Section 1: Derivatives of Polynomials and Exponentials

1. Find the derivative of the following:
   Note: In problem (g), a, b, c and d are constants.

   (a) $f(x) = 1356$

   (b) $g(t) = \frac{2}{5}t^2 - \pi^2$

   (c) $y = \sqrt[5]{x^2} + \sqrt[5]{x^7}$

   (d) $h(r) = 4\pi r^2$

   (e) $G(x) = (x-3)(4x+1)$

   (f) $g(x) = \dfrac{x^3 - 2x + 1}{\sqrt{x}}$

   (g) $y = \dfrac{a}{x^4} + bx^{3/2} - c\sqrt{x} + de^x$

   (h) $y = \left(x + \dfrac{1}{\sqrt{x}}\right)^2$

2. Find the equation of the tangent and normal lines to the curve at the given point:

   (a) $f(x) = x^2 + \dfrac{3}{x}$ at $x = 1$

   (b) $f(x) = x + 4e^x$ at $x = 0$

3. The position of a particle is given by $s(t) = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 2t$.

   (a) Find the velocity as a function of time.

   (b) Find the acceleration as a function of time.

   (c) When is the velocity zero?

   (d) What is the acceleration when the velocity is zero?

4. For what x-values does the function $f(x) = 2x^3 - 3x^2 - 12x + 4$ have horizontal tangents?

Section 1: Answers to Selected Problems

1a) 0 1b) $\frac{4}{5}t$ 1c) $\frac{1}{5}x^{-3/5}(2+7x)$ 1d) $8\pi r$ 1e) $8x-11$

1f) $\frac{5}{2}x^{3/2} - x^{-1/2} - \frac{1}{2}x^{-3/2}$ 1g) $-\frac{4a}{x^5} + \frac{3}{2}bx^{1/2} - \frac{1}{2}cx^{-1/2} + de^x$ 1h) $2x + \frac{1}{\sqrt{x}} - \frac{1}{x^2}$

2a) Tangent: $(y-4) = -1(x-1)$ Normal: $y-4 = x-1$

2b) Tangent: $(y-4) = 5x$ Normal: $y-4 = -\frac{1}{5}x$

3a) $v(t) = t^2 - 3t + 2$ 3b) $a(t) = 2t - 3$ 3c) $t = 1$ and $t = 2$ 3d) $a(1) = -1$ and $a(2) = 1$

4) $x = 2$ and $x = -1$

Section 2: The Product and Quotient Rules

1. Find the derivative of the following:
   Note: In problem (e), a, b and c are constants.

   (a) $y = (x^3 + \sqrt[3]{x})e^x$ Simplify.

   (b) $f(x) = \left(\frac{1}{x} + x^2\right)\left(x^3 + 4x^2 + 9\right)$

   (c) $s(t) = \dfrac{t}{(t-1)^2}$ Simplify.

   (d) $p(t) = \dfrac{3t+2}{4t-5}$ Simplify.

   (e) $F(x) = \dfrac{ax}{bx + cx^2}$

   (f) $G(x) = \dfrac{xe^x}{1+x}$ Simplify.

2. Find the equation of the tangent line to $y = xe^x$ at $x = 0$

3. Find the equation of the tangent line to $y = \dfrac{1}{x^2+1}$ at $x = 1$

4. Suppose $f(3) = -1$, $f'(3) = 1$, $g(3) = 3$ and $g'(3) = 2$. Find the following:

   (a) $(fg)'(3)$.

   (b) $\left(\dfrac{f}{g}\right)'(3)$.

Section 2: Answers to Selected Problems

1a) $y' = (3x^2 + \frac{1}{3}x^{-2/3})e^x + (x^3 + \sqrt[3]{x})e^x = e^x(x^3 + 3x^2 + \sqrt[3]{x} + \frac{1}{3}x^{-2/3})$

1b) $f'(x) = \left(\frac{-1}{x^2} + 2x\right)(x^3 + 4x^2 + 9) + \left(\frac{1}{x} + x^2\right)(3x^2 + 8x)$

1c) $s'(t) = \frac{1(t-1)^2 - (2t-2)t}{(t-1)^4} = \frac{-(t+1)}{(t-1)^3}$

1d) $p'(t) = \frac{3(4t-5) - 4(3t+2)}{(4t-5)^2} = -\frac{23}{(4t-5)^2}$

1e) $F'(x) = \frac{a(bx + cx^2) - (b + 2cx)ax}{(bx + cx^2)^2}$

1f) $G'(x) = \frac{(e^x + xe^x)(1+x) - (1)(xe^x)}{(1+x)^2} = \frac{e^x(1 + x + x^2)}{(1+x)^2}$

2) $y = x$

3) $y - \frac{1}{2} = -\frac{1}{2}(x - 1)$

4a) 1 4b) 5/9

Section 3: Derivatives of Trigonometric Functions

1. Find the derivative of the following:

   (a) $y = e^x - \frac{2}{3}\cot x$

   (b) $f(x) = x\sin x + \csc x$

   (c) $h(t) = \frac{\tan t}{t^2}$ Simplify.

   (d) $f(\theta) = \sin\theta\cos\theta$ Simplify.

   (e) $y = e^x \sec x$ Simplify.

   (f) $y = \frac{\sec(x)}{1+\sec(x)}$ Simplify.

   (g) $y = \sqrt{x}\, e^x \cot x$

2. Find the equation of the tangent line to the curve $y = x + \sin x$ at $x = \frac{\pi}{2}$

3. Where, on the interval $(0, 2\pi)$, does the curve $f(x) = x + 2\cos x$ have horizontal tangents?

4. A mass on the end of a spring is undergoing simple harmonic motion has position given by $s(t) = 3\cos(t)$. Find the velocity and acceleration of the mass.

5. Consider $f(x) = \cos x$. Find $f'(x)$, $f''(x)$, and$f^{(3)}(x)$. Notice that the fourth derivative is your original function, that is the derivatives cycle through. Use this information to find $f^{(26)}(x)$.

6. Evaluate the following limits:

   (a) $\lim_{x\to 0} \frac{\sin 5x}{x}$

   (b) $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

   (c) $\lim_{t\to 0} \frac{\tan 3t}{\sin 8t}$

   (d) $\lim_{t\to 0} \frac{\sin^2 \sqrt{2}\, t}{t^2}$

Section 3: Answers to Selected Problems

1a) $y' = e^x + \frac{2}{3}\csc^2(x)$

1b $f'(x) = \sin(x) + x\cos(x) - \csc(x)\cot(x)$

1c) $h'(t) = \dfrac{t\sec^2(t) - 2\tan(t)}{t^3}$

1d) $f'(\theta) = \cos^2(\theta) - \sin^2(\theta)$

1e) $y' = e^x\sec(x) + e^x\sec(x)\tan(x) = e^x\sec(x)(1+\tan(x))$

1f) $y' = \dfrac{\sec(x)\tan(x)}{(1+\sec(x))^2}$

1g) $y' = \dfrac{1}{2\sqrt{x}}e^x\cot(x) + \sqrt{x}e^x\cot(x) - \sqrt{x}e^x\csc^2(x)$

2) $y - (\frac{\pi}{2}+1) = x - \frac{\pi}{2}$, or simplified: $y = x+1$.

3) $\frac{\pi}{6}$, $\frac{5\pi}{6}$

4) $v(t) = -3\sin(t)$, $a(t) = -3\cos(t)$

5) $f^{(26)}(x) = -\cos(x)$

6a) 5 6b) 3/5 6c) 3/8 6d) 2

Section 4: The Chain Rule

1. Find the derivative of the following:

   (a) $y = e^{3x}$

   (b) $y = \sin 7x$

   (c) $y = 3^x + \cos(4x)$

   (d) $y = \tan\left(\frac{1}{x}\right)$

   (e) $f(x) = e^{\sqrt{x}}$

   (f) $p(t) = (t^5 - t^3)^{45}$

   (g) $f(x) = xe^{-x^2/2}$ Simpify.

   (h) $f(x) = \dfrac{3}{(x^2+6)^7}$

   (i) $y = (x^2 + 3x - 4)^7(x^2 + 2x - 5)^4$ Simplify.

   (j) $H(z) = e^{z \sin z}$ Simplify.

   (k) $y = \dfrac{x}{\sqrt{x^2+1}}$ Simplify.

   (l) $y = \sec^2 x + \sin^2(3x)$

   (m) $y = 5^{\tan\left(\frac{x}{2}\right)}$

   (n) $y = \sqrt{x^2+1}(x+3)$ Simplify.

2. Consider $f(x) = \sin(3x)$. Find $f'(x)$, $f''(x)$, $f^{(3)}(x)$, and $f^{(4)}(x)$. Using the pattern you see from these derivatives, find $f^{(31)}(x)$

3. Given $y = xe^{ax}$ find y' and y''.

4. Find the equation of the tangent line to $f(x) = \cos\left(\frac{x}{3}\right)$ at $x = \pi$.

5. Let $H(x) = f(g(x))$ and $D(x) = g(g(x))$. Use the table below to find $H'(1)$ and $D'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	1	2	-2	-1
2	0	1	3	4

Section 4: Answer to Selected Problems

1a) $y' = 3e^{3x}$

1b) $y' = 7\cos(7x)$

1c) $y' = 3^x \ln(3) - 4\sin(4x)$

1d) $y' = -\dfrac{\sec^2\left(\frac{1}{x}\right)}{x^2}$

1e) $f'(x) = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

1f) $p'(t) = 45(t^5 - t^3)^{44}(5t^4 - 3t^2)$

1g) $f'(x) = e^{-x^2/2}(1 - x^2)$

1h) $f'(x) = \dfrac{-42x}{(x^2+6)^8}$

1i) $y' = (x^2+2x-5)^3(x^2+3x-4)^6[7(2x+3)(x^2+2x-5) + 4(2x+2)(x^2+3x-4)]$ I factored common terms out here.

1j) $H'(z) = e^{z\sin(z)}(\sin(z) + z\cos(z))$

1k) $y' = (x^2+1)^{-3/2}$

1l) $y' = 2\sec^2(x)\tan(x) + 6\sin(3x)\cos(3x)$

1m) $\dfrac{\ln(5)}{2} 5^{\tan(x/2)} \sec^2\left(\dfrac{x}{2}\right)$

1n) $y' = \dfrac{2x^2+3x+1}{\sqrt{x^2+1}}$

2) $-(3^{31})\cos(3x)$

3) $y' = e^{ax} + axe^{ax}$, $y'' = 2ae^{ax} + a^2xe^{ax}$

4) $y - \frac{1}{2} = -\frac{\sqrt{3}}{6}(x - \pi)$

5) $H'(x) = -3$, $D'(x) = -4$

Section 5: Implicit Differentiation

1. Find the derivative of the following:

   (a) $y = \tan^{-1}(3x)$

   (b) $y = x \tan^{-1} \sqrt{x}$

   (c) $y = \sin^{-1}(x^2)$

   (d) $y = \sin^{-1}(3^x)$

2. Use Implicit Differentiation to find $\dfrac{dy}{dx}$ of the following:

   (a) $x^4 + y^3 = 5$

   (b) $2xy + x^2 = y + y^2$

   (c) $e^{xy} + \cos x = y^2$

   (d) $\sin x \cos y = x$

   (e) $\sin(xy) = x^2 + y$

3. Find the equation of the tangent line to $x^2 y + \sin y = 2\pi$ at $(1, 2\pi)$

Section 5: Answers to Selected Problems

1a) $y' = \frac{3}{1+9x^2}$

1b) $y' = \tan^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2(1+x)}$

1c) $y' = \frac{2x}{\sqrt{1-x^4}}$

1d) $y' = \frac{3^x \ \ln 3}{\sqrt{1-3^{2x}}}$

Remember for problem 2, if **every** term in your answer differs in sign to what is listed here, your answer is the same as mine. It just means we took things to the opposite side of the equation to solve.

2a) $y' = \frac{-4x^3}{3y^2}$

2b) $y' = \frac{2y+2x}{1+2y-2x}$

2c) $y' = \frac{ye^{xy}-\sin(x)}{2y-xe^{xy}}$

2d) $y' = \frac{1-\cos(x)\cos(y)}{-\sin(x)\sin(y)}$

2e) $y' = \frac{y\cos(xy)-2x}{1-x\cos(xy)}$

3) $y - 2\pi = -2\pi(x-1)$

Section 6: Derivatives of Logarithmic Functions

1. Find the derivative of the following:

   (a) $y = \sqrt[3]{\ln(x)}$

   (b) $f(x) = \log_4(\sin(x))$

   (c) $p(t) = \ln\left(\dfrac{3^t + t^2}{\sin t}\right)$

   (d) $y = \ln(\sec(x)\csc(x))$

   (e) $f(x) = \dfrac{\log_6(\sqrt{x})}{x}$

   (f) $h(t) = \cos(\ln(t))$

2. Find the equation of the tangent line to $y = x\ln(x)$ at $(1, 0)$.

3. Find the equation of the tangent line to $f(x) = \ln(x^2 + 2)$ at $x = 1$.

4. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ of the following:

   (a) $y = (x^2 + 3)^5(5x - 4)^7$

   (b) $y = x^{\sqrt{x}}$

   (c) $y = (\tan(x))^x$

Section 6: Answers to Selected Problems

1a) $y' = \frac{1}{3x(\ln x)^{2/3}}$

1b) $f'(t) = \frac{\cot(x)}{\ln 4}$

1c) $p'(t) = \frac{3^t \ln(3) + 2t}{3^t + t^2} - \cot(t)$

1d) $y' = \tan(x) - \cot(x)$

1e) $f'(x) = \frac{\frac{1}{2 \ln 6} - \log_6(\sqrt{x})}{x^2}$

1f) $h'(t) = \frac{-\sin(\ln t)}{t}$

2) $y = x - 1$

3) $y - \ln(3) = \frac{2}{3}(x - 1)$

4a) $y' = (x^2+3)^5(5x-4)^7 \left(\frac{10x}{x^2+3} + \frac{35}{5x-4} \right)$

4b) $y' = x^{\sqrt{x}} \left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$

4c) $y' = (\tan x)^x \left(\ln(\tan(x)) + \frac{x \sec^2(x)}{\tan(x)} \right)$

Section 7: Rates of Change in the Natural Sciences

1. The position, in feet, of a particle as a function of time, in seconds, is given by $s(t) = t^3 - 6t^2 + 9t + 4$. Consider the time interval $[0, 6]$. Find the following:
   (a) The velocity as a function of time.
   (b) The velocity at $t = 2$.
   (c) When the particle is at rest.
   (d) When the particle is moving to the right.
   (e) When the particle is moving to the left.
   (f) The **total** distance traveled between $t = 0$ and $t = 6$
   (g) The acceleration as a function of time.

2. A rock is thrown upward from the surface of Mars with initial velocity $5m/s$. The height as a function of time for the rock is given by $s(t) = 5t - 1.9t^2$. Find the following:
   (a) The velocity as a function of time of the rock.
   (b) The acceleration as a function of time of the rock.
   (c) When the rock reaches its highest point. (2 decimals)
   (d) The highest point the rock reaches. (2 decimals)
   (e) When the rock hits the ground. (2 decimals)

3. A bacteria culture can be modeled by exponential growth. The culture initially contains 50 cells. After an hour there are 270 cells. Find the following:
   (a) The equation that models the number of bacteria after t hours.
   (b) The number of bacteria after 3 hours.
   (c) When there will be 1000 bacteria. (2 decimals)
   (d) The instantaneous rate of change in growth after 3 hours. (2 decimals)

4. When a bowl of soup is brought to your dinner table its temperature is 90°C. Engrossed in dinner conversation, your soup cools for five minutes to a temperature of 85°C before you start to eat it. The room temperature is 24°C. Find the following:
   (a) The temperature after 15 minutes. (2 decimals)
   (b) How long it takes for the soup to reach 55°C. (2 decimals)

Section 7: Answers to Selected Problems

1a) $v(t) = 3t^2 - 12t + 9$ 1b) -3 1c) $t = 1,\ 3$ 1d) $(0,1) \cup (3,6)$ 1e) $(1,3)$ 1f) 62 feet

1g) $a(t) = 6t - 12$

2a) $v(t) = 5 - 3.8t$ 2b) $a(t) = -3.8$ 2c) $t = 1.32$ 2d) 3.29 m 2e) 2.63

3a) $y = 50(5.4)^t$ 3b) $50(5.4)^3 = 7873.2$ 3c) $t = \dfrac{\ln(20)}{\ln(5.4)} = 1.78$ 3d) Remember, the instantaneous rate of change is the derivative! $50\ln(5.4)\,(5.4)^3 = 13277.36$

4a) $T = 24 + 66\left(\frac{61}{66}\right)^{15/5} = 76.11°\text{C}$ 4b) $t = \dfrac{5\ln\left(\frac{31}{66}\right)}{\ln\left(\frac{61}{66}\right)} = 47.96$ min.

Section 8: Related Rates-Day One

1. Let x and y depend on time. Suppose $x^2y^3 = 8$ and when $x = 1$ and $\frac{dy}{dt} = -4$. Find $\frac{dx}{dt}$.

2. An oil spill spreads in a perfect circle. The radius is increasing at a rate of 3 miles per hour. How fast is the area increasing when the area covered by the spill is 10000 mi^2?

3. A cylindrical water storage tank, with radius 50 feet, is being filled with water at a rate of 10 ft^3/min. How fast is the height changing when 1000 ft^3 of water have been pumped in?

4. A plane is flying at an altitude of 2 miles and a speed of 400 mi/h as it passes directly over you. Find the rate at which the (diagonal) distance between you and the plane is increasing when the plane is 4 miles (diagonally) away from you.

5. A 20 foot ladder is leaning against the side of a building. The top of the ladder slides down the building at a rate of 5 ft/s. How fast is the bottom of the ladder sliding away from the building when the bottom of the ladder is 12 feet from the building? **Hint:** be careful with signs and make sure your answer makes sense.

6. Consider a rectangular box with a top whose sides are changing at the following rates: $\frac{dx}{dt} = 3\ cm/s$, $\frac{dy}{dt} = -2\ cm/s$, and $\frac{dz}{dt} = 4\ cm/s$. Find the following when the sides of the box are $x = 3\ cm$, $y = 2\ cm$ and $z = 5\ cm$ long:

   (a) The rate of change of the volume.

   (b) The rate of change of the surface area.

Section 8: Answers to Selected Problems

1) $\frac{dx}{dt} = 3$

2) $\frac{dA}{dt} = 1063.47$

3) $\frac{dh}{dt} = \frac{1}{250\pi}$

4) $\frac{dz}{dt} = 100\sqrt{12}$

5) $\frac{dx}{dt} = \frac{20}{3}$

6a) $\frac{dV}{dt} = 24$ 6b) $\frac{dSA}{dt} = 50$

Section 9: Related Rates - Day 2

1. Two people start at the same point. One travels North at 3 mi/h and the other travels East at 2 mi/h. How fast is the diagonal distance between them changing after 4 hours?

2. A triangle's area is increasing at a rate of 4 cm^2/min. The base of the triangle is increasing at a rate of 1 cm/min. How fast is the height of the triangle changing when the base is 5 cm and the area is 30 cm^2

3. Gravel is being poured at a rate of 10 ft^3/min into a conical pile. The height of the cone is always the same as the diameter of the base. How fast is the radius increasing when the pile is 6 feet high?

4. You are out flying a kite at Crown Point. Your kite is 150 feet above the ground and moving horizontally at a rate of 3 ft/s. At what rate is the angle between the string and the horizontal decreasing when you have let out 300 feet of string?

5. A 20 foot ladder is leaning against the side of a building. The top of the ladder slides downward at a rate of 5 ft/s. How fast is the angle the ladder makes with the building changing when the top of the ladder is 4 feet from the ground?

Section 9: Answers to Selected Problems

1) $\frac{dz}{dt} = 3.61$

2) $\frac{dh}{dt} = -\frac{4}{5}$

3) $\frac{dr}{dt} = \frac{5}{9\pi}$

4) $\frac{d\theta}{dt} = -0.005$

5) $\frac{d\theta}{dt} = .255$

Section 10: Linear Approximation and Differentials

1. Find the linearization of $f(x) = (2x+1)^3$ at $x = 1$
2. Find the linearization of $f(x) = \sin(2x)$ at $x = \frac{\pi}{2}$
3. Find the differential dy and then compute dy for the given values of x and dx for the following:
   (a) $y = e^{x^2}$, $x = 1$, $dx = 0.01$
   (b) $y = \dfrac{1}{3x-1}$ $x = 2$, $dx = -0.1$
4. Consider $y = 2x^2 + x$
   (a) Find Δy, if x changes from 1 to 1.05.
   (b) Find the differential dy if $x = 1$ and $dx = 0.05$
   (c) Find $\Delta y - dy$ to see the error in approximation using a differential.
5. Use linearization to approximate $(1.01)^6$ and to find the approximation (dy) for the change in y when x changes from 1 to 1.01.
6. Use differentials to approximate $\sqrt{35.7}$ and to find the approximation (dy) for the change in y when x changes from 36 to 35.7.

Section 10: Answers to Selected Problems

1) $y - 27 = 54(x - 1)$

2) $y = -2(x - \frac{\pi}{2})$

3a) .0544 3b) .012

4a) .255 4b) .25 4c) .005

5) $(1.01)^6 \approx 1.06$ and $dy = .06$

6) $\sqrt{35.7} \approx 5.975$ and $dy = -0.025$

Section 11: Hyperbolic Functions

1. Find the derivative of the following:

   (a) $f(x) = \tanh(3x)$

   (b) $g(x) = \sinh(x + e^{4x})$

   (c) $y = \sinh(x)\cosh(x)$

   (d) $y = \operatorname{sech}(\ln(x))$

   (e) $h(t) = \coth(t)(t - \ln(\sinh(t))$

   (f) $f(x) = \operatorname{csch}(\tanh(3x))$

   (g) $y = \tanh^3(t^5)$

Section 11: Answers to Selected Problems

1a) $f'(x) = 3\text{sech}^2(3x)$

1b) $g'(x) = (1 + 4e^{4x}) \cosh(x + e^{4x})$

1c) $y' = \cosh^2(x) + \sinh^2(x)$

1d) $y' = -\frac{\text{sech}(\ln(x))\tanh(\ln(x))}{x}$

1e) $h'(t) = -(t - \ln(\sinh(t)))\text{csch}^2(t) + (1 - \coth(t)) \coth(t)$

1f) $f'(x) = -3\text{csch}(\tanh(3x)) \coth(\tanh(3x))\text{sech}^2(3x)$

1g) $y' = 15t^4 \tanh^2(t^5)\text{sech}^2(t^5)$