LOVE AFFAIRS AND LINEAR DIFFERENTIAL EQUATIONS

EMMA COATES, ANN DESPAIN, CARSON MILLER

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I INTRODUCTION

"Love Affairs and Linear Differential Equations" by Fabian Dablander and inspired by Strogatz uses linear differential equations as a means to study the types of love affairs two people might find themselves in. In this study, we reproduce three select figures from this publication and detail the analytical and numerical methods behind each figure. Specifically, we study how communication styles affect love affairs using coupled linear differential equations, and analytical and numerical solutions.

II BIOLOGY

Attraction is mediated by neurotransmitters including dopamine and norepinephrine cortisol. It takes only a fifth of a second for these chemicals to act on the brain and make you fall in love. Differential equations can be used to study different types of love affairs people might find themselves in. These relationships include the attraction of opposites, lack of communication, and when the lovers are out of touch with their own feelings. Using mathematical modeling and graphical techniques, we can see how different conditions may lead to different outcomes in a relationship.

We consider Romeo and Juliet relationship dynamics when modeling love affairs. Let R be Romeo's feelings towards Juliet, and let J be Juliet's feelings for Romeo. We can write a linear system of two differential equations, one for Romeo and one for Juliet,

$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ.$$

Let a and d represent the level of emotional affection Romeo and Juliet feel respectively. If a and d are positive, the lovers increase their feelings the more they feel. If negative, they dampen their feelings the stronger they feel. Let b represent how Romeo communicates with Juliet, and c represent how Juliet communicates with Romeo. If b and c are positive, the lovers feelings will follow whichever way the other feels. If negative, their feelings will move the opposite way of their lover. With these equations, we can model how Romeo and Juliet's feelings towards one another change over time in various relationship dynamics.

III THE SADDLE OF LOVE

We begin with a relationship dynamic in which Romeo dampens his feelings the more strongly he feels, while Juliet increases her feelings the more strongly she feels. Romeo listens to Juliet, such that however Juliet feels, Romeo's feelings follow. Juliet does not listen to Romeo, such that however Romeo feels, Juliet's feelings diverge from Romeo's feelings. We begin with the coupled linear differential equations,

$$\frac{dR}{dt} = -2R + J$$
$$\frac{dJ}{dt} = -R + 2J$$

with parameters $a = -2$, $b = 1$, $c = -1$, $d = 2$. In this dynamic, Romeo and Juliet are essentially opposites.

Solving the Saddle of Love. To solve this system, we use the general solution to a system of two coupled linear differential equations

$$\vec{x} = \vec{v_1} C_1 e^{1t} + \vec{v_2} C_2 e^{2t}.$$

Let A be the matrix representing the system,

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}.$$

We first solve for the eigenvalues.

$$\begin{vmatrix} -2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = (-2-\lambda)(2-\lambda) - (1)(-1)$$
$$= \lambda^2 - 3 = 0,$$

implying

$$\lambda = \pm\sqrt{3}.$$

We now solve for the eigenvectors. We begin with finding the eigenvector for $\lambda = \sqrt{3}$,

$$\begin{pmatrix} -2-\sqrt{3}\lambda & 1 \\ -1 & 2-\sqrt{3}\lambda \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solutions,

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}.$$

Finding the eigenvector for $\lambda = -\sqrt{3}$,

$$\begin{pmatrix} -2+\sqrt{3}\lambda & 1 \\ -1 & 2+\sqrt{3}\lambda \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

with solutions,

$$\vec{v_2} = \begin{pmatrix} 1 \\ 2-\sqrt{3} \end{pmatrix}.$$

Plugging the eigenvalues and eigenvectors into the general solution form yields

$$\vec{x} = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix} C_1 e^{\sqrt{3}t} + \begin{pmatrix} 1 \\ 2-\sqrt{3} \end{pmatrix} C_2 e^{-\sqrt{3}t}.$$

Assuming Romeo and Juliet have positive, similar feelings towards one another initially, our initial conditions become $(R,T)^T = (1,1)^T$ at $t=0$. With this condition, we solve for the constants,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{3} & 1-\sqrt{3} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1+\sqrt{3} & 1-\sqrt{3} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{pmatrix} 0.21 \\ 0.79 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Thus, we have solutions

$$\begin{aligned} R(t) &= 0.21 \cdot e^{\sqrt{3}t} + 0.79 \cdot e^{-\sqrt{3}t} \\ J(t) &= 0.79 \cdot e^{\sqrt{3}t} + 0.21 \cdot e^{-\sqrt{3}t}. \end{aligned}$$

We now find equilibrium,

$$\begin{aligned} \frac{dR}{dt} &= -2R + J = 0 && \Rightarrow && J = 2R \\ \frac{dJ}{dt} &= -R + 2J = 0 && \Rightarrow && R = 2J, \end{aligned}$$

yielding an equilibrium of $(R^*, J^*) = (0,0)$. We now classify the stability of our steady state by computing the eigenvalues of the Jacobian.

$$\begin{aligned} \begin{vmatrix} -2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} &= (-2-\lambda)(2-\lambda) - (1)(-1) \\ &= \lambda^2 - 3 = 0 \quad \Rightarrow \quad \lambda = \pm\sqrt{3}. \end{aligned}$$

Thus, $(R^*, J^*) = (0,0)$ is unstable. We will see this graphically in the next section.

Numerical Methods. We use R in order to visual the trajectories of Romeo and Juliet in this specific relationship dynamic. The program consists of four functions that solves linear system and plots vector fields along with trajectories and solutions. The following method solves linear systems given matrix A and initial conditions *inits*:

```
# Solves linear systems of differential equations given matrix 'A' and
#  intial conditions inits
solve_linear <- function(A, inits, tmax = 50, n = 500) {
    # compute eigenvectors and eigenvalues
    eig <- eigen(A)
    E <- eig$vectors
    lambdas <- eig$values

    # solve for the initial conditon
    C <- solve(E) %*% inits

    # create time steps
    ts <- seq(0, tmax, length.out = n)
    x <- matrix(0, nrow = n, ncol = ncol(A))

    for (i in seq(n)) {
    t <- ts[i]
    x[i, ] <- E %*% (C * exp(lambdas * t))
}

    # Re drops the imaginary part
    Re(x)
}
```

The function computes the system's eigenvalues and eigenvectors, and solves for the solutions constants. Matrix x stores the values at time step t_s. The solution at time step t_s for both $R(t)$ and $J(t)$ is stored in x during the for loop using the general solution formula. In a later figure, we consider a system with imaginary solutions, so we drop the imaginary part of the solution.

The following function plots a vector field for a linear system give matrix A and the desired title of the plot:

```
# Plots a vector field given matrix 'A' and plot title
plot_vector_field <- function(A, title = '', ...) {
    x <- seq(-4, 4, .50)
    y <- seq(-4, 4, .50)

     RJ <- as.matrix(expand.grid(x, y))
     dRJ <- t(A %*% t(RJ))

     plot(
        x, y, type = 'n', axes = FALSE,
        xlab = '', ylab = '', main = title, cex.main = 1.5, ...
     )

    arrow.plot(
         RJ, dRJ,
         arrow.ex = .1, length = .05, lwd = 1.5, col = 'gray82'
    )

    text(3.9, -.2, 'R', cex = 1.25, font = 2)
    text(-.2, 3.9, 'J', cex = 1.25, font = 2)
    lines(c(-4, 4), c(0, 0), lwd = 1)
    lines(c(0, 0), c(4, -4), lwd = 1)
}
```

The next function plots the eigenvectors of a given system given matrix A on an existing plot:

```
plot_eigenvectors <- function(A, ...) {
    E <- eigen(A)$vectors * 4
    arrows(-E[1, 1], -E[2, 1], E[1, 1], E[2, 1], length = 0, lty = 2, ...)
    arrows(-E[1, 2], -E[2, 2], E[1, 2], E[2, 2], length = 0, lty = 2, ...)
}
```

The final function plots the trajectories of a system given matrix A and intial conditions $inits$:

```
add_line <- function(A, inits) {
    lines(solve_linear(A, inits = inits), col = 'red', lwd = 1.5)
}
```

We then use a combination of the above functions to plot the vector field and select trajectories of Romeo and Juliet when they are opposites.

```
# The following lines reproduce the figure "The Saddle of Love"
A <- cbind(c(-2, -1), c(1, 2))
plot_vector_field(A, title = 'The Saddle of Love')
plot_eigenvectors(A)

add_line(A, c(3.5, 1))
add_line(A, c(3.5, .75))
add_line(A, -c(3.5, 1))
add_line(A, -c(3.5, .75))

points(0, 0, cex = 1.7)
points(0, 0, cex = 2.2, col = 'white', pch = 20)
```

Finally, this yields the Saddle of Love. This phase plane verifies the equilibrium node $(0, 0)$ is unstable, since it is a saddle node.

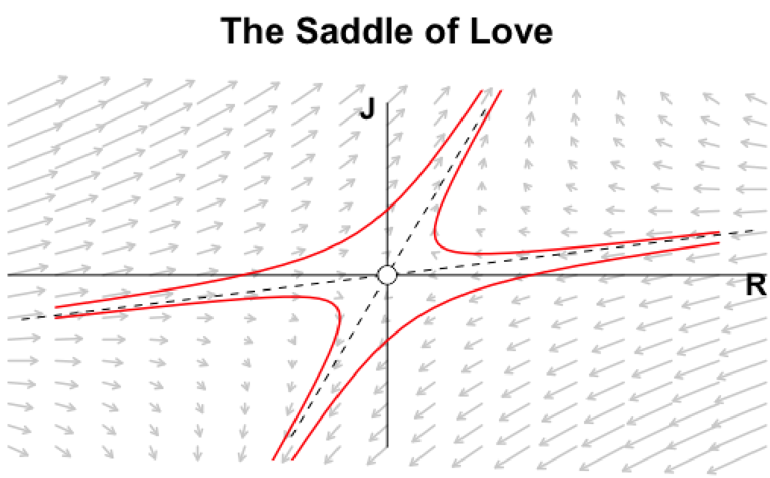

Figure 1: The figure above visualizes the resulting vector field, the eigenvectors (dashed lines), and four possible trajectories. If Romeo and Juliet start in the top right or top left eigenquadrant, their love grows exponentially. If they start in the bottom left or bottom right eigenquadrant, their hate grows exponentially. Note that we have a saddle point, as there is exponential decay along one eigenvector and exponential growth along the other. In other words, only if Romeo and Juliet's initial feelings are exactly on the decaying eigenvector do we end up in a state of indifference.

IV THE STABLE SPIRAL OF LOVE

We now consider a relationship dynamic in which Romeo dampens his own feelings slightly, and feels more love when Juliet hates him and more hate if Juliet loves him. Juliet, however, does not listen to her own feelings, and follows Romeo's feelings. We begin with the coupled linear differential equations,

$$\frac{dR}{dt} = -0.20R - J$$
$$\frac{dJ}{dt} = R$$

with parameters $a = -0.20$, $b = -1$, $c = 1$, $d = 0$.

Solving the Stable Spiral of Love. This system, as it so happens, has complex eigenvalues, and therefore complex solutions. We know the eigenvalues of a linear system to be

$$\lambda = \frac{\tau}{2} \pm \frac{\sqrt{\tau^2 - 4\Delta}}{2},$$

where τ is the trace of the system's matrix and Δ is the determinant of the system's matrix. The eigenvalues will be complex is $\tau^2 - 4\delta < 0$, in which case we can write the eigenvalues as

$$\begin{aligned}\lambda &= \frac{\tau}{2} \pm \frac{\sqrt{-1}\sqrt{4\Delta - \tau^2}}{2} \\ &= \alpha \pm i\omega,\end{aligned}$$

where $\alpha = \tau/2$ and $\omega = \sqrt{4\Delta - \tau^2}/2$. While the general solution to the system still remains as before, when we plug in our complex eigenvalues, we write the solution as

$$\begin{aligned}\vec{x} &= \vec{v_1}C_1e^{{}_1t} + \vec{v_2}C_2e^{{}_2t} \\ &= \vec{v_1}C_1e^{(\alpha+i\omega)t} + \vec{v_2}C_2e^{(\alpha-i\omega)t} \\ &= \vec{v_1}C_1e^{\alpha t}e^{i\omega t} + \vec{v_2}e^{\alpha t}e^{-i\omega t} \\ &= \vec{v_1}C_1e^{\alpha t}\left(\cos\omega + i\sin\omega\right) + \vec{v_2}e^{\alpha t}\left(\cos\omega - i\sin\omega\right).\end{aligned}$$

Note that when $\alpha < 0$ and $\omega \neq 0$, the solution produces damped oscillations, whereas when $\alpha > 0$ and $\omega \neq 0$, the solution produces amplifying oscillations. For the current relationship dynamics,

$$A = \begin{pmatrix} -0.20 & -1 \\ 1 & 0 \end{pmatrix},$$

$\tau = -0.20$, so $\alpha = -0.10$, and $\Delta = 1$, so $\omega \neq 0$. Furthermore, we know our solution should have damped oscillations. We will only consider the real part of our solution when plotting.

Since our system has complex eigenvalues, we know the system will have a spiral node at equilibrium. Finding equilibrium,

$$\frac{dR}{dt} = -0.20R - J = 0 \qquad \Rightarrow \qquad J = -0.20R$$
$$\frac{dJ}{dt} = -R = 0 \qquad \Rightarrow \qquad R = 0,$$

yielding an equilibrium of $(R^*, J^*) = (0, 0)$. We compute the eigenvalues of the Jacobian at equilibrium to classify the stability of our spiral node,

$$\begin{aligned}\begin{vmatrix} -0.20 - \lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= (-0.20 - \lambda)(-\lambda) + 1 \\ &= \lambda^2 + 0.20 + 1 \\ &= 0 \quad \Rightarrow \quad \lambda = \frac{-0.20}{2} \pm \frac{i\sqrt{4.04}}{2}.\end{aligned}$$

Since $Re(\lambda) < 0$, our spiral node is stable, resulting in the Stable Spiral of Love!

Numerical Methods. We visual the Stable Spiral of Love and the real part of the system's solution using R. We use a combination of the four functions from the Saddle of Love section. We begin with plotting the Stable Spiral of Love.

```
# The following lines reproduce the figure "The Stable Spiral of Love"
A <- cbind(c(-0.20, -1), c(1, 0))
plot_vector_field(A, title = 'The Stable Spiral of Love')
add_line(A, c(2,2))
```

The above lines of code create the matrix, plot the vector field, and add a trajectory using functions previously introduced. This results in the following figure:

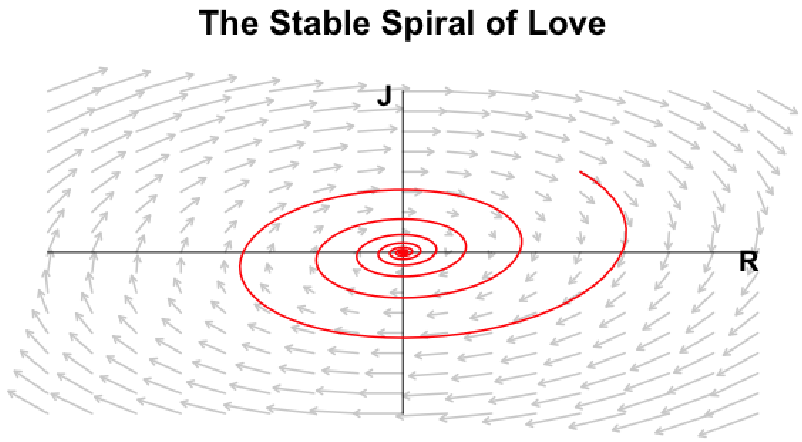

Figure 2: The figure above visualizes the resulting vector field and a possible trajectory. Although both lovers start at mutual affection, over the course of their relationship, they feel happy, then sad, then happy, then sad, until they don't feel anymore.

We now consider the solution to the system with the following lines of code:

```
# The following lines reproduce the figure "Damped Oscillations"
plot(seq(1,500,by=1), solve_linear(A,c(2,2))[,1], type = 'l', col='red',
       xlab="Time", ylab="Feelings", ylim=c(-3,3))
lines(seq(1,500,by=1), solve_linear(A,c(2,2))[,2], type = 'l', col='blue')
title("Damped Oscillations")
legend("topright", c("Romeo", "Juliet"), col=c("red","blue"), lty=c(1,1),
           cex=0.75)
```

Using the function that solves linear systems, we plot the output. Note that this only plots the real part of the solution, resulting in the following figure:

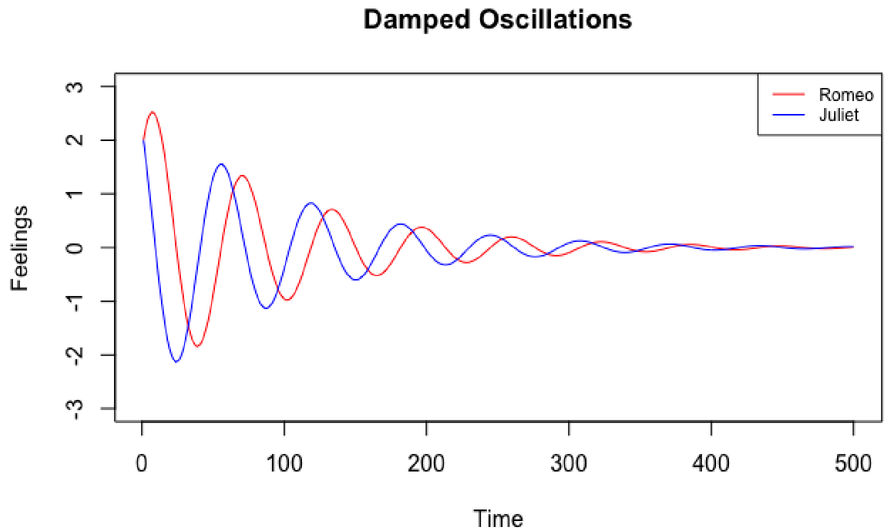

Figure 3: The figure above visualizes the real part of the solution. Although both lovers start at mutual affection, over the course of their relationship, they feel happy, then sad, then happy, then sad, until they don't feel anymore.

V THE UNSTABLE SPIRAL OF LOVE

We now consider a relationship dynamic in which Romeo increases his own feelings slightly, and feels more love when Juliet hates him and more hate if Juliet loves him. Juliet, however, does not listen to her own feelings, and follows Romeo's feelings. We begin with the coupled linear differential equations,

$$\frac{dR}{dt} = 0.10R - J$$
$$\frac{dJ}{dt} = R$$

with parameters $a = 0.10$, $b = -1$, $c = 1$, $d = 0$. This relationship dynamic is very similar to the previous dynamic, however Romeo now listens to his own feelings.

Solving the Stable Spiral of Love. This system, as it so happens, also has complex eigenvalues, and therefore complex solutions. We know from the previous dynamic, the system has a solution of the form

$$\vec{x} = \vec{v_1}C_1e^{\alpha t}\left(\cos\omega + i\sin\omega\right) + \vec{v_2}e^{\alpha t}\left(\cos\omega - i\sin\omega\right).$$

Remember that when $\alpha < 0$ and $\omega \neq 0$, the solution produces damped oscillations, whereas when $\alpha > 0$ and $\omega \neq 0$, the solution produces amplifying oscillations. For the current relationship dynamics,

$$A = \begin{pmatrix} 0.10 & -1 \\ 1 & 0 \end{pmatrix},$$

$\tau = 0.10$, so $\alpha = 0.05$, and $\Delta = 1$, so $\omega \neq 0$. Furthermore, we know our solution should have amplifying oscillations. We will only consider the real part of our solution when plotting.

Since our system has complex eigenvalues, we know the system will have a spiral node at equilibrium. Finding equilibrium,

$$\begin{aligned} \frac{dR}{dt} &= 0.10R - J = 0 & \Rightarrow \qquad & J = 0.10R \\ \frac{dJ}{dt} &= -R = 0 & \Rightarrow \qquad & R = 0, \end{aligned}$$

yielding an equilibrium of $(R^*, J^*) = (0, 0)$. We compute the eigenvalues of the Jacobian at equilibrium to classify the stability of our spiral node,

$$\begin{aligned} \begin{vmatrix} 0.10 - \lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= (0.10 - \lambda)(-\lambda) + 1 \\ &= \lambda^2 - 0.10 + 1 \\ &= 0 \quad \Rightarrow \quad \lambda = \frac{0.10}{2} \pm \frac{i\sqrt{3.99}}{2}. \end{aligned}$$

Since $Re(\lambda) > 0$, our spiral node is unstable, resulting in the Unstable Spiral of Love!

Numerical Methods. We visual the Unstable Spiral of Love and the real part of the system's solution using R. Our approach is similar to the Stable Spiral of Love. We begin with plotting the Unstable Spiral of Love.

```
# The following lines reproduce the figure "The Stable Spiral of Love"
# The following lines reproduce the figure "The Unstable Spiral of love"
A <- cbind(c(0.10, -1), c(1,0))
plot_vector_field(A, title = 'The Unstable Sprial of Love')
add_line(A, c(0.1,0.1))
```

The above lines of code create the matrix, plot the vector field, and add a trajectory using functions previously introduced. This results in the following figure:

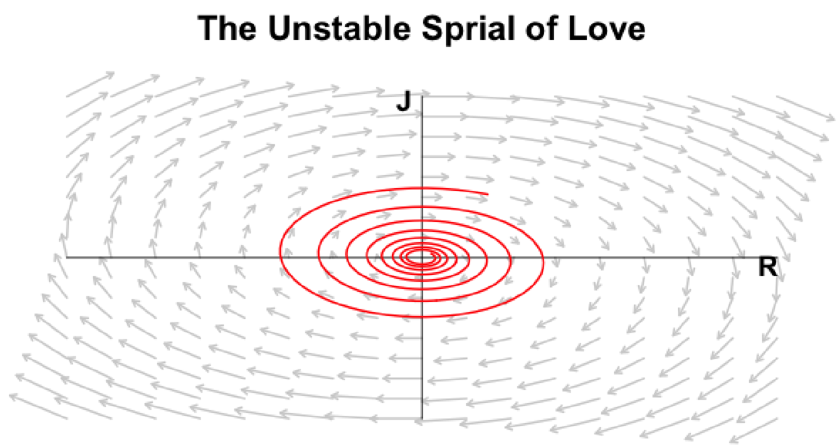

Figure 4: The figure above visualizes the resulting vector field and a possible trajectory. Both lovers initially have only an ounce of mutual affection for one another (0.1, 0.1), resulting in the couple spiralling forever, with feelings always growing, always changing.

We now consider the solution to the system with the following lines of code:

```
# The following lines reproduce the figure "Damped Oscillations"
plot(seq(1,500,by=1), solve_linear(A,c(2,2))[,1], type = 'l', col='red',
       xlab="Time", ylab="Feelings", ylim=c(-3,3))
lines(seq(1,500,by=1), solve_linear(A,c(2,2))[,2], type = 'l', col='blue')
title("Damped Oscillations")
legend("topright", c("Romeo", "Juliet"), col=c("red","blue"), lty=c(1,1),
          cex=0.75)
```

Using the function that solves linear systems, we plot the output. Note that this only plots the real part of the solution, resulting in the following figure:

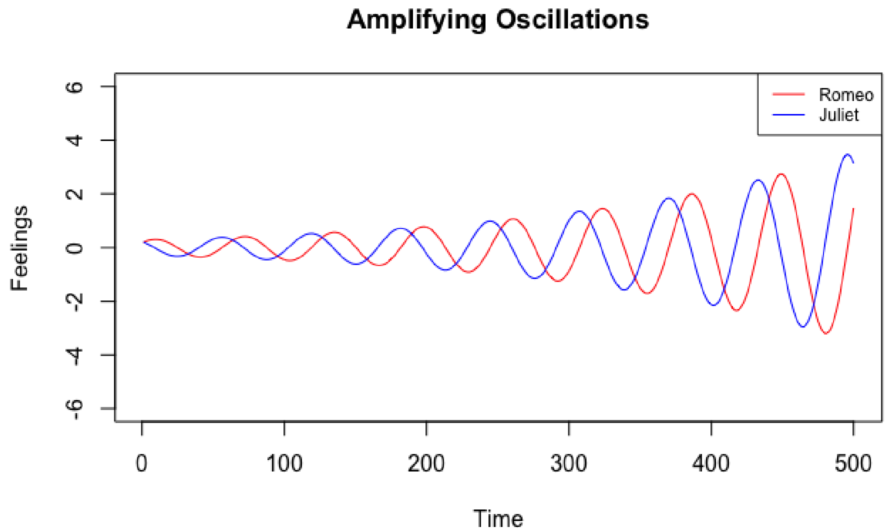

Figure 5: The figure above visualizes the real part of the solution. Both lovers initially have only an ounce of mutual affection for one another (0.1, 0.1), resulting in the couple spiralling forever, with feelings always growing, always changing.

VI CONCLUSION

In conclusion, we have shown that differential equations and graphical methods can be used to model the frequent, biological experience of love. Communication between lovers is critical in a relationship. As seen in the spirals of love, when two people communicate in opposition, their feelings will constantly be changing between happy and sad. Whether or not they listen to their own feelings is just as critical. With such a small change between dampening his feelings, and listening to his feelings, Romeo and Juliet's relationship outcome changed from ending with no feelings towards each other to extreme, alternating feelings. It was also shown that initial conditions, how much affection towards each other, can change the whole outcome of a relationship. Take the saddle of love: depending on the quadrant they started in, Romeo and Juliet's love for each other would have either grown or decreased exponentially. Self-awareness, communication, and initial affection all play important roles in the development of a relationship. Love is a very complex idea, but with differential equations, we can better examine the inner workings of what makes or breaks a romantic affair.

REFERENCES

Dablander, Fabian. "Love Affairs and Linear Differential Equations." Fabian Dablander, 29 Aug. 2019, https://fabiandablander.com/r/Linear-Love.html.