

# Oblig 1. matinf3100- Linear optimization

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## Problem 1

a)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & -3 & 4 \\ 1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Now the LP problem given can be written as  $\max(\mathbf{c}^T \mathbf{x})$  subject to  $\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$

b)

Introduce slack variables  $w_1, w_2, w_3$ , and get the following dictionary.

$$\begin{aligned} \eta &= & - & 7x_1 & & + & 2x_3 \\ w_1 &= & 1 & & + & 3x_2 & - & 4x_3 \\ w_2 &= & 2 & - & x_1 & + & x_2 \\ w_3 &= & & 3x_1 & & - & x_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 &\geq 0 \end{aligned}$$

Since  $2x_3$  is the only variable that we can increase to make the objective value increase. This means  $x_3$  is the entering variable. And by observation on the dictionary we see that  $w_3$  is the leaving variable and  $x_3$  can not increase at all. After change we get the following dictionary.

$$\begin{aligned} \eta &= & - & x_1 & & - & 2w_3 \\ w_1 &= & 1 & - & 12x_1 & + & 3x_2 & + & 4w_3 \\ w_2 &= & 2 & - & x_1 & + & x_2 \\ x_3 &= & & 3x_1 & & - & w_3 \\ x_1, x_2, x_3, w_1, w_2, w_3 &\geq 0 \end{aligned}$$

Now we can not increase the objective value any more so the optimal solution is  $x_1 = 0, x_2 = 0, x_3 = 0$  with the objective value 0.

c)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \mathbf{A}' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & \ddots & \\ \vdots & & & \ddots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{A} = [\mathbf{A}' \quad \mathbf{I}_m]$$

Now the LP problem given can be written as  $\max(\mathbf{c}^T \mathbf{x})$  subject to  $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$

## Problem 2

a)

Start by introducing slack variables and get the following initial dictionary.

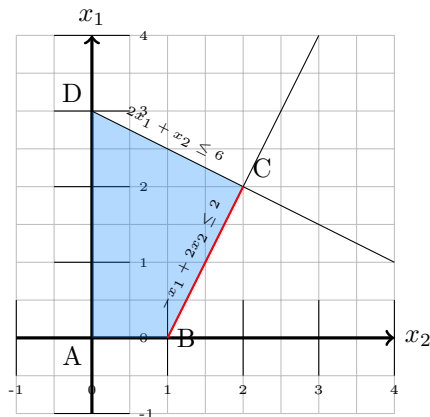
$$\begin{aligned} \eta &= -3x_1 + 6x_2 \\ w_1 &= 6 - 2x_1 - x_2 \\ w_2 &= 2 + x_1 - 2x_2 \\ x_1, x_2, w_1, w_2 &\geq 0 \end{aligned}$$

To increase  $\eta$  let  $x_2$  be the entering variable and let  $w_2$  be the leaving variable. Get the new dictionary.

$$\begin{aligned} \eta &= 6 - 3w_2 \\ x_2 &= 1 + \frac{x_1}{2} - \frac{w_2}{2} \\ w_1 &= 5 - \frac{5x_1}{2} + \frac{w_2}{2} \\ x_1, x_2, w_1, w_2 &\geq 0 \end{aligned}$$

It is impossible to increase  $\eta$ , so we have reached an optimal value,  $\eta = 6$ . Since  $x_1$  is not in the basis, but missing from the objective function this can have any value as long as it doesn't contradict the restrictions. Thus all the optimal solutions are given by  $w_2 = 0$  and  $x_1 \leq 2$ , this implies that  $x_2 \in [1, 2]$ .

b)



The red line symbolizes all the optimal solution. The reason there are multiple solutions is because the objective function is parallel to one of the restrictions.

c)

Start by introducing slack variables and get the following initial dictionary.

$$\eta = \quad \quad 3x_1 \quad + \quad 2x_2$$

$$w_1 = 3 \quad - \quad x_1 \quad + \quad x_2$$

$$w_2 = 2 \quad - \quad x_1$$

$$x_1, x_2, w_1, w_2 \geq 0$$

To increase  $\eta$  let  $x_1$  be the entering variable and let  $w_2$  be the leaving variable. Get the new dictionary.

$$\eta = 6 \quad + \quad 2x_2 \quad - \quad 3w_2$$

$$x_1 = 2 \quad \quad \quad - \quad w_2$$

$$w_1 = 1 \quad + \quad x_2 \quad + \quad w_2$$

$$x_1, x_2, w_1, w_2 \geq 0$$

To increase  $\eta$  further let  $x_2$  be the entering variable, but since we can increase  $x_2$  as much as we want and still not break any restrictions we have that the problem is unbounded.

## Problem 3

a)

$$\text{minimize:} \quad \sum_i |b_i - \sum_j a_{ij}x_j|$$

The minimalization problem above can easily be written as the below by introducing  $t_i = |b_i - \sum_j a_{ij}x_j|$  for  $i = 1, 2, \dots, n$ :

$$\begin{aligned} \text{minimize:} \quad & \sum_i t_i \\ \text{subject to:} \quad & t_i - |b_i - \sum_j a_{ij}x_j| = 0, \quad \text{for } i = 1, 2, \dots, n \end{aligned}$$

but then  $b_i - \sum_j a_{ij}x_j$  has to either be equal to  $t_i$  or  $-t_i$  hence we can say  $b_i - \sum_j a_{ij}x_j$  lies in between the two, for all  $i = 1, 2, \dots, n$ . Thus we can rewrite the previous to:

$$\begin{aligned} \text{minimize:} \quad & \sum_i t_i \\ \text{subject to:} \quad & -t_i \leq b_i - \sum_j a_{ij}x_j \leq t_i, \quad \text{for } i = 1, 2, \dots, n \end{aligned}$$

b)

Using the following code to compute  $\mathbf{x}$  using  $L_2$ -regression.

```
b = [2, 2, 4, 9, 6, 10, 11, 20, 16, 20];
a = [0: 9];
A = [0, 1; 1, 1; 2, 1; 3, 1; 4, 1; 5, 1; 6, 1; 7, 1; 8, 1; 9, 1];

x = (inv((A.')*A))*(A.')(b. ');

scatter(a, b, 'filled', 'r')
hold on

plot(a, A*x, 'b')
hold on

plot(a, A*[2; 0], 'm')

xlabel('a')
ylabel('b')
legend('datapoints', 'regression_using_L2', 'regression_using_L1', ...
'Location', 'northwest')
```

Here  $\mathbf{x} = [2.1212, 0.4545]$

Using the followin code to compute  $\mathbf{x}$  usin  $L_2$ -regression.

```

/*****
 * OPL 12.5 Model
 * Author: emmestl
 * Creation Date: Feb 5, 2017 at 3:55:31 PM
 *****/
int m = ...;
int n = ...;

range rows = 1..n;
range cols = 1..m;

float b[rows] = ...;
float A[rows][cols] = ...;

dvar float+ x[cols];
dvar float+ t[rows];

minimize sum(i in rows) t[i];

subject to {
    forall (i in rows){
        b[i] - (sum(j in cols) A[i][j] *x[j]) <= t[i];
        -b[i] + (sum(j in cols) A[i][j] *x[j]) <= t[i];
    }
}

```

With the given data:

```

/*****
 * OPL 12.5 Data
 * Author: emmestl
 * Creation Date: Feb 5, 2017 at 3:55:31 PM
 *****/
m = 2;
n = 10;

b = [2, 2, 4, 9, 6, 10, 11, 20, 16, 20];
A = [[0, 1], [1, 1], [2, 1], [3, 1], [4, 1], [5, 1], [6, 1], [7, 1], ...
[8, 1], [9, 1]];

```

Here  $\mathbf{x} = [2, 0]$

And by comparing them to the original data we get the following plot: