Sequences and Series - Lesson 2 (Linear sequences and series)

Grade 12 Mathematics

Linear Sequences

A linear sequence is a sequence in which the difference between consecutive terms is constant. This constant difference is called the *common difference*.

Example

Find the difference between the terms following sequence and is the sequence linear: 3, 7, 11, 15, ...

Solution: To find the common difference, subtract the first term from the second term and check if it is the same for other consecutive terms?

$$d = 7 - 3 = 4$$

Checking other terms:

$$11 - 7 = 4 \quad \text{and} \quad 15 - 11 = 4$$

Thus, the common difference d is 4. Since the difference is constant, the sequence is linear.

Take away If a the terms of a sequence have a constant difference, then the sequence is linear.

0.1 Checkpoint Exercise

Determine if the following sequences are linear. If they are, find the common difference.

a. 2, 5, 8, 11, 14, ...

b. 10, 20, 30, 40, 50, ...

c. 1, 4, 9, 16, 25, ...

d. 100, 90, 80, 70, 60, ...

1 Deriving the nth term of a linear sequence

To derive the nth term of a linear sequence, first let's denote the first term of the sequence as a and the common difference as d. Then, each term in the sequence can be expressed as follows:

$$\begin{aligned}
T_1 &= a \\
T_2 &= a + d \\
T_3 &= a + 2d \\
T_4 &= a + 3d \\
&\vdots \\
T_n &= a + (n-1)d
\end{aligned}$$

Therefore, the nth term of a linear sequence is given by:

Nth term of a linear sequence

$$T_n = a + (n-1)d \tag{1}$$

Where:

- T_n is the nth term of the sequence
- a is the first term of the sequence
- d is the common difference
- n is the term number.

Example

Find the nth term of the linear sequence: 5, 8, 11, 14, ...

Solution: Here, the first term $a = 5$ and the common difference $d = 8 - 5 = 3$. Using the formula:

$$T_n = a + (n-1)d = 5 + (n-1) \cdot 3 = 5 + 3n - 3 = 3n + 2$$

Therefore, the nth term is $T_n = 3n + 2$.

1.1 Checkpoint Exercise

Find the nth term of the following linear sequences:

a. 4, 9, 14, 19, ...

b. 12, 15, 18, 21, ...

c. 20, 18, 16, 14, ...

d. 7, 14, 21, 28, ...

2 Linear Series and sum of n terms

A linear series is the sum of the terms of a linear sequence.

$$T_1, T_2, T_3, ..., T_n \tag{2}$$

The sum of the first n terms of a linear series is given by:

$$S_n = T_1 + T_2 + T_3 + ... + T_n \tag{3}$$

Where S_n is the sum of the first n terms of the series.

Example

Find the sum of the first 10 terms of the linear sequence: 2, 5, 8, 11, ...

Solution: We can find the sum of the this linear sequence by adding the first 10 terms:

$$S_{10} = T_1 + T_2 + T_3 + ... + T_{10}$$

Then by substituting the values:

$$S_{10} = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29$$

Calculating the sum:

$$S_{10} = 155$$

Therefore, the sum of the first 10 terms is 155.

Take away The sum of the first n terms of a linear sequence can be found by directly adding the terms which forms a series.

2.1 Checkpoint Exercise

Find the sum of the first 15 terms of the following linear sequences:

a. 3, 6, 9, 12, ...

b. 10, 15, 20, 25, ...

c. 5, 10, 15, 20, ...

d. 8, 12, 16, 20, ...

3 Deriving the sum of n terms of a linear series

To derive the sum of the first n terms of a linear series. We must recall that

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n \tag{4}$$

We can further express equation (4) as:

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] \tag{5}$$

Now, we can create another equation by writing equation (5) in reverse order:

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \ldots + (a + d) + a \tag{6}$$

Adding these two equations (5) and (6) together:

$$\begin{aligned} S_n =& a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] \\ S_n =& [a + (n-1)d] + [a + (n-2)d] + \ldots + (a + d) + a \end{aligned}$$

We get:

$$\begin{aligned} 2S_n =& [a + (n-1)d] + [a + (n-1)d] + [a + (n-1)d] + \ldots + [a + (n-1)d] \\ 2S_n =& n[a + (n-1)d] \end{aligned}$$

Dividing both sides by 2, we obtain the formula for the sum of the first n terms of a linear series:

Sum of n terms of a linear series

$$S_n = \frac{n}{2}[2a + (n-1)d] \tag{7}$$

Where:

- S_n is the sum of the first n terms of the series.
- a is the first term of the sequence.
- d is the common difference.
- n is the number of terms to be summed.

Example

Find the sum of the first 20 terms of the linear sequence: 4, 7, 10, 13, ...
Solution: Here, the first term $a = 4$, the common difference $d = 7 - 4 = 3$, and the number of terms $n = 20$. Using the formula:

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Substituting the values:

$$S_{20} = \frac{20}{2}[2 \cdot 4 + (20-1) \cdot 3]$$

Calculating the inside of the brackets:

$$= 10[8 + 57] = 10 \cdot 65 = 650$$

Therefore, the sum of the first 20 terms is 650.

3.1 Checkpoint Exercise

Find the sum of the first 12 terms of the following linear sequences:

a. 6, 11, 16, 21, ...

b. 15, 20, 25, 30, ...

c. 9, 14, 19, 24, ...

d. 7, 13, 19, 25, ...

Summary

In this lesson, we explored linear sequences and series. We learned how to identify linear sequences by their constant common difference. We derived the formula for the nth term of a linear sequence and practiced finding it for various sequences. Additionally, we examined linear series and derived the formula for the sum of the first n terms of a linear series. Through examples and exercises, we reinforced our understanding of these concepts.