simplified to

$$X_e(T) = \frac{-1 + \sqrt{1 + 4X_0\left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right]}}{2X_0\left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right]},$$

and if we define

$$f = X_0\left(\frac{2\pi T}{m_e}\right)^{3/2} \exp\left[\frac{E_i}{T}\right],$$

the final solution take the form

$$X_e(T) = \frac{-1 + \sqrt{1 + 4f}}{2f},$$

which is much ”prettier”.

Problem 2f: I append a Mathematica Notebook with the plot.

Problem 2g: The temperature of recombination can be calculated by

$$X_e(T) = 0.5,$$

which I solved numerically, with the result given by

$$T_{rec} = 0.323887\text{eV}.$$

Problem 2h: For a_{rec} we have

$$T = \frac{T_0}{a} \implies a_{rec} = \frac{T_0}{T_{rec}},$$

and if we use $T_0 = 2.3 \times 10^{-4}$eV we have

$$a_{rec} = 0.000710124.$$

Problem 2i: Finally, we need to compute t_{rec}, where t is given by

$$t(a) = \frac{1}{H_0}\int_0^a \frac{dx}{x\sqrt{\Omega_{M,0}/x^3 + \Omega_{R,0}/x^4}},$$

and if we use $a = a_{rec}$ together with $\Omega_{M,0} = 0.3$, $\Omega_{R,0} = 8.6 \times 10^-5$ and $H_0 = 0.7 \times 10^{-10}$/years, we have

$$t_{rec} = 243884 \text{ years}.$$