ME 364: Final exam notes

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This document is meant to provide a thorough equation sheet and brief notes of the entire ME 364 course taught at UW-Madison. PLEASE NOTE: there are some miscellaneous equations in the notes section that are not in the equation section. I tried to explain what each variable meant, but there are definitely some that I left out.

I welcome feedback on this document. Please feel free to reach out to me by email at egalles@wisc.edu to share your thoughts or concerns.

"It is not the critic who counts; not the man who points out how the strong man stumbles, or where the doer of deeds could have done them better. The credit belongs to the man who is actually in the arena, whose face is marred by dust and sweat and blood; who strives valiantly; who errs, who comes short again and again, because there is no effort without error and shortcoming; but who does actually strive to do the deeds; who knows great enthusiasms, the great devotions; who spends himself in a worthy cause; who at the best knows in the end the triumph of high achievement, and who at the worst, if he fails, at least fails while daring greatly, so that his place shall never be with those cold and timid souls who neither know victory nor defeat."

- Theodore Roosevelt

Table of Contents

Equations **2**

1-D Steady State Conduction 2

Thermal resistances 2

Generation 3

Quasi 1-D conduction (fins) 4

Transient conduction 5

Radiation 7

Convection (external flow) 9

Convection (internal flow) 11

Heat exchangers 13

Miscellaneous 14

Brief notes **15**

1-D Steady State Conductionn 15

Thermal resistances 15

Generation 16

Quasi 1-D conduction (fins) 16

Transient conduction 16

Radiation 17

Convection (external flow) 20

Convection (internal flow) 20

Heat exchangers 21

Equations

1-D Steady State Conduction

Heat flux units: $[\frac{W}{m^2}]$

Conduction heat flux:

$$q_x'' = -k\frac{dT}{dx} \tag{1}$$

Convection heat flux:

$$q_x'' = h(T_s - T_\infty) \tag{2}$$

Radiation heat flux:

$$q_x'' = \varepsilon\sigma(T_s^4 - T_{surr}^4) \tag{3}$$

General solution for plane wall:

$$T(x) = C_1x + C_2 \tag{4}$$

General solution for a hollow cylindrical shell:

$$T(r) = C_1ln(r) + C_2 \tag{5}$$

Thermal resistances

General equation with thermal resistances:

$$qR = \Delta T \tag{6}$$

Conductive thermal resistance (planar):

$$R_{cond} = \frac{L}{kA_c} \tag{7}$$

Conductive thermal resistance (cylindrical; $r_2 > r_1$):

$$R_{cond} = \frac{ln(\frac{r_2}{r_1})}{2\pi Lk} \tag{8}$$

Conductive thermal resistance (spherical; $r_2 > r_1$):

$$R_{cond} = \frac{1}{4\pi k}\Big(\frac{1}{r_1} - \frac{1}{r_2}\Big) \tag{9}$$

Convective thermal resistance:

$$R_{conv} = \frac{1}{hA_s} \tag{10}$$

Radiative thermal coefficient:

$$h_r = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr}) \tag{11}$$

Radiative thermal resistance:

$$R_{rad} = \frac{1}{h_r A_s} \tag{12}$$

Generation

Some notation for fin problems:

$$g''' = \text{volumetric generation rate}$$

Conduction through a plane wall with generation:

$$T(x) = \frac{-g'''}{2k}x^2 + C_1 x + C_2 \tag{13}$$

Conduction through a cylinder with generation:

$$T(r) = \frac{-g'''}{4k}r^2 + C_1 ln(r) + C_2 \tag{14}$$

Quasi 1-D conduction (fins)

Some notation for fin problems:

$$P = \text{Perimeter of fin}$$
$$m^2 = \frac{hP}{kA_c} \text{ (units: } [\tfrac{1}{m^2}])$$
$$\theta = T(x) - T_\infty$$
$$q_f = \text{Fin heat transfer rate}$$
$$a = \text{Fin spacing}$$
$$w = \text{Fin width}$$

Biot number ($\perp$ is in direction we want to neglect):

$$Bi = \frac{\Delta T_{cond,\perp}}{\Delta T_{conv}} \approx \frac{R_{cond,\perp}}{R_{conv}} \tag{15}$$

General solution for fin problem:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \tag{16}$$

Common boundary conditions for fins:

$$\text{Fixed temp at base: } \theta(x=0) = T_b - T_\infty \equiv \theta_b$$
$$\text{Convection at tip: } -k\tfrac{d\theta}{dx}|_{x=L} = h\theta(L)$$
$$\text{Adiabatic tip: } \tfrac{d\theta}{dx}|_{x=L} = 0$$
$$\text{Prescribed temperature at tip: } \theta(L) = \theta_L$$
$$\text{Infinitely long fin } (L \to \infty)\text{: } \theta(L) = 0$$

General expression for fin efficiency:

$$\eta_f = \frac{q_f}{hA_s(T_b - T_\infty)} \tag{17}$$

Fin resistance:

$$R_f = \frac{1}{hA_s\eta_f} \tag{18}$$

Equivalent fin resistance:

$$R_{f,eq} = \frac{R_f}{\frac{L}{a+w}} \tag{19}$$

I will not be solving writing down equations for numerical solutions for conduction problems. This topic was covered in lecture on 02/12 and 02/19.

Transient conduction

Some notation for transient problems:

$$m = \text{mass}$$
$$c = \text{specific heat}$$
$$\alpha = \tfrac{k}{\rho c}$$
$$M = \text{number of time nodes}$$

General expression for conduction length:

$$\text{conduction length} = \frac{V}{A_s} \tag{20}$$

Internal resistance to conduction:

$$R_{cond,int} = \frac{\text{conduction length}}{k(\text{area for conduction})} \tag{21}$$

Lumped capacitance Biot number:

$$Bi = \frac{R_{cond,int}}{R_{surr}} \tag{22}$$

For a sphere, we have the following:

$$\text{conduction length} \approx Rad$$
$$\text{conduction area} \approx A_s$$
$$Bi \approx \tfrac{hRad}{k}$$

Rate of energy storage:

$$E_{st} = mc\frac{dT}{dt} \tag{23}$$

Solution for an arbitrary object in bath ($T_{obj} > T_\infty$):

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{\frac{-hA_s}{mc}t} \tag{24}$$

Lumped capacitance time constant:

$$\tau_{LC} = \frac{mc}{hA_s} \tag{25}$$

Thermal penetration depth:

$$\delta_T = 2\sqrt{\alpha t} \tag{26}$$

Conductive thermal resistance (transient):

$$R_{cond,trans} = \frac{\delta_T}{kA_c} \tag{27}$$

Diffusive time constant:

$$\tau_{diff} = \frac{L^2}{4\alpha} \tag{28}$$

Euler method for numerical solution:

$$T_{j+1} = T_j + \left.\frac{dT}{dt}\right|_{t_j} \cdot \Delta t \qquad \text{for } j \in \{1, 2, ..., M-1\} \tag{29}$$

Implicit technique for numerical solution:

$$T_{j+1} = T_j + \left.\frac{dT}{dt}\right|_{t_{j+1}} \cdot \Delta t \qquad \text{for } j \in \{1, 2, ..., M-1\} \tag{30}$$

Radiation

Some notation for radiation problems:

$$\alpha = \text{absorption}$$
$$\rho = \text{reflection}$$
$$\tau = \text{transmission (for this class, } \tau = 0\text{)}$$
$$G = \text{irradiation}$$

Emissive power for black bodies (per unit area):

$$E_b = \sigma T^4 \tag{31}$$

Wien's displacement law:

$$\lambda_{max} T = 2898 \tag{32}$$

View factor reciprocity rule:

$$A_i F_{ij} = A_j F_{ji} \tag{33}$$

View factor summation rule (enclosure required):

$$\sum_{j=1}^{N} F_{ij} = 1 \tag{34}$$

NET heat flow between two BLACK surfaces:

$$q_{ij} = A_i F_{ij} (T_i^4 - T_j^4) \tag{35}$$

Emissivity:

$$\varepsilon = \frac{E}{E_b} \tag{36}$$

Reflection, absorption, transmission identity:

$$\alpha + \rho + \tau = 1 \tag{37}$$

Gray surface identity:

$$\varepsilon = \alpha \qquad \text{for gray surfaces} \tag{38}$$

Total radiation leaving a surface (per unit area):

$$J_i = E_i + \rho_i G_i \tag{39}$$

NET heat flow leaving a surface:

$$q_i = J_i A_i - G_i A_i \tag{40}$$

Space resistance (NOT A THERMAL RESISTANCE):

$$R_{ij} = \frac{1}{A_i F_{ij}} \tag{41}$$

Surface resistance (NOT A THERMAL RESISTANCE):

$$R_{s,i} = \frac{1 - \varepsilon_i}{\varepsilon_i A_i} \tag{42}$$

Convection (external flow)

Some notation for convection (external flow) problems:

$$\rho = \text{density}$$
$$\alpha = \text{thermal diffusivity} = \tfrac{k}{\rho c}$$
$$\mu = \text{dynamic viscosity}$$
$$\nu = \text{kinematic viscosity} = \tfrac{\mu}{\rho}$$
$$k_f = \text{fluid conductivity}$$
$$Re_{x,crit} = \text{when turbulence occurs} = 5 \cdot 10^5$$

Thermal boundary layer thickness:

$$\delta_T \approx 2\sqrt{\frac{\alpha x}{u_\infty}} \tag{43}$$

Momentum boundary layer thickness:

$$\delta \approx 2\sqrt{\frac{\nu x}{u_\infty}} \tag{44}$$

Prandtl number:

$$Pr = \frac{\nu}{\alpha} = \frac{\mu c}{k_f} \tag{45}$$

Relationship between Prandtl number and boundary layers:

$$\frac{\delta_T}{\delta} = \frac{1}{\sqrt{Pr}} \tag{46}$$

Reynolds number:

$$Re_x = \frac{\rho u_\infty x}{\mu} \tag{47}$$

Intuitive approximation of heat transfer coefficient:

$$h \approx \frac{k_f}{\delta_T} \tag{48}$$

Average heat transfer coefficient:

$$\overline{h} = \frac{1}{L}\int_0^L h\,dx \tag{49}$$

Nusselt number:

$$Nu_L = \frac{L_{char}h}{k_f} \tag{50}$$

Film temperature:

$$T_f = \frac{1}{2}(T_s + T_\infty) \tag{51}$$

Case 1: flow over a flat, isothermal plate

Local Nusselt number (laminar, $Pr \geq 0.6$):

$$Nu_x = \frac{h_x x}{k_f} = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} \tag{52}$$

Average Nusselt number (laminar, $Pr \geq 0.6$):

$$\overline{Nu_x} = \frac{\overline{h_x} x}{k_f} = 0.664 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} \tag{53}$$

Local Nusselt number (turbulent, $0.6 \leq Pr \leq 60$)

$$Nu_x = 0.0296 Re_x^{\frac{4}{5}} Pr^{\frac{1}{3}} \tag{54}$$

Case 2: flow over a flat plate with constant heat flux

Local Nusselt number (laminar, $Pr \geq 0.6$):

$$Nu_x = 0.453 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} \tag{55}$$

Local Nusselt number (turbulent, $0.6 \leq Pr \leq 60$)

$$Nu_x = 0.0308 Re_x^{\frac{4}{5}} Pr^{\frac{1}{3}} \tag{56}$$

Surface temperature:

$$T_s(x) = T_\infty + \frac{q_s''}{h_x} \tag{57}$$

Case 3: flow over a curved surface

Reynolds number:

$$Re_D = \frac{\rho u_\infty D}{\mu} \tag{58}$$

Average Nusselt number:

$$\overline{Nu_D} = \frac{\overline{h}D}{k_f} \tag{59}$$

Convection (internal flow)

Some notation for convection (internal flow) problems:

$$\rho = \text{density}$$
$$\alpha = \text{thermal diffusivity} = \tfrac{k}{\rho c}$$
$$\mu = \text{dynamic viscosity}$$
$$\nu = \text{kinematic viscosity} = \tfrac{\mu}{\rho}$$
$$k_f = \text{fluid conductivity}$$
$$Re_{D,crit} = \text{when turbulence occurs} = 2800$$

Mean velocity:

$$u_m = \frac{\dot{m}}{\rho A_c} \tag{60}$$

Hydraulic diameter:

$$D_h = \frac{4A_c}{Per} \tag{61}$$

Reynolds number:

$$Re_D = \frac{\rho u_m D_h}{\mu} \tag{62}$$

See (45) for Prandtl number equation (calculation is the same regardless of flow type).

Fully developed entry length (laminar Re_D):

$$x_{fd,h} = 0.063 Re_D D_h \tag{63}$$

Fully developed entry length (turbulent Re_D):

$$x_{fd,turb} = 3.8 Re_D^{\frac{1}{6}} D_h \tag{64}$$

Friction factor (laminar):

$$f = \frac{64}{Re_D} \tag{65}$$

Pressure drop:

$$\Delta P = \frac{f \rho u_m^2 (x_2 - x_1)}{2D_h} \tag{66}$$

Thermal entry length:

$$x_{fd,t} = 0.063 Re_D Pr D_h \tag{67}$$

Mean temperature (constant surface heat flux):

$$T_m(x) = \frac{q_s'' Per}{\dot{m}c} x + T_{in} \tag{68}$$

Mean temperature (constant surface temperature):

$$T_m(x) = T_s - (T_s - T_{in}) e^{\frac{-Per\overline{h_x}}{\dot{m}c} x} \tag{69}$$

Heat flux:

$$q_s'' = h(T_s - T_m) \tag{70}$$

Nusselt number:

$$Nu_D = \frac{hD_h}{k_f} \tag{71}$$

Nusselt number (smooth circular tube, $T_s > T_m$)

$$Nu_D = 0.023 Re_D^{\frac{4}{5}} Pr^{0.4} \tag{72}$$

Nusselt number (smooth circular tube, $T_s < T_m$)

$$Nu_D = 0.023 Re_D^{\frac{4}{5}} Pr^{0.3} \tag{73}$$

Average Nusselt number:

$$\overline{Nu_D} = Nu_{D,fd} \tag{74}$$

Heat exchangers

Some notation for convection (internal flow) problems:

$$\dot{m}_h = \text{mass flow on hot side}$$
$$\dot{m}_c = \text{mass flow on cold side}$$
$$c_h = \text{specific heat on hot side}$$
$$c_c = \text{specific heat on cold side}$$
$$UA = \text{conductance; inverse of total thermal resistance}$$

Heat capacity rate:

$$\dot{C}_i = \dot{m}_i c_i \tag{75}$$

Energy balance (hot side):

$$q = \dot{m}_h c_h (T_{h,in} - T_{h,out}) = \dot{C}_h (T_{h,in} - T_{h,out}) \tag{76}$$

Energy balance (cold side):

$$q = \dot{m}_c c_c (T_{c,out} - T_{c,in}) = \dot{C}_c (T_{c,out} - T_{c,in}) \tag{77}$$

Log-mean temperature difference method:

$$ln\left|\frac{T_{h,out} - T_{c,in}}{T_{h,in} - T_{c,out}}\right| = -UA\left[\frac{1}{\dot{C}_h} - \frac{1}{\dot{C}_c}\right] \tag{78}$$

Maximum heat transfer:

$$q_{max} = \dot{C}_{min}(T_{h,in} - T_{c,in}) \tag{79}$$

Effectiveness:

$$\varepsilon = \frac{q}{q_{max}} \tag{80}$$

Number of transfer units:

$$NTU = \frac{UA}{\dot{C}_{min}} \tag{81}$$

Capacity ratio:

$$Cr = \frac{\dot{C}_{min}}{\dot{C}_{max}} \tag{82}$$

Effectiveness and capacity ratio identity:

$$\lim_{Cr \to 0} \varepsilon = 1 - e^{-NTU} \tag{83}$$

Miscellaneous

Required pump power:

$$\dot{W} = \frac{\Delta P \dot{m}}{\rho} = \Delta P \dot{V} \tag{84}$$

Brief notes

1-D Steady State Conduction

(4) and (5) are just some examples of common situations with conduction. If there is a problem with a different configuration, please use an energy balance ($E_{in} - E_{out} + E_{gen} = E_{st}$).

Thermal resistances

Thermal resistances, mainly conductive thermal resistances, are NOT well defined in regions that have generation. Do NOT try to define a resistance through a material that is experiencing generation.

Recall that when we're considering which resistor is most/least significant, large resistors in series dominate while small resistors in parallel dominate.

Suppose we have a composite structure of two materials 1 and 2 and $k_1 > k_2$. This means that the temperature profile in material 1 will be much flatter (experience less of a temperature change) than material 2. Recall that metals have high k values while something like styrofoam has a low k value. This can be understood by looking at where k appears in (7), (8), and (9).

If we encounter a contact resistance, this will typically be a resistance in series and will cause a pseudo jump in the temperature profile plot. Please look at the units of contact resistance info! Things get weird with whether you divide or multiply by an area term.

Generation

Once again, do NOT try to define a thermal resistance through a material that has generation!

Some cases may require us to solve for the volumetric heat generation in a material. This can usually be done by $g''' = \frac{q}{Vol}$.

Quasi 1-D conduction (fins)

Recall that in order to use a 1-D approximation, the Biot number from (15) must be much less than 1.

Fins are more effective when h is small, so if we're asked which side to put fins on or something like that, figure out which side is exposed to a smaller h.

Transient conduction

Recall that in order to use a 0-D approximation, the Biot number from (22) must be much less than 1.

Note that when $t = 5\tau_{LC}$ (τ_{LC} from (25)), the object has equilibrated with its surroundings. In other words, the exposed surfaces approach T_∞.

Note that $t = 5\tau_{diff}$ is not an important quantity.

Please refer to Figure 1 for tips on plotting temperature profiles if we know time constants.

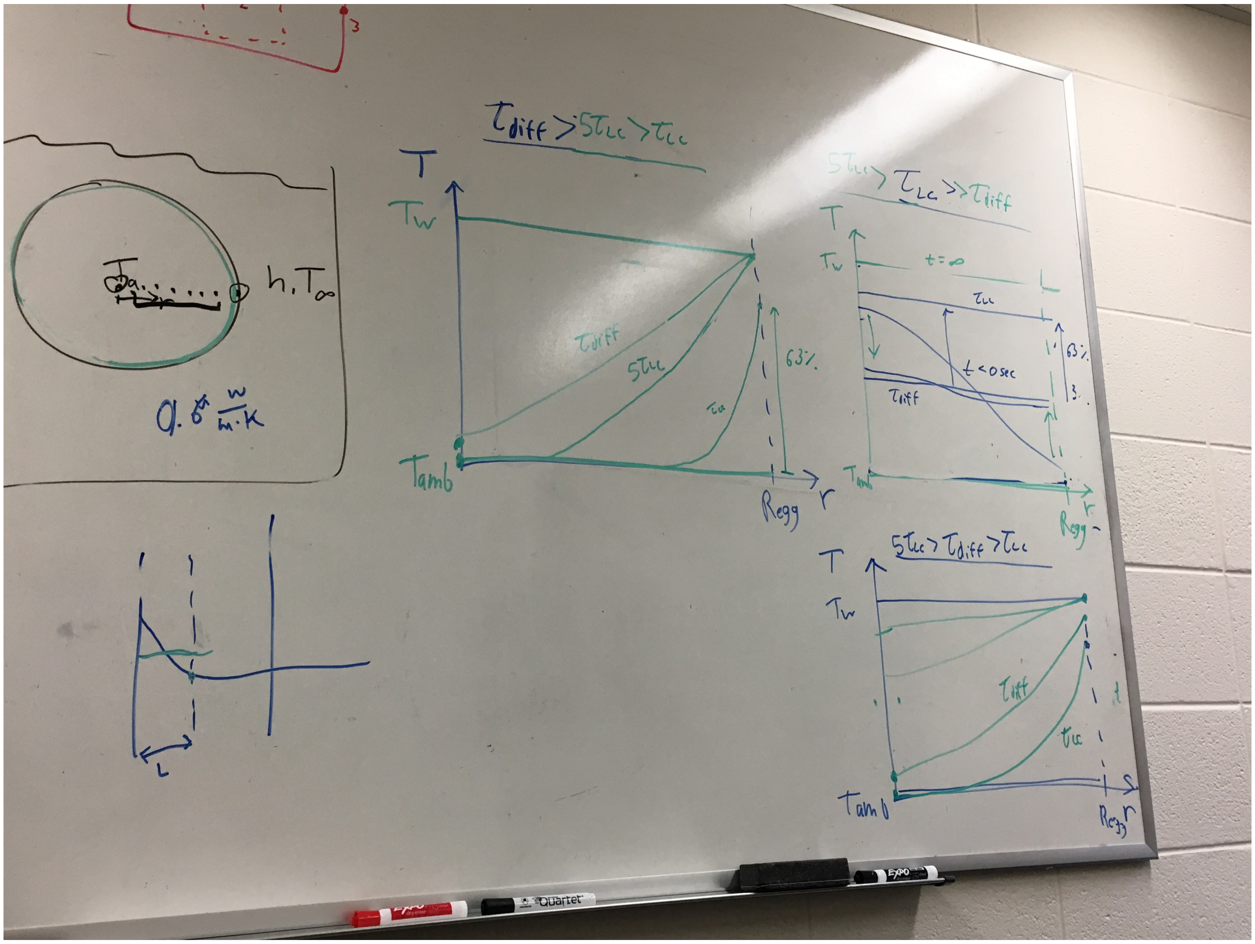

Figure 1: Plots of temperature profiles with different time constant relationships (taken during office hours with Professor Dakotah Thompson).

Radiation

In this class, please note that we assume $\tau = 0$ for all situations (no transmission).

If we need to know how what fraction of the emissive power is between two specific wavelengths, refer to Table 12.2 of the textbook to find $F_{0\to\lambda_1}$ values. Recall that $F_{0\to\lambda_1}$ is the fraction of emissive power between $\lambda = 0$ and $\lambda = \lambda_1$.

View factors that are referenced in (33), (34), and (35) are pure geometric quantities; they have NO dependence on temperature or surface properties.

Also keep in mind that it may be advantageous to define a pseudo surface to create an enclosure in order to use (34).

With three radioactive surfaces, the first thing we need to do is set up a space resistance network between the surfaces; we'll have R_{12}, R_{13}, and R_{23}. Note that $R_{ij} = R_{ji}$. Note that if $F_{ij} = 0$, then $R_{ij} \to \infty$, and we have the equivalent of an open circuit.

After space resistances, we look at the surface resistance at each node (node $\equiv$ surface). This resistors will emanate from each surface of our triangle of space resistances, where each point of the triangle is at potential J_i and the other end of the surface resistance is at $E_{b,i}$ Recall that surface resistances account for the inefficiency between a real surface and a black surface. Therefore, if a surface i has a very low emissivity, $R_{s,i}$ will be extremely high and there will be a large difference between $E_{b,i}$ and J_i. On the other hand, if a surface j has an emissivity equal to 1, then $R_{s,j} = 0$ and $E_{b,j} = J_j$.

From here, we are ready to analyze our system. We typically find our view factors in order to quantify the space resistances, and then we can do the same with our surface resistances. It is not uncommon to have the $E_{b,i}$ and J_i values undefined. These can be found with creating energy balances at nodes. Remember that $E_b = \sigma T^4$.

Special cases

There are two special cases that frequently arise:

1. Black surfaces ($\varepsilon = 1$): when we have a black surface i, there is no surface resistance, so the node at that surface will be equal to the black body emissive power ($E_{b,i}$). **NOTE**: although $J = E_b$, there can still be heat flow in/out of this node! Rather than having $q_i = \frac{E_{b,i}-J_i}{R_{s,i}}$, we solve for q_i with an energy balance around node i, since $q_i = \frac{E_{b,i}-J_i}{R_{s,i}}$ would lead us to

divide by zero ($\frac{0}{0}$). An analogy is trying to use Ohm's law over a wire in circuits. The wire is at the same potential everywhere, and there is no resistance, so the current cannot be solved for.

2. Re-radiating surfaces (or insulated surfaces): at a re-radiating surface j, $q_j = 0$. This means that whatever heat flow is entering the surface is the heat flow that is leaving the surface. This means that $J_j = E_{b,j}$.

Pay no attention to the numbers, but please see Figure 2 on how to set up a radiative resistance network.

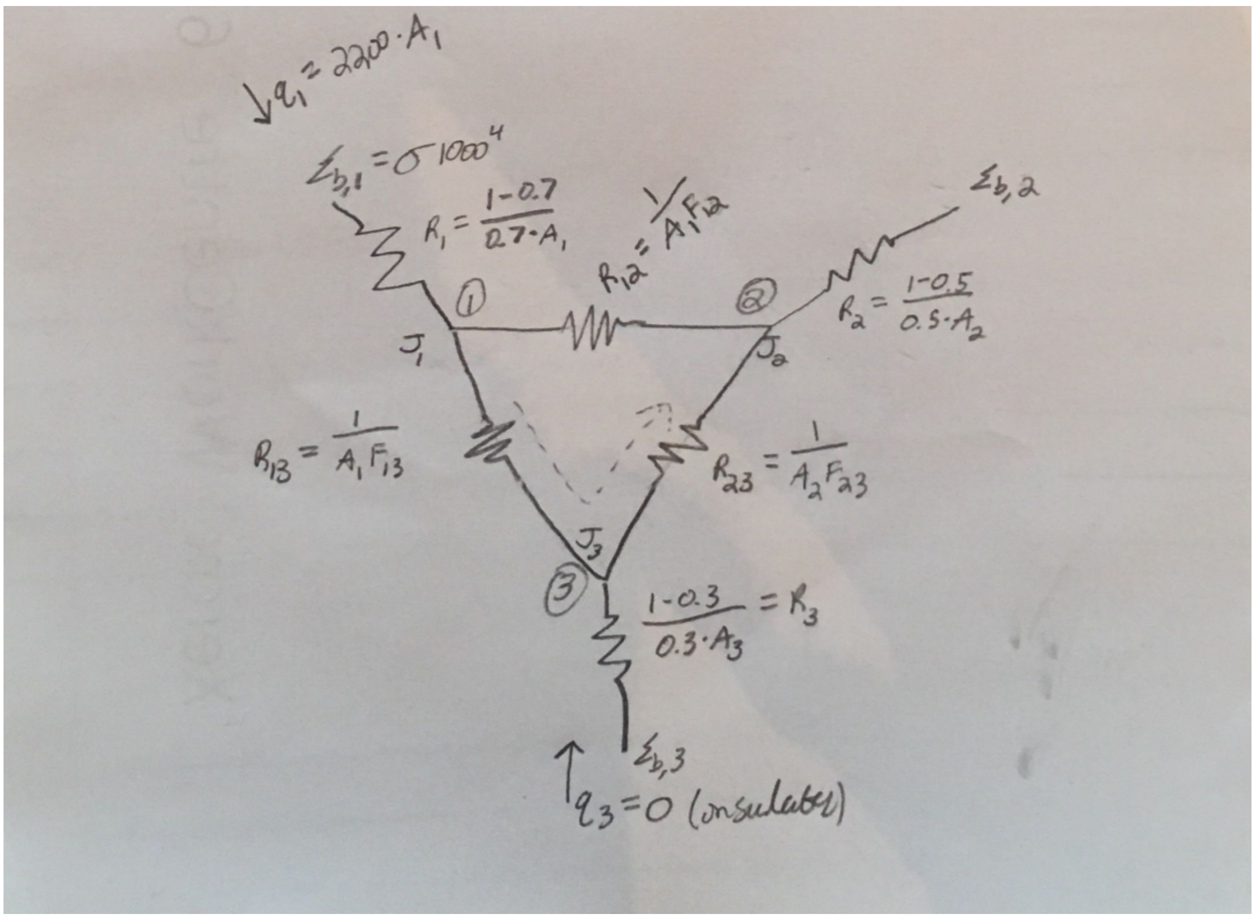

Figure 2: An example of how to set up a resistance network; specific numbers are not important.

Please realize that terms like E_b and J are all per unit area. However, since our resistance term is also per unit area, we end up with correct units. Think of E_b and G as nodes, and then we divide by resistance to get q.

Convection (external flow)

The Prandtl number that we define in (45) is an intrinsic fluid property and is essentially a measure of whether momentum or heat energy transfer dominates. When $Pr << 1$, δ_T grows faster than δ, and when $Pr >> 1$, δ grows faster than δ_T.

Realize that when we talk about q this is the heat transfer INTO the fluid.

If we are looking to minimize h (and therefore find the highest temperature on our surface), find the location where $Re_x = Re_{x,crit}$. h decreases with x, but experiences a large jump when turbulence occurs. The boundary layer δ also experiences a jump.

The Nusselt number we define in (50) is a tool to help us find h. Don't look too far into it.

When we are finding fluid properties, always evaluate at the film temperature that we define in (51).

Convection (internal flow)

It is important to note that once flow becomes fully developed thermally and hydraulically, h and the shear stress τ are constants! However, please note that as $L \to \infty$, $T_m \to T_\infty$.

No matter what the Reynolds number is, there will be some turbulence in the entry region. However, it is often considered negligible. Given that flow enters as laminar, we assume that the turbulent regime ends once $x > x_{fd,h}$. We define $x_{fd,h}$ in (63).

Heat exchangers

In general, a counterflow heat exchanger is better since there is a larger driving potential ΔT along the length of the heat exchanger.

Please note that when we calculate UA, we just find the thermal equivalent resistance from fluid to fluid, which will usually include two convection terms and possibly a conduction term. Then, we take the reciprocal of this equivalent resistance. Note that the equivalent resistance represents the resistance that q experiences between fluids at a certain location along the heat exchanger.

If we are given a heat transfer coefficient h and asked to use it to help find temperatures of a heat exchanger, we can use it to find UA and then NTU.

NTU, which was defined in (81), is basically a dimensionless way of measuring how good a heat exchanger is. The higher the NTU, the better the heat exchanger (more effective). As $NTU \to 0$, we can say $\varepsilon \approx NTU$.

Please note that if the capacity ratio Cr, which was defined in (82), approaches 0, the configuration of the heat exchanger is NOT important, and it is a sign that a fluid is experiencing a phase change. Furthermore, when $Cr = 1$, both streams experience the same ΔT. Also note that when Cr is small, we could use (83) as an approximation.

Similarly, when $NTU \to 0$, configuration becomes irrelevant.

Correlations exist to retrieve ε from Cr and NTU. I believe they will be given on the exam, but Table 11.3 in the textbook has some as well.

We may be asked to plot temperature profiles of the streams. Please note that if we have two streams with $\dot{C}_1 > \dot{C}_2$, then the temperature change in stream 1 will be smaller than in stream 2. Please see (76) or (77) to see how this is true. $\dot{C}$ is to heat exchangers as k is to conduction.

Please note that $T_{c,out} < T_{h,out}$ for parallel flow configuration, but it is possible for $T_{c,out} > T_{h,out}$ with a counterflow design.

Don't have a good explanation as to why, but in practice exam solutions it has been noted that when flow is laminar, roughness becomes irrelevant when finding exit temperatures.