Mandatory assignment 1

Class	MAT-MEK4270
Due Date	@October 11, 2024

1.2.1 Finding an expression for ω

We have a function u given by $u(t,x,y) = \sin(k_x x)\sin(k_y y)\cos(\omega t)$ that we know satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$. From here, we get:

$$-\omega^2 \sin(k_x x)\sin(k_y y)\cos(\omega t) = \\ c^2(-k_x^2 \sin(k_x x)\sin(k_y y)\cos(\omega t) - k_y^2 \sin(k_x x)\sin(k_y y)\cos(\omega t))$$

which can be simplified to

$$-\omega^2 = c^2(-k_x^2 - k_y^2) \\ \omega^2 = c^2(k_x^2 + k_y^2) \\ \omega = c\sqrt{k_x^2 + k_y^2}$$

This expression is used in the code implementation, as well as in the following pen and paper calculation.

1.2.3 Exact solution

Show that $u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$ satisfies the wave equation.

The wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$, where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

$$\begin{aligned} c^2\nabla^2 u &= c^2\nabla^2(e^{i(k_x x + k_y y - \omega t)}) \\ &= c^2\nabla^2(e^{ik_x x}\cdot e^{ik_y y}\cdot e^{-i\omega t}) \\ &= c^2[(ik_x)^2(e^{ik_x x}\cdot e^{ik_y y}\cdot e^{-i\omega t}) + (ik_x)^2(e^{ik_x x}\cdot e^{ik_y y}\cdot e^{-i\omega t})] \\ &= c^2(-k_x^2 - k_y^2)(e^{ik_x x}\cdot e^{ik_y y}\cdot e^{-i\omega t}) \\ &= -\omega^2(e^{ik_x x}\cdot e^{ik_y y}\cdot e^{-i\omega t}) \\ &= e^{ik_x x}\cdot e^{ik_y y}\cdot(-\omega^2)e^{-i\omega t} \\ &= e^{ik_x x}\cdot e^{ik_y y}\cdot(-i\omega)^2 e^{-i\omega t} \\ &= \frac{\partial^2 u}{\partial t^2} \end{aligned}$$

1.2.4 Dispersion coefficient

Show that for CFL number $C = \frac{1}{\sqrt{2}}$, we get $\tilde{\omega} = \omega$.

First, we need a discrete version of the function u that we are working with. For simplicity, I will assume $k_x = k_y$ and call this variable k instead, such that $\omega^2 = 2c^2k^2 \Rightarrow \omega = \sqrt{2}ck$ (from the expression we found in 1.2.1). Also of note is that we write $\imath$ for $\sqrt{-1}$, while i and j are indices in the x and y directions respectively. The discrete function is then given by:

$$u_{ij}^n = e^{\imath(khi + khj - \tilde{\omega}n\Delta t)} = e^{\imath(kh(i+j) - \tilde{\omega}n\Delta t)}$$

We also need the discretized wave equation, which is:

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right)$$

Now, we insert the expression we just found for u_{ij}^n, which will result in a horribly long expression:

$$\frac{e^{\imath(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} - 2e^{\imath(kh(i+j) - \tilde{\omega}n\Delta t)} + e^{\imath(kh(i+j) - \tilde{\omega}(n-1)\Delta t)}}{\Delta t^2} = \\ c^2\left(\frac{e^{\imath(kh(i+1+j) - \tilde{\omega}n\Delta t)} - 2e^{\imath(kh(i+j) - \tilde{\omega}n\Delta t)} + e^{\imath(kh(i-1+j) - \tilde{\omega}n\Delta t)}}{h^2} + \frac{e^{\imath(kh(i+j+1) - \tilde{\omega}n\Delta t)} - 2e^{\imath(kh(i+j) - \tilde{\omega}n\Delta t)} + e^{\imath(kh(i+j-1) - \tilde{\omega}n\Delta t)}}{h^2}\right)$$

Let's reinsert u_{ij}^n for $e^{\imath(kh(i+j) - \tilde{\omega}n\Delta t)}$ wherever we can:

$$\frac{u_{ij}^n e^{-\imath\tilde{\omega}\Delta t} - 2u_{ij}^n + u_{ij}^n e^{\imath\tilde{\omega}\Delta t}}{\Delta t^2} = c^2(\frac{u_{ij}^n(e^{\imath kh} - 2 + e^{-\imath kh})}{h^2} + \frac{u_{ij}^n(e^{\imath kh} - 2 + e^{-\imath kh})}{h^2})$$

$$\frac{u_{ij}^n(e^{-\imath\tilde{\omega}\Delta t} - 2 + e^{\imath\tilde{\omega}\Delta t})}{\Delta t^2} = \frac{2u_{ij}^n c^2(e^{\imath kh} - 2 + e^{-\imath kh})}{h^2}$$

Now we can use the nifty trick $2\cos(x) = e^{\imath x} + e^{-\imath x}$:

$$\frac{u_{ij}^n(2\cos(\tilde{\omega}\Delta t) - 2)}{\Delta t^2} = 2c^2\frac{u_{ij}^n(2\cos(kh) - 2)}{h^2}$$

$$\frac{\cos(\tilde{\omega}\Delta t) - 1}{\Delta t^2} = 2c^2\frac{\cos(kh) - 1}{h^2}$$

$$\cos(\tilde{\omega}\Delta t) - 1 = 2\frac{c^2\Delta t^2}{h^2}(\cos(kh) - 1)$$

We recall that the CFL number is given by $C = \frac{c\Delta t}{h}$, so we can substitute $\frac{c^2\Delta t^2}{h^2}$ for C^2:

$$\cos(\tilde{\omega}\Delta t) - 1 = 2C^2(\cos(kh) - 1)$$

Now, we can consider the case where the CFL number $C = \frac{1}{\sqrt{2}}$:

$$\cos(\tilde{\omega}\Delta t) - 1 = 2(\frac{1}{\sqrt{2}})^2(\cos(kh) - 1)$$

$$\cos(\tilde{\omega}\Delta t) - 1 = \cos(kh) - 1$$

$$\cos(\tilde{\omega}\Delta t) = \cos(kh)$$

$$\tilde{\omega}\Delta t = kh$$

We need expressions for k and h at this point, which we can find using expressions we already have for ω and C respectively:

$$\omega = \sqrt{2}ck \Rightarrow k = \frac{\omega}{\sqrt{2}c}$$

$$\text{and}$$

$$C = \frac{1}{\sqrt{2}} = \frac{c\Delta t}{h} \Rightarrow h = \sqrt{2}c\Delta t$$

Picking up where we left off:

$$\tilde{\omega}\Delta t = kh$$

$$\tilde{\omega}\Delta t = \frac{\omega}{\sqrt{2}c} \cdot \sqrt{2}c\Delta t$$

$$\tilde{\omega}\Delta t = \omega\Delta t$$

$$\tilde{\omega} = \omega$$

Thus, we have shown that the numerical approximation of $\tilde{\omega}$ is exact when the CFL number has the value $\frac{1}{\sqrt{2}}$.