

Project 1 - FYS3150

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Project 1a)

$$\begin{aligned} -\frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} &= f_i \\ -(u_{i+1} + u_{i-1} - 2u_i) &= f_i h^2 \\ 2u_i - u_{i+1} - u_{i-1} &= f_i h^2 \end{aligned}$$

This expands to

$$\begin{aligned} 2u_1 - u_0 - u_2 &= f_1 h^2 \\ 2u_2 - u_1 - u_3 &= f_2 h^2 \\ &\vdots \\ 2u_n - u_{n-1} - u_{n+1} &= f_n h^2 \end{aligned}$$

The boundary conditions give us $u_{n+1} = u(1) = 0$ and $u_0 = u(0) = 0$. We also introduce $f_i h^2 = g_i$. We can then write this expression as

$$\begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-1} \\ g_n \end{bmatrix}$$

Project 1b)

The forward substitution algorithm is as followed.

$$b_1 u_1 + c_1 u_2 = \tilde{g}_1 \tag{1}$$

$$a_1 u_1 + b_2 u_2 + c_2 u_3 = \tilde{g}_2 \tag{2}$$

$$a_2 u_2 + b_3 u_3 + c_3 u_4 = \tilde{g}_3 \tag{3}$$

$$\vdots$$

$$a_{n-1} u_{n-1} + b_n u_n = \tilde{g}_n \tag{4}$$

Multiplying equation (1) with $\frac{a_1}{b_1}$, which gives us.

$$a_1 u_1 + \frac{a_1 c_1}{b_1} u_2 = \tilde{g}_1 \frac{a_1}{b_1}$$

We then set equation (2) minus equation (1)

$$\begin{aligned} a_1 u_1 - a_1 u_1 + b_2 u_2 - \frac{a_1 c_1}{b_1} u_2 + c_2 u_3 &= g_2 - g_1 \frac{a_1}{b_1} \\ \left(b_2 - \frac{a_1 c_1}{b_1} \right) u_2 + c_2 u_3 &= g_2 - g_1 \frac{a_1}{b_1} \\ \tilde{b}_2 u_2 + c_2 u_3 &= \tilde{g}_2 \end{aligned}$$

The general expressions is

$$\tilde{b}_i = b_i - \frac{c_{i-1} a_{i-1}}{\tilde{b}_{i-1}}, \quad \tilde{g}_i = g_i - g_{i-1} \frac{a_{i-1}}{\tilde{b}_{i-1}}$$

Where $\tilde{b}_1 = b_1$ and $\tilde{g}_1 = g_1$

We can then use this to compute the vector \hat{u} . This has the general solution

$$u_i = \frac{\tilde{g}_i - a_i u_{i+a}}{\tilde{b}_i}$$