Remarkable representations of the 2 + 2 de sitter group

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I. SOME IDEAS ABOUT THE $\mathfrak{su}(2,2)$ TWO OSCILLATOR REALIZATION

Dirac's remarkable representation of $\mathfrak{su}(2,2)$ [1, 2]

$$m_{02} = L_1 = \frac{i}{4}\left(a^{\dagger 2} - a^2 - b^{\dagger 2} + b^2\right) \tag{1a}$$
$$m_{2-1} = L_2 = \frac{1}{4}\left(a^2 + a^{\dagger 2} - b^2 - b^{\dagger 2}\right) \tag{1b}$$
$$m_{12} = L_3 = \frac{1}{2i}\left(a^\dagger b - b^\dagger a\right) \tag{1c}$$
$$-m_{1-1} = K_1 = -\frac{1}{2}\left(a^\dagger b^\dagger + ab\right) \tag{1d}$$
$$-im_{01} = K_2 = -\frac{i}{2}\left(ab - a^\dagger b^\dagger\right) \tag{1e}$$
$$m_{-10} = K_3 = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 1\right) \tag{1f}$$

The K_i operators form an $\mathfrak{su}(2,1)$ algebra. The commutation relations:

$$[K_i, K_j] = i\epsilon_{ijk}K^k \tag{2a}$$
$$[L_i, L_j] = i\epsilon_{ijk}K^k \tag{2b}$$
$$[K_i, L_j] = i\epsilon_{ijk}L^k \tag{2c}$$
$$\tag{2d}$$

Where the metric tensor is $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1,1,1)$.

We could introduce the operators:

$$I_i = \frac{1}{2}\left(K_i + L_i\right) \tag{3a}$$
$$J_i = \frac{1}{2}\left(K_i - L_i\right) \tag{3b}$$

Both of them form an $\mathfrak{su}(2,1)$ algebra, with the commutation relations of (2a), meaning that we SO(2,2) group is a direct sum of two SO(2,1)-s: $SO(2,2) \cong SO(2,1) \oplus SO(2,1)$.

Other elements that complete $\mathfrak{su}(2,3)$:

$$m_{23} = -\frac{1}{2}\left(a^\dagger a - b^\dagger b\right) \tag{4a}$$
$$m_{31} = -\frac{1}{2}\left(a^\dagger b + b^\dagger a\right) \tag{4b}$$
$$m_{3-1} = \frac{i}{4}\left(a^{\dagger 2} - a^2 + b^{\dagger 2} - b^2\right) \tag{4c}$$
$$m_{03} = \frac{1}{4}\left(a^{\dagger 2} + a^2 + b^{\dagger 2} + b^2\right) \tag{4d}$$

A. Constructing quantum gates from the algebra elements

The entangling gates constructed from the K_i $\mathfrak{su}(2,1)$ operators are well known, they map the uncoupled vacuum state to the SU(1,1) coherent state:

$$\begin{aligned}&\exp\left[i\theta\left(-\sin\varphi K_1 + \cos\varphi K_2\right)\right]|0\rangle_1 \otimes |0\rangle_2 = \\ &= \cosh^{-1}\frac{\theta}{2}\sum_{n=0}^{\infty}\left(-\tanh\frac{\theta}{2}e^{-i\varphi}\right)^n |n\rangle_1 \otimes |n\rangle_2\end{aligned} \tag{5}$$

With the special $\varphi = \pi$ case used in most physically relevant cases, e.g. in quantum optics the Two-mode squeezed states of light.

$$e^{i\theta K_2} = \cosh^{-1}\frac{\theta}{2}\sum_{n=0}^{\infty}\tanh^n\frac{\theta}{2}|n\rangle_1 \otimes |n\rangle_2 \tag{6}$$

We build the remaining quantum gates from the $\mathfrak{su}(2,3)$ operators and investigate their impact on the uncoupled vacuum state based on the idea of the SU(1,1) coherent state generating operator. This idea is based on the paper of Jefferson and Myers [5].
Looking at the L_i operators, L_3 acts trivially on the vacuum state just like m_{23} and m_{31}.

$$\begin{aligned}e^{i\epsilon L_3}|0\rangle_1 \otimes |0\rangle_2 &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\epsilon}{2}\right)^n\left(a^\dagger b - b^\dagger a\right)|0\rangle_1 \otimes |0\rangle_2 = \\ &= |0\rangle_1 \otimes |0\rangle_2\end{aligned} \tag{7}$$

Since the operators L_1, L_2, m_{3-1} and m_{03} are composed of sums of operators that act on one of the sub-Hilbert spaces, they do not entangle the uncoupled vacuum state. To demonstrate it we write down the effect of L_1 as an example.

$$\begin{aligned}&e^{i\epsilon\frac{i}{4}\left(a^{\dagger 2} - a^2 - b^{\dagger 2} + b^2\right)}|0\rangle_1 \otimes |0\rangle_2 = \\ &= e^{-\frac{\epsilon}{4}\left(a^{\dagger 2} - a^2\right)}|0\rangle_1 \otimes e^{\frac{\epsilon}{4}\left(b^{\dagger 2} - b^2\right)}|0\rangle_2\end{aligned} \tag{8}$$

B. Entanglement of standard (Weyl-Heisenberg) coherent states

The standard oscillator coherent states of the Heisenberg-Weyl group W_1:

$$|\alpha\rangle = D(\alpha)\,|0\rangle = e^{-|\alpha|/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,, \quad \alpha\in\mathbb{C} \tag{9a}$$

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{9b}$$

$$a^\dagger\,|\alpha\rangle = \left(\frac{\partial}{\partial\alpha}+\frac{\overline{\alpha}}{2}\right)|\alpha\rangle \tag{9c}$$

Form an overcomplete set on their respective one-particle Hilbert space. The displacement operator $D(\alpha)$ have the following properties:

$$D(\alpha) = \exp\left(\alpha a^\dagger - \overline{\alpha}a\right) = \tag{10a}$$

$$= \exp\left(-|\alpha|^2/2\right)\exp\left(\alpha a^\dagger\right)\exp\left(-\overline{\alpha}a\right) \tag{10b}$$

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \tag{10c}$$

$$D^\dagger(\alpha)aD(\alpha) = a+\alpha \tag{10d}$$

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger+\overline{\alpha} \tag{10e}$$

$$D(\alpha)D(\beta) = D(\alpha+\beta)\exp\left(i\mathrm{Im}\{\alpha\overline{\beta}\}\right) \tag{10f}$$

The standard coherent states are the closest possible states to classical ones, meaning that the Heisenberg uncertainty relation calculated in these states gives equality.

$$\Delta q\cdot\Delta p = \frac{\hbar}{2} \tag{11}$$

Let us now investigate the action of the $SO(2,1)$ displacement operator $U(\theta,\varphi)$ on two uncoupled standard coherent states.

$$U(\theta,\varphi)\,|\alpha\rangle_1\otimes|\beta\rangle_2 = \\ \exp\left[\overline{\eta}K_+\right]\exp\left[\log\left(1-|\eta|^2\right)K_3\right]\exp\left[-\eta K_-\right]|\alpha\rangle_1\otimes|\beta\rangle_2 \tag{12}$$

Where $K_+ = -a^\dagger b^\dagger$ and $K_- = -ab$.

$$\exp\left(\eta ab\right)|\alpha\rangle_1\otimes|\beta\rangle_2 = \exp\left(\eta\alpha\beta\right)|\alpha\rangle_1\otimes|\beta\rangle \tag{13}$$

$$(a^\dagger a)^n\,|\alpha\rangle = \left(\alpha\frac{\partial}{\partial\alpha}+\frac{|\alpha|^2}{2}\right)^n|\alpha\rangle \tag{14}$$

$$e^{\log\left(1-|\eta|^2\right)K_3}\,|\alpha\rangle_1\otimes|\beta\rangle = \\ \cosh^{-1}\frac{\theta}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\log^n\left(\operatorname{sech}\frac{\theta}{2}\right)\left(\alpha\frac{\partial}{\partial\alpha}+\frac{|\alpha|^2}{2}\right)^n|\alpha\rangle_1\otimes \\ \sum_{m=0}^{\infty}\frac{1}{m!}\log^m\left(\operatorname{sech}\frac{\theta}{2}\right)\left(\beta\frac{\partial}{\partial\beta}+\frac{|\beta|^2}{2}\right)^n|\beta\rangle_2 = \\ \cosh^{-1}\frac{\theta}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\log^n\left(\operatorname{sech}\frac{\theta}{2}\right)e^{-|\alpha|/2}\sum_{n'=0}^{\infty}\frac{\alpha^{n\prime}}{\sqrt{n'!}}n'\,|n'\rangle_1\otimes \\ \sum_{m=0}^{\infty}\frac{1}{m!}\log^m\left(\operatorname{sech}\frac{\theta}{2}\right)e^{-|\beta|/2}\sum_{m'=0}^{\infty}\frac{\beta^{m\prime}}{\sqrt{m'!}}m'\,|m'\rangle_2 \tag{15}$$

Another form of the previous operator's action

$$e^{\log\left(1-|\eta|^2\right)K_3}\,|\alpha\rangle_1\otimes|\beta\rangle_2 = \\ e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,m=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!m!}}\left(\cosh\frac{\theta}{2}\right)^{-1-n-m}|n\rangle_1\otimes|m\rangle_2 \tag{16}$$

$$U(\theta,\varphi)\,|\alpha\rangle_1\otimes|\beta\rangle_2 = \\ e^{\eta\alpha\beta}e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,m=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!m!}}\left(\cosh\frac{\theta}{2}\right)^{-1-n-m} \\ \sum_{k=0}^{\infty}\sqrt{\binom{n+k}{k}\binom{m+k}{k}}\left(-\overline{\eta}\right)^k|n+k\rangle_1\otimes|m+k\rangle_2 \tag{17}$$

Let's look at the special case of $\beta=\overline{\alpha}$, $\varphi=0$:

$$|\theta;\alpha\rangle := U(\theta,0)\,|\alpha\rangle_1\otimes|\overline{\alpha}\rangle_2 = \\ \frac{e^{(\eta-1)|\alpha|^2}}{\cosh\frac{\theta}{2}}\sum_{n,m=0}^{\infty}\frac{\alpha^n\overline{\alpha}^m}{\sqrt{n!m!}}\left(\cosh\frac{\theta}{2}\right)^{-(n+m)} \\ \sum_{k=0}^{\infty}\sqrt{\binom{n+k}{k}\binom{m+k}{k}}\left(-\overline{\eta}\right)^k|n+k\rangle_1\otimes|m+k\rangle_2 \tag{18}$$

One state where we believe it's possible to calculate the entanglement entropy is where we average over the $|\alpha|$ radius states in the complex plane.

$$|\theta;|\alpha|\rangle := \frac{1}{2\pi}\int_0^{2\pi}d\phi\,|\theta;\alpha(\phi)\rangle\,, \quad \alpha(\phi)=|\alpha|e^{i\phi} \tag{19}$$

This turns the double sum in equation 18 into one. The physical meaning of this state if that we take the superposition of infinite coherent states in a way that their position and momentum is zero.

$$\langle\alpha|\,x\,|\alpha\rangle \propto \mathrm{Re}(\alpha) \tag{20a}$$

$$\langle\alpha|\,p\,|\alpha\rangle \propto \mathrm{Im}(\alpha) \tag{20b}$$

$$|\alpha|^2 = \langle x\rangle^2+\langle p\rangle^2 \tag{20c}$$

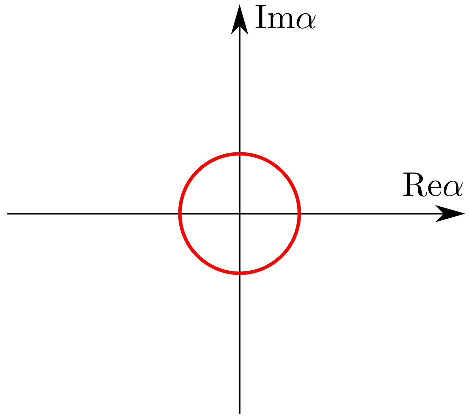

FIG. 1: The graphical representation of our state in the complex plane.

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} d\phi \alpha^n \overline{\alpha}^m = \frac{|\alpha|^{n+m}}{2\pi}\int_0^{2\pi} d\phi e^{i(n-m)\phi} = \\ = |\alpha|^{2n}\delta_{n,m}\end{aligned} \tag{21}$$

$$\begin{aligned}|\theta;|\alpha|\rangle =& \frac{e^{(\tanh\frac{\theta}{2}-1)|\alpha|^2}}{\cosh\frac{\theta}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{|\alpha|}{\cosh\frac{\theta}{2}}\right)^{2n} \\ &\sum_{k=0}^{\infty}\binom{n+k}{k}\left(-\tanh\frac{\theta}{2}\right)^k |n+k\rangle_1 \otimes |n+k\rangle_2 = \\ =& \frac{e^{(\tanh\frac{\theta}{2}-1)|\alpha|^2}}{\cosh\frac{\theta}{2}}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\frac{1}{(s-k)!}\left(\frac{|\alpha|}{\cosh\frac{\theta}{2}}\right)^{2(s-k)} \\ &\binom{s}{k}\left(-\tanh\frac{\theta}{2}\right)^k |s\rangle_1 \otimes |s\rangle_2\end{aligned} \tag{22}$$

Cauchy Product? Carliz-formula?

II. SPACELIKE GEODESICS OF ADS_3 AND THEIR CONNECTION TO THE TWO OSCILLATOR REALIZATION

The three-dimensional anti-de Sitter space (AdS_3) is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. In other words it is the projectivization of the following set.

$$AdS_3 = \mathbb{P}\left\{X \in \mathbb{R}^{2,2}\middle| X \cdot X < 0\right\} \tag{23}$$

Where product is defined $X \cdot X = \eta_{ab} X^a X^b$ with the metric $\eta_{ab} = \text{diag}(-1,-1,1,1)$. The boundary of AdS_3 is

$$\partial AdS_3 = \mathbb{P}\left\{U \in \mathbb{R}^{2,2}\middle| U \cdot U = 0\right\} \tag{24}$$

The global coordinates of the AdS space are the following

$$X^a = \begin{bmatrix} X^{-1} \\ X^0 \\ X^1 \\ X^2 \end{bmatrix} = L \begin{bmatrix} \cosh\theta\cos\sigma \\ \cosh\theta\sin\sigma \\ \sinh\theta\sin\varphi \\ \sinh\theta\cos\varphi \end{bmatrix} \tag{25}$$

Where L is the AdS radius. The Poincaré coordinates (t, x, z) give a more commonly used parametrization of the AdS space.

$$X^a = \begin{bmatrix} X^{-1} \\ X^0 \\ X^1 \\ X^2 \end{bmatrix} = \frac{1}{2z} \begin{bmatrix} L^2 - t^2 + x^2 + z^2 \\ 2Lt \\ 2Lx \\ L^2 + t^2 - x^2 - z^2 \end{bmatrix} \tag{26}$$

These are local coordinates restricted to the Poincaré Patch where $z > 0$. Also we define $\mathbf{x} = \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} t \\ x \end{bmatrix}$, $\mathbf{x} \bullet \mathbf{x} = \eta_{\mu\nu}\mathbf{x}^\mu\mathbf{x}^\nu = -t^2 + x^2$ $(\mu,\nu = 0,1)$ is a $1 \oplus 1$ dimensional Minkowski spacetime (Here lives the CFT). The Poincaré coordinates of the boundary points

$$U^a = \frac{1}{2\Delta_u} \begin{bmatrix} L^2 - t_u^2 + x_u^2 \\ 2Lt_u \\ 2Lx_u \\ L^2 + t_u^2 - x_u^2 \end{bmatrix} \tag{27}$$

describe the lightlike vectors of AdS_3, formally can be constructed by setting $z = 0$. The mapping between the two coordinates is the following:

$$t = L\frac{X^0}{X^+}, \quad x = L\frac{X^1}{X^+}, \quad z = \frac{L^2}{X^+} \tag{28}$$

With $X^\pm = X^2 \pm X^{-1}$.

Theorem 1 *The spacelike geodesics of the AdS_n space ($n \in \mathbb{N}$) have the following form*

$$X^a(\lambda) = \frac{L}{\sqrt{-2U\cdot V}}\left(U^a e^{\sqrt{D}\lambda} + V^a e^{-\sqrt{D}\lambda}\right) \tag{29}$$

Where $U \cdot U = V \cdot V = 0$ are lightlike vectors ($U, V \in \partial AdS_3$) with $U \cdot V < 0$ and D is defined later in equation 33.

One can define the two ends of the geodesic at $\lambda \to \pm\infty$.

$$\lim_{\lambda\to\infty} X^\mu(\lambda) = L\frac{U^\mu e^{\sqrt{D}\lambda}}{\sqrt{-2U\cdot V}} \sim X_U^\mu = L\frac{U^\mu}{U^+} \tag{30a}$$

$$\lim_{\lambda\to-\infty} X^\mu(\lambda) \sim X_V^\mu = L\frac{V^\mu}{V^+} \tag{30b}$$

with $\mu = 0, 1$ and $U^\pm = U^2 \pm U^{-1}$. Theorem 1 can be calculated from the Lagrangian:

$$L(X, \dot{X}, \alpha) = \dot{X}\cdot\dot{X} + \alpha\left(X \cdot X + L^2\right) \tag{31}$$

Here $\dot{X}$ denotes the differentiation w.r.t. the λ parameter. The following matrix is equivalent to an angular momentum vector

$$M^{ab} = \frac{1}{L^2}\left(X^a\dot{X}^b - X^b\dot{X}^a\right) \tag{32}$$

Which is a conserved quantity for the spacelike geodesics: $\dot{M}^{ab} = 0$, it tells us that the geodesics will be inside a plane. Another conserved quantity can be constructed from the M tensor:

$$D = -\frac{1}{2}M^{ab}M_{ab} \implies \dot{D} = 0 \tag{33}$$

The length of the geodesics would be infinity but it can be regularized with the definition of horocyles (limit circles).

$$HORO_- = \left\{X \in AdS_3 \middle| U \cdot X = -\frac{L^2}{\sqrt{2}}\right\} \tag{34a}$$

$$HORO_+ = \left\{X \in AdS_3 \middle| V \cdot X = -\frac{L^2}{\sqrt{2}}\right\} \tag{34b}$$

These circles will cross the geodesics at the parameters $\lambda_\pm = \pm\frac{1}{\sqrt{D}}\log\left(\frac{\sqrt{-U\cdot V}}{L}\right)$. The length of the spacelike geodesic is then

$$l(\lambda_-, \lambda_+) = \int_{\lambda_-}^{\lambda_+} |\dot{x}\cdot\dot{x}|^{1/2} d\lambda = L\log\left(\frac{|U\cdot V|}{L^2}\right) \tag{35}$$

Defining the difference between the endpoints of the geodesics: $\mathbf{\Delta}^\mu = \mathbf{X_V}^\mu - \mathbf{X_U}^\mu$ ($\mu = 0, 1$, $\eta_{\mu\nu} = \text{diag}(-1, 1)$) the following equation holds.

Theorem 2

$$\boldsymbol{\Delta} \bullet \boldsymbol{\Delta} = -2L^2 \frac{U \cdot V}{U^+ V^+} \tag{36}$$

an other parametrization

$$e^{2\sqrt{D}\lambda} = \frac{1-s}{1+s}\frac{V^+}{U^+}, \quad s \in [-1,1] \tag{37a}$$
$$\mathbf{x}^\mu(s) = \frac{1}{2}\boldsymbol{\Delta}^\mu s + \mathbf{x_0}^\mu \tag{37b}$$
$$\mathbf{x_0}^\mu = \frac{1}{2}\left(\mathbf{x_V}^\mu + \mathbf{x_U}^\mu\right) \tag{37c}$$
$$\mathbf{x}^\mu(s=-1) = \mathbf{x_U}^\mu \tag{37d}$$
$$\mathbf{x}^\mu(s=1) = \mathbf{x_V}^\mu \tag{37e}$$

One can prove

$$\left(\mathbf{x} - \mathbf{x_0}\right)^2 + z^2 = \frac{1}{4}\boldsymbol{\Delta} \bullet \boldsymbol{\Delta} \tag{38a}$$
$$\left(\mathbf{x} - \mathbf{x_0}\right)^2 = \frac{1}{4}\boldsymbol{\Delta} \bullet \boldsymbol{\Delta} s^2 \implies \tag{38b}$$
$$s^2 + \frac{z}{\rho^2} = 1, \quad \rho^2 = \frac{1}{4}\boldsymbol{\Delta} \bullet \boldsymbol{\Delta} \tag{38c}$$

Where the $\rho^2 > 0 \leftrightarrow U^+V^+ > 0$ spacelike separated geodesics are ellipses and the $\rho^2 < 0 \leftrightarrow U^+V^+ < 0$ timelike separated geodesics are hyperbolas. A special type of geodesic is where $\mathbf{x_U}$ is outside of the Poincaré Patch at $z = \infty$, these are the generalizations of the straight line geodesics of the upper half-plane or in other words the static slice of AdS_3.

$$U^a = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{39a}$$
$$0 = U \cdot X = \frac{L^2}{z} \implies z = \infty \tag{39b}$$

Theorem 3 *The spacelike separated spacelike geodesics ($U^+V^+ > 0$) are the following 2 codimensional surfaces*

$$U \cdot X = 0 \tag{40a}$$
$$V \cdot X = 0 \tag{40b}$$

These two equations describe two cones with the same orientation and opening angle, their intersection of them will give us an ellipse. An equivalent pair of equations are

$$\varphi_1 = \boldsymbol{\Delta} \bullet (\mathbf{x} - \mathbf{x_0}) = 0 \tag{41a}$$
$$\varphi_2 = z - \sqrt{\rho^2 - (\mathbf{x} - \mathbf{x_0}) \bullet (\mathbf{x} - \mathbf{x_0})} = 0 \tag{41b}$$

Theorem 4 *One can define two normal unit vectors to the surfaces $\varphi_{1,2}$ in the following way $g^{\mu\nu} = z^2 diag(-1,1,1)$.*

$$s^\mu_{1,2} = \frac{\mathbf{n_{1,2}}^\mu}{|\mathbf{n_{1,2}}^\nu \mathbf{n_{1,2}}_\nu|^{1/2}}$$

$$s_1 \bullet \mathbf{s_1} = +1, \quad \mathbf{s_2} \bullet \mathbf{s_2} = -1, \quad \mathbf{s_1} \bullet \mathbf{s_2} = 0$$

$$s^\mu_1 = \frac{z}{\sqrt{-\boldsymbol{\Delta}\bullet\boldsymbol{\Delta}}} \begin{bmatrix} 2(t - x_{0t}) \\ 2(x - x_{0x})) \\ 2\sqrt{\rho^2 - (\mathbf{x} - \mathbf{x_0})^2} \end{bmatrix}$$

$$s^\mu_2 = \frac{z}{\sqrt{-\boldsymbol{\Delta}\bullet\boldsymbol{\Delta}}} \begin{bmatrix} \boldsymbol{\Delta}_\mathbf{t} \\ \boldsymbol{\Delta}_\mathbf{x} \\ 0 \end{bmatrix}$$

One can define the light-cone coordinates $\mathbf{N}_\pm{}^\mu = \frac{a}{\sqrt{2}}(\mathbf{s}^\mu_2 \pm \mathbf{s}^\mu_1)$ with the property $\mathbf{N}_\pm \bullet \mathbf{N}_\pm = 0$. From this, we could define the light sheets

$$\theta_\pm = g^{\mu\nu} h^{\cdot\varrho}_\mu h^{\cdot\sigma}_\nu \nabla_\varrho N_{\pm\sigma} = 0 \tag{43}$$
$$\nabla_\varrho N_{\pm\sigma} = \partial_\varrho N_{\pm\sigma} - \Gamma^\lambda_{\varrho\sigma} N_{\pm\lambda} \tag{44}$$
$$h_{\mu\nu} = g_{\mu\nu} + \mathbf{s_2}_\mu \mathbf{s_2}_\nu - \mathbf{s_1}_\mu \mathbf{s_1}_\nu \tag{45}$$

A. Killing-vectors of AdS$_3$

The Killing vector field of AdS_3 is a vector field the preserves the AdS metric $ds^2 = -\left(dX^{-1}\right)^2 - \left(dX^0\right)^2 - \left(dX^1\right)^2 + \left(dX^2\right)^2$. It can be easily seen that this field is $\binom{4}{2} = 6$ dimensional and has the following basis called the Killing vectors

$$K^{01} = X^0 \frac{\partial}{\partial X^1} + X^1 \frac{\partial}{\partial X^0} \tag{46a}$$
$$K^{-12} = X^{-1} \frac{\partial}{\partial X^2} + X^2 \frac{\partial}{\partial X^{-1}} \tag{46b}$$
$$K^{-11} = X^{-1} \frac{\partial}{\partial X^1} + X^1 \frac{\partial}{\partial X^{-1}} \tag{46c}$$
$$K^{02} = X^0 \frac{\partial}{\partial X^2} + X^2 \frac{\partial}{\partial X^0} \tag{46d}$$
$$K^{12} = X^1 \frac{\partial}{\partial X^2} - X^2 \frac{\partial}{\partial X^1} \tag{46e}$$
$$K^{-10} = X^{-1} \frac{\partial}{\partial X^0} - X^0 \frac{\partial}{\partial X^{-1}} \tag{46f}$$

The first four equations are Lorentz-boost in their given plane, while the final two are rotations.

The metric of the three-dimensional anti-de Sitter space has the following form in the Poincaré coordinates

$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2} \tag{47}$$

Using equation (28) one can prove that the Killing vectors expressed in terms of the Poincaré coordinates are

$$\begin{aligned}
K^{01} &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} \\
K^{-12} &= x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} \\
K^{-11} &= \frac{1}{2L}\left(L^2 - t^2 - x^2 + z^2\right)\frac{\partial}{\partial x} - \frac{xt}{L}\frac{\partial}{\partial t} - \frac{xz}{L}\frac{\partial}{\partial z} \\
K^{02} &= -\frac{xt}{L}\frac{\partial}{\partial x} + \frac{1}{2L}\left(L^2 - t^2 - x^2 - z^2\right)\frac{\partial}{\partial t} - \frac{tz}{L}\frac{\partial}{\partial z} \\
K^{12} &= -\frac{1}{2L}\left(L^2 + t^2 + x^2 - z^2\right)\frac{\partial}{\partial x} - \frac{xt}{L}\frac{\partial}{\partial t} - \frac{xz}{L}\frac{\partial}{\partial z} \\
K^{-10} &= \frac{xt}{L}\frac{\partial}{\partial x} + \frac{1}{2L}\left(L^2 + t^2 + x^2 + z^2\right)\frac{\partial}{\partial t} + \frac{tz}{L}\frac{\partial}{\partial z}
\end{aligned}$$

If I choose to have:

$$G_1 = K^{02}, \quad G_2 = -iK^{12}, \quad G_3 = K^{01} \tag{49}$$

$$F_1 = -iK^{-11}, \quad F_2 = K^{-10}, \quad F_3 = -iK^{-12} \tag{50}$$

Then

$$[G_k, G_l] = i\epsilon_{klm}G_m, \quad [G_k, F_l] = i\epsilon_{klm}F_m, \quad [F_k, F_l] = -i\epsilon_{klm}G_m$$

For $(j, k, l = 1, 2, 3)$. With the relabeling of

$$K_1 = iG_1 = iK^{02}, \quad L_1 = iF_1 = K^{-11} \tag{51}$$

$$K_2 = iG_2 = K^{12}, \quad L_2 = iF_2 = iK^{-10} \tag{52}$$

$$K_3 = G_3 = K^{01}, \quad L_3 = F_3 = -iK^{-12} \tag{53}$$

The K_i and L_i operators satisfy the commutation relations of the $\mathfrak{su}(2,2)$ algebra, seen in equation (2a). In summary:

$$\begin{aligned}
K_1 &= i\left[-\frac{xt}{L}\frac{\partial}{\partial x} + \frac{1}{2L}\left(L^2 - t^2 - x^2 - z^2\right)\frac{\partial}{\partial t} - \frac{tz}{L}\frac{\partial}{\partial z}\right] \\
K_2 &= -\frac{1}{2L}\left(L^2 + t^2 + x^2 - z^2\right)\frac{\partial}{\partial x} - \frac{xt}{L}\frac{\partial}{\partial t} - \frac{xz}{L}\frac{\partial}{\partial z} \\
K_3 &= t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t} \\
L_1 &= \frac{1}{2L}\left(L^2 - t^2 - x^2 + z^2\right)\frac{\partial}{\partial x} - \frac{xt}{L}\frac{\partial}{\partial t} - \frac{xz}{L}\frac{\partial}{\partial z} \\
L_2 &= i\left[\frac{xt}{L}\frac{\partial}{\partial x} + \frac{1}{2L}\left(L^2 + t^2 + x^2 + z^2\right)\frac{\partial}{\partial t} + \frac{tz}{L}\frac{\partial}{\partial z}\right] \\
L_3 &= -i\left[x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z}\right]
\end{aligned}$$

Comparing these equations with equation (1a) we get a two-oscillator realization of the AdS_3 Killing vectors. With this, we can see the connection between the entanglement entropy of the certain quantum states of the two-oscillator realization and the geodesics of the AdS_3 space.

B. Static slice and hyperbolic models

The static slice of AdS_3 is $X^0 = \tau = t = 0$ case. Choosing $L = 1$, the global coordinates (25) parameterize the upper sheet of a hyperboloid. The Poincaré disk model ($\mathbb{D}$) maps this hyperboloid into the complex unit disk with the following transformation:

$$\eta = \tanh\frac{\theta}{2}e^{i\varphi} = \frac{X^2 + iX^1}{1 + X^{-1}} \in \mathbb{D} \tag{54}$$

$$\mathbb{D} = \left\{\eta \in \mathbb{C} \;\middle|\; |\eta| \le 1\right\} \tag{55}$$

Meanwhile, the inverse of the Cayley transformation maps the Poincaré disk model into the upper-half plane model that we are most interested in. The Poincaré upper half-plane model ($\mathbb{U}$) is the static slice of the AdS_3's Poincaré Patch where we set $t = 0$. The mapping is the following

$$\tau = x + iz = \frac{X^1 + i}{X^+} = i\frac{1-\eta}{1+\eta} \in \mathbb{U} \tag{56}$$

$$\mathbb{U} = \left\{\tau = x + iz \in \mathbb{C} \middle| z > 0\right\} \tag{57}$$

C. The two-oscillator realization in the Poincaré Patch

A general point in the Poincaré Patch can be described by a split-quaternion algebra $\tau = x + iz + jt$ on the four basis element $(1, i, j, k = ij = -ji)$ satisfying the following product and conjugation rules.

$$1^2 = 1, \quad i^2 = -1, \quad j^2 = 1, \quad k^2 = 1 \tag{58}$$

$$\bar{1} = 1, \quad \bar{i} = -i, \quad \bar{j} = -j, \quad \overline{wz} = \bar{z}\,\bar{w} \tag{59}$$

This algebra is not commutative but still associative. Assume here that all distances are measured in units of L. ($L \coloneqq 1$) Here there lives the two oscillation realization of the AdS_3 space, where the separable state stays at $\tau = i \leftrightarrow X^a = [1, 0, 0, 0]^T$ (By definition? We need a Hamiltonian for defining a vacuum state but GT says that it behaves just like the excited ones. Do we need a Hamiltonian for describing entanglement?). The Hamiltonian of the separable two-oscillator state is K_3.

A Lorentz boost in the $X^{-1} - X^2$ plane is described by the $\exp(i\theta L_3)$ operator, this maps the separable two-oscillator state $(\theta = 1, \sigma = 0, \varphi = 0) \to (\theta, 0, 0)$. A boost on the upper part f the two-sheeted hyperboloid is equivalent to a dilatation in the Poincaré Patch since

$$L_3(\tau = iz) = -i\left[X^{-1}\frac{\partial}{\partial X^2} + X^2\frac{\partial}{\partial X^{-1}}\right] = z\frac{\partial}{\partial z} \quad (60)$$

$$\exp\left(i\theta L_3\right) = \exp\left(i\theta z\frac{\partial}{\partial z}\right) \quad (61)$$

So one parameterized geodesic is

$$\tau_0(\theta) = \exp\left(i\theta L_3\right) i = i\exp\left(\theta\right), \quad \theta\in\mathbb{R} \quad (62)$$

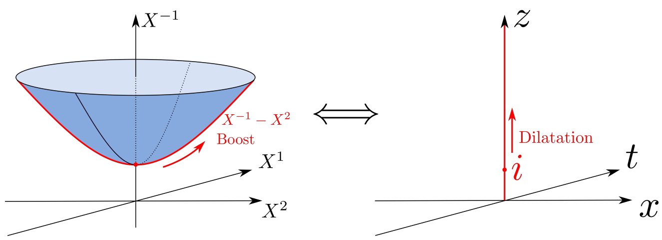

FIG. 2: A unit speed spacelike geodesic in the different hyperbolic models

Now that we have found one geodesic, we would like to construct all of them with the transformations of the Poincaré Patch's isometry group $Sl(2,\mathbb{B})$ where

$$\mathbb{B} = \left\{b = b_0 + jb_1 \in \mathbb{C}\middle| b^0, b^1\in\mathbb{R}, j^2 = 1, \bar{j} = -j\right\} \quad (63)$$

$$A = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \in Sl(2,\mathbb{B}) \quad (64)$$

We translate $\tau_0(\theta)$ around $Sl(2,\mathbb{B})$ to obtain a description of all space-like geodesics in this model. Consider

$$\tau(\theta) = A\tau_0(\theta) = (ae^\theta i + b)(ce^\theta i + d)^{-1} = \\ = \frac{a\bar{c}e^\theta + b\bar{d}e^{-\theta} + i}{|c|^2e^\theta + |d|^2e^{-\theta}}$$

Where $\tau(-\infty) = bd^{-1} = x_V^\mu$ and $\tau(\infty) = ac^{-1} = x_U^\mu$ in a sense that the $1\oplus 1$ dimensional Minkowski inner product is defined on the Binarios as

$$\bullet : \mathbb{B}\times\mathbb{B}\to\mathbb{B} \quad (65)$$

$$(a,b)\mapsto a\bullet b = -\frac{1}{2}(a\bar{b} + \bar{a}b) = -a^0b^0 + a^1b^1 \quad (66)$$

Some other values expressed as $Sl(2,\mathbb{B})$ matrix elements

$$\mathbf{x_0}^\mu = \frac{ac^{-1} + bd^{-1}}{2}$$

$$\boldsymbol{\Delta}^\mu = bd^{-1} - ac^{-1} = -\frac{1}{cd}$$

$$\frac{1}{4}\boldsymbol{\Delta}\bullet\boldsymbol{\Delta} = \frac{1}{4|c|^2|d|^2} = (\mathbf{x} - \mathbf{x_0})^2 + z^2 \\ = \left|\tau(\theta) - \frac{ac^{-1} + bd^{-1}}{2}\right|^2$$

III. IDEA: ADS$_3$ AS A CLEBSCH-GORDAN COEFFICIENT PROBLEM

The isometry group of AdS_3 is $SO(2,2)$ which is locally isomorphic with the direct product of two $SU(1,1)$ groups

$$SO(2,2) \cong SU(1,1)\times SU(1,1) \quad (67)$$

In our case both $SU(1,1)$ algebras will be represented with the Schwinger-type two-boson realization (doli).

$$I_1^{(1)} = -\frac{1}{2}\left(a^\dagger b^\dagger + ab\right) \qquad I_1^{(2)} = -\frac{1}{2}\left(c^\dagger d^\dagger + cd\right) \quad (68a)$$

$$I_2^{(1)} = -\frac{i}{2}\left(ab - a^\dagger b^\dagger\right) \qquad I_2^{(2)} = -\frac{i}{2}\left(cd - c^\dagger d^\dagger\right) \quad (68b)$$

$$I_3^{(1)} = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 1\right) \quad I_3^{(2)} = \frac{1}{2}\left(c^\dagger c + d^\dagger d + 1\right) \quad (68c)$$

Where a, b, c and d are bosonic annihilation operators. The idea is to describe the $SO(2,2)$ CS-s as a CGC problem.

First let's summarize all the relevant information about the $SU(1,1)$ CS-s[3, 6]. They are known to be in the UIR's known as the discrete positive series $\mathcal{D}^+(k)$, where k is the so-called Bargmann index. It appears in the eigenvalues of the Casimir-operator.

$$C_2 = I_3^2 - I_1^2 - I_2^2 = I_3^2 - \frac{1}{2}\left(I_+I_- + I_-I_+\right) = \frac{1}{4}\left(\Delta^2 - 1\right) \quad (69)$$

Where $I_\pm = \pm i(I_1 \pm iI_2)$ and $\Delta = a^\dagger a - b^\dagger b$ is the occupation number difference operator between the two modes. In $\mathcal{D}^+(k)$ the Bargmann index is $k = 1/2, 1, 3/2, ...$, and the I_3, C_2 common basis elements are indexed by $|m,k\rangle$ where $(m = 0,1,2,...)$. The operators act the following way on the eigenstates:

$$C_2\left|m,k\right\rangle = k\left(k-1\right)\left|m,k\right\rangle \quad (70a)$$

$$I_3\left|m,k\right\rangle = \left(m+k\right)\left|m,k\right\rangle \quad (70b)$$

$$I_+\left|m,k\right\rangle = \sqrt{(m+1)(m+2k)}\left|m+1,k\right\rangle \quad (70c)$$

$$I_-\left|m,k\right\rangle = \sqrt{m(m+2k-1)}\left|m-1,k\right\rangle \quad (70d)$$

Therefore the general $SU(1,1)$ basis element can be constructed by acting on the ground state $|0,k\rangle$ with the I_+ operator

$$|m,k\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}(I_+)^m\,|0,k\rangle \quad (71)$$

It can be seen that in the Schwinger-type two-boson realization the indices are the following

$$k = \frac{1}{2}\left(|n_1-n_2|+1\right) \in \frac{1}{2}\mathbb{Z}^+ \quad (72a)$$

$$m = \frac{1}{2}\left(n_1+n_2-|n_1-n_2|\right) \in \mathbb{N} \quad (72b)$$

The $SU(1,1)$ CS-s according to Perelomov are

$$|\eta,k\rangle = U(\alpha)\,|0,k\rangle \quad (73a)$$

$$U(\alpha) = \exp\left[\alpha I_+ - \overline{\alpha} I_-\right], \qquad \alpha = \frac{\theta}{2}e^{-i\varphi} \quad (73b)$$

Where $\eta = -\tanh\frac{\theta}{2}e^{-i\phi}$

$$|\eta,k\rangle = (1-|\eta|)^k\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(m+2k)}{m!\Gamma(2k)}}\eta^m\,|m,k\rangle \quad (74a)$$

$$|\eta,q\rangle = (1-|\eta|)^{(1+q)/2}\sum_{n_2=0}^{\infty}\sqrt{\frac{(n_2+q)!}{n_2!q!}}\eta^{n_2}\,|n_2+q,n_2\rangle \quad (74b)$$

Where $q=|n_1-n_2|$ is the occupation number difference.

The CGC problem of $\mathcal{D}^+(k_1)\otimes\mathcal{D}^+(k_2)$ is the following:

$$|M,K\rangle = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}C(m_1,k_1;m_2,k_2|M,K)\,|m_1,k_1\rangle\otimes|m_2,k_2\rangle \quad (75)$$

The operators

$$\mathfrak{I}_i = I_i^{(1)}\otimes\mathbb{1}+\mathbb{1}\otimes I_i^{(2)} \quad (76)$$

$$\mathfrak{C}_2 = C_2^{(1)}+C_2^{(2)}+2I_3^{(1)}I_3^{(2)}-(I_+^{(1)}I_-^{(2)}+I_-^{(1)}I_+^{(2)}) \quad (77)$$

also satisfy the $\mathfrak{su}(1,1)$ algebra and and generate the direct product group $SU(1,1)\times SU(1,1)$ Acting with $\mathfrak{I}_3$ on the state $|M,K\rangle$ yields

$$M+K = m_1+k_1+m_2+k_2 \quad (78)$$

Furthermore, acting with $\mathfrak{C}_2$ on the state $|0,K\rangle$ yields

$$K_{\min}(K_{\min}-1) = (k_1+k_2)(k_1+k_2-1) \quad (79)$$

So $K_{\min}=k_1+k_2$ and since $k_{1,2}>0$ there is no upper limit and for $K_{\min}$: $M=m_1+m_2$

$$D_{k_1}^+\otimes D_{k_2}^+ = \sum_{K=k_1+k_2}^{\infty}D_K^+ \quad (80)$$

One can prove that the GS ($M=0$) is

$$|0,K\rangle = \sum_{q=0}^{l}C(q,k_1;l-q,k_2|0,k)\,|q,k_1\rangle\otimes|l-q,k_2\rangle \quad (81)$$

For which we know that $K=k_1+k_2+l$, $l=m_1+m_2=0,1,2,....$ With equation (71) we can construct the $SU(1,1)\times SU(1,1)$ basis of the coupled modes out of the products of the individual states of the two modes $|m_1,k_1\rangle\otimes|m_2,k_2\rangle$

$$|M,K\rangle = \sum_{m=0}^{l+M}C(m,k_1,l+M-m,k_2|M,K)\,|m,k_1\rangle\otimes|l+M-m,k_2\rangle \quad (82)$$

Where the coefficient $C(m,k_1,l+M-m,k_2|M,K)$ is given in appendix A [4]. In the Schwinger-type two-boson realization of $SU(1,1)$ the CGS problem gives a 4 oscillator realisation of $SU(1,1)\times SU(1,1)$ where the following equations hold for the occupation numbers

$$k_1 = \frac{1}{2}\left(|n_1-n_2|+1\right) \quad (83a)$$

$$m = \frac{1}{2}\left(n_1+n_2-|n_1-n_2|\right) \quad (83b)$$

$$k_2 = \frac{1}{2}\left(|n_3-n_3|+1\right) \quad (83c)$$

$$l+M-m = \frac{1}{2}\left(n_3+n_4-|n_3-n_4|\right) \in \mathbb{N} \quad (83d)$$

The $SU(1,1)\times SU(1,1)$ CS is

$$\begin{aligned}|\eta,K\rangle &= U(\alpha)\,|0,K\rangle \\ &= (1-|\eta|)^K\sum_{M=0}^{\infty}\sqrt{\frac{\Gamma(M+2K)}{M!\Gamma(2K)}}\eta^M\,|M,K\rangle \\ &= \sum_{M+0}^{\infty}\sum_{m=0}^{l+M}A(l,M,m,\eta)\,|m,k_1\rangle\otimes|l+M-m,k_2\rangle\end{aligned} \quad (84)$$

With $\alpha = -\frac{1}{2}\theta e^{-i\varphi}$ and $A(l,M,m,\eta) = (1-|\eta|)^K\eta^M\sqrt{\frac{\Gamma(M+2K)}{M!\Gamma(2K)}}C(m,k_1,l+M-$

$m, k_2|M, K)$. Since the $U(\alpha)$ operator factors into product of two $SU(1,1)$ squeeze operators the $SU(1,1)\times SU(1,1)$ Perelomov CS factors into a product of two two-mode $SU(1,1)$ CS:

$$|\eta, K\rangle = |\eta, k_1\rangle \otimes |\eta, k_2\rangle\,, \quad K = k_1 + k_2 \tag{85}$$

So there is no entanglement between the two subsystems if we separate the oscillators this way. What if we do a different 2-2 oscillator separation? What if we do a 1-3 oscillator separation? How does that correspond to the AdS_3 geodesics? How do the $\mathfrak{I}_i$ operators correspond to the AdS_3 Killing vectors?

Idea: in AdS_3

$$K_i = I_i^{(1)} + I_i^{(2)} \sim \mathfrak{I}_i \tag{86}$$

$$L_i = I_i^{(1)} - I_i^{(2)} \tag{87}$$

Remark: This can be furthter generalised to N (even) oscillatior realisations of $SU(1,1) \times SU(1,1) \times ... \times SU(1,1)$.

A. AdS$_3$ geodesics in the 4 oscillator realization

The $SU(1,1)\times SU(1,1)$ squeeze operator is a product of $SU(1,1)$ squeeze operators since $\mathfrak{I}_\pm = \pm i(\mathfrak{I}_1 \pm i\mathfrak{I}_2) = I_\pm^{(1)} + I_\pm^{(2)}$

$$\begin{aligned}
U(\theta,\varphi) =& \exp\left(-\frac{\theta}{2}e^{-i}\mathfrak{I}_+ + \frac{\theta}{2}e^{i}\mathfrak{I}_-\right)\\
=& \exp\left(-\frac{\theta}{2}e^{-i}I_+^{(1)} + \frac{\theta}{2}e^{i}I_-^{(1)}\right)\\
&\times \exp\left(-\frac{\theta}{2}e^{-i}I_+^{(2)} + \frac{\theta}{2}e^{i}I_-^{(2)}\right)\\
=& \exp\left[i\theta\left(-\sin\varphi I_1^{(1)} + \cos\varphi I_2^{(1)}\right)\right]\\
&\times \exp\left[i\theta\left(-\sin\varphi I_1^{(2)} + \cos\varphi I_2^{(2)}\right)\right]\\
=& \exp\left[\eta I_+^{(1)}\right] \exp\left[\log\left(1-|\eta|^2\right) I_3^{(1)}\right] \exp\left[-\overline{\eta} I_-^{(1)}\right]\\
&\times \exp\left[\eta I_+^{(2)}\right] \exp\left[\log\left(1-|\eta|^2\right) I_3^{(2)}\right] \exp\left[-\overline{\eta} I_-^{(2)}\right]\\
=& U(\theta,\varphi)^{(1)} U(\theta,\varphi)^{(2)}
\end{aligned} \tag{88}$$

Where it is understood that the $I_\pm^{(i)}$ $(i = 1, 2)$ operators act on their respective Hilbert spaces. This operator only produces the K_i operators. this is expected since $\eta = -\tanh\frac{\theta}{2}e^{-i\varphi}$ is still just two parameters. Idea: introduce the operator

$$\begin{aligned}
&U(\theta,\varphi)^{(1)}\overline{U(\theta,\varphi)^{(2)}}\\
&= \exp\left[i\theta\left(-\sin\varphi(I_1^{(1)} - I_1^{(2)}) + \cos\varphi(I_2^{(1)} - I_2^{(2)})\right)\right]\\
&= \exp\left[i\theta\left(-\sin\varphi L_1 + \cos\varphi L_2\right)\right]
\end{aligned} \tag{89}$$

Which will produce the $SU(1,1)\times SU(1,1)$ state $|\eta, k_1\rangle \otimes |-\eta, k_2\rangle$. Although, it is not a CS of the direct product group, but still made of proper $SU(1,1)$ CS-s since θ can be negative. Other idea: generalise η into the AdS_3 version of the Poincaré disk ($\eta \mapsto \eta + jt$??).

If we separate the state $|\eta, K\rangle$ into the subsystems $\{x_1, x_3\} + \{x_2, x_4\}$ it turns out that the entanglement entropy changes trivially from the case of the $SU(1,1)$ CS-s. It has the form:

$$\begin{aligned}
&S_{SU(2,2)}(q_1, q_2) = S_{SU(1,1)}(q_1) S_{SU(1,1)}(q_2)\\
=&2(q_1+1)\left[\cosh^2\frac{\theta}{2}\log\left(\cosh\frac{\theta}{2}\right) - \sinh^2\frac{\theta}{2}\log\left(\sinh\frac{\theta}{2}\right)\right]\\
&-\left(1-\tanh^2\frac{\theta}{2}\right)^{q_1+1}\sum_{n_2=0}^{\infty}\frac{(q_1+n_2)!}{q_1!n_2!}\tanh^{2n_2}\frac{\theta}{2}\log\left[\frac{(q_1+n_2)!}{q_1!n_2!}\right]\\
&+2(q_2+1)\left[\cosh^2\frac{\theta}{2}\log\left(\cosh\frac{\theta}{2}\right) - \sinh^2\frac{\theta}{2}\log\left(\sinh\frac{\theta}{2}\right)\right]\\
&-\left(1-\tanh^2\frac{\theta}{2}\right)^{q_2+1}\sum_{n_4=0}^{\infty}\frac{(q_2+n_4)!}{q_2!n_4!}\tanh^{2n_4}\frac{\theta}{2}\log\left[\frac{(q_2+n_4)!}{q_2!n_4!}\right]
\end{aligned} \tag{90}$$

where it is understood that $q_1 = |n_1 - n_2|$ and $q_2 = |n_3 - n_4|$. The same thing holds if we calculate the entropy of the state $U(\theta,\varphi)^{(1)}\overline{U(\theta,\varphi)^{(2)}}\,|0, K\rangle$.

Idea: Since the operators don't correspond to the desired AdS_3 Killing vectors one can choose $I_k^{(1)} = I_k$ and $I_k^{(2)} = J_k$ from equation [3a] in order to get Dirac's remarkable representation of the 2+2 de sitter group. The basis elements in the CS will have a different correspondence to the occupation number this way but the desired mapping to the Killing vectors will be achieved and we will have properly defined SU(2,2) CS-s this way.

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Appendix A: Details for the $SU(1,1) \times SU(1,1)$ CGC problem

The coefficient in equation (82) is

$$
\begin{aligned}
&C(m, k_1, l+M-m, k_2|M, K) \\
&= \sum_{q=0}^{l} \sum_{p=0}^{M} \delta_{m,p+q} (-1)^q \frac{1}{q!(l-q)!\Gamma(2k_1+q)\Gamma(2k_2+l-2)} \binom{M}{p} \\
&\times \sqrt{\frac{l!\Gamma(2K)\Gamma(2k_1)\Gamma(2k_2+l)(p+q)!(l+M-p-q)!\Gamma(2k_1+p+q)\Gamma(2k_2+l+M-p-q)}{M!\Gamma(2K+M)}} \\
&\times \left[\sum_{r=0}^{l} \binom{l}{r} \frac{\Gamma(2k_1)\Gamma(2k_2+l))}{\Gamma(2k_1+r)\Gamma(2k_2+l-r)} \right]^{-\frac{1}{2}}
\end{aligned}
\tag{A1}
$$

where, of course, it is understood that $K = k_1 + k_2 + l$.