Quantum Geometric Tensor in the manifold of quantum states

TDK paper

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November 15, 2023

Abstract

The Quantum Geometric Tensor (QGT) is an important quantity for describing the local geometry of quantum states. It is a complex quantity that can be separated into two parts with vastly different meanings: For pure states, the real part of it can be identified as the pullback of the Fubini-Study metric from the projective Hilbert space (Space of rays) to the parameter space of the system. On the other hand, its imaginary part is the familiar Berry curvature. If we consider mixed states the real and imaginary parts generalize to the Bures metric and the Uhlmann form, respectively. In this paper, we consider systems whose parameter spaces are the coset spaces $SU(2)/U(1)$ and $SU(1,1)/U(1)$ of the Lie groups $SU(2)$ and $SU(1,1)$ corresponding to classical phase spaces for coherent states (CSs). We calculate the QGT of Perelomovian CSs [1] with a method that relies heavily on group theoretical considerations. Then we investigate the determinant of the QGT which we show to be related to the Heisenberg uncertainty principle. We also introduce the generalized QGT and show how it is related to entanglement.

1 Motivation

This paper can be regarded as a result of many fruitful discussions with Péter Lévay and Balázs Hetényi, while they were working on their paper [2] about the geometry of quantum state manifolds. That original paper considered many geometric tensor quantities generalized to the quantum regime e.g. the metric tensor, Christoffel symbols, and the four index Riemann curvature and they explicitly calculated them for generalized coherent states (CSs) of the Heisenberg-Weyl, $SU(2)$ and the $SU(1,1)$ groups. The present paper aims to reproduce and elaborate on their results in an alternative manner eligible for further possible investigations. Instead of treating all the previously mentioned tensors, I will focus on one quantity, the second cumulant of the generating overlap amplitude function more commonly known in the literature as the quantum geometric tensor (QGT) proposed by Provost and Vallee [3]. I elaborate in detail on the geometric meaning of this quantity and show how it can be related to entanglement entropy for bipartite systems.

Over the last couple of years, there has been a new wave of interest concerning the geometry of quantum states of many-body systems in the fields of quantum statistical mechanics, atomic, molecular, condensed matter, and quantum information physics. The quantity at the heart of this approach is the so-called quantum geometric tensor. Geometric invariants based on it (e.g. fidelity) were used extensively to classify and understand quantum phase transitions [4–6] and to create optimal adiabatic ground-state preparation protocols [7] for example. The Provost-Valee metric and the Berry curvature, which makes up this tensor's real and imaginary parts, come up in countless areas of physics e.g. the Berry curvature becomes the Chern number in topological insulators [8], or the gravitational symplectic form in the bulk for holographic field theories [9]. In conformal field theories deformed by an exactly marginal perturbation, the QGT appears in the literature as the fidelity-susceptibility and it was shown to be related to the volume of a maximal surface in the corresponding AdS bulk [10]. This same correspondence was also found by Stanford and Susskind [11], but they related this quantity to computational complexity in quantum field theory. Furthermore, fidelity-susceptibility has also been present in concrete realizations of the complexity=volume conjecture [9]. These ideas within the field of holographic models of emergent spacetime structures go way further than the scope of this paper, but I plan to explore them deeper in my MSc thesis. This work can be considered as a prelude for further investigations.

The first idea that quantum mechanics can be reformulated as a geometric theory comes from the work of Kibble in 1979 [12]. Later it was considered in the light of the theory of geometric phases. In particular, in 1990 Anandan and Aharonov were able to reformulate the non-relativistic Schrödinger picture of quantum mechanics within a framework of a geometric theory [13]. In this context, Aharonov and Anandan were able to prove that the time evolution of the quantum system is driven by its energy fluctuations and it is independent from the Hamiltonian. In this geometric setting, the basic role is played not by the Hilbert space of states but its space of rays, i.e. the corresponding complex projective space where the equivalent states differ by a $U(1)$ local phase factor. In the center of this geometrical paradigm shift of quantum mechanics, stands M. V. Berry's ever-celebrated seminal 1984 paper [14] concerning the adiabatic cyclic evolution of quantum states in a parameter space $\mathcal{M}$. The geometric part of this evolution has been identified as the parallel transport of the states parametrized by the points in $\mathcal{M}$ and the associated anholonomy was found to be a special case of the geometric phase. Such ideas have given rise to the appearance of gauge potentials, and their associated curvatures and metrics living on $\mathcal{M}$. They have made dramatic effects on the dynamics of parameters (both classical and quantum), yielding gauge forces of electric and magnetic types.

2 A brief introduction to the quantum geometric tensor

The idea of defining a metric structure on the parameter space $\mathcal{M}$ was introduced in an interesting paper by Provost and Vallee [3] as a part of the quantum geometric tensor. In this section, we will briefly introduce the standard way [15–17] of deriving this central quantity of our paper. Let us assume that our quantum system's Hamiltonian is explicitly dependent on the parameters $X \in \mathcal{M}$. This occurs frequently in physical systems, one example could be the case of a single spin in a magnetic field: $H(\mathbf{X}) = -\omega_0 \mathbf{X} \cdot \mathbf{J}$, Where the parameter $\mathbf{X}$ appears as the magnetic field in three dimensions, which could be varied adiabatically then but our considerations will be more general than the adiabatic limit. In any case, the state of the system will inherit this parameter dependence, expressed as $|\psi(X)\rangle$. A common possibility is when the system is perturbed and all the parameters will change infinitesimally: $X \mapsto X + dX$ therefore, the original state evolves

into $|\psi(X+dX)\rangle$. The fidelity is defined as the absolute value of this event's transition amplitude.

$$\mathcal{F}(X, X+dX) = |\langle\psi(X+dX)|\psi(X)\rangle| \tag{1}$$

Now take the Taylor expansion of the perturbed state up to the second order in dX.

$$|\psi(X+dX)\rangle = |\psi(X)\rangle + |\partial_\mu\psi(X)\rangle\, dX^\mu + \frac{1}{2}\,|\partial_\mu\partial_\nu\psi(X)\rangle\, dX^\mu dX^\nu + \mathcal{O}\left(dX^3\right) \tag{2}$$

Then the transition amplitude's expansion is:

$$\langle\psi(X)|\psi(X+dX)\rangle = 1 + \langle\psi|\partial_\mu\psi(X)\rangle\, dX^\mu + \frac{1}{2}\,\langle\psi|\partial_\mu\partial_\nu\psi(X)\rangle\, dX^\mu dX^\nu + \mathcal{O}\left(dX^3\right) \tag{3}$$

If we assume our state $|\psi(X)\rangle \in \mathcal{S}$ to be normalized. Now substituting this equation back to the definition of fidelity and using the identities $|z| = \sqrt{\bar{z}z}$ for complex $z \in \mathbb{C}$ numbers and also that $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, we get that

$$\mathcal{F}(X, X+dX) = 1 + \frac{1}{2}G_{\mu\nu}(X)dX^\mu dX^\nu + \mathcal{O}\left(dX^3\right) \tag{4}$$

where $G_{\mu\nu}$ is the complex-valued metric tensor, the quantum geometric tensor (QGT):

$$G_{\mu\nu}(X) = \langle\partial_\mu\psi(X)|\partial_\nu\psi(X)\rangle - \langle\partial_\mu\psi(X)|\psi(X)\rangle\,\langle\psi(X)|\partial_\nu\psi(X)\rangle \tag{5}$$

The real and imaginary parts of the QGT play quite separate roles. The former is symmetric due to the Hermitian nature of the inner product on the Hilbert space and it serves as a Riemannian metric living on $\mathcal{M}$. Since it is also dependent on a particular quantum state $|\psi(X)\rangle$ it is called the quantum metric tensor (QMT).

$$\begin{aligned} g_{\mu\nu}(X) = \mathrm{Re}\left\{G_{\mu\nu}(X)\right\} =& \frac{1}{2}\left(\langle\partial_\mu\psi(X)|\partial_\nu\psi(X)\rangle + \langle\partial_\nu\psi(X)|\partial_\mu\psi(X)\rangle\right) \\ &- \langle\partial_\mu\psi(X)|\psi(X)\rangle\,\langle\psi(X)|\partial_\nu\psi(X)\rangle \end{aligned} \tag{6}$$

In Sec. 2.2 we will show that this quantity is closely related to the notion of a "distance" in the space of rays i.e. the projective Hilbert space $\mathcal{PH} = \mathcal{H}/U(1)$.

The imaginary part of the QGT is antisymmetric and it is the famous Berry curvature multiplied by a factor of -2.

$$-\frac{1}{2}F_{\mu\nu}(X) = \omega_{\mu\nu}(X) = \frac{-i}{2}\left(\langle\partial_\mu\psi(X)|\partial_\nu\psi(X)\rangle - \langle\partial_\nu\psi(X)|\partial_\mu\psi(X)\rangle\right) \tag{7}$$

It is gauge-invariant by construction. Its physical origin and derivation are discussed in a later chapter.

Due the the Hermitian nature of the inner product, the Taylor expansion for the fidelity only contains the real part of $G_{\mu\nu}$ leaving any ambiguity where one should think that a complex coefficient would appear in the Taylor expansion of a real quantity.

$$\mathcal{F}(X, X+dX) = 1 + \frac{1}{2}g_{\mu\nu}(X)dX^\mu dX^\nu + \mathcal{O}\left(dX^3\right) \tag{8}$$

To summarize, the most important equation we derived in this chapter is that the QGT contains the symmetric quantum metric tensor (QMT) and the antisymmetric Berry curvature as its real and imaginary parts respectively.

$$G_{\mu\nu}(X) = g_{\mu\nu}(X) - \frac{i}{2}F_{\mu\nu}(X) \tag{9}$$

2.1 The Fubini-Study metric

To arrive at a better understanding of the origin of the quantities defined above, we introduce the Fubini-Study metric [18] which is closely related to the quantum metric tensor by a pullback operation from the projective Hilbert space to the parameter space. Let us consider a quantum system with states represented by vectors $|\psi\rangle \in \mathcal{H} \simeq \mathbb{C}^{n+1}$ of some Hilbert space which we assume to be finite $n+1$ dimensional for simplicity. The complex coefficients characterizing the state $|\psi\rangle$ in an arbitrary basis are denoted by Z^α, $\alpha = 0, 1, ..., n$. The normalization condition then can be written as:

$$\langle\psi|\psi\rangle = \delta_{\alpha\beta}\overline{Z^\alpha}Z^\beta = \overline{Z_\alpha}Z^\alpha = 1 \tag{10}$$

Where we used the Einstein convention for repeated indices. Normalized states are elements of the unit sphere $\mathcal{S} \simeq S^{2n+1}$. As always, one can define an equivalence relation between nonzero states $|\psi\rangle$ and $|\phi\rangle$ if they are related by a complex multiplicative factor.

$$|\psi\rangle \sim |\phi\rangle \iff |\psi\rangle = \lambda\,|\phi\rangle\,, \quad \lambda \in \mathbb{C}/\{0\} \tag{11}$$

The set of equivalence classes $\mathcal{P} \equiv \mathcal{H}/\sim$ is called the space of rays. They contain the distinct physical states of our system. For finite-dimensional Hilbert spaces $\mathcal{H} \simeq \mathbb{C}^{n+1}$ we have the isometry $\mathcal{P} \simeq \mathbb{CP}^n$ between the space of rays and the n-dimensional projective Hilbert space. For normalized states there is an alternative construction for $\mathcal{P}$, in this case the equivalent vectors are related by a $U(1)$ phase factor. This means that in (11) $\lambda \in U(1)$. It is also known that $\mathcal{S}$ can be regarded as the total space of a principal bundle over $\mathcal{P}$ with the structure group $U(1)$. This just means that there is an $\mathcal{S} \to \mathcal{P}$ projection

$$\pi : |\psi\rangle \in \mathcal{S} \longmapsto |\psi\rangle\langle\psi| \in \mathcal{P} \tag{12}$$

Hence $\mathcal{P}$ can also be regarded as the space of rank one projectors in $\mathcal{H}$.

In the $Z^0 \neq 0$ coordinate patch of $\mathcal{P}$ we can define the local coordinates for a ray based on the ones defined on $\mathcal{H}$

$$w^j = \frac{Z^j}{Z^0}, \quad j = 1, ..., n \tag{13}$$

In this context, Z^js are called the homogeneous coordinates for $\mathbb{C}^{n+1}$ and the w^js are the inhomogeneous coordinates for $\mathbb{CP}^n$ on the local coordinate patch $\mathcal{U}_0$

$$\mathcal{U}_0 = \left\{ w^j = \frac{Z^j}{Z^0} \Big| Z^0 \neq 0 \right\} \tag{14}$$

To motivate the existence of the Riemannian metric g^{FS} on $\mathcal{P}$ induced by the scalar product we define the distance between two normalized rays $|\psi\rangle\langle\psi|\,, |\varphi\rangle\langle\varphi| \in \mathcal{P}$ denoted by $s(\psi, \varphi)$.

$$\cos\left(\frac{s(\psi,\varphi)}{2}\right) := |\langle\psi|\varphi\rangle| \tag{15}$$

$\mathcal{P}$ with this given distance is indeed a metric space and the right-hand side of the equation is always in the $[0, 1]$ interval due to the Cauchy-Schwartz inequality. The maximal distance between two rays is $s = \pi$ in the case of orthogonal states. To get the Fubini-Study metric from this distance one has to consider an infinitesimal step in the space of rays. This can be done by choosing $|\psi\rangle\,, |\varphi\rangle = |\psi\rangle + |d\psi\rangle \in \mathcal{S}$. Then we can take the

Taylor expansion of the left side up to second order in the infinitesimal distance ds.

$$\cos\left(\frac{ds(\psi,\varphi)}{2}\right) = 1 - \frac{1}{8}ds^2 + \mathcal{O}(ds^4) \tag{16}$$

Similarly, on the right side, we can take the expansion up to $\mathcal{O}(d\psi^2)$

$$1 - \frac{1}{2}\langle d\psi|\left(1 - |\psi\rangle\langle\psi|\right)|d\psi\rangle + \mathcal{O}(d\psi^4) \tag{17}$$

The next step is to express $|\psi\rangle$ and $|d\psi\rangle$ with the previously defined coordinates.

$$|\psi\rangle = Z^0\begin{bmatrix}1\\ w^j\end{bmatrix} \implies |d\psi\rangle = \frac{dZ^0}{Z^0}|\psi\rangle + Z^0\begin{bmatrix}0\\ dw^j\end{bmatrix} \tag{18}$$

As we write down the expression for $\langle d\psi|\left(1 - |\psi\rangle\langle\psi|\right)|d\psi\rangle$ the first term proportional to $|\psi\rangle$ will be projected out.

$$\begin{aligned}\langle d\psi|\left(1 - |\psi\rangle\langle\psi|\right)|d\psi\rangle &\approx \overline{Z_0}Z^0\begin{bmatrix}0 & \overline{dw_j}\end{bmatrix}\begin{bmatrix}1-\overline{Z_0}Z^0 & -\overline{Z_0}Z^0\overline{w_k}\\ -\overline{Z_0}Z^0w^j & \delta^j_k - \overline{Z_0}Z^0w^j\overline{w_k}\end{bmatrix}\begin{bmatrix}0\\ dw^k\end{bmatrix}\\ &= \overline{Z_0}Z^0\left[d\overline{w_j}dw^j - \overline{Z_0}Z^0\left(d\overline{w_j}w^j\right)\left(\overline{w_k}dw^k\right)\right]\end{aligned} \tag{19}$$

Now by using the identity coming from the normalization condition: $\overline{Z_0}Z^0 = \left(1+\overline{w_k}w^k\right)^{-1}$ we can conclude that

$$ds^2 = 4g^{FS}_{j\bar{k}}dw^j d\overline{w^k}, \quad j,k = 1,2,...,n \tag{20a}$$

$$g^{FS}_{j\bar{k}} = \frac{\left(1+\overline{w_l}w^l\right)\delta_{jk} - \overline{w_j}w_k}{\left(1+\overline{w_m}w^m\right)^2} \tag{20b}$$

$g^{FS}_{j\bar{k}}$ is giving the components of the Fubini-Study metric on $\mathcal{U}_0 \subset \mathcal{P}$ expressed in terms of the inhomogeneous coordinates. Notice that $\mathcal{P}$ is an n complex dimensional manifold with w^ks are the n complex coordinates. Clearly, $\mathcal{P}$ can alternatively be regarded as a $2n$ dimensional real manifold, by considering the real and imaginary parts of the w^ks. Being an even-dimensional manifold one suspects that $\mathcal{P}$ can as well be regarded as a phase space provided one can endow it by a symplectic form. This is indeed the case.

2.2 Pullback from the Fubini-Study metric to the QMT

To see how the Fubini-Study metric is related to the QMT, let's compare Eq. (17) and (6) both describing notions of distances on the projective Hilbert space and the parameter space respectively. The former one looks as

$$ds^2_{\mathcal{P}} = 4\left\langle d\psi\right|\left(1-\left|\psi\right\rangle\left\langle\psi\right|\right)\left|d\psi\right\rangle = 4\frac{\left(1+\overline{w_l}w^l\right)\delta_{jk}-\overline{w_j}w_k}{\left(1+\overline{w_m}w^m\right)^2}dw^j d\overline{w^k} \tag{21}$$

Where we expressed the ray $\left|\psi\right\rangle\left\langle\psi\right| \in \mathcal{P}$ with the complex inhomogeneous coordinates $w^j \in \mathcal{U}_0$. We now argue that if we have a parameter space $\mathcal{M}$ describing our physical system then there exists a smooth map f between the two.

$$\begin{array}{rccc} f: & \mathcal{M} & \longrightarrow & \mathcal{P} \\ & \cup & & \cup \\ & X & \longmapsto & \left|\psi(X)\right\rangle\left\langle\psi(X)\right| \end{array}$$

Figure 1: The map f between the parameter space and the space of rays.

This mapping depends on the quantum state $\left|\psi(X)\right\rangle\left\langle\psi(X)\right|$ just how the QMT depends on the state we constructed it from, i.e. if the state depends on some quantum number n $\left|\psi(X)\right\rangle\left\langle\psi(X)\right| \equiv \left|\psi_n(X)\right\rangle\left\langle\psi_n(X)\right|$ then the mapping f inherits this dependence: $f \equiv f^{(n)}$. The parameter dependence for $\left|\psi(X)\right\rangle\left\langle\psi(X)\right|$ appears in its inhomogeneous coordinates.

$$\left|\psi(X)\right\rangle\left\langle\psi(X)\right| = \frac{1}{1+\overline{w_m}(X)w^m(X)}\begin{bmatrix} 1 & \overline{w_k}(X) \\ w^j(X) & \overline{w_k}(X)w^j(X) \end{bmatrix} \tag{22}$$

To see how Eq. (21) is related to the QMT, let's study the form of the infinitesimal distance in the space of rays as we move in the parameter space with a small step: $X \mapsto X + dX$.

$$\begin{aligned} ds^2_{\mathcal{M}} &= ||\psi(X+dX)-\psi(X)||^2 = \left\langle\partial_\mu\psi|\partial_\nu\psi\right\rangle dX^\mu dX^\nu \\ &= \left(\gamma_{\mu\nu}(X)+i\omega_{\mu\nu}(X)\right)dX^\mu dX^\nu = \gamma_{\mu\nu}(X)dX^\mu dX^\nu \end{aligned} \tag{23}$$

Where the real and imaginary parts are

$$\gamma_{\mu\nu}(X) = \frac{1}{2}\left(\langle\partial_\mu\psi(X)|\partial_\nu\psi(X)\rangle + \langle\partial_\nu\psi(X)|\partial_\mu\psi(X)\rangle\right) \tag{24a}$$

$$\omega_{\mu\nu}(X) = \frac{-i}{2}\left(\langle\partial_\mu\psi(X)|\partial_\nu\psi(X)\rangle - \langle\partial_\nu\psi(X)|\partial_\mu\psi(X)\rangle\right) \tag{24b}$$

Because the inner product is Hermitian, it follows that $\gamma_{\mu\nu}(X) + i\omega_{\mu\nu}(X) = \gamma_{\nu\mu}(X) - i\omega_{\nu\mu}(X)$ so $\gamma_{\mu\nu}(X)$ is symmetric and $\omega_{\mu\nu}(X)$ is antisymmetric. That's the reason we could simplify our expression in the last line of the Eq. (23).

Careful reasoning [16] shows us that $\gamma_{\mu\nu}$ is not a metric yet because it is not gauge-invariant. The appropriate metric is exactly what we got from the real part of the QGT as we subtract the extra term $\beta_\mu(X)\beta_\nu(X)$

$$g_{\mu\nu}(X) = \gamma_{\mu\nu}(X) - \beta_\mu(X)\beta_\nu(X) \tag{25a}$$

$$\beta_\mu(X) = i\,\langle\psi(X)|\partial_\mu\psi(X)\rangle \tag{25b}$$

The finite quantum distance between two rays at the X_i and X_f parameters is

$$|\langle\psi(X_f)|\psi(X_i)\rangle| = 1 - \frac{1}{2}\int_{X_i}^{X_f} g_{\mu\nu}(X)dX^\mu dX^\nu \tag{26}$$

Where the integral in Eq. (26) is the length of a geodesic in the space of rays between the rays $|\psi(X_i)\rangle\langle\psi(X_i)|$ and $|\psi(X_f)\rangle\langle\psi(X_f)|$.

We got back our expression for the QMT from Eq. (6) and we proved that it is indeed a metric on $\mathcal{M}$. Furthermore, it gives the same notion of distance as the Fubini-Study metric written in Eq. (21). The operation that relates these two quantities is the pullback f^*, so we can express the central result of this section in one short equation

$$g = f^* g^{FS} \tag{27}$$

The exact form of the map f or the pullback can be quite complicated since we have to express the complex-valued inhomogeneous coordinates as a function of the real-valued coordinates in the parameter space $w^j(X^\mu)$. Hence we clarified the relationship between the QMT and the Fubini-Study metric.

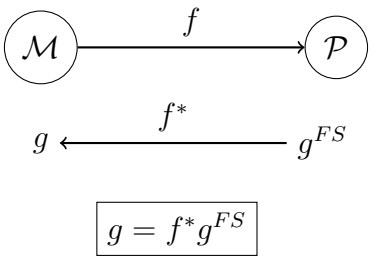

Figure 2: The pullback of f creates the connection between the Fubini-Study metric and the QMT.

2.3 The Berry phase and curvature

Now to find out a bit more about the imaginary part of the QGT, I would like to present an idea proposed by Berry in his 1984 paper [14, 19] where he discussed the parallel transport of quantum states in a parameter space $\mathcal{M}$. To understand his work, we have to discuss two crucial concepts. Adiabaticity and anholonomy.

Anholonomy is a geometrical phenomenon where the nonintegrability of a dynamical system causes some variable to change from its original value when others that drive it are altered in a cycle. The most obvious example of anholonomy is the parallel transport of vectors. The most frequently mentioned examples here are the change in the direction of the swing of a Foucault pendulum as the Earth takes a full rotation around its axis and the change in the direction of linear polarization of light along a twisting ray. The anholonomy Berry discovered was inherently of quantum in origin, and it manifested itself as a phase factor for a state that was parallel-transported around a cycle in a parameter space.

Adiabaticity tells us that this transport was made infinitely slowly, at the border between statics and dynamics. Then the adiabatic theorem will guarantee that the system will return to its initial state as we close the cycle. More explicitly, if we are concerned with the eigenstates of a non-degenerate Hamiltonian then the time-dependent Schrödinger equation will only multiply this state with a phase factor throughout its evolution. To see why we will present the details of this theorem [20] in Appendrix A.

In this section, we'll derive the Berry phase based on these concepts. The time-dependent Schrödinger equation is the following

$$i\hbar\partial_t \left|\psi(X(t))\right\rangle = H(X(t/\tau)) \left|\psi(X(t))\right\rangle \tag{28}$$

Where the adiabatic limit is reached in the $\tau \to \infty$ case. We'll be only concerned with

the eigenstates of the Hamiltonian

$$H(X(t))\,|n, X(t)\rangle = E_n(X(t))\,|n, X(t)\rangle \tag{29}$$

and our initial state will be $|\psi(X(0))\rangle = |n, X(0)\rangle$. The adiabatic theorem tells us that the time-evolved state will have the following form

$$|\psi(X(t))\rangle = c_n(t)\,|n, X(t)\rangle\,, \quad |c_n(t)| = 1 \tag{30}$$

Substituting this form into Eq. (29), it's straightforward to get

$$i\hbar\left(\frac{d}{dt}\log c_t(t) + \langle n, X(t)|\frac{d}{dt}|n, X(t)\rangle\right) = E_n(X(t)) \tag{31}$$

Which can be solved by choosing the ansatz $c_n(t) \equiv a_n(t)b_n(t)$. With this choice, we have two independent differential Eq.s:

$$\frac{\dot{a}_n(t)}{a_n(t)} = -\frac{i}{\hbar}E_n(X(t)) \tag{32a}$$

$$\frac{\dot{b}_n(t)}{b_n(t)} = -\langle n, X(t)|\frac{d}{dt}|n, X(t)\rangle \tag{32b}$$

So the solution is

$$c_n(t) = \exp\left[-\frac{i}{\hbar}\int_0^t E_n(X(t'))dt'\right]\exp\left[-\int_0^t \langle n, X(t')|\frac{d}{dt'}|n, X(t')\rangle\,dt'\right] \tag{33}$$

Observe that taking the derivative of the equation. $\langle n|n\rangle = 1$ we get $\langle\dot{n}|n\rangle + \langle n|\dot{n}\rangle = 0$. Furthermore it is a general property of an inner product that $\langle\dot{n}|n\rangle = \overline{\langle n|\dot{n}\rangle}$ so we can conclude that $\langle n|\dot{n}\rangle$ is purely imaginary and the second term in Eq. (33) is a phase factor as well.

Now if we consider a cyclic evolution, where the system moves adiabatically on a closed loop in the parameter space $\mathcal{M}$ we get that $0 < t < T$ and $X(T) = X(0)$. Furthermore, if we choose single valued eigenvectors over the parameter space then we have $|n, X(T)\rangle = |n, X(0)\rangle$, so for our final state after time T the adiabatic theorem gives the result

$$|\psi(T)\rangle = \exp\left[-\frac{i}{\hbar}\int_0^T E_n(X(t))dt\right]\exp\left[-\int_0^T \langle n, X(t)|\frac{d}{dt}|n, X(t)\rangle\, dt\right]|n, X(0)\rangle \quad (34)$$

The total phase change of $|\psi\rangle$ includes a dynamical part as well as the Berry phase:

$$\langle\psi(T)|\psi(0)\rangle = \exp\left[i\left(\gamma_d^{(n)} + \gamma^{(n)}(C)\right)\right] \quad (35a)$$

$$\gamma_d^{(n)} = -\frac{1}{\hbar}\int_0^T E_n(X(t))dt \quad (35b)$$

$$\gamma^{(n)}(C) = i\int_0^T \langle n, X(t)|\frac{d}{dt}|n, X(t)\rangle\, dt \quad (35c)$$

The two phases tell us different things about the system's evolution in $\mathcal{M}$, the dynamical phase $\gamma_d^{(n)}$ explains how long the journey took and the Berry phase $\gamma^{(n)}(C)$ tell us where did we go. To further develop our geometric view of the Berry phase we should take note that the time dependence of the eigenstate $|n, X(t)\rangle$ is only implicit through the parameter $X(t)$

$$\frac{d}{dt}|n, X(t)\rangle = \dot{X}(t)\nabla_X|n, X(t)\rangle \quad (36)$$

So we could rewrite the time integral into an integral over the closed loop $C \subset \mathcal{M}$ by defining the Berry vector potential $A_\mu^{(n)}(X)$ also known as the Berry connection responsible for the notion of parallel transport in $\mathcal{M}$.

$$A_\mu^{(n)}(X) = i\langle n, X(t)|\,\partial_\mu|n, X(t)\rangle \quad (37a)$$

$$\gamma^{(n)}(C) = \oint_C A^{(n)}(X) \quad (37b)$$

Where in the first equation $\partial_\mu = \frac{\partial}{\partial X^\mu} = (\nabla_X)_\mu$ and in the last one we have created the one-form $A^{(n)}(X) = A_\mu^{(n)}(X)dX^\mu$. To see why it is called a vector potential we can see that it has a $U(1)$ gauge symmetry by making the gauge transformation $|n, X\rangle \mapsto e^{i\alpha(X)}|n, X\rangle$. Then the Berry connection transforms as

$$A_\mu^{(n)}(X) \longmapsto A_\mu^{(n)}(X) - \partial_\mu\alpha(X) \quad (38)$$

Also, the Berry phase, which is a physical observable remains unchanged since the closed loop integral of a gradient is zero. By making use of the Stokes-theorem we can rewrite the Berry phase to be an integral over a Σ area enclosed by the loop C: $\partial\Sigma = C$.

$$\gamma^{(n)}(C) = \int_{\Sigma} F^{(n)}(X) \tag{39a}$$

$$F^{(n)}(X) = dA^{(n)}(X) = \frac{1}{2} F^{(n)}_{\mu\nu}(X) dX^{\mu} \wedge dX^{\nu} \tag{39b}$$

$$F^{(n)}_{\mu\nu}(X) = \partial_{\mu} A^{(n)}_{\nu}(X) - \partial_{\nu} A^{(n)}_{\mu}(X) \tag{39c}$$

The quantity $F^{(n)}(X)$ defined the equation above is a two-form and $F^{(n)}_{\mu\nu}(X)$ are the components of the Berry-curvature two-form. $dA^{(n)}(X)$ denotes the exterior derivative of the Berry connection. It is important to realize that the superscript n showing up in the curvature two-form refers to the important fact that $F^{(n)}$ should directly be calculated from the spectral projector $P_n(X) := |n(X)\rangle\langle n(X)|$. These projectors are the same ones that are shown in Fig. 2. With the corresponding map, this time is to be called $f^{(n)}$.

To see this and also to see how the Berry curvature relates to the imaginary part of the QGT. We proceed as follows. In Eq. (7) we can calculate the effect of the exterior derivative.

$$F^{(n)}(X) = dA^{(n)}(X) = id\,\langle n|dn\rangle = i\,\langle dn| \wedge |dn\rangle = -\mathrm{Im}\left\{\langle dn| \wedge |dn\rangle\right\} \tag{40}$$

Moreover, an alternative calculation shows that

$$F^{(n)}(X) = i\mathrm{Tr}\left(P_n(X) dP_n(X) \wedge dP_n(X)\right) \tag{41}$$

Here we used the notation $|dn\rangle \equiv \partial_{\mu} |n, X\rangle\, dX^{\mu}$. Eq.(41) shows that Berry curvature can indeed be directly expressed in terms of the spectral projectors defining the map $f^{(n)}$ which was used in our pullback procedure of Sec. 2.2 Moreover, referring back to Figure 1. from these calculations one can also show that Berry curvature can be regarded as the pullback of the symplectic form associated with the Fubini-Study metric.

We mention in closing this section that this revolutionary idea proposed by Berry can further be extended to non-adiabatic evolution as well. Here we are concerned with closed loops in the projective Hilbert space $\mathcal{P}$. This generalization was the work of Aharonov

and Anandan [21], an informative introduction can be read here [22].

3 QGT on a generalized CS Manifold

Let's reformulate how we think about the parameterized states from Sec. 2 a little bit. Also instead of taking any general state $|\psi(X)\rangle$, we will consider generalized CSs denoted by $|n, X\rangle \equiv |n\rangle$ where we will sometimes omit the parameter dependence for convenience. These states are not necessarily eigenstates of a Hamiltonian, this just means that they depend on a quantum number n and parameters X which will describe a point in a manifold corresponding to a group's coset space. The reason that we denote this state with an index n is that it is constructed from an initial state $|n, 0\rangle$ by acting on it with an operator $D(X)$. This operator is what solely carries the parameter dependence, $|n, 0\rangle$ will be an extremal state of our system like the $|\pm j, j\rangle$ basis states in the irrep j of the $SU(2)$ group. The $D(X)$ operator corresponds to the displacement operator well-known in quantum optics and it creates the generalized CSs for an arbitrary group in the Perelomovian definition [1]. With this convention, the QGT can be written up in the following form

$$G^{(n)}_{\mu\nu}(X) = \sum_{m\neq n} \langle \partial_\mu n|m\rangle \langle m|\partial_\nu n\rangle \tag{42a}$$

$$|n\rangle = |n, X\rangle = D(X)\,|n, 0\rangle \tag{42b}$$

We have explicitly denoted that the QGT depends on the state $|n\rangle$ we construct it from or in other words it depends on where we are in the space of rays $\mathcal{P}$ and on the parameter space $X \in \mathcal{M}$. This is a consequence of the mapping $f^{(n)} : \mathcal{M} \to \mathcal{P}$ being dependent on the quantum numbers that we argued for in Sec. 2.2. This is the same expression that we have written up in Eq. (5) but we have written an identity operator $\mathbb{1} = \sum_m |m\rangle\langle m|$ in the middle of the first term where $\{|m\rangle\}$ is an ONB containing $|n\rangle$.

To be more precise, we can construct CSs on an arbitrary Lie-group G by having an extremal state $|\psi\rangle_0$ and a unitary irrep of G denoted by D on the Hilbert space. Consider all the elements $h \in G$ for which $D(h)$ only multiplies $|\psi\rangle_0$ by a phase factor. These elements make up the stability group H. The extremal state is then an eigenstate of the

operators in the stability group with definite quantum numbers and the complement operators of H in G are ladder operators, responsible for raising and lowering these quantum numbers. Construction shows that the vectors $|\psi\rangle_g = D(g)\,|\psi\rangle_0$ for all $g \in G$ belonging to a left coset class G/H differ only by a phase factor. Therefore we can take a representative $g(x) \in G$ for all the cosets $x \in G/H$, and one gets the set of states $\{|\psi\rangle_x\} \equiv \{|x\rangle\}$. This system of states is called a CS system. The coset space will then respond to a manifold e.g. it will be an $\mathcal{S}^2 = SU(2)/U(1)$ three-sphere for $SU(2)$, which will give us our parameter space $\mathcal{M} = \mathcal{S}^2$. A point in $\mathcal{S}^2$ can be determined by a unit vector $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and it also determines a CS.

$$|\mathbf{n}\rangle = e^{i\alpha(\mathbf{n})} D(g_{\mathbf{n}})\,|\psi\rangle_0 \equiv D(X)\,|\psi\rangle_0 \tag{43}$$

where the phase factor is chosen to be unity and therefore the displacement operator for the $SU(2)$ group is

$$D(X) \equiv D(g_{\mathbf{n}}) = \exp\left[i\theta\left(\sin\varphi J_1 - \cos\varphi J_2\right)\right] \tag{44}$$

To see explicitly that the QGT is indeed gauge invariant we shall express it with the use of the rank one density operators $\varrho = |n\rangle\langle n| \in \mathcal{P}$.

$$G^{(n)}_{\mu\nu}(X) = \langle \partial_\mu n | Q | \partial_\nu n \rangle\,, \quad Q = \mathbb{1} - \varrho \tag{45}$$

Furthermore, since $\varrho|n\rangle = |n\rangle$ one has $\partial_\nu \varrho|n\rangle = Q|\partial_\nu n\rangle$. Moreover, $Q^2 = Q$ then we have

$$G^{(n)}_{\mu\nu}(X) = \langle \partial_\mu n | QQ | \partial_\nu n \rangle = \langle n | \partial_\mu \varrho \partial_\nu \varrho | n \rangle = \mathrm{Tr}(\varrho \partial_\mu \varrho \partial_\nu \varrho) \tag{46}$$

The imaginary part of this expression is to be compared with the Berry curvature formula of Eq.(41). Therefore we see that the QGT is a trace over an operator constructed by the density operator ϱ and its derivatives.

The mathematically inclined readers should point out our abuse of notation here since we should have written $G^{(\varrho)}_{\mu\nu}(X)$ to denote where we have calculated the QGT in the space of rays, but we will only work with pure states from now onwards so let us use it for the sake of clarity.

4 Generalized QGT as an entanglement measure

In this section, we shall show how the QGT is related to entanglement through a mapping between the quantum correlation matrix and the QMT [23]. Let us now start from the original definition of Eq. (5) as we choose our state to be an eigenstate of the systems Hamiltonian: $H(X)\,|n,X\rangle = E_n(X)\,|n,X\rangle$, which will be the following

$$G^{(n)}_{\mu\nu} = \langle \partial_\mu n|\partial_\nu n\rangle - \langle \partial_\mu n|n\rangle\,\langle n|\partial_\nu n\rangle \tag{47}$$

We have to rewrite this original definition of the QGT into a perturbative form. To find this, we have to express the transition amplitude from $|\partial_\mu n\rangle$ to an eigenstate $|m\rangle \neq |n\rangle$. Start from the eigenstate Eq. of H and do the steps as shown below.

$$H\,|n\rangle = E_n\,|n\rangle \tag{48a}$$

$$(\partial_\mu H)\,|n\rangle + H\,|\partial_\mu n\rangle = (\partial_\mu E_n)\,|n\rangle + E_n\,|\partial_\mu n\rangle \tag{48b}$$

$$\langle m|\partial_\mu H|n\rangle + E_m\,\langle m|\partial_\mu n\rangle = (\partial_\mu E_n)\,\langle m|n\rangle + E_n\,\langle m|\partial_\mu n\rangle \tag{48c}$$

Since $\langle m|n\rangle = 0$ we can conclude that

$$\langle m|\partial_\mu n\rangle = \frac{\langle m|\partial_\mu H|n\rangle}{E_n - E_m} \tag{49}$$

Therefore the QGT can be written in the following form [24]

$$G^{(n)}_{\mu\nu} = \sum_{m\neq n} \frac{\langle n|\partial_\mu H|m\rangle\langle m|\partial_\nu H|n\rangle}{(E_m - E_n)^2} \tag{50}$$

One useful trait of this form is that it shows that at the stationary points of a quantum phase transition, which are characterized by the ground-state level crossing, the components of the QGT (and hence also the components of the QMT and the Berry curvature) are singular.

Now to relate this quantity to the quantum correlation matrix, we have to generalize it so that the $a, b = 1, 2, ..., \dim(\mathcal{M})$ indices will be extended to include $\alpha, \beta = 1, ..., 2N$ indexing the phase space coordinates as well, where N is the dimension of the configuration space. A point in the phase space can be described with an operator-valued vector

$$\xi_\alpha = \begin{bmatrix} q_1 & q_2 & ... & q_N & p_1 & p_2 & ... & p_N \end{bmatrix}^T \tag{51}$$

Now this redefinition will completely change the meaning of the QGT, its connection to the QMT and the Berry curvature is only relevant if we only take the derivatives concerning the parameter space coordinates in Eq. (50). Luckily the off-diagonal elements of the block matrix $G^{(n)}_{a\beta} = G^{(n)}_{\alpha b} = 0$ are zero, so the generalized QGT will have the block diagonal form.

$$G = \left[\begin{array}{c|c} G_{\mu\nu} & 0 \\ \hline 0 & G_{\alpha\beta} \end{array}\right] \tag{52}$$

To relate the phase space part of the generalized QGT ($G_{\alpha\beta}$) to the correlation matrix, let's use coordinate representation, where the eigenvalue Eq. of the Hamiltonian read as

$$H\left(q, -i\hbar\partial_q\right)\langle q|n\rangle = E_n\langle q|n\rangle \tag{53}$$

Now apply the operator $-i\hbar\partial_k = -i\hbar\frac{\partial}{\partial q_k}$ on this Eq. and multiply with $\langle m|q\rangle \equiv \overline{\varphi}_m(q)$ and integrate over q

$$\begin{aligned} \langle m| - i\hbar\partial_k H|n\rangle &= -i\hbar\int d^N q\overline{\varphi}_m(q)\frac{\partial H}{\partial q_k}\varphi_n(q) \\ &= (E_n - E_m)\int d^N q\overline{\varphi}_m(q)\left(-i\hbar\frac{\partial\varphi_n(q)}{\partial q_k}\right) \\ &= (E_n - E_m)\langle m|p_k|n\rangle \end{aligned} \tag{54}$$

Substituting this result into Eq. (50) yields

$$G^{(n)}_{q_k q_l} = \sum_{m\neq n}\frac{1}{\hbar^2}\langle n|p_k|m\rangle\langle m|p_l|n\rangle \tag{55}$$

Adding and subtracting the $m = n$ term gives the following expression

$$G^{(n)}_{q_k q_l} = \frac{1}{\hbar^2}\left[\langle p_k p_l\rangle_n - \langle p_k\rangle_n\langle p_l\rangle_n\right] \tag{56}$$

Taking the real part of this expression then gives the QMT for the position coordinates

$$g^{(n)}_{q_k q_l} = \frac{1}{\hbar^2}\left[\frac{1}{2}\left(\langle p_k p_l\rangle_n + \langle p_l p_k\rangle_n\right) - \langle p_k\rangle_n \langle p_l\rangle_n\right] \tag{57}$$

We get similar results for the other indices, in summary, we get that

$$g^{(n)}_{q_k p_l} = \frac{1}{\hbar^2}\left[\frac{1}{2}\left(\langle p_k q_l\rangle_n + \langle q_l p_k\rangle_n\right) - \langle p_k\rangle_n \langle q_l\rangle_n\right] \tag{58}$$

$$g^{(n)}_{p_k p_l} = \frac{1}{\hbar^2}\left[\frac{1}{2}\left(\langle q_k q_l\rangle_n + \langle q_l q_k\rangle_n\right) - \langle q_k\rangle_n \langle q_l\rangle_n\right] \tag{59}$$

Which is quite similar in form to the quantum correlation matrix

$$C_{\alpha\beta} = \frac{1}{2}\langle\{\xi_\alpha, \xi_\beta\}\rangle_n - \langle\xi_\alpha\rangle_n \langle\xi_\beta\rangle_n \tag{60}$$

The two matrices are essentially the same, if we shift the indices of one by modulo N, we get the other one. The relation of the correlation matrix to the entanglement entropy of Gaussian states is well-known [25] and it has the following form

$$S = \sum_{i=1}^{n_{\text{sub}}}\left[\left(\sigma_i + \frac{1}{2}\right)\log\left(\sigma_i + \frac{1}{2}\right) - \left(\sigma_i - \frac{1}{2}\right)\log\left(\sigma_i - \frac{1}{2}\right)\right] \tag{61}$$

Where $\{\sigma_i\}$ denote so-called symplectic eigenvalues of C and the index i runs over the modes of one of the subsystems after taking a bipartition of the system. The symplectic eigenvalues appear in the Williamson-form of C, denoted by C_w

$$S_w C S_w^T = C_w \tag{62}$$

Where $C_w = \text{diag}\left(\sigma_1, \sigma_2, ..., \sigma_N, \sigma_1, \sigma_2, ..., \sigma_N\right)$ is diagonal and $S_w \in Sp(2N, \mathbb{R})$ is a symplectic matrix that diagonalizes C. Due to Williamson's theorem, such a matrix always exists. A general $S \in Sp(2N, \mathbb{R})$ symplectic matrix has the defining property that

$$S\Omega S^T = \Omega, \quad \Omega = \begin{bmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{bmatrix} \tag{63}$$

Finding the symplectic eigenvalues of C can be reduced to the problem of finding the usual eigenvalues of the matrix $i\Omega C$ [26]. One can prove that the eigenvalue problem of this matrix is

$$i\Omega C = U D U^{-1} \tag{64a}$$

$$D = \text{diag}\left(-\sigma_1, \sigma_1, -\sigma_2, \sigma_2, ..., -\sigma_N, \sigma_N\right) \tag{64b}$$

$$U = \frac{1}{\sqrt{2}} \left(S_w^T\right)^{-1} \begin{bmatrix} \mathbb{1}_N & \mathbb{1}_N \\ i\mathbb{1}_N & -i\mathbb{1}_N \end{bmatrix} \tag{64c}$$

For our purposes, it's enough to say that the GS of any Hamiltonian that is quadratic in its canonical operators is Gaussian.

$$H = \sum_{\alpha,\beta=1}^{2N} \xi_\alpha H_{\alpha\beta} \xi_\beta \implies \text{GS of } H \text{ is Gaussian!} \tag{65}$$

5 QGT of the SU(1,1) CSs

The $\mathfrak{su}(1,1)$ Lie-algebra consists of 3 generators denoted by K_i $(i = 1, 2, 3)$. They obey the following commutation relations

$$[K_1, K_2] = -iK_3, \qquad [K_2, K_3] = iK_1, \qquad [K_3, K_1] = iK_2 \tag{66}$$

We can also define ladder operators using the construction $K_\pm = \pm i(K_1 \pm iK_2)$. They will commute as such

$$[K_3, K_\pm] = \pm K_\pm, \qquad [K_-, K_+] = 2K_3 \tag{67}$$

There also exists a Casimir operator for the algebra that commutes with all the elements.

$$K^2 = K_3^2 - K_1^2 - K_2^2 = K_3^2 - \frac{1}{2}\left(K_+K_- + K_-K_+\right) \tag{68}$$

To summarize all the relevant information about the $SU(1,1)$ CS-s[1, 27], we only need to consider the UIRs known as the discrete positive series $\mathcal{D}^+(k)$. Here, k denotes the so-called Bargmann index, a c-number that appears in the eigenvalues of the Casimir

operator. In $\mathcal{D}^+(k)$ the Bargmann index is $k = 1/2, 1, 3/2, ...$, and the common eigenstates of the K_3 and K^2 operators are indexed by $|m, k\rangle$, where $m = 0, 1, 2, ...$ The operators act the following way on these states:

$$K^2 |m, k\rangle = k (k - 1) |m, k\rangle \tag{69a}$$

$$K_3 |m, k\rangle = (m + k) |m, k\rangle \tag{69b}$$

$$K_+ |m, k\rangle = \sqrt{(m + 1)(m + 2k)} |m + 1, k\rangle \tag{69c}$$

$$K_- |m, k\rangle = \sqrt{m(m + 2k - 1)} |m - 1, k\rangle \tag{69d}$$

Therefore the general $SU(1, 1)$ basis element can be constructed by acting on the extremal state $|0, k\rangle$ with the K_+ operator

$$|m, k\rangle = \sqrt{\frac{m!\Gamma(2k)}{\Gamma(2k + m)}} (K_+)^m |0, k\rangle \tag{70}$$

For each irrep in $\mathcal{D}^+(k)$, the extremal state is $|0, k\rangle$, so the algebra of the stability group consists of one element: $\mathfrak{h} = \{K_3\}$. The coset space then corresponds to a hyperboloid $G/H = SU(1, 1)/U(1) \equiv \mathbb{H}$ that will parameterized by (ρ, φ). This hyperboloid will be the parameter space of our physical system.

The $SU(1, 1)$ CS-s according to Perelomov [1] are constructed with the displacement operator $D(\alpha)$.

$$|k, \eta\rangle = D(\alpha) |0, k\rangle \tag{71a}$$

$$\begin{aligned} D(\alpha) &= \exp [\alpha K_+ - \overline{\alpha} K_-] \\ &= \exp [-i\rho (\cos\varphi K_1 + \sin\varphi K_2)] \\ &= \exp [\eta K_+] \exp \left[\log \left(1 - |\eta|^2\right) K_3\right] \exp [-\overline{\eta} K_-] \end{aligned} \tag{71b}$$

Where $\alpha = -\frac{\rho}{2} e^{-i\varphi}$ and $\eta = -\tanh \frac{\rho}{2} e^{-i\varphi}$. The first rewriting of the displacement operator in Eq. (71b) is trivial but the second one is not so much and we detail its derivation in Appendix B. Using the completeness of the UIRs, the CSs can be expressed on their basis.

$$|k,\eta\rangle = (1-|\eta|)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(m+2k)}{m!\Gamma(2k)}} \eta^m |m,k\rangle \tag{72}$$

We can get these coefficients by using the third form of the displacement operator in Eq. (71b).

Now we know all there is about the $SU(1,1)$ CSs, what's left is to calculate the QGT on them! Our task is to get

$$G^{(k)}_{\mu\nu}(\rho,\varphi) = \sum_{l\neq k} \left(\partial_\mu \langle k,\eta|\right) |l,\eta\rangle\langle l,\eta| \left(\partial_\nu |k,\eta\rangle\right) \tag{73}$$

Where $\mu,\nu = \rho,\varphi$. First, let's write the expression inside the sum of the QGT in the following way

$$\langle l,\eta| \left(\partial_\nu |k,\eta\rangle\right) = \langle 0,l| D^{-1}(\rho,\varphi)\partial_\nu D(\rho,\varphi) |0,k\rangle \tag{74}$$

This form of our expression contains the extrinsic construction of the Mauer-Cartan form $D^{-1}(\rho,\varphi)\partial_\nu D(\rho,\varphi)$ expressed on the local coordinates (ρ,φ). We can expand this operator on the basis of the Lie-algebra $\{K_i\}$.

$$D^{-1}(\rho,\varphi)\partial_\nu D(\rho,\varphi) = E^1_\nu(\rho,\varphi)K_1 + E^2_\nu(\rho,\varphi)K_2 + A^3_\nu(\rho,\varphi)K_3 \quad (\nu = \rho,\varphi) \tag{75}$$

The E^1_ν, E^2_ν and A^3_ν coefficients are independent of representation so we introduce a different representation of $SU(1,1)$ constructed from the Pauli matrices.

$$K_1 = \frac{i}{2}\sigma_1 \quad K_1 = \frac{i}{2}\sigma_2 \quad K_3 = \frac{1}{2}\sigma_3 \tag{76}$$

Since this is a finite-dimensional representation of $SU(1,1)$ the algebra elements are not Hermitian since $SU(1,1)$ is not a compact group. Therefore, $D(\rho,\varphi)$ won't be unitary anymore. In this representation, one can prove that the displacement operator has the following form.

$$D(\rho,\varphi) = \begin{bmatrix} \cosh\frac{\rho}{2} & \sinh\frac{\rho}{2}e^{-i\varphi} \\ \sinh\frac{\rho}{2}e^{i\varphi} & \cosh\frac{\rho}{2} \end{bmatrix} \tag{77}$$

This is again detailed in Appendix B. Since this is a simple 2×2 matrix it's easy to calculate

its inverse or take its derivative w.r.t. ρ or φ. The results of these short calculations are the following.

$$D^{-1}(\rho,\varphi)\partial_\rho D(\rho,\varphi) = \frac{i}{2}\begin{bmatrix} 0 & e^{-i\varphi} \\ -e^{i\varphi} & 0 \end{bmatrix} = -i\sin\varphi K_1 + i\cos\varphi K_2 \tag{78a}$$

$$\begin{aligned} D^{-1}(\rho,\varphi)\partial_\varphi D(\rho,\varphi) &= \begin{bmatrix} -i\sinh^2\frac{\rho}{2} & \frac{1}{2}\sinh\rho e^{-i\varphi} \\ i\sinh^2\frac{\rho}{2} & \frac{1}{2}\sinh\rho e^{i\varphi} \end{bmatrix} \\ &= -i\sinh\rho\cos\varphi K_1 - i\sinh\rho\sin\varphi K_2 - 2i\sinh^2\frac{\rho}{2}K_3 \end{aligned} \tag{78b}$$

This can be summarised in the coefficients

$$E^1_\rho(\rho,\varphi) = -i\sin\varphi \qquad E^1_\varphi(\rho,\varphi) = -i\sinh\rho\cos\varphi \tag{79a}$$

$$E^2_\rho(\rho,\varphi) = i\cos\varphi \qquad E^2_\varphi(\rho,\varphi) = -i\sinh\rho\sin\varphi \tag{79b}$$

$$A^3_\rho(\rho,\varphi) = 0 \qquad A^3_\varphi(\rho,\varphi) = -2i\sinh^2\frac{\rho}{2} \tag{79c}$$

Since $|0,k\rangle$ is an eigenstate of K_3 and $\langle 0,l|$ is orthogonal to it since it is in a different irrep, the A^3_ν coefficients won't appear in the QGT. The final result can be rewritten into the following form using the identity $\mathbb{1} = \sum_l |0,l\rangle\langle 0,l|$

$$\begin{aligned} G^{(k)}_{\mu\nu}(\rho,\varphi) &= -\sum_{l\neq k}\sum_{i,j=1,2} E^i_\mu(\rho,\varphi)E^j_\nu(\rho,\varphi)\langle 0,k|K_i|0,l\rangle\langle 0,l|K_j|0,k\rangle \\ &= -\sum_{i,j=1,2} E^i_\mu(\rho,\varphi)E^j_\nu(\rho,\varphi)\langle 0,k|K_iK_j|0,k\rangle \end{aligned} \tag{80}$$

The overall minus sign comes from the identity that $(\partial_\nu\langle l,\eta|)|k,\eta\rangle = -\langle l,\eta|(\partial_\nu|k,\eta\rangle)$. We can add the $l=k$ term to the summation without any effect in the second line of Eq. (80) since the K_1 and K_2 operators map the $|0,k\rangle$ state to orthogonal ones. As we have argued in Sec. 2, $G^{(k)}_{\mu\nu}(\rho,\varphi)$ is Hermitian, so it can be written up with the real-valued matrices $g^{(k)}_{\mu\nu}(\rho,\varphi)$ and $\omega^{(k)}_{\mu\nu}(\rho,\varphi)$

$$G^{(k)}_{\mu\nu}(\rho,\varphi) = g^{(k)}_{\mu\nu}(\rho,\varphi) + i\omega^{(k)}_{\mu\nu}(\rho,\varphi) \tag{81a}$$

$$g^{(k)}_{\nu\mu}(\rho,\varphi) = g^{(k)}_{\mu\nu}(\rho,\varphi) \tag{81b}$$

$$\omega^{(k)}_{\nu\mu}(\rho,\varphi) = -\omega^{(k)}_{\mu\nu}(\rho,\varphi) \tag{81c}$$

$\omega^{(k)}_{\mu\nu}(\rho,\varphi)$ being antisymmetric, corresponds to the antisymmetric commutator part, while $g^{(k)}_{\mu\nu}(\rho,\varphi)$ corresponds to the symmetric anticommutator part of the resolution in Eq. (80).

$$K_\mu K_\nu = \frac{1}{2}\left[K_\mu, K_\nu\right] + \frac{1}{2}\{K_\mu, K_\nu\} \tag{82}$$

After some calculations, we can see that $g^{(k)}_{\mu\nu}(\rho,\varphi)$ and $\omega^{(k)}_{\mu\nu}(\rho,\varphi)$ have the following form.

$$g^{(k)}_{\mu\nu}(\rho,\varphi) = (\Delta K_1)^2_0 \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2\rho \end{bmatrix} \tag{83a}$$

$$(\Delta K_1)^2_0 = (\Delta K_2)^2_0 = \langle k,0|\, K_1^2\, |k,0\rangle - \langle k,0|\, K_1\, |k,0\rangle^2 = \frac{1}{4}\langle k,0|\, K_- K_+\, |k,0\rangle \tag{83b}$$

$$\omega^{(k)}_{\mu\nu}(\rho,\varphi) = \langle K_3\rangle_0 \begin{bmatrix} 0 & -\frac{1}{2}\sinh\rho \\ \frac{1}{2}\sinh\rho & 0 \end{bmatrix} \tag{84a}$$

$$\langle K_3\rangle_0 = \langle k,0|\, K_3\, |k,0\rangle \tag{84b}$$

It turns out that the QGT is independent of φ. The benefit of writing the QGT in this form is that we can see explicitly that the only terms that are dependent on the representation we're using are $(\Delta K_1)^2_0$ and $\langle K_3\rangle_0$ coefficients, everything else depends on the geometry of the $SU(1,1)$ group. In particular, if we disregard the multiplicative term $(\Delta K_1)^2_0$ at the beginning of the QMT it's simply the metric structure on a hyperboloid $\mathbb{H} = SU(1,1)/U(1)$. To see in detail how the QGT is related to quantum fluctuations, we calculate the determinant.

$$\det G^{(k)}_{\mu\nu}(\rho,\varphi) = \left[(\Delta K_1)^2_0(\Delta K_2)^2_0 - \left|\frac{1}{2}\langle K_3\rangle_0\right|^2\right]\sinh^2\rho \tag{85}$$

So the determinant of the QGT The Heisenberg uncertainty principle tells us that the

expression inside the square bracket has to be greater than zero.

$$(\Delta K_1)_0(\Delta K_2)_0 \geq \frac{1}{2}\left|\langle K_3\rangle_0\right| \tag{86}$$

So the $SU(1,1)$ QGT has a non-negative determinant and it's even a positive semidefinite matrix. One way to get a zero determinant is to choose the $\rho = \eta = 0$ point in the parameter space, this corresponds to the displacement operator being the identity: $(\rho = 0) = \mathbb{1}$, so our coherent state is our original extremal state $|0,k\rangle$. If we calculate the representation-dependent expectation values, it turns out that

$$(\Delta K_1)_0^2 = (\Delta K_1)_0^2 = \frac{1}{2}k \tag{87a}$$

$$\langle K_3\rangle_0 = k \tag{87b}$$

Therefore the determinant of the QGT is

$$\det G_{\mu\nu}^{(k)}(\rho,\varphi) = \frac{1}{4}\left[k^2 - k^2\right]\sinh^2\rho = 0 \tag{88}$$

So the generalized CSs will always have a 0 determinant! This is due to them being closest to the classical states in the sense that the Heisenberg uncertainty principle will become an equality for these states.

At this point, it's easy to calculate and interesting to see what would change if we started from a general $|m,k\rangle$ state instead of the extremal $|0,k\rangle$. The only difference is in the expectation values, it turns out that the uncertainty principle won't be an equality anymore.

$$(\Delta K_1)_m^2 = (\Delta K_1)_m^2 = \frac{1}{4}\langle m,k|\left\{K_+,K_-\right\}|m,k\rangle = \frac{1}{2}\left[m(m+2k)+k\right] \tag{89a}$$

$$\langle K_3\rangle_m = \langle m,k|K_3|m,k\rangle = m+k \tag{89b}$$

Therefore the determinant wont be zero for all points on the parameter space

$$\det G_{\mu\nu}^{(k)}(\rho,\varphi) = \frac{1}{4}m\,(m+1)\,(m+2k-1)\,(m+2k)\sinh^2\rho \tag{90}$$

Since $m = 0,1,2,...$, the only values where the determinant is zero, therefore we have

minimal uncertainty states is when $m = 0$ or when $\rho = 0$.

5.1 QGT in the two-oscillator realization of SU(1,1)

In the following subsection we deal with a two-boson realization of $SU(1,1)$ [28] which is a projective representation of the group, they only cover the positive discrete series $\mathcal{D}^+(k)$ and not all the possible irreps of the group. The generators are realized by bosonic creation and annihilation operators a, b.

$$K_+ = a^\dagger b^\dagger, \qquad K_- = ab, \qquad K_3 = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 1\right) \tag{91a}$$

$$K_1 = -\frac{i}{2}\left(a^\dagger b^\dagger - ab\right), \qquad K_2 = -\frac{1}{2}\left(a^\dagger b^\dagger + ab\right) \tag{91b}$$

Where $\left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1$. This is an infinite-dimensional representation of $SU(1,1)$ and the generators are Hermitian. The Casimir operator will have the following form.

$$K^2 = K_3^2 - K_1^2 - K_2^2 = K_3^2 - \frac{1}{2}\left(K_+K_- + K_-K_+\right) = \frac{1}{4}\left(\Delta^2 - 1\right) \tag{92}$$

Where $\Delta = a^\dagger a - b^\dagger b$ is the occupation number difference operator between the two modes. The different UIRs in the positive discrete series will be indexed by the expectation value of it on the states $|n_1, n_2\rangle \equiv |n_1\rangle_a \otimes |n_2\rangle_b$. These states are products of excited harmonic oscillator modes with excitation numbers $n_1 = a^\dagger a$, $n_2 = b^\dagger b$. Let's see how these operators act on these states to see how the excitation numbers correspond to the Bargmann index k and m from the previous Section.

$$K^2 |n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{4}\left(|n_1 - n_2|^2 - 1\right) |n_1\rangle_a \otimes |n_2\rangle_b \tag{93a}$$

$$K_3 |n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{2}\left(n_1 + n_2 + 1\right) |n_1\rangle_a \otimes |n_2\rangle_b \tag{93b}$$

$$K_+ |n_1\rangle_a \otimes |n_2\rangle_b = \sqrt{(n_1+1)(n_2+1)}\, |n_1+1\rangle_a \otimes |n_2+1\rangle_b \tag{93c}$$

$$K_- |n_1\rangle_a \otimes |n_2\rangle_b = \sqrt{n_1 n_2}\, |n_1-1\rangle_a \otimes |n_2-1\rangle_b \tag{93d}$$

It can be deduced that in the two-boson realization $SU(1,1)$, the indices denoting the states spanning the different irreps are the following

$$k = \frac{1}{2}\left(|n_1 - n_2| + 1\right) \in \frac{1}{2}\mathbb{Z}^+ \tag{94a}$$

$$m = \frac{1}{2}\left(n_1 + n_2 - |n_1 - n_2|\right) \in \mathbb{N} \tag{94b}$$

The CSs are constructed in the same way as in Eq. (72) and they have the following form.

$$|q,\eta\rangle = D(\alpha)\,|n_1\rangle_a \otimes |n_2\rangle_b\,, \qquad (n_1 = n_2 + q) \tag{95a}$$

$$|q,\eta\rangle = (1-|\eta|)^{(1+q)/2} \sum_{n_2=0}^{\infty} \sqrt{\frac{(n_2+q)!}{n_2!q!}}\eta^{n_2}\,|n_2+q\rangle_a \otimes |n_2\rangle_b \tag{95b}$$

Where $q = |n_1 - n_2|$ is the occupation number difference. Please take note that instead of starting from the extremal state $|0\rangle_a \otimes |0\rangle_b$, we use $|n_1\rangle_a \otimes |n_2\rangle_b$ from the beginning to get a general result corresponding to Eq. (90). At this point, it's important to mention, that the $n_1 = n_2 = 0$ case, so the $SU(1,1)$ CSs have an extensive meaning in quantum optics in this realization. They appear as the two-mode squeezed vacuum state (TMSVS), which is a laboratory standard, routinely produced through spontaneous parametric down convertion [29–31]. Getting the QGT for the Schwinger realization means calculating the representation-dependent expectation values

$$\frac{1}{4}\,{}_a\langle n_1| \otimes {}_b\langle n_2|\,\{K_+, K_-\}\,|n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{4}\left[n_1 n_2 + (n_1+1)\,(n_2+1)\right] \tag{96a}$$

$${}_a\langle n_1| \otimes {}_b\langle n_2|K_3\,|n_1\rangle_a \otimes |n_2\rangle_b = \langle k,0|\,K_3\,|k,0\rangle = \frac{1}{2}\left(n_1+n_2+1\right) \tag{96b}$$

We can get these equations by actually calculating the expectation values or by substituting Eq. (94) into (89). So the determinant of the QGT will be the following.

$$\det G_{\mu\nu}^{(n_1,n_2)}(\rho,\varphi) = \frac{1}{4} n_1 n_2\left(n_1 n_2 + n_1 + n_2 + 1\right)\sinh^2\rho \tag{97}$$

Which equals zero if either of the two oscillators is in the ground state, meaning $n_1 = 0$

or $n_2 = 0$, or in the case of the $\rho = 0$ separable case. So the Perelomovian CSs, where $n_1 = n_2 = 0$ but $\rho \neq 0$, the QMT, $g_{\mu\nu}^{(n_1,n_2)} = \mathrm{Re}\left\{G_{\mu\nu}^{(n_1,n_2)}\right\}$ is non-invertible, therefore it is not a proper metric.

5.2 Entanglement entropy of the SU(1,1) CSs in the two-oscillator realization

In this section, we calculate the entanglement entropy of the $SU(1,1)$ CSs using the definition of the generalized QGT introduced in Sec. 4. To do that, first we have to argue that the $SU(1,1)$ CSs are Gaussian states in this representation. This means that they are the GS of a parameter-dependent Hamiltonian, quadratic in q_1, q_2, p_2, p_2, at each point in $\mathcal{M}$. One possible choice is

$$H = X \cdot K = \eta_{ab} X^a K^b \tag{98}$$

Where the parameter space is the $\mathbb{H} = SU(1,1)/U(1)$ hyperboloid

$$X^1 = \sinh\rho\cos\varphi \tag{99a}$$

$$X^1 = \sinh\rho\sin\varphi \tag{99b}$$

$$X^3 = \cosh\rho \tag{99c}$$

and the metric tensor in Eq. (98) is $\eta_{ab} = \eta^{ab} = \mathrm{diag}(-1,-1,1)$. By introducing the phase space coordinates

$$a = \frac{1}{\sqrt{2}}(q_1 + ip_1) \qquad a^\dagger = \frac{1}{\sqrt{2}}(q_1 - ip_1) \tag{100a}$$

$$b = \frac{1}{\sqrt{2}}(q_2 + ip_2) \qquad b^\dagger = \frac{1}{\sqrt{2}}(q_2 - ip_2) \tag{100b}$$

We can rewrite the algebra elements using them and see that $H(X)$ is indeed quadratic in its canonical operators.

$$K_1 = -\frac{1}{2}\left(q_1 p_2 + q_2 p_1\right) \quad K_2 = -\frac{1}{2}\left(q_1 q_2 - p_1 p_2\right) \quad K_3 = \frac{1}{4}\left(p_1^2 + p_2^2 + q_1^2 + q_2^2\right) \tag{101}$$

By using the Baker-Campbell-Hausdorff formula on $D(\rho, \varphi)$ one can see that the Hamiltonian has the following form

$$H(\rho, \varphi) = D(\rho, \varphi) K_3 D(\rho, \varphi)^\dagger \tag{102}$$

Therefore, we can see that $|0, \eta\rangle = D(\alpha) |0\rangle_a \otimes |0\rangle_b$ is really the GS of it.

Calculating the real part of the generalized QGT means calculating the correlation matrix, and since it is a bit less tedious, we will follow that path. The expectation value of any canonical operator will be zero in the $|0, \eta\rangle$ state, due to it being Gaussian so we only need to find the values $\langle 0, \eta | \frac{1}{2} \{\xi_\alpha, \xi_\beta\} |0, \eta\rangle$, where $\xi_\alpha = [q_1, q_2, p_1, p_2]^T$. As an example let's take

$$\begin{aligned}\langle 0, \eta | q_1^2 | 0, \eta \rangle &= \frac{1 - |\eta|^2}{2} \left(\sum_{n=0}^{\infty} \overline{\eta}^n \, {}_a\langle n| \otimes {}_b\langle n| \right) \left(a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2}\right) \left(\sum_{m=0}^{\infty} \eta^m |m\rangle_a \otimes |m\rangle_b \right) \\ &= \frac{1 - |\eta|^2}{2} \sum_{n=0}^{\infty} |\eta|^{2n} (1 + 2n) = \frac{1}{2} \cdot \frac{1 + |\eta|^2}{1 - |\eta|^2}\end{aligned} \tag{103}$$

Very similar calculations yield the result

$$C_{\alpha\beta} = \begin{bmatrix} \mathcal{A} & \mathcal{B} & -\frac{i}{2} & \mathcal{C} \\ \mathcal{B} & \mathcal{A} & \mathcal{C} & -\frac{i}{2} \\ -\frac{i}{2} & \mathcal{C} & \mathcal{A} & -\mathcal{B} \\ \mathcal{C} & -\frac{i}{2} & -\mathcal{B} & \mathcal{A} \end{bmatrix} \implies g^{(0)}_{\alpha\beta} = \begin{bmatrix} \mathcal{A} & -\mathcal{B} & -\frac{i}{2} & \mathcal{C} \\ -\mathcal{B} & \mathcal{A} & \mathcal{C} & -\frac{i}{2} \\ -\frac{i}{2} & \mathcal{C} & \mathcal{A} & \mathcal{B} \\ \mathcal{C} & -\frac{i}{2} & \mathcal{B} & \mathcal{A} \end{bmatrix} \tag{104}$$

$$\mathcal{A} = \frac{1}{2} \cdot \frac{1 + |\eta|^2}{1 - |\eta|^2} \qquad \mathcal{B} = \frac{\mathrm{Re}\{\eta\}}{1 - |\eta|^2} \qquad \mathcal{C} = -i \frac{\mathrm{Im}\{\eta\}}{1 - |\eta|^2} \tag{105}$$

On the basis $\xi_\alpha = [q_1, q_2, p_1, p_2]^T$. From this, it's straightforward to calculate the

entanglement entropy using Eq. (61) after getting the symplectic eigenvalues of $C_{\alpha\beta}$.

$$S(\rho,\varphi) = 2\left[\cosh^2\frac{\rho}{2}\log\left(\cosh\frac{\rho}{2}\right) - \sinh^2\frac{\rho}{2}\log\left(\sinh\frac{\rho}{2}\right)\right] \tag{106}$$

It is a monotonically increasing function of ρ and independent of φ.

5.3 QGT in the one-oscillator realization of SU(1,1)

For a single-mode field described by the annihilation and creation operators a and a, respectively, the $\mathfrak{su}(1,1)$ Lie-algebra is realized by the operators

$$K_3 = \frac{1}{2}(a^\dagger a + \frac{1}{2}) \qquad K_- = \frac{1}{2}a^2 \qquad K_+ = \frac{1}{2}a^{\dagger 2} \tag{107}$$

Furthermore, the extremal state is $|0\rangle$, the GS of K_3, the Hamiltonian of a single harmonic oscillator. Just like in the previous subsection, instead of the GS we will consider $|n\rangle$, a general eigenstate of K_3, where $n = a^\dagger a$. Once again, the only things that differ from the previous sections are the expectation values

$$\frac{1}{4}\langle n|\{K_+,K_-\}|n\rangle = \frac{1}{8}\left(n^2+n+1\right) \tag{108}$$

$$\langle n|K_3|n\rangle = \frac{1}{2}(n+\frac{1}{2}) \tag{109}$$

Therefore the determinant of the QGT is

$$\det G^{(n)}_{\mu\nu}(\rho,\varphi) = \frac{1}{64}(n-1)n(n+1)(n+2)\sinh\rho \tag{110}$$

Which is surprisingly equal to zero in the case of $n = 0$, $n = 1$, and $\rho = 0$. The states generated from the $n = 1$ excited states will still give an equality for the Heisenberg uncertainty principle. This unusual result is because the one-oscillator representation of $SU(1,1)$ is a projective one, where the Bargmann index can be $k = \frac{1}{4}, \frac{3}{4}$. This can be seen from the Casimir operator being $K^2 = -\frac{3}{16}$. The CSs constructed with this method appear in quantum optics as the odd ($k = \frac{1}{4}$) and even ($k = \frac{3}{4}$) single-mode squeezed vacuum states [2, 31].

6 QGT of the SU(2) CSs

The $SU(2)$ group is much more well-known in the literature than $SU(1,1)$, due to it being the universal cover of $SO(3)$, the 3D rotation group. It's algebra contains 3 generators denoted by J_i $(i = 1, 2, 3)$ that commute as

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{111}$$

It also has a Casimir operator

$$J^2 = J_1^2 + J_2^2 + J_3^2 \tag{112}$$

The common eigenbasis of J^2 and J_3 is denoted with $|m, j\rangle$ where $j = 0, 1/2, 1, ...$ accounts for the different irreps and $m = -j, -j+1, ..., j-1, j$ denotes the vectors in one. Each irrep is $2j + 1$ dimensional.

$$J^2 |m, j\rangle = j(j+1) |m, j\rangle \tag{113a}$$

$$J_3 |m, j\rangle = m |m, j\rangle \tag{113b}$$

We can also define ladder operators $J_\pm = J_1 \pm iJ_2$ that map between different states for a given j subspace. The algebra of the stability group is $\mathfrak{h} = \{J_3\}$ and the parameter space of our interest is the group manifold $\mathcal{S}^2 = SU(2)/SU(1)$, the three-sphere parametrized by (θ, ϕ) in the following sections.

Since the irreps are finite-dimensional, we have the freedom to choose an extremal state to construct the CSs since the $m = -j$ and $m = j$ states are equally good choices since both will lead to the same CS system, but once again we will use the general $|m, j\rangle$ state and act on it with the $SU(2)$ displacement operator. We can always get the QGT for the actual CSs by substituting either $m = \pm j$ in the result. The CS construction is then

$$|j,\xi\rangle = D(\beta)\,|\pm j, j\rangle \tag{114a}$$

$$\begin{aligned} D(\beta) &= \exp\left[\beta J_+ - \overline{\beta} J_-\right] \\ &= \exp\left[i\theta\left(\sin\phi J_1 - \cos J_2\right)\right] \\ &= \exp\left[\xi J_+\right]\exp\left[\log\left(1+|\xi|^2\right) J_3\right]\exp\left[-\overline{\xi} J_-\right] \end{aligned} \tag{114b}$$

Where $\beta = \frac{i\theta}{2}e^{i\phi}$ and $\xi = -\tan\frac{\theta}{2}e^{-i\phi}$. Getting the different forms of the displacement operator is from a completely analogous calculation of the $SU(1,1)$ case, as seen in Appendix B. Using the normal form of the displacement operator, we can construct the $SU(2)$ CSs.

$$|0,\xi\rangle = D(\theta,\phi)\,|j,-j\rangle = \ = \left(1+|\xi|^2\right)^{-j}\sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\xi^{j+m}\,|j,m\rangle \tag{115}$$

The dynamical generation of an $SU(2)$ CS can be achieved with the well-known example of $N = 2j$ two-level atoms that are coupled to coherent monochomatic laser pulses [31]

6.1 QGT in the Schwinger realization of SU(2)

In the Schwinger realization for the $SU(2)$ algebra, the elements can reconstructed as

$$J_i = \sum_{n,m=1}^{2}\frac{1}{2}a_n^\dagger\left(\sigma_i\right)_{nm}a_m \tag{116}$$

Where $a_1 \equiv a$ and $a_2 \equiv b$ are bosonic annihilation operators. The result is the following.

$$J_3 = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right) \qquad J_+ = a^\dagger b \qquad J_- = ab^\dagger \tag{117a}$$

$$J_1 = \frac{1}{2}\left(a^\dagger b + ab^\dagger\right) \qquad J_2 = \frac{1}{2i}\left(a^\dagger b - ab^\dagger\right) \tag{117b}$$

In this representation of $SU(2)$ the states spanning the UIRs are products of harmonic oscillator eigenstates with occupation numbers $n_1 = a^\dagger a$, $n_2 = b^\dagger b$.

$$|m,j\rangle = |n_1\rangle_a \otimes |n_2\rangle_b \implies j = \frac{1}{2}(n_1+n_2), \quad m = \frac{1}{2}(n_1-n_2) \tag{118}$$

The action of these operators on the state $|n_1, n_2\rangle$ is the following.

$$\begin{aligned} J_1 |n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{2}(&\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\rangle_a \otimes |n_2-1\rangle_b \\ &+ \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\rangle_a \otimes |n_2+1\rangle_b) \end{aligned} \tag{119a}$$

$$\begin{aligned} J_2 |n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{2i}(&\sqrt{n_1+1}\sqrt{n_2}\,|n_1+1\rangle_a \otimes |n_2-1\rangle_b \\ &- \sqrt{n_1}\sqrt{n_2+1}\,|n_1-1\rangle_a \otimes |n_2+1\rangle_b) \end{aligned} \tag{119b}$$

$$J_3 |n_1\rangle_a \otimes |n_2\rangle_b = \frac{1}{2}(n_1-n_2)\,|n_1\rangle_a \otimes |n_2\rangle_b \tag{119c}$$

We want to calculate the QGT of the state $D(\theta,\phi)\,|n_1\rangle_a \otimes |n_2\rangle_b$ assuming that our system has an $SU(2)$ symmetry. For that, we use Eq. (42a). So we need to calculate $D^\dagger \partial_\theta D$ and $D^\dagger \partial_\varphi D$. Since $SU(2)$ is a compact group, there exists a finite-dimensional representation for its algebra that is Hermitian, for example, $J_i = \frac{1}{2}\sigma_i$. Therefore $D(\theta,\phi)$ is a unitary operator even in this case unlike for $SU(1,1)$. Very similar calculations that we have done in Sec. 5 give us the following expressions for the $SU(2)$ Mauer-Cartan form.

$$D^\dagger \partial_\theta D = i\sin\varphi J_1 - i\cos\varphi J_2 \tag{120a}$$

$$D^\dagger \partial_\varphi D = 2i\sin^2\frac{\theta}{2} J_3 + i\sin\theta\,(\cos\varphi J_1 + \sin\varphi J_2) \tag{120b}$$

From this, it's straightforward to calculate its following matrix elements on the oscillator basis.

$$\begin{aligned} &{}_a\langle m_1| \otimes {}_b\langle m_2| D^\dagger \partial_\theta D\,|n_1\rangle_a \otimes |n_2\rangle_b = \\ &= -\frac{1}{2}e^{-i\varphi}\sqrt{n_1+1}\sqrt{n_2}\,\delta_{m_1,n_1+1}\delta_{m_2,n_2-1} \\ &+ \frac{1}{2}e^{i\varphi}\sqrt{n_1}\sqrt{n_2+1}\,\delta_{m_1,n_1-1}\delta_{m_2,n_2+1} \end{aligned} \tag{121}$$

$$
\begin{aligned}
&{}_a\langle m_1| \otimes {}_b\langle m_2| D^\dagger \partial_\varphi D \left|n_1\right\rangle_a \otimes \left|n_2\right\rangle_b = \\
&= \frac{i}{2} \sin\theta e^{-i\varphi} \sqrt{n_1+1}\sqrt{n_2} \delta_{m_1,n_1+1}\delta_{m_2,n_2-1} \\
&+ \frac{i}{2} \sin\theta e^{i\varphi} \sqrt{n_1}\sqrt{n_2+1} \delta_{m_1,n_1-1}\delta_{m_2,n_2+1}
\end{aligned} \tag{122}
$$

And also from Eq. (42a), we can conclude that the real and imaginary part of the QGT $G_{\mu\nu}^{(n_1,n_2)}(\theta,\phi) = g_{\mu\nu}^{(n_1,n_2)}(\theta,\phi) + i\omega_{\mu\nu}^{(n_1,n_2)}(\theta,\phi)$ are the following.

$$
g_{\mu\nu}^{(n_1,n_2)}(\theta,\phi) = \frac{1}{4}\left(n_1+n_2+2n_1n_2\right) \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \tag{123a}
$$

$$
\omega_{\mu\nu}^{(n_1,n_2)}(\theta,\phi) = -\left(n_1-n_2\right) \begin{bmatrix} 0 & -\frac{1}{2}\sin\theta \\ \frac{1}{2}\sin\theta & 0 \end{bmatrix} \tag{123b}
$$

As we can see, the representation of independent parts in the metric and the Berry curvature are the same as the $SU(1,1)$ case but the $\sinh\rho$ function is replaced by a $\sin\theta$. This is due to the similarities between hyperbolic and spherical geometries. The determinant of the SU(2) Schwinger realization's QGT is the following.

$$
\det G_{\mu\nu}^{(n_1,n_2)}(\theta,\phi) = \frac{1}{4} n_1 n_2 \left(n_1n_2+n_1+n_2+1\right)\sin^2\theta \tag{124a}
$$

$$
= \frac{1}{16}\left[\left(2n_1n_2+n_1+n_2\right)^2 - \left(2n_1n_2+n_1+n_2\right)^2\right]\sin^2\theta \tag{124b}
$$

$$
= \frac{1}{16}\left[j(j+1)-m(m+1)\right]\left[j(j+1)-m(m-1)\right]\sin^2\theta \tag{124c}
$$

In the last line, we have used Eq. (118) to determine how the occupation numbers correspond to the vectors in the $SU(2)$ irreps and express the determinant with j and m. As expected, in the case of either $m = \pm j$, i.e. when the QGT is calculated on the Perelomovian CSs it will be a singular matrix. In the oscillator realization, this corresponds to either one of the oscillators being in their GS: $n_1 = 0$ or $n_2 = 0$. The only other possible way to achieve a singular QGT is to set $\theta = 0$. This is the case of the displacement operator being identity.

6.2 Entanglement entropy of the SU(2) CSs in the Schwinger realization

Since the Schwinger representation of $SU(2)$ consists of two bosonic modes, it makes sense to have a bipartition of the system. Therefore, in this section, we will calculate the entanglement entropy of the $SU(2)$ CSs. The states have the following form

$$|0,\xi\rangle \equiv D(\theta,\phi)\,|0\rangle_a \otimes |2j\rangle_b = \left(1+|\xi|^2\right)^{-j} \sum_{n=0}^{2j} \sqrt{\binom{2j}{n}} \xi^n\, |n\rangle_a \otimes |2j-n\rangle_b \tag{125}$$

Instead of calculating the generalized QGT or the correlation matrix of this state as we did in Sec. 5.2, we'll use a more traditional method. First, we construct the reduced density operator of the subsystem "a", corresponding to the oscillator with ladder operators $a, a^\dagger$.

$$\varrho_a = \mathrm{Tr}_b\, |0,\xi\rangle\langle 0,\xi| = \left(1+|\xi|^2\right)^{-2j} \sum_{n=0}^{2j} \binom{2j}{n} |\xi|^{2n}\, |n\rangle_a\langle n| \tag{126}$$

Since this matrix is already diagonal, we don't have to do any extra work we can just read off its eigenvalues.

$$p_n = \left(1+|\xi|^2\right)^{-2j} \binom{2j}{n} |\xi|^{2n} \tag{127}$$

Knowing this, it's straightforward to calculate the entanglement entropy of subsystem "a".

$$\begin{aligned} S_j(\theta) = -\mathrm{Tr}\varrho_a \log \varrho_a &= -\sum_{n=0}^{2j} p_n \log p_n \\ &= -2j\left[\cos^2\frac{\theta}{2}\log\left(\cos^2\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2}\log\left(\sin^2\frac{\theta}{2}\right)\right] \\ &\quad -\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{2j} \sum_{m=-j}^{j} \binom{2j}{j+m} \left(\tan\frac{\theta}{2}\right)^{2m} \log\binom{2j}{j+m} \end{aligned} \tag{128}$$

We can do a cross-check on this quantity. The so-called atomic state corresponds to the $\xi = 1/2$ choice of parameters. In this special case, the entropy has the following form

$$S_j\left(\frac{\pi}{2}\right) = 2j\log 2 - \left(\frac{1}{2}\right)^{2j}\sum_{m=-j}^{j}\binom{2j}{j+m}\log\binom{2j}{j+m} \tag{129}$$

For $j = \frac{1}{2}$ the atomic state simplifies to an EPR state of two qubits as shown in Eq. (135). Now the entropy is simply

$$S_{1/2}\left(\frac{\pi}{2}\right) = \log 2 \tag{130}$$

This can be explicitly calculated from the EPR state. The density matrix will be the following.

$$\varrho_{1/2}\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(|10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01|\right) \tag{131}$$

Therefore the reduced density matrix of subsystem "a" is

$$\varrho_{1/2a}\left(\frac{\pi}{2}\right) = \frac{1}{2}\left(|0\rangle_a\langle 0| + |1\rangle_a\langle 1|\right) = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{132}$$

So the entropy is indeed

$$S_{1/2}\left(\frac{\pi}{2}\right) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \log 2 \tag{133}$$

7 QGT of the SU(2) extremal states

7.1 The embedded qubit EPR state

In this section, we consider a generalization of the $SU(2)$ CSs. In the original construction, the states are generated by considering either extremal state $|\pm j, j\rangle$ and "displace" them in the parameter space with the operator $D(\theta, \phi)$. Now instead of choosing one of the two extremal states, we take the uniformly distributed linear combination of them:

$$|j, 0\rangle^+ \equiv \frac{1}{\sqrt{2}}\left(|-j, j\rangle + |j, j\rangle\right) \tag{134}$$

In the Schwinger representation, this state will manifest itself as an entangled 2 qubit EPR state for the $j = \frac{1}{2}$ irrep.

$$|\psi^+_{1/2}\rangle_0 = \frac{1}{\sqrt{2}}\left(|0\rangle_a \otimes |1\rangle_b + |1\rangle_a \otimes |0\rangle_b\right) \tag{135}$$

We want to calculate the QGT of this state $|\psi^+_{1/2}\rangle(\theta,\phi) = D(\theta,\phi)\,|\psi^+_{1/2}\rangle_0$. To do that, we have to consider the action of the algebra elements on the state $|\psi^+_{1/2}\rangle_0$.

$$J_1\,|\psi^+_{1/2}\rangle_0 = \frac{1}{2}\,|\psi^+_{1/2}\rangle_0 \tag{136a}$$
$$J_2\,|\psi^+_{1/2}\rangle_0 = \frac{1}{2i}\,|\psi^-_{1/2}\rangle_0 \tag{136b}$$
$$J_3\,|\psi^+_{1/2}\rangle_0 = \frac{1}{2}\,|\psi^-_{1/2}\rangle_0 \tag{136c}$$
$$|\psi^\pm_{1/2}\rangle_0 = \frac{1}{\sqrt{2}}\left(|1\rangle_a \otimes |0\rangle_b \pm |0\rangle_a \otimes |1\rangle_b\right) \tag{136d}$$

The sum in the definition of $G^{(+)}_{\mu\nu}(\theta,\phi)$ runs over the orthocomplement of the span of the $|\psi\rangle_0$ state, which is

$$\left\{|\psi^-_{1/2}\rangle_0, |m_1\rangle_a \otimes |m_2\rangle_b \,\middle|\, (m_1,m_2) \neq (0,1),(1,0)\right\} \tag{137}$$

These terms are all orthogonal to each other and are elements of the different irreps of $SU(2)$ since in the Schwinger realization

$$|j,m\rangle = |n_1\rangle_a \otimes |n_2\rangle_b \implies j = \frac{1}{2}(n_1+n_2), \quad m = \frac{1}{2}(n_1-n_2) \tag{138}$$

From all this, it is straightforward to calculate the matrix elements:

$$G^{(+)}_{1/2\,\theta\theta}(\theta,\phi) = \frac{1}{4}\cos^2\phi \tag{139a}$$
$$G^{(+)}_{1/2\,\theta\phi}(\theta,\phi) = -\frac{1}{2}\left(\frac{1}{2}\sin\theta\sin\phi + i\sin^2\frac{\theta}{2}\right)\cos\phi \tag{139b}$$
$$G^{(+)}_{1/2\,\phi\theta}(\theta,\phi) = -\frac{1}{2}\left(\frac{1}{2}\sin\theta\sin\phi - i\sin^2\frac{\theta}{2}\right)\cos\phi \tag{139c}$$
$$G^{(+)}_{1/2\,\phi\phi}(\theta,\phi) = \sin^2\frac{\theta}{2} - \frac{1}{4}\sin^2\theta\cos^2\phi \tag{139d}$$

The result is much more complicated than what we got for the usual CSs since the QGT of the embedded qubit EPR state is ϕ dependent. To know more about its uncertainty, let's evaluate the tensors determinant.

$$\begin{aligned}\det G^{(+)}_{1/2} =&\frac{1}{4}\cos^2\phi\left(\sin^2\frac{\theta}{2}-\frac{1}{4}\sin^2\theta\cos^2\frac{\theta}{2}\right)-\frac{1}{4}\cos^2\phi\left(\frac{1}{4}\sin^2\theta\sin^2\phi+\sin^2\frac{\theta}{2}\right)\\ =&\frac{1}{4}\cos^2\phi\left(\sin^2\frac{\theta}{2}-\frac{1}{4}\sin^2\theta-\sin^4\frac{\theta}{2}\right)=0\end{aligned} \tag{140}$$

The result is exactly zero. This is quite surprising and one could think that it's a property of extremal states in any irrep. The next section will prove us otherwise.

7.2 QGT of the general SU(2) extremal state

The the generalized $SU(2)$ CSs constructed from the $j=\frac{1}{2}$ extremal state $|\psi^+_{1/2}\rangle_0$ turned out to have minimal uncertainty. Now we want to investigate if this is a property for any j. The general $SU(2)$ extremal state in the Schwinger representation is the following.

$$|\psi^{(+)}_j\rangle_0=\frac{1}{\sqrt{2}}\left(|0\rangle_a\otimes|2j\rangle_b+|2j\rangle_a\otimes|0\rangle_b\right),\quad j\in\frac{1}{2}\mathbb{N} \tag{141}$$

The $SU(2)$ Schwinger operators act on this state in the following manner.

$$J_1|\psi^{(+)}_j\rangle_0=\frac{1}{2}\sqrt{j}\left(|1\rangle_a\otimes|2j-1\rangle_b+|2j-1\rangle_a\otimes|1\rangle_b\right) \tag{142}$$

$$J_2|\psi^{(+)}_j\rangle_0=\frac{1}{2i}\sqrt{j}\left(|1\rangle_a\otimes|2j-1\rangle_b-|2j-1\rangle_a\otimes|1\rangle_b\right) \tag{143}$$

$$J_3|\psi^{(+)}_j\rangle_0=j|\psi^{(-)}_j\rangle_0 \tag{144}$$

$$|\psi^{(-)}_j\rangle_0=\frac{1}{\sqrt{2}}\left(|2j\rangle_a\otimes|0\rangle_b-|0\rangle_a\otimes|2j\rangle_b\right) \tag{145}$$

These equations won't hold for the $j=\frac{1}{2}$ and $j=1$ case. The former situation was already dealt with in the previous section but we still have to treat the latter. If $j=1$, then the first two operators give peculiar results: $J_1|\psi^{(+)}_1\rangle_0=|1\rangle_a\otimes|1\rangle_b$, $J_2|\psi^{(+)}_1\rangle_0=0$. It is then straightforward to calculate

$$\begin{aligned}{}_a\langle m_1|_0\otimes{}_b\langle m_2|D^\dagger\partial_\nu U|\psi^{(+)}_1\rangle =&i\sin\phi\delta^{m_1,m_2}_{1,1}\partial_{\nu,\theta}\\ &+\left(2i\sin^2\frac{\theta}{2}\delta^{m_1,m_2}_{\psi^{(-)}}+i\sin\theta\cos\phi\delta^{m_1,m_2}_{1,1}\right)\partial_{\nu,\phi}\end{aligned} \tag{146}$$

and the QGT is the following:

$$G_1^{(+)}{}_{\theta\theta}(\theta,\phi) = \sin^2\phi \tag{147}$$

$$G_1^{(+)}{}_{\theta\phi}(\theta,\phi) = \sin\theta\sin\phi\cos\phi \tag{148}$$

$$G_1^{(+)}{}_{\phi\phi}(\theta,\phi) = 4\sin^2\frac{\theta}{2} + \sin^2\theta\cos^2\phi \tag{149}$$

so for the $j = 1$ case, the determinant is the following.

$$\det G_1^{(+)}(\theta,\phi) = 4\sin^2\frac{\theta}{2}\sin^2\phi \tag{150}$$

It's not necessarily zero. This seems surprising. However, at the end of this section, we will explain it. Let's get back to the general case where $j \geq \frac{3}{2}$. One has to calculate the matrix element in the usual way.

$$\begin{aligned} {}_a\langle m_1|_0\otimes {}_b\langle m_2|D^\dagger\partial_\nu D\,|\psi_j^{(+)}\rangle = {} & \frac{1}{2}\sqrt{j}\left(-e^{-i\phi}\delta_1^{m_1}\delta_{2j-1}^{m_2} + e^{i\phi}\delta_{2j-1}^{m_1}\delta_1^{m_2}\right)\delta_{\nu\theta} \\ & + \left[2ij\sin^2\frac{\theta}{2}\delta_{\psi_j^{(-)}}^{m_1,m_2} + \frac{i}{2}\sqrt{j}\sin\theta\left(e^{-i\phi}\delta_1^{m_1}\delta_{2j-1}^{m_2} + e^{i\phi}\delta_{2j-1}^{m_1}\delta_1^{m_2}\right)\right]\delta_{\nu\phi} \end{aligned} \tag{151}$$

Therefore the matrix elements are the following.

$$G_j^{(+)}{}_{\theta\theta}(\theta,\phi) = \frac{1}{2}j \tag{152}$$

$$G_j^{(+)}{}_{\theta\phi}(\theta,\phi) = -G_j^{(+)}{}_{\phi\theta}(\theta,\phi) = 0 \tag{153}$$

$$G_j^{(+)}{}_{\phi\phi}(\theta,\phi) = 4j\sin^2\frac{\theta}{2} + \frac{1}{2}j\sin^2\theta \tag{154}$$

With the determinant of

$$\det G_{j\geq\frac{3}{2}}^{(+)}(\theta,\phi) = \frac{1}{4}j^2\left(8j\sin^4\frac{\theta}{2} + \sin^2\theta\right) \tag{155}$$

In summary, the generalized CSs constructed from the $j \geq 1$ extremal states won't have minimal uncertainty. The reason the $j = \frac{1}{2}$ case had this property is because the extremal state was also an $SU(2)$ CS to begin with. Therefore the construction of the generalized CSs corresponded to acting twice on the state $|-1/2, 1/2\rangle$ with the displacent operator $D(\theta,\varphi)$. Displacing the same extremal state twice still results in an $SU(2)$ CS but at a

different point in the parameter space.

8 Summary

In this paper, we have extended and reformulated the ideas presented in [2]. The different geometric quantities appearing in the QGT were reconsidered and explained. The most important conclusion was that the QMT, the real part of the QGT is related to the Fubini-Study metric by a pullback from the projective Hilbert space onto the parameter space. This point is rarely mentioned in the literature and is even less frequently explained on a satisfactory level for mathematically inclined readers.

The QGT was calculated for the $SU(1,1)$ and $SU(2)$ generalized CSs using group theoretical considerations that can be used for any Lie group. Its determinant was shown to be connected to quantum fluctuations, i.e. it was proven to be connected to the Heisenberg uncertainty principle. We have also calculated the so-called generalized QGT for a certain representation of $SU(1,1)$ where the algebra elements are realized by two bosonic creation and annihilation operator pairs. The generalized QGT is intimately related to the quantum correlation matrix which can be used to calculate the entanglement entropy for Gaussian states. This calculation was made for the $SU(1,1)$ two-mode queezed vacuum state. The entanglement entropy was also calculated for the $SU(2)$ CSs with a more traditional method concerning the reduced density matrix of the state and summing over its eigenvalues. Finally, we have looked at the QGT for different states in the Schwinger realization of $SU(2)$, namely the "displaced" two-qubit EPR state and the more general $SU(2)$ extremal state. We conjectured that the extremal states have minimal uncertainties but it turned out to be an exclusive property for the $j = 1/2$ case.

This work can be regarded as a prelude to investigations we plan to carry out within the field of holography and AdS/CFT. Of course, since such investigations should include field theoretic generalizations of the considerations presented here, they are of a more sophisticated kind than the examples of this work.

A Adiabatic Theorem

We suppose that the Hamiltonian of our system changes continuously in time as we change e.g. the external field. The corresponding eigenfunctions and eigenvalues will then depend

intimately on the time T at which the evolution of H takes place. Let us say that our Hamiltonian takes its initial value H_0 at time t_0 and it reaches its final H_1 value at time t_1. We then define.

$$T = t_1 - t_0, \quad s = \frac{t - t_0}{T} \tag{156}$$

The latter is a reparametrization of $H(t) \to H(s) = H(t_0 + stT)$. The time evolution of the system between t_0 and t_1 is described by the unitary operator $U(t, t_0) \coloneqq U_T(s)$ and our problem boils down to determining it. This task becomes quite simple in the two limiting cases.

First in the $T \to 0$ limit, in the case of an infinitely rapid change, the system stays unmoved, i.e.

$$\lim_{T \to 0} U_T(1) = 1 \tag{157}$$

The other limit is when $T \to \infty$ is called the adiabatic evolution. In this extremely slow development, the Hamiltonian applies the so-called Adiabatic theorem [20].

Theorem 1 *(Adiabatic theorem) Assume our Hamiltonian $H(s)$ has a discrete spectrum with the elements denoted by $\epsilon_j(s)$ and projectors $P_j(s)$ mapping onto their respective eigenspaces both continuous in s. We make a couple of assumptions:*

1. *The energy levels never cross i.e.*

$$\epsilon_j(s) \neq \epsilon_k(s) \tag{158}$$

   for any $j, k \in \mathbb{N}^+$ and all $s \in [0, 1]$.

2. *The time evolution operator obeys the Schrödinger equation*

$$i\hbar \frac{d}{ds} U_T(s) = T H(s) U_T(s) \tag{159}$$

Then the statement is that the time evolution operator has the asymptotic property:

$$\lim_{T \to \infty} U_T(s) P_j(0) = P_j(s) \lim_{t \to \infty} U_T(s) \tag{160}$$

for any $j \in \mathbb{N}^+$.

The theorem has far-reaching consequences, e.g. that if the system starts in one of the eigenspace of the Hamiltonian $H(0)$, it will remain in its instantaneous eigenspace throughout its evolution. Then the time evolution operator can be calculated in the case of unchanged subspaces: $(P_J(s) \equiv P_j)$

$$U_T(s) = \exp\left(-\frac{i}{\hbar}T\int_0^s d\tilde{s}H(\tilde{s})\right) = \sum_{j\in\mathbb{N}^+}\exp\left(-\frac{i}{\hbar}T\phi_j(s)\right) \tag{161}$$

with the following phase factor

$$\phi_j(s) = \int_0^s d\tilde{s}\epsilon_j(\tilde{s}) \tag{162}$$

We therefore have that if the initial state of the system is an eigenvector of H_0 with eigenvalue $\epsilon_j(0)$ then at time t_1, it differs only by a phase factor

$$|\phi_j(0)\rangle \longmapsto |\phi_j(1)\rangle = |\phi_j(0)\rangle\exp\left(-\frac{i}{\hbar}T\phi_j(1)\right) \tag{163}$$

B The normal form of the $SU(1,1)$ displacement operator

Here we will derive the non-trivial "normal-ordered" form of the displacement operator shown in the third line of the Eq. (71b) for the $SU(1,1)$ CSs. Let's use the representation introduced in Eq. (76) and express $D(\theta,\varphi)$ in it.

$$D(\theta,\varphi) = \exp\left(\frac{\theta}{2}\begin{bmatrix}0 & e^{-i\varphi}\\ e^{i\varphi} & 0\end{bmatrix}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\theta}{2}\begin{bmatrix}0 & e^{-i\varphi}\\ e^{i\varphi} & 0\end{bmatrix}\right)^n \tag{164}$$

As we write out the terms in the series, it becomes evident that the solution is a sum of two matrices, and the coefficients multiplying them are the Taylor series of $\cosh\frac{\theta}{2}$ and $\sinh\frac{\theta}{2}$.

$$
\begin{aligned}
D(\theta,\varphi) &= \left(1+\frac{1}{2!}\left(\frac{\theta}{2}\right)^2+...\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\left(\frac{\theta}{2}+\frac{1}{3!}\left(\frac{\theta}{2}\right)^3+...\right)\begin{bmatrix}0 & e^{-i\varphi}\\ e^{i\varphi} & 0\end{bmatrix}\\
&= \begin{bmatrix}\cosh\frac{\theta}{2} & \sinh\frac{\theta}{2}e^{-i\varphi}\\ \sinh\frac{\theta}{2}e^{i\varphi} & \cosh\frac{\theta}{2}\end{bmatrix}
\end{aligned}
\tag{165}
$$

Thus we have proved Eq. (77) so far!

Now to relate this to Eq. (71b), the so-called "normal-form" of the displacement operator, we construct the matrices:

$$
\exp\left(\alpha K_+\right)=\exp\left(\alpha\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\right)=\mathbb{1}+\alpha K_+=\begin{bmatrix}1 & -\alpha\\ 0 & 1\end{bmatrix} \tag{166a}
$$

$$
\exp\left(\beta K_3\right)=\begin{bmatrix}e^{\beta/2} & 0\\ 0 & e^{-\beta/2}\end{bmatrix} \tag{166b}
$$

$$
\exp\left(\gamma K_-\right)=\exp\left(\gamma\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)=\mathbb{1}+\gamma K_-=\begin{bmatrix}1 & 0\\ \gamma & 1\end{bmatrix} \tag{166c}
$$

Then multiply them together we get that

$$
D(\theta,\varphi)=e^{\alpha K_+}e^{\beta K_3}e^{\gamma K_-}=\begin{bmatrix}e^{\beta/2}-\alpha\gamma e^{-\beta/2} & -\alpha e^{-\beta/2}\\ \gamma e^{-\beta/2} & e^{-\beta/2}\end{bmatrix} \tag{167}
$$

Comparing the matrix elements with (165), we conclude that

$$
\alpha=-\tanh\frac{\rho}{2}e^{-i\varphi}=\eta \tag{168}
$$

$$
\beta=\log\left(1-\tanh^2\frac{\rho}{2}\right)=\log\left(1-|\eta|^2\right) \tag{169}
$$

$$
\gamma=\tanh\frac{\rho}{2}e^{i\varphi}=-\overline{\eta} \tag{170}
$$

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