Mandatory Assignment 1
mat2410 - Introduction to Complex analysis

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Exercise 1

a)

Start by writing the formula to cartesian form

$$\frac{1+i}{\sqrt{3}+i} = \frac{(1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{\sqrt{3}+1+i(\sqrt{3}-1)}{4}$$

Rewrite to polar cordinates:

$$\begin{aligned} r &= \sqrt{\left(\frac{\sqrt{3}+1}{4}\right)^2 + \left(\frac{\sqrt{3}-1}{4}\right)^2} \\ &= \sqrt{\frac{3+2\sqrt{3}+1+3-2\sqrt{3}+1}{16}} \\ &= \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} \theta &= \cos^{-1}\left(\frac{\frac{\sqrt{3}+1}{4}}{\frac{\sqrt{2}}{2}}\right) = \cos^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \\ \theta &= \sin^{-1}\left(\frac{\frac{\sqrt{3}-1}{4}}{\frac{\sqrt{2}}{2}}\right) = \sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \\ \theta &= \frac{\pi}{12} \end{aligned}$$

the roots are :

$$z^6 = \frac{\sqrt{2}}{2}e^{i\left(\frac{\pi}{12}+2\pi m\right)} \implies z = \frac{\sqrt[12]{2^{11}}}{2}e^{i\left(\frac{\pi}{72}+\frac{\pi m}{3}\right)}, \qquad m = 0,1,2,3,4,5$$

b)

Can not use the same strategy as before because θ is not exact. But what we can do is complete the square and get:

$$(7+24i)^{\frac{1}{2}} = (7+9-9+24i)^{\frac{1}{2}} = (16-9+24i)^{\frac{1}{2}} = \left(16+9i^2+24i\right)^{\frac{1}{2}} = (4+3i)$$

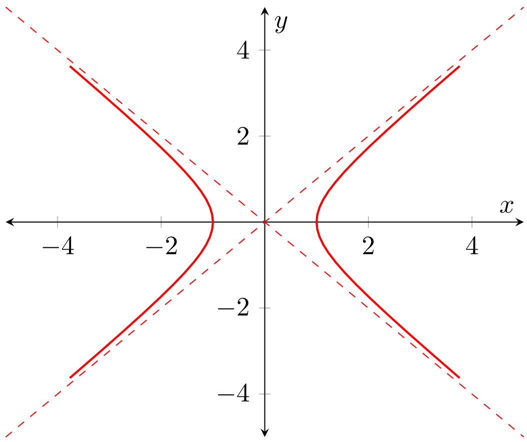
The other solution is rotated π around the unit circle and the solutions are thus:

$$4+3i, \quad -4-3i$$

c)

$$\{z \mid z^2+\overline{z}^2=2\}=\{z \mid x^2+2ixy-y^2+x^2-2ixy-y^2=2\}=\{z \mid x^2-y^2=1\}$$

Is the formula of a hyperbola and gives the following plot.



d)

Insert into the bc-formula and get:

$$z=\frac{1+i}{2}\pm\sqrt{\left(\frac{1+i}{2}\right)^2-\frac{i}{4}}=\frac{1+i}{2}\pm\sqrt{\frac{i}{2}-\frac{i}{4}}=\frac{1+i}{2}\pm\sqrt{\frac{i}{4}}$$

Rewrite the terms separately to polar coordinates

$$=\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}\pm\sqrt{\frac{1}{4}e^{i\frac{\pi}{2}}}=\frac{\sqrt{2}}{2}e^{i\frac{\pi}{4}}\pm\frac{1}{2}e^{i\frac{\pi}{4}}=\frac{e^{i\frac{\pi}{4}}}{2}(\sqrt{2}\pm 1)$$

e)

$|f(z_1)-f(z_2)|=|\alpha z_1+\beta-\alpha z_2-\beta|=|\alpha(z_1-z_2)|=|\alpha||z_1-z_2|$
From this we can easily see that if $|\alpha|=1$ then $|f(z_1)-f(z_2)|=|z_1-z_2|$

Exercise 2

a)

Show that if U and V are convex then U $\cap$ V is convex.

Let U and V be convex sets. And let z_1, z_2 be two arbitary points in U $\cap$ V. Need to show that there exists a straight line from z_1 to z_2 that lies in U $\cap$ V.

But since $z_1, z_2 \in$ U $\cap$ V then $z_1, z_2 \in$ U, $z_1, z_2 \in$ V and since U and V are convex sets, that is there is a straight line between the points where the line itself lies in the set, i.e. for $t \in [0, 1]$:

$$tz_1 + (1-t)z_2 \in \mathrm{U}, \quad tz_1 + (1-t)z_2 \in \mathrm{V}$$

This implies that

$$tz_1 + (1-t)z_2 \in \mathrm{U} \cap \mathrm{V}$$

And we have that U $\cap$ V is convex.

$\square$

b)

Show that if U is convex then U $\cup \partial$U is convex.

Let z_1, z_2 bet two arbitary points in U$\cup \partial$U and U a convex set. Then there exists two sequences $\{w_n\}, \{v_n\} \subset$ U that converge respectly to z_1 and z_2.

And since $\{w_n\}, \{v_n\} \subset$ U we know that for every i, j there is a straight line between w_i and v_j. i.e.

$$tw_i + (1-t)v_j \in \mathrm{U}$$

And as $\{w_n\} \to r_1$ and $\{v_n\} \to r_2$

$$tr_1 + (1-t)r_2 \in \mathrm{U} \cup \partial \mathrm{U}$$

as $n \to \infty$.

Thereby we have that U$\cup \partial$U is a convex set

Exercise 3

$$\langle z, w \rangle = \mathrm{Re}(z\overline{w}), \quad z = a + ib,\ w = x + iy$$

a)

Show that $|z| = \langle r, r \rangle^{1/2}$

$$\langle r, r \rangle^{1/2} = \mathrm{Re}(z\overline{z})^{1/2} = \mathrm{Re}((a+ib)(a-ib))^{1/2} = \mathrm{Re}(a^2+b^2)^{1/2} = (a^2+b^2)^{1/2} = |z|$$

$\square$

Show the Cauchy-Schwartz ineqiuality i.e.
Show that $|\langle z, w \rangle| \leq |z||w|$

Uses polar coordinates

$$\begin{aligned}|\langle z, w \rangle| &= |\mathrm{Re}(z\overline{w})| = |\mathrm{Re}(r_z e^{i\mathrm{arg(z)}} r_w e^{-i\mathrm{arg(w)}})| = |\mathrm{Re}(r_z r_w e^{i(\mathrm{arg(z)\text{- } arg(w)})})| = |r_z r_w \cos(\mathrm{arg(w)\text{ - }arg(w)})| \\ &\leq |r_z r_w| = |r_z||r_w| = ||z||||w|| = |z||w|\end{aligned}$$

$\square$

Show the triangle inequality i.e.
Show $|z+w| \leq |z| + |w|$

$$\begin{aligned}|z+w| &= \langle z+w, z+w \rangle^{1/2} = \mathrm{Re}((z+w)\overline{(z+w)})^{1/2} = \mathrm{Re}((z+w)(\overline{z}+\overline{w}))^{1/2} \\ &= \mathrm{Re}(z\overline{z} + w\overline{w} + z\overline{w} + \overline{z}w)^{1/2} \\ &= \mathrm{Re}(r_z^2 e^{i\mathrm{arg(z)}-i\mathrm{arg(z)}} + r_w^2 e^{i\mathrm{arg(w)}-i\mathrm{arg(w)}} + r_z r_w e^{i(\mathrm{arg(z)\text{ - }arg(w)})} + r_z r_w e^{-i(\mathrm{arg(z)\text{-}arg(w)})})^{1/2} \\ &= \mathrm{Re}(r_z^2 + r_w^2 + 2r_z r_w \cos(\mathrm{arg(z)\text{ - }arg(w)}))^{1/2} \leq (r_w^2 + r_z^2 + 2r_w r_z)^{1/2} = ((r_w + r_z)^2)^{1/2} = |z| + |w|\end{aligned}$$

$\square$

b)

Need to show the implication in both directions:

$\Rightarrow$ Show that if $\langle z, w \rangle = 0$ then z/w is pure imaginary:

That $\langle z, w \rangle = 0$ implies that $\mathrm{Re}(z\overline{w}) = 0$ wich means $\mathrm{Re}(z)\,\mathrm{Re}(w) + \mathrm{Im}(z)\,\mathrm{Im}(w) = 0$

$$\begin{aligned}
z/w &= \frac{\operatorname{Re}(z) + i\operatorname{Im}(z)}{\operatorname{Re}(w) + i\operatorname{Im}(w)} = \frac{(\operatorname{Re}(z) + i\operatorname{Im}(z))(\operatorname{Re}(w) - i\operatorname{Im}(w))}{\operatorname{Re}(w)^2 + Im(w)^2} \\
&= \frac{(\operatorname{Re}(z)\operatorname{Re}(w) + \operatorname{Im}(z)\operatorname{Im}(w) + i(\operatorname{Re}(w)\operatorname{Im}(z) - i\operatorname{Im}(w)\operatorname{Re}(z))}{\operatorname{Re}(w)^2 + \operatorname{Im}(w)^2} \\
&= \frac{i(\operatorname{Re}(w)\operatorname{Im}(z) - i\operatorname{Im}(w)\operatorname{Re}(z))}{\operatorname{Re}(w)^2 + \operatorname{Im}(w)^2}
\end{aligned}$$

And we have that z/w has no real part

$\Leftarrow$ Show that if z/w is pure imaginary then $\langle z, w\rangle = 0$

That z/w is pure imaginary means that $\operatorname{Re}(z)\operatorname{Re}(w) + \operatorname{Im}(z)\operatorname{Im}(w) = 0$

$$\langle z, w\rangle = \operatorname{Re}(z\overline{w}) = \operatorname{Re}(z)\operatorname{Re}(w) + \operatorname{Im}(z)\operatorname{Im}(w)$$

Therfore $\langle z, w\rangle$ must be equal to 0 if z/w is pure imaginary.

And since both implications hold, we know the statement to be true

$\square$

Exercise 4

a)

$$\lim_{n\to\infty} \left(\frac{i}{2}\right)^n = \lim_{n\to\infty} \left(\frac{e^{i\pi/2}}{2}\right)^n = \lim_{n\to\infty} \frac{e^{in\pi/2}}{2^n} = 0, \qquad \text{Since } e^{in\pi/2} \text{ is periodic}$$

b)

$$\lim_{z\to\infty} \frac{z^2+1}{z+i} = \lim_{z\to\infty} \frac{(z+i)(z-i)}{(z+i)} = \lim_{z\to\infty} z - i = \infty$$

c)

$$\lim_{n\to\infty} \left(1 + \frac{i}{n}\right)^{n\pi} = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{i}{n}\right)^{n\pi}\right)} = \lim_{n\to\infty} e^{n\pi\cdot\ln\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} e^{\frac{\pi\ln\left(1+\frac{i}{n}\right)}{1/n}} = \lim_{n\to\infty} e^{\pi\frac{\frac{\frac{-i}{n^2}}{1+\frac{i}{n}}}{\frac{-1}{n^2}}} = \lim_{n\to\infty} e^{\pi\frac{i}{1+\frac{i}{n}}} = e^{i\pi}$$

Exercise 5

a)

i.

$$\frac{\partial f}{\partial z} = \frac{\partial z^2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}(x^2+2ixy-y^2) - i\frac{\partial}{\partial y}(x^2+2ixy-y^2)\right) = \frac{1}{2}\left(\frac{\partial x^2-y^2}{\partial x} + i\frac{\partial 2xy}{\partial x} - i\frac{\partial x^2-y^2}{\partial y} + \frac{\partial 2xy}{\partial y}\right)$$
$$= \frac{1}{2}(2x+i2y+i2y+2x) = 2x+2iy = 2z$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial z^2}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(x^2+2ixy-y^2) + i\frac{\partial}{\partial y}(x^2+2ixy-y^2)\right) = \frac{1}{2}\left(\frac{\partial x^2-y^2}{\partial x} + i\frac{\partial 2xy}{\partial x} + i\frac{\partial x^2-y^2}{\partial y} - \frac{\partial 2xy}{\partial y}\right)$$
$$= \frac{1}{2}(2x+i2y-i2y-2x) = 0$$

ii.

$$\frac{\partial f}{\partial z} = \frac{\partial e^z}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}(e^{x+iy}) - i\frac{\partial}{\partial y}(e^{x+iy})\right) = \frac{1}{2}\left(\frac{\partial e^x\cos(y)}{\partial x} + i\frac{e^x\sin(y)}{\partial x} - i\frac{\partial e^x\cos(y)}{\partial y} + \frac{e^x\sin(y)}{\partial y}\right)$$
$$= \frac{1}{2}\left(e^x\cos(y)+ie^x\sin(y)+ie^x\sin(y)+e^x\cos(y)\right) = e^x\cos(y)+ie^x\sin(y) = e^x(e^{iy}) = e^{x+iy} = e^z$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial e^z}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(e^{x+iy}) + i\frac{\partial}{\partial y}(e^{x+iy})\right) = \frac{1}{2}\left(\frac{\partial e^x\cos(y)}{\partial x} + i\frac{e^x\sin(y)}{\partial x} + i\frac{\partial e^x\cos(y)}{\partial y} - \frac{e^x\sin(y)}{\partial y}\right)$$
$$= \frac{1}{2}\left(e^x\cos(y)+ie^x\sin(y)-ie^x\sin(y)-e^x\cos(y)\right) = 0$$

iii.

$$\frac{\partial f}{\partial z} = \frac{\partial |z|^2}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}(x^2+y^2) - i\frac{\partial}{\partial y}(x^2+y^2)\right) = x-iy = \overline{z}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial |z|^2}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(x^2+y^2) + i\frac{\partial}{\partial y}(x^2+y^2)\right) = x+iy = z$$

iv.

$$\frac{\partial f}{\partial z} = \frac{\partial\,\mathrm{Im}(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}(y) - i\frac{\partial}{\partial y}(y)\right) = -i$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial\,\mathrm{Im}(z)}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}(y) + i\frac{\partial}{\partial y}(y)\right) = i$$

b)

i

Have that:

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \frac{\partial \overline{f}}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}$$

By summing the formulas we get

$$\frac{\partial f}{\partial x} + \frac{\partial \overline{f}}{x} = 2\frac{\partial u}{\partial x} \implies \frac{\partial u}{\partial x} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial \overline{f}}{\partial x}\right)$$

Have that:

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \qquad \frac{\partial \overline{f}}{\partial y} = \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y}$$

By summing the formulas we get

$$\frac{\partial f}{\partial y} + \frac{\partial \overline{f}}{\partial y} = 2\frac{\partial u}{\partial y} \implies \frac{\partial u}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial y} + \frac{\partial \overline{f}}{\partial y}\right)$$

ii

By subtracting the formulas from above we get

$$\frac{\partial f}{\partial x} - \frac{\partial \overline{f}}{x} = 2i\frac{\partial v}{\partial x} \implies \frac{\partial v}{\partial x} = \frac{1}{2i}\left(\frac{\partial f}{\partial x} - \frac{\partial \overline{f}}{\partial x}\right)$$

$$\frac{\partial f}{\partial y} - \frac{\partial \overline{f}}{y} = 2i\frac{\partial v}{\partial y} \implies \frac{\partial v}{\partial y} = \frac{1}{2i}\left(\frac{\partial f}{\partial y} - \frac{\partial \overline{f}}{\partial y}\right)$$

iii

Have that:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Summing the formulas we get

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(2\frac{\partial f}{\partial x}\right) \implies \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}$$

Subtracting the formulas we get

$$\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(-2i\frac{\partial f}{\partial y}\right) \implies \frac{\partial f}{\partial y} = i\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \overline{z}}\right)$$

c)

Let $D(f) = \frac{\partial f}{\partial z}$ or $D(f) = \frac{\partial f}{\partial \bar{z}}$. Will show all the properties with $D(f) = \frac{\partial f}{\partial z}$ but the argument is the same for $D(f) = \frac{\partial f}{\partial \bar{z}}$ but with a change in signs.

Show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any complex constant α, β

$$\begin{aligned} D(\alpha f + \beta g) &= \frac{\partial(\alpha f + \beta g)}{\partial z} = \frac{1}{2}\left(\frac{\partial(\alpha f + \beta g)}{\partial x} - i\frac{\partial(\alpha f + \beta g)}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial(\alpha u_f + \beta u_g)}{\partial x} + i\frac{\partial(\alpha v_f + \beta v_g)}{\partial x} - i\frac{\partial(\alpha u_f + \beta u_g)}{\partial y} + \frac{\partial(\alpha v_f + \beta v_g)}{\partial y}\right) \end{aligned}$$

Since u, v are real valued functions we allready know the rule holds for them, and get:

$$\begin{aligned} &= \frac{1}{2}\left(\frac{\alpha\partial u_f + \beta\partial u_g}{\partial x} + i\frac{\alpha\partial v_f + \beta\partial v_g}{\partial x} - i\frac{\alpha\partial u_f + \beta\partial u_g}{\partial y} + \frac{\alpha\partial v_f + \beta\partial v_g}{\partial y}\right) \\ &= \frac{1}{2}\left(\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial g}{\partial x} - i\left(\alpha\frac{\partial f}{\partial y} + \beta\frac{\partial g}{\partial y}\right)\right) = \alpha\frac{\partial f}{\partial z} + \beta\frac{\partial g}{\partial z} = \alpha D(f) + \beta D(g) \end{aligned}$$

$\square$

Show that the Leibnitz rule holds:

$$\begin{aligned} D(fg) &= \frac{\partial(fg)}{\partial z} = \frac{1}{2}\left(\frac{\partial fg}{\partial x} - i\frac{\partial fg}{\partial y}\right) \\ &= \frac{1}{2}\left(\frac{\partial(u_f u_g - v_f v_g)}{\partial x} + i\frac{\partial(u_f v_g + u_g v_f)}{\partial x} - i\frac{\partial(u_f u_g - v_f v_g)}{\partial y} + \frac{\partial(u_f v_g + u_g v_f)}{\partial y}\right) \end{aligned}$$

Since u,v are real valued functions we know the Leibnitz rule holds

$$\begin{aligned} &= \frac{1}{2}\left(u_f\frac{\partial u_g}{\partial x} + u_g\frac{\partial u_f}{\partial x} - \left(v_f\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_f}{\partial x}\right) + (\cdots) + u_f\frac{\partial v_g}{\partial y} + v_g\frac{\partial u_f}{\partial y} + \left(v_f\frac{\partial u_g}{\partial y} + u_g\frac{\partial v_f}{\partial y}\right)\right) \\ &= \frac{1}{2}\left(u_f\frac{\partial g}{\partial x} + u_g\frac{\partial f}{\partial x} + i\left(v_f\frac{\partial g}{\partial x} + v_g\frac{\partial f}{\partial x}\right) - i\left(u_f\frac{\partial g}{\partial y} + u_g\frac{\partial f}{\partial y} + i\left(v_f\frac{\partial g}{\partial y} + v_g\frac{\partial f}{\partial y}\right)\right)\right) \\ &= u_f\frac{\partial g}{\partial z} + u_g\frac{\partial f}{\partial z} + i\left(v_f\frac{\partial g}{\partial z} + v_g\frac{\partial f}{\partial z}\right) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial z} = fD(g) + gD(f) \end{aligned}$$

$\square$

Show that the quotient rule holds:

$$D(\frac{f}{g}) = \frac{\partial(f/g)}{\partial z} = \frac{\partial f g^{-1}}{\partial z} = f\frac{\partial g^{-1}}{\partial z} + g^{-1}\frac{\partial f}{\partial z} = \frac{f}{2}\left(\frac{\partial \frac{u_g - iv_g}{u_g^2+v_g^2}}{\partial x} - i\frac{\partial \frac{u_g - iv_g}{u_g^2+v_g^2}}{\partial y}\right) + \frac{1}{g}\frac{\partial f}{\partial z}$$
$$= \frac{f}{2}\left(\frac{\partial \frac{u_g}{u_g^2+v_g^2}}{\partial x} - i\frac{\partial \frac{v_g}{u_g^2+v_g^2}}{\partial x} - i\left(\frac{\partial \frac{u_g}{u_g^2+v_g^2}}{\partial y} - i\frac{\partial \frac{v_g}{u_g^2+v_g^2}}{\partial y}\right)\right) + \frac{gD(f)}{g^2}$$

Know the quotient rule holds for real valued functions. Get:

$$= \frac{gD(f)}{g^2} + \frac{f}{2}\left(\frac{u_g^2+v_g^2}{(u_g^2+v_g^2)^2}\frac{\partial u_g}{\partial x} - \frac{u_g}{(u_g^2+v_g^2)^2}\frac{\partial u_g^2+v_g^2}{\partial x} - i\left(\frac{u_g^2+v_g^2}{(u_g^2+v_g^2)^2}\frac{\partial v_g}{\partial x} - \frac{v_g}{(u_g^2+v_g^2)^2}\frac{\partial u_g^2+v_g^2}{\partial x}\right)\right)$$
$$- i\frac{f}{2}\left(\frac{u_g^2+v_g^2}{(u_g^2+v_g^2)^2}\frac{\partial u_g}{\partial y} - \frac{u_g}{(u_g^2+v_g^2)^2}\frac{\partial u_g^2+v_g^2}{\partial y} - i\left(\frac{u_g^2+v_g^2}{(u_g^2+v_g^2)^2}\frac{\partial v_g}{\partial y} - \frac{v_g}{(u_g^2+v_g^2)^2}\frac{\partial u_g^2+v_g^2}{\partial y}\right)\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left((u_g^2+v_g^2)\frac{\partial u_g}{\partial x} + (-u_g+iv_g)\frac{\partial u_g^2+v_g^2}{\partial x} - i(u_g^2+v_g^2)\frac{\partial v_g}{\partial x}\right)$$
$$- i\frac{f}{2|g|^4}\left((u_g^2+v_g^2)\frac{\partial u_g}{\partial y} + (-u_g+iv_g)\frac{\partial u_g^2+v_g^2}{\partial y} - i(u_g^2+v_g^2)\frac{\partial v_g}{\partial y}\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left((u_g^2+v_g^2)\frac{\partial u_g}{\partial x} + (-u_g+iv_g)\left(2u_g\frac{\partial u_g}{\partial x} + 2v_g\frac{\partial v_g}{\partial x}\right) - i(u_g^2+v_g^2)\frac{\partial v_g}{\partial x}\right)$$
$$- i\frac{f}{2|g|^4}\left((u_g^2+v_g^2)\frac{\partial u_g}{\partial y} + (-u_g+iv_g)\left(2u_g\frac{\partial u_g}{\partial y} + 2v_g\frac{\partial v_g}{\partial y}\right) - i(u_g^2+v_g^2)\frac{\partial v_g}{\partial y}\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left(\frac{\partial u_g}{\partial x}(u_g^2+v_g^2-2u_g^2+2iu_gv_g) + i\frac{\partial v_g}{\partial x}(2iv_gu_g+2v_g^2-u_g^2-v_g^2)\right)$$
$$- i\frac{f}{2|g|^4}\left(\frac{\partial u_g}{\partial y}(u_g^2+v_g^2-2u_g^2+2iu_gv_g) + i\frac{\partial v_g}{\partial y}(2iv_gu_g+2v_g^2-u_g^2-v_g^2)\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left(\frac{\partial u_g}{\partial x}(v_g+iu_g)^2 + i\frac{\partial v_g}{\partial x}(v_g+iu_g)^2 - i\left(\frac{\partial u_g}{\partial y}(v_g+iu_g)^2 + i\frac{\partial v_g}{\partial y}(v_g+iu_g)^2\right)\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left(\frac{\partial g}{\partial x}(i(u_g-iv_g))^2 - i\frac{\partial g}{\partial y}(i(u_g-iv_g))^2\right) = \frac{gD(f)}{g^2} + \frac{f}{2|g|^4}\left(\frac{\partial g}{\partial x}(i\overline{g})^2 - i\frac{\partial g}{\partial y}(i\overline{g})^2\right)$$
$$= \frac{gD(f)}{g^2} + \frac{f(i\overline{g})^2}{|g|^4}\left(\frac{\partial g}{\partial z}\right) = \frac{gD(f)}{g^2} + \frac{-f\overline{g}^2}{|g|^4}D(g) = \frac{gD(f)}{g^2} + \frac{-f(r_g e^{-i\,\mathrm{Arg}(g)})^2}{r_g^4}D(g)$$
$$= \frac{gD(f)}{g^2} + \frac{-f}{r_g^2 e^{2i\,\mathrm{Arg}(g)}}D(g) = \frac{gD(f) - fD(g)}{g^2}$$

□

d)

⇒ Show that if $\frac{\partial f}{\partial \bar{z}} = 0$ then the Cauchy-Riemann equations hold

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \implies \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \implies \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$
$$\implies \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\Leftarrow$ Show that if the Cauchy-Riemann equations hold then $\frac{\partial f}{\partial \overline{z}} = 0$ Have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Then

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) = 0$$

Since both implications hold the statements are equivalent.

$\square$

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$$

Where at the last two steps we know the limit needs to be the same if we go along one or both axis.

e)

Know that f is analytic if and only if the Cauchy-Riemann equations hold. And know that the Cauchy-Riemann equations hold if and only if $\frac{\partial f}{\partial \overline{z}} = 0$. This means that f is analytic if and only if $\frac{\partial f}{\partial \overline{z}} = 0$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)$$

i.

$$f(z) = \mathrm{Re}(z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial\,\mathrm{Re}(z)}{\partial x} = \frac{\partial x}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = \frac{\partial 0}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} = \frac{\partial\,\mathrm{Re}(z)}{\partial y} = \frac{\partial x}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial 0}{\partial x} = 0$$

The Cauchy-Riemann equations do not hold and Re(z) is therefore not analytic

ii.

$$f(z) = (x^2 - y^2) + 2xyi = z^2$$

Know from exercise 5ai that $\frac{\partial z^2}{\partial \overline{z}} = 0$ and f is therefore analytic

iii.

$$f(z) = e^{iy} = \cos(y) + i\sin(y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial \cos(y)}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = \frac{\partial \sin(y)}{\partial y} = \cos(y)$$
$$\frac{\partial u}{\partial y} = \frac{\partial \cos(y)}{\partial y} = -\sin(y) \qquad \frac{\partial v}{\partial x} = \frac{\partial \sin(y)}{\partial x} = 0$$

The Cauchy-Riemann equations do not hold, and f is not analytic

iv.

$$f(z) = z(z + \overline{z}^2) = z^2 + |z|^2\overline{z} = x^2 + 2ixy - y^2 + \quad (x^2 + y^2)(x + iy) = x^3 + x^2 + y^2x - y^2 + i(2xy + x^2y + y^3)$$

$$\frac{\partial u}{\partial x} = \frac{\partial(x^3 + x^2 + y^2x - y^2)}{\partial x} = 3x^2 + 2x \qquad \frac{\partial v}{\partial y} = \frac{\partial(2xy + x^2y + y^3)}{\partial y} = 2x + x^2 + 3y^2$$
$$\frac{\partial u}{\partial y} = \frac{\partial x^3 + x^2 + y^2x - y^2}{\partial y} = 2yx - 2y \qquad \frac{\partial v}{\partial x} = \frac{\partial(2xy + x^2y + y^3)}{\partial x} = 2y + 2xy$$

Cauchy-Riemann doesn't hold so the function is not analytic

Exercise 6

Let

$$f(z) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

a)

For any $\epsilon > 0$ there exists a $\delta > 0$ such that when $|f(0) - f(w)| < \epsilon$ then $|0 - w| = |w| < \delta$

$$\begin{aligned} |f(0) - f(w)| &= \left| -\left(\frac{u^3 - v^3}{u^2 + v^2} + i\frac{u^3 + v^3}{u^2 + v^2} \right) \right| \leq \left| \frac{u^3 - v^3}{|w|^2} \right| + |i| \left| \frac{u^3 + v^3}{|w|^2} \right| \\ &\leq \frac{2(|u| + |v|)(u^2 + v^2)}{||w|^2|} + \frac{2(|u| + |v|)(u^2 + v^2)}{||w|^2|} \\ &= \frac{4(|u| + |v|)|w|^2}{|w|^2} = 4(|u| + |v|) \leq 4(|w| + |w|) = 8|w| < 8\delta = 8\frac{\epsilon}{8} = \epsilon \end{aligned}$$

By choosing $\delta = \epsilon/8$
And we have that f is continuous at 0 □

b)

Need to check the Cauchy-Riemann equations, for $z = 0$ this is trivial and the answer is yes. For $z \neq 0$ we get:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial \left(\frac{x^3-y^3}{x^2+y^2} \right)}{\partial x} = \frac{3x^2(x^2 + y^2) - (x^3 - y^3)(2x)}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2} \\ \frac{\partial v}{\partial y} &= \frac{\partial \left(\frac{x^3+y^3}{x^2+y^2} \right)}{\partial y} = \frac{3y^2(x^2 + y^2) - (x^3 + y^3)(2y)}{(x^2 + y^2)^2} = \frac{y^4 + 3x^2y^2 - 2x^3y}{(x^2 + y^2)^2} \end{aligned}$$

See that the Cauchy-Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ does not hold. And have that f is thereby not analytic