Project 1 - FYS3150

Emil Helland Broll, Benedicte Allum Pedersen,
Fredrik Oftedal Forr

Introduction

Poisson's equation is a classical equation from electromagnetism, in three dimensions the equation is:

$$\nabla^2 \Phi = -4\pi\rho(\mathbf{r}).$$

where Φ is the electrostatic potential generated by a localized charge distribution $\rho(\mathbf{r})$. If Φ and $\rho(\mathbf{r})$ are spherical symmetrical, and we do the substitution $\Phi(r) = \phi(r)/r$, the equation simplifies to:

$$\frac{d^2\phi}{dr^2} = -4\pi r\rho(r).$$

If we let $f = -4\pi r\rho(\mathbf{r})$, and by let $\phi \to u$ and $r \to x$, the general one-dimensional Poisson equation will read:

$$-u''(x) = f(x).$$

Project 1a)

We have solved the one-dimensional Poisson equation with Dirichlet boundary conditions and by rewriting it as a set of linear equations.
We let the discretized approximation to u be defined as v_i. The second derivative of u is then defined as:

$$-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i$$

where h is the step length and is defined as $h = 1/(n+1)$ and where $f_i = f(x_i)$. We can rewrite this equation to a set of linear equations like this:

$$\begin{aligned}
-\frac{v_{i+1} + v_{i-1} - 2v_i}{h^2} &= f_i \\
-(v_{i+1} + v_{i-1} - 2v_i) &= f_i h^2 \\
2v_i - v_{i+1} - v_{i-1} &= f_i h^2
\end{aligned}$$

Wich expands to

$$\begin{aligned}
2v_1 - v_0 - v_2 &= f_1 h^2 \\
2v_2 - v_1 - v_2 &= f_2 h^2 \\
&\vdots \\
2v_n - v_{n-1} - v_n + 1 &= f_3 h^2
\end{aligned}$$

The boundary conditions give us $v_{n+1} = u(1) = 0$ and $v_0 = u(0) = 0$. We also introduce $f_i h^2 = g_i$. We can then write this expression as

$$\begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-1} \\ g_n \end{bmatrix}$$

Project 1b)

We rewrite our matrix A in terms of one-dimensional vectors $a,\ b,\ c$ of length $1 : n$;

$$\mathbf{A} = \begin{bmatrix} b_1 & c_1 & 0 & \dots & \dots & 0 \\ a_1 & b_2 & c_2 & \dots & \dots & 0 \\ 0 & a_2 & b_3 & c_3 & \dots\dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & 0 & \dots & \dots & a_{n-1} & b_n \end{bmatrix}$$

The algorithm for the forward substitution will then be as followed.

$$b_1 v_1 + c_1 v_2 = \tilde{g}_1 \tag{1}$$
$$a_1 v_1 + b_2 v_2 + c_2 v_3 = \tilde{g}_2 \tag{2}$$
$$a_2 v_2 + b_3 v_3 + c_3 v_4 = \tilde{g}_3 \tag{3}$$
$$\vdots$$
$$a_{n-1} v_{n-1} + a_n v_n = \tilde{g}_n \tag{4}$$

Multiplying equation (1) with $\frac{a_1}{b_1}$, wich gives us.

$$a_1 v_1 + \tfrac{a_1 c_1}{b_1} v_2 = \tilde{g}_1 \tfrac{a_1}{b_1}$$

We then set equation (2) minus equation (1)

$$\begin{aligned} a_1 v_1 - a_1 v_1 + b_2 v_2 - \frac{a_1 c_1}{b_1} v_2 + c_2 v_3 &= g_2 - g_1 \frac{a_1}{b_1} \\ \left(b_2 - \frac{a_1 c_1}{b_1} \right) v_2 + c_2 v_3 &= g_2 - g_1 \frac{a_1}{b_1} \\ \tilde{b}_2 v_2 + c_2 v_3 &= \tilde{g}_2 \end{aligned}$$

The general expressions is

$$\tilde{b}_i = b_i - \frac{c_{i-1}a_{i-1}}{\tilde{b}_{i-1}}, \qquad \tilde{g}_i = g_i - g_{i-1}\frac{a_{i-1}}{\tilde{b}_{i-1}}$$

Where $\tilde{b}_1 = b_1$ and $\tilde{g}_1 = g_1$

We can then use this to compute the vector $\hat{u}$. This has the general solution

$$v_i = \frac{\tilde{g}_i - a_i v_{i+a}}{\tilde{b}_i}$$

We have made a code for the algortihm and solved the problem for matrices of the size 10 x 10, 100 x 100 and 1000 x 1000. To reduce the problem and to save memory we only use the vectors a, b and c, since the rest of the matrix only consist of zeros. Then the number of floating points are $O(9n)$.

Project 1c)

Our matrix now have identical elements along the diagonal and identical values for the non diagonal elements, it will look like this:

$$\begin{bmatrix} b & a & 0 & \dots & 0 & 0 \\ a & b & a & \dots & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b & a \\ 0 & 0 & 0 & \dots & a & b \end{bmatrix}$$

We develope the algortihm for the forward substitution in the same way as earlier, so the algorithm will be as followed. Here $\tilde{b}_1 = b = 2$ and the algorithm will run with i starting from 2 and going up to n.

$$\tilde{b_i} = b - \frac{a^2}{b + (i-2)} = 2 - \frac{a^2}{i} \quad , \qquad \tilde{g_i} = g_i - \frac{a}{\tilde{b}_{i-1}} g_{i-1}$$

where $\tilde{b}_1 = b$ and $\tilde{g}_1 = g$.

Likewise the algorithm for the bacward substitution will be:

$$u_i = \frac{\tilde{g}_i - a}{\tilde{b}_i} v_{i+1},$$