Universitetet i Oslo

Project 5

Making a model for the solar system using ordinary differential equations

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24. november 2019

Abstract

We have used the velocity Verlet algorithm and Euler's forward algorithm for solving coupled ordinary differential equations and object orientation for making a model for the soslar system. The equations we have used for this is based on Newton's law of motion due to the gravitational force. Our results confirm that the energy is conserved at all times. We experienced that Verlet gives a more efficient calculation than Euler. We also found that the initial velocity for the Earth to obtain a circular orbit around the sun is 2π and the initial velocity when the Earth start to escape the sun is $\sqrt{8}\pi$ AU/yr. We changed the gravitational force of the Earth and saw that this affected the Earths orbit around the Sun. We added Jupiter to the system and looked at how Jupiters mass affects the gravitational force.

Contents

1 Introduction

We have made a model for the solar system by starting with a hypothetical solar system with only two bodies, the Sun and the Eart. We have used Newtons second law of motion to obtain the motion of the planets. Further we have added more planets to obtain a model for the whole solar system. We have also looked at how a relativistic constant affect the perihelion of Mercury.

2 Method

For a hypothetical solar system with only the Earth and the Sun Newton's law is given by one force F_G

$$F_G = \frac{GM_\odot M_E}{r^2} \tag{1}$$

In the above equation $M_\odot$ and M_E is the mass of the Sun and the Earth. G is the gravitational constant and r is the distance between the Earth and the Sun. We neglect the motion of the Sun and only look at the motion of the Earth relative to the Sun. We can do this because we assume that the mass of the Sun is much larger than the mass of the Earth. The gravitiational force consits of a x and y component $F_{G,x}$ and F_{Gy}, we then use Newton's second law and obtain:

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{F_{G,y}}{M_E} \tag{2}$$

We use that the average distance bewteen the Earth and the Sun is $1.5 \cdot 10^{11}$ m and we call this one astronomical unit (1 AU). We use the masses of the different planets including the Sun that are given in the description of Project 5.

We introduce $x = r\ \cos(\theta), y = r\sin(\theta)$ and $r = \sqrt{(}x^2 + y^2)$. We can then rewrite

$$F_x = -\frac{GM_\odot M_E}{r^2}cos(\theta) = -\frac{GM_\odot M_E}{r^3}x$$

and

$$F_y = -\frac{GM_\odot M_E}{r^2}sin(\theta) = -\frac{GM_\odot M_E}{r^3}y$$

for the x and y direction of the gravitational force. These to equations can again be rewritten so we obtain four first-order coupled differential equations

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x, \qquad \frac{dx}{dt} = v_x,$$
$$\frac{dv_y}{dt} = -\frac{GM_\odot}{r^3}y, \qquad \frac{dy}{dt} = v_y$$

By introducing astronomical units, described above(r = 1AU) and with the equation for circular motion, with $a = v^2/r$ we can set F=ma equal to the expression for the gravitational force.

$$\frac{M_E v^2}{r} = F = \frac{GM_\odot M_E}{r^2}$$

We then solve for $v^2 r$, where v is the velocity of Earth and when we assume circular motion we have $v = 2/yr = 2/yr$

$$v^2 r = GM_\odot = 4\pi^2 \quad AU^3/(yr)^2 \tag{3}$$

where yr is short for years, which we use instead of seconds to decribe the motions in the solar system.

We can discretize the four coupled differential equations by using Euler's method with step length h.

$$v_{x,i+1} = v_{x,i} - h\frac{4\pi^2}{r_i^3}x_i, \qquad x_{i+1} = x_i + hv_{x,i},$$
$$v_{y,i+1} = v_{y,i} - h\frac{4\pi^2}{r_i^3}y_i, \qquad y_{i+1} = y_i + hv_{y,i}$$

2.1 The Verlet-velocity method

The Verlet method is a different numerical algorithm, where we again consider the second-order differential equation like Newton's second law, in one dimension this reads

$$m\frac{d^2x}{dt} = F(x,t)$$

which can be rewritten in terms of two coupled differential equations

$$\frac{dx}{dt} = v(x,t) \quad and \quad \frac{dv}{dt} = F(x,t)/m = a(x,t)$$

We perform a Taylor expansion of the discretized equations with step length h

$$x(t+h) = x(t) + hx^{(1)}(t) + \frac{h^2}{2}x^{(2)}(t) + O(h^3)$$

We know the second derivative from Newton's second law where $x^{(2)}(t) = a(x,t)$. By adding the Taylor expansion for $x(t-h)$ and by using the discretized expressions $x(t_i \pm h) = x_{i\pm1}$ and $x_i = x(t_i)$ we gets

$$x_{i+1} = 2x_i - x_{i-1} + h^2 x_i^{(2)} + O(h^4)$$

where the truncation error goes like $O(h^4)$. We can also compute the velocity which has a truncation error $O(h^2)$

$$x_i^{(2)} = \frac{x_{i+1} - x_{i-1}}{2h} + O(h^2) \tag{4}$$

The Taylor expansion for the velocity is given by

$$v_{i+1} = v_i + hv_i^{(1)} + \frac{h^2}{2}v_i^{(2)} + O(h^3)$$

With Newton's second law we have

$$v_i^{(1)} = \frac{d^2 x_i}{dt_i^2} = \frac{F(x_i, t_i)}{m}$$

We add this to the derivative of the velocity

$$v_{i+1}^{(1)} = v_i^{(1)} + hv_i^{(2)} + O(h^2)$$

Since our errors goes as $O(h^3)$ we obtain $hv_i^{(2)} \approx v_{i+1}^{(1)} - v_{(1)}$. We rewrite the final equations for the position and the velocity and obtain

$$x_{i+1} = x_i + hv_i + \frac{h^2}{2}v_i^{(2)} + O(h^3)$$

and

$$v_{i+1} = v_i + \frac{h}{2}\left(v_{i+1}^{(1)} + v_i^{(1)}\right) + O(h^3)$$

2.2 Adding more planets to the binary system

For a system with more than one planet we need to add the magnitude of the force between Earth and the other planet, for example jupiter which is the most massive planet in the solar system. This force is given by

$$F_{Eart-Jupiter} = \frac{GM_{Jupiter}M_E}{r^2_{Earth-Jupiter}}$$

where $M_{Jupiter}$ is the mass of Jupiter and $r_{Earth-Jupiter}$ is the distance between Earth and Jupiter. To extend the model for all the planets in the solar system we add the other planets in the same way and choose the initial positions and velocities for all the planets from NASA.

2.3 Perihelion of Mercury

The perihelion of a planet is the point where the planet in orbit around the Sun is closest to the Sun. The general theory of relativity explains the anomalous perihelion precession of the planet Mercury. The observed value for the perihelion precession when the classical effects are subtracted is 43"(43 arc seconds) per century. The behavior of the orbit around the sun will not be the same for each round, this means that the perihelion will slowly precess around the Sun and we would need to add a general relativistic correction to the Newton gravitational force. The force then becomes:

$$F_G = \frac{GM_{Sun}M_{Mercury}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right] \tag{5}$$

where $M_{Mercury}$ is the mass of Mercury and r is the distance bewteen the Sun and Mercury. The magnitude of the angular momentum is given by $l = |\vec{r}X\vec{v}|$ and c is the speed of light in vacuum. We obtain the perihelion angle $_p$ from the positions when Mercury is at its closest to the Sun, i.e the perihelion positions, x_p and y_p. θ_p is then given by

$$tan\theta_p = \frac{y_p}{x_p} \tag{6}$$

2.4 Object orientation

We make a system which contains different classes, including the planet class which is a own class in it self

3 Results

We have solved equation 3 using both Euler's forward algorithm and the velocity Verlet method. We saw that Euler's algorithm requiers a lot more calculation steps to become accurate than the Verlet algorithm. Euler's algorithm requires $5 \cdot 10^6$ steps while the Verlet algorithm requires only 500 steps. If we run our code with 100 000 integration points Euler takes 23.8 ms and uses 17 N flops while Verlet uses 35.7 ms with 38 N flops. We have therefore used the velocity Verlet method to calculate and plot the position, the energy and the angular momentum of the Earth. The results can be found in Figure 1, 2 and 3. The plot of the energy(Figure2) is obtained by a sum of the potential and the kinetic energy. From these figures we can see that the energy and the angular momentum is conserved.

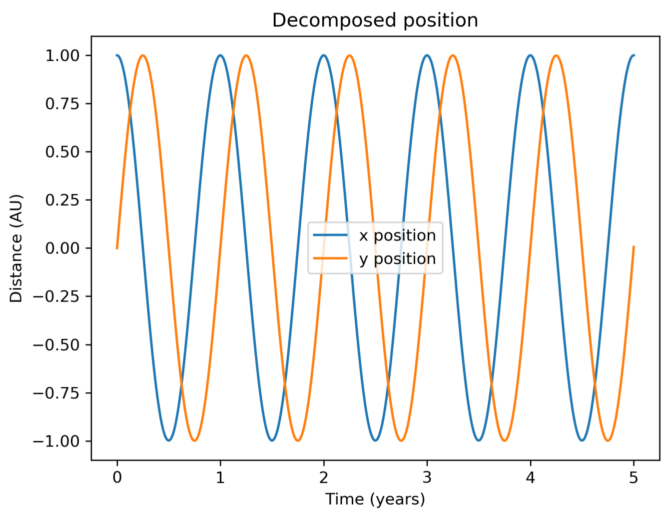


Figure 1: x and y position of the Earth relative to the Sun.

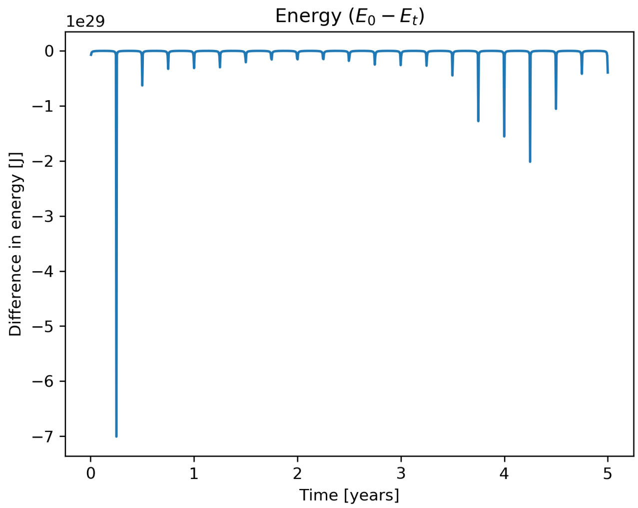


Figure 2: Energy of the Earth plottet against time.

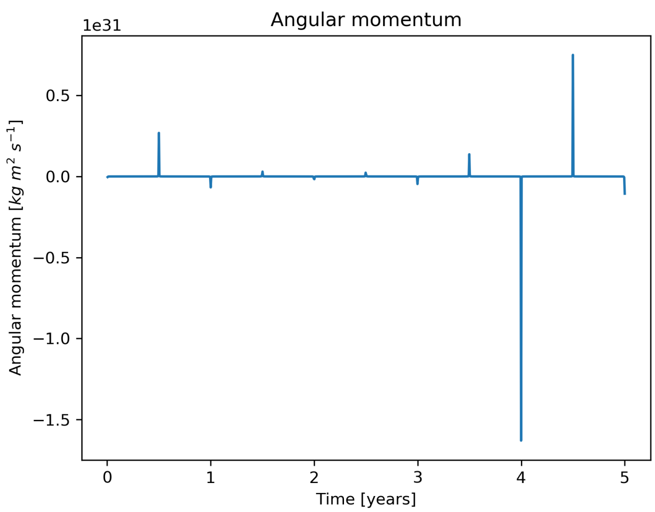


Figure 3: Angular Momentum

3.1 Initial velocity for circular orbit

The initial velocity needs to be minimum 2π AU/(yr) to obtain a circular orbit of the Earth around the Sun. The plot of the Earth's orbit around the Sun is shown in Figure 4

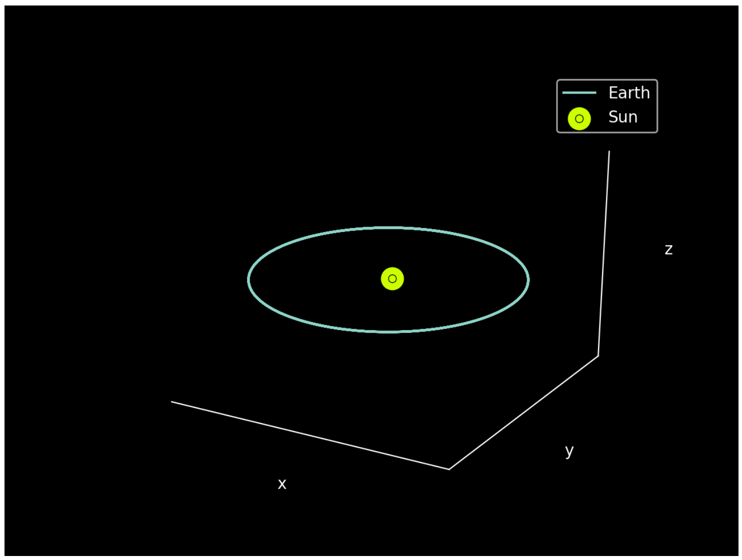


Figure 4: Plot of the orbit of the Earth around the Sun.

3.2 Escape velocity

We then consider a planet which begins at a distance 1 AU from the Sun, the inital velocity at when the planets begin to escape from the Sun is at about 2.82π AU/(yr), this means that at this inital velocity the planets orbits is no longer circular. Figure 5 shows the last orbit with this stable initial velocity before the Earth escapes from the Sun.

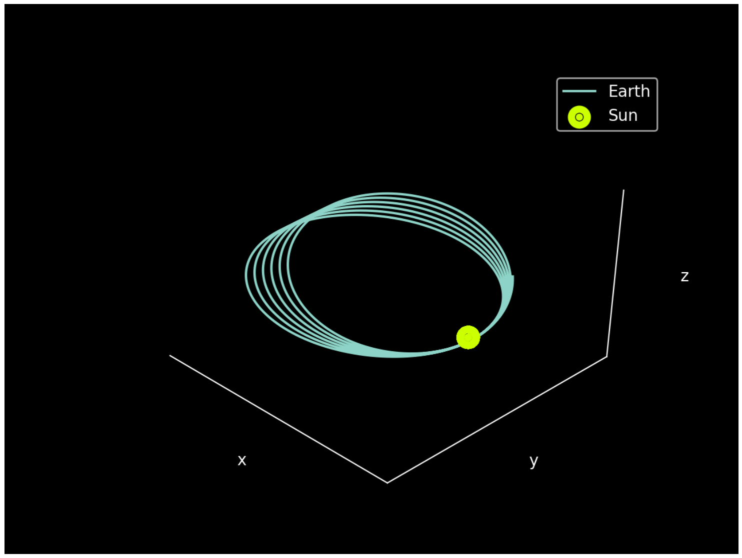


Figure 5: Plot of Earth's orbit around the sun before it escapes the sun.

3.3 Changing the gravitational force

If we try to replace the gravitational force from Equation 1 with

$$F_G = \frac{GM_\odot M_E}{r^\beta}$$

with $\beta \in [2, 3]$. The results is shown in Figure 6. These figures shows that when β approches 3 the Earth begins to escape the sun.

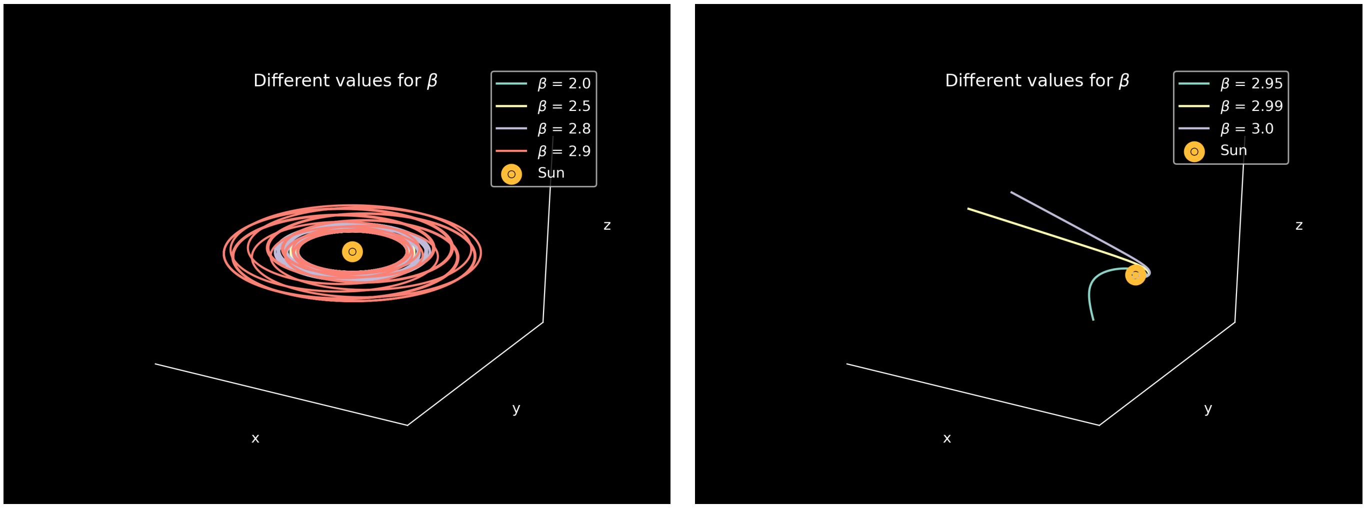


Figure 6: The Earth-Sun system for different values of β.

3.4 Adding Jupiter to the system

We then add Jupiter to our system and we evaluate how it affects our system when we increase Jupiters mass. The result when Jupiter has it's own mass is shown in Figure 7. When we increase the mass with a factor of 10 the result does not differ from what we see in Figure 7. Figure 8 shows what will happen when we increase the mass of Jupiter with a factor of respectively 950 and 1000. These figures shows that when the mass of Jupiter gets large it will attract the Earth away from its normal orbit around the Sun. When we increase the mass with a factor of 1000 the Earth will escape the sun because the repulsive force bewteen Jupiter and the Earth become really big. When Jupiter's mass is increased with a factor of 1000 we needed about 20 000 steps for the system to be stable for 20 years.

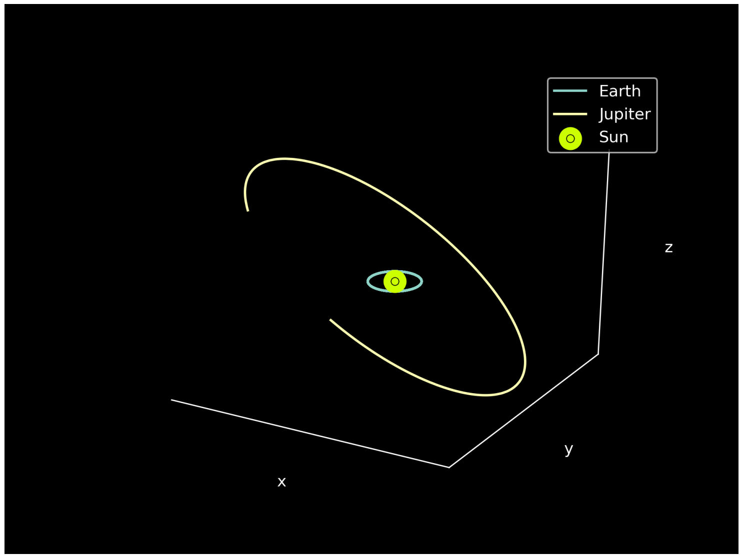
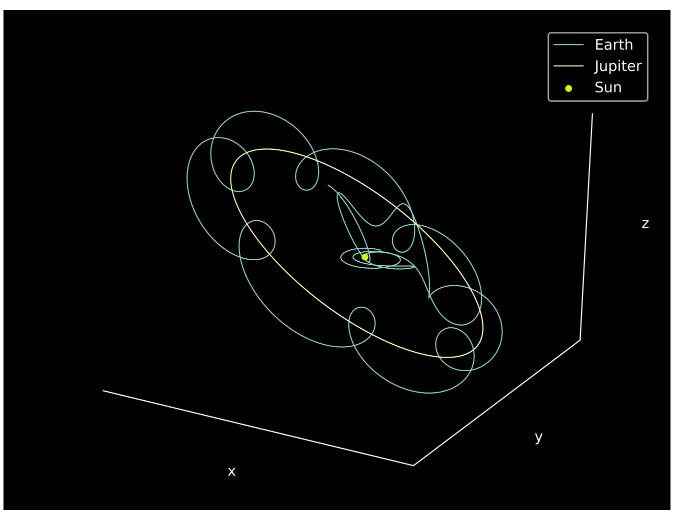
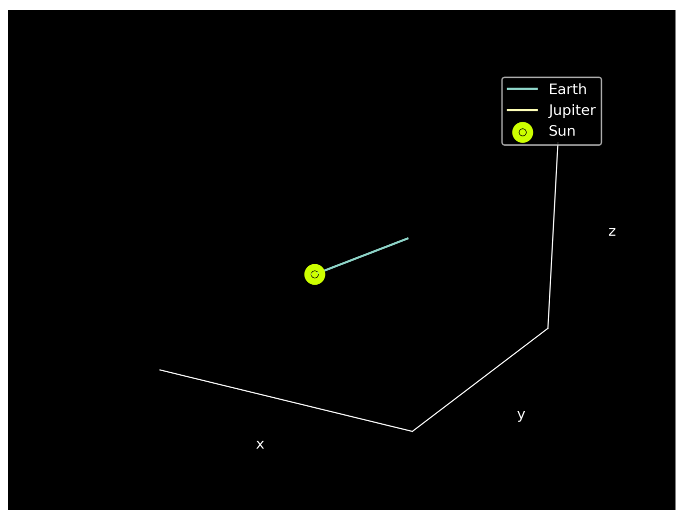


Figure 7: Solar system including Earth and Jupiter.



(a) Mass of Jupiter increased with a factor of 950



(b) Mass off Jupiter increased with a factor of 1000

Figure 8: Solar system including Earth and Jupiter, with the mass of Jupiter increased.

3.5 Finished solar system

Further we have completed the whole solar system by adding more planets, we have plottet this with 500 000 integration points and over 164 years. The results is represented in Figure 9

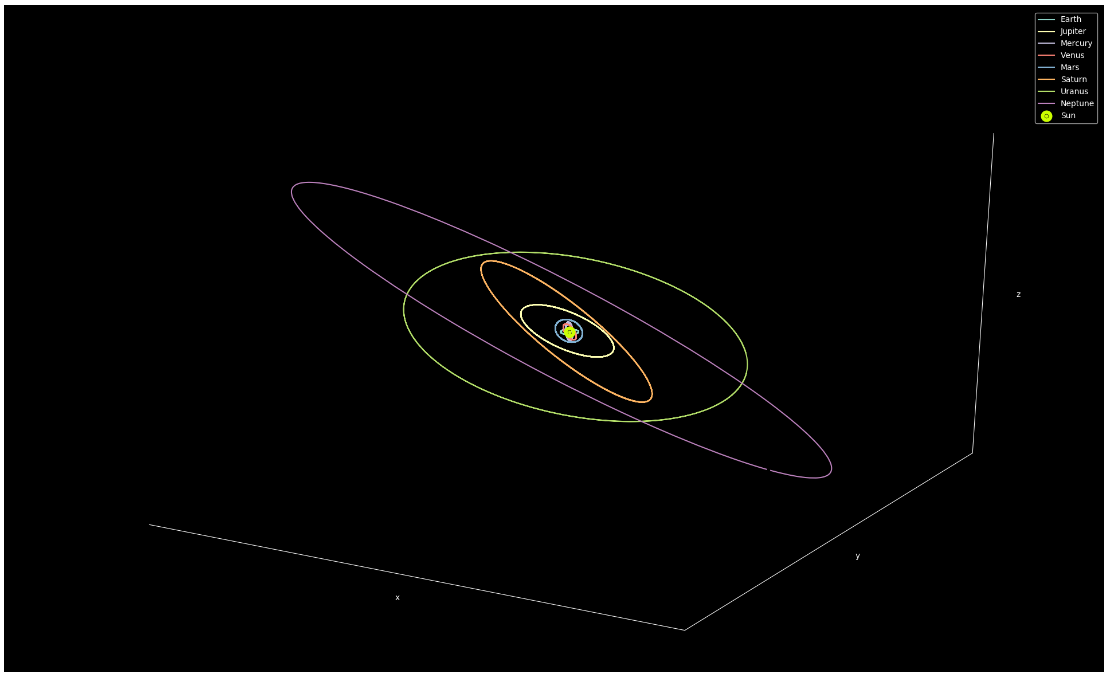


Figure 9: Plot of the solar system.

3.6 Perihelion of Mercury

The speed of Mercury at perihelion is 12.44 AU/yr, and the distance to the Sun at perihelion is 0.3075 AU. With the realtivistic constant from Equation 5 we get the results shown in Table 1.

Table 1: θ_p of Mercury when the distance is 0.3075 AU

	Numerical angle	Real angle
Relativistic	0.02524	0.01194
Newtonian	0.01426	0

4 Discussion

We have seen that the velocity Verlet method is more efficient for solving the coupled differential equations. The plot of the Earth's x and y position relative to the Sun follows a respectively a cosine and a sinus curve which shows that the Earth moves in a circular orbit around the Sun.

We also see from our results that the kinetic and potential energy as well as the angular momentum is conserved. These properties is consereved because if they were not conserved the Earth would not be in orbit around the Sun.

We saw from our numerical results that the inital value for the Earth to have a circular orbit around the Sun needs to be 2π. We can also see this analyticaly with the acceleration in the circular motion equal to v^2/r.

$$
\begin{aligned}
\frac{mv^2}{r} &= \frac{GM_\odot M_E}{r^2} \\
v^2 &= \frac{GM_\odot}{r} \\
v &= \sqrt{\frac{4\pi^2}{1}} = 2\pi
\end{aligned}
$$

We found that the inital escape velocity was between 2.82π and 2.83π AU/yr. This numerical value is quite accurate compared to the exact value. We can calculate this escape velocity exact by considering the work needed to move a planet over a small distance dr against the attractive force the planet feels. This work is given by:

$$
\begin{aligned}
dW &= F_G dr = -\frac{GM_\odot M_E}{r^2} \\
W &= \int_r {}_0^\infty - G\frac{M_\odot M_E}{r^2} = -G\frac{M_\odot M_E}{r_0}
\end{aligned}
$$

This gives the minimal kinetic energy to be able to reach infinity, therefore the escape velocity v_0 satisfies

$$
\begin{aligned}
W + K = 0 \rightarrow \frac{1}{2}M_E v_0^2 &= G\frac{M_\odot M}{r_0} \\
v_0 &= \sqrt{\frac{2GM_\odot}{r_0}} = \sqrt{2 \cdot 4\pi^2} AU/yr = 2.828\pi AU/yr
\end{aligned}
$$

So we see that the difference in our estimated escape velcoity and the analytical one is very small. From these results we observe that if the Earth would have a higher velocity around the Sun than it does we would be travelling away from the Sun, and we dont think that would have been appreciated here on the Earth.

Further we examined how a change in the gravitational force would affect the Earth orbit around the Sun. We saw that when β in Equation **??** approches 3 the Earth starts to escape the Sun. This would have been the case if the gravitational force worked different, so we are lucky since the gravitational force keeps us in orbit around the Sun.

When we added Jupiter to the system and examined what would happen if we increased Jupiter's mass to be approximately the mass of the Sun. We saw that when Jupiter's mass was increased with a factor of 950 the system would be stable up to 20 years, and we can see from the plot that the Earth starts to escape the system after 20 years. When Jupiter's mass gets this large our results show that it will affect the Earth's orbit becuse the Earth will be attracted to Jupiter. When we increase the mass with a factor of 1000 we see that the Earth will immediatley escape the system, while Jupiter will stay in its orbit around the Sun.

When we simmulate all the planets inn the Solar system we use the initial-values from NASA, this means realistic initalvalues. Our model shows that the Earth still goes in a circular orbit around the Sun, which indicates that our model gives a good representation of the real

5 Conclusion

6 Bibliography

Link to our GitHub repository

Lecture Slides on Differential Equations, Hjorth-Jensen Morten

Escape velocity from Wikipedia