1 What is the Paris-Harrington Theorem?

The Paris-Harrington Theorem shows that a specific result in Finite Ramsey Theory is not provable in Peano Arithmetic (PA). This result is the first natural example of a statement about natural numbers that is not provable in PA. Godel's Second Incompleteness Theorem told us that such statements must exist.

So what statement will we be working with? First, a quick definition. We say a finite set $H \subset \mathbb{N}$ is *relatively large* if $|H| \geq \min(H)$. We write

$$M \underset{*}{\rightarrow} (k)^e_r$$

to say that for every partition $P : [M]^e \rightarrow r$, there is a *relatively large* $H \subseteq M$ which is homogeneous for P and has cardinality $|H| \geq k$. The statement of Ramsey theory that we will be working with says the following:

$$\forall e, r, k \in \mathbb{N},\ \exists M \text{ such that } M \underset{*}{\rightarrow} (k)^e_r \tag{1}$$

In plain words this means that for any e, r, k there is a partition $[M]^e \rightarrow r$ such that there is a relatively large set $H \subseteq M$ that is homogeneous for P and has cardinality $\geq k$. Theorem 1 is sometimes known as the Strengthened Finite Ramsey Theorem. Note that if we remove the relatively large condition on H, then we have the Finite Ramsey Theorem, which we know is provable in PA.

Now we can state the Paris-Harrington Theorem:

$$\textit{Theorem 1 is not provable in Peano Arithmetic} \tag{2}$$

2 The Proof

Proof Outline: We will first prove Theorem 1. Then we will develop a system T with the property that the consistency of T implies the consistency of PA. We will then show that the combinatorial principle of Theorem 1 implies the consistency of T. If we prove in PA both that $\text{Con}(T) \rightarrow \text{Con(PA)}$ and that Theorem 1 $\rightarrow \text{Con}(T)$, then by Gödel's Second Incompleteness Theorem, we will have proved Theorem 1: that you can't prove Theorem 2 in PA since you can't prove a system's consistency using that system.

Proof of Theorem 1: We need will need to use a stronger system than PA to prove Theorem 1. We will fix e, r, and k. Assume, for the sake of contradiction, that there is no M that satisfies the claim. Let's say P is a partition that exhibits a counterexample for M. That is P partitions $[M]^e$ into r sets such that none of them are relatively large homogeneous sets of cardinality $\geq k$. Let's view all the counterexamples as a "finitely branching infinite tree" where P is below P' in the tree iff, for P a counterexample for M and P' a counterexample for M', $M < M'$ and "P is the restriction of P' to $[M]^e$". By König's Lemma, we know thee is a partition $P : [\omega]^e \to r$ such that $\forall M$, the restriction of P to M is a counterexample for M. We also know from the Infinite Ramsey Theorem that $\exists H \subset \omega$ that is homogeneous for P. However, now we see that if we choose a M large enough compared to k and $\min(H)$, $H \cap M$ will be a relatively large homogeneous sets of size $\geq k$. $\square$

The System $\boldsymbol{T}$: We will now define a system T and then show that both $\text{Con}(T) \to \text{Con}(\text{PA})$ and Theorem 1 $\to \text{Con}(T)$. T uses the language of PA plus new symbols $c_0, c_1, ...$ that represent constants. T has the following axioms:

i. The usual equations for $+, \times, <$ and the induction axioms *but only for limited formulas.*

ii. For each $i \in \mathbb{N}$, the axiom $(c_i)^2 < c_{i+1}$.

iii. For each finite subset $\boldsymbol{i} = i_1, i_2, ..., i_r$ of ω, let $c(\boldsymbol{i}) = c_{i_1}, c_{i_2}, ..., c_{i_r}$. For each $i < \boldsymbol{k}, \boldsymbol{k}'$ and each limited formula $\psi(\boldsymbol{y}; \boldsymbol{z})$ where $\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{z}$ have the same length, we have the axiom $\forall \boldsymbol{y} < c_i,\ [\psi(\boldsymbol{y}; c(\boldsymbol{k})) \leftrightarrow \psi(\boldsymbol{y}; c(\boldsymbol{k}'))]$.

Con($\boldsymbol{T}$) $\to$ Con(PA): The first step in proving Theorem 2 is to prove (in PA) that $\text{Con}(T) \to \text{Con}(\text{PA})$. First, we will let $\mathfrak{A} \models T$ and we will let I be the initial segment of $\mathfrak{A}$ of those $a < c_i$ for some $i \in \omega$. By the second axiom of T, we know that I is closed under addition and multiplication. To show that the consistency of T implies the consistency of PA we need to show the following two things:

i. $\mathfrak{I} = \langle I, +, \times, < \rangle$ is a model of PA.

ii. Given $i < \boldsymbol{k}, \boldsymbol{a} < c_i$, and $\theta(y)$ where $\boldsymbol{k}$, $\boldsymbol{a}$, and y are all of appropriate length, $\mathfrak{I} \models \theta(\boldsymbol{a}) \leftrightarrow \mathfrak{A} \models \theta^*(\boldsymbol{a}; c(\boldsymbol{k}))$ where θ^* is defined as follows. For every formula $\theta(y) = \exists x_1 ... \forall x_r, \varphi(x; y)$ in of PA where φ is quantifier free, we can define a limited formula $\theta^*(y; z_1, ..., z_r) = \exists x_1 < z_1 ... \forall x_r < z_r, \varphi(x; y)$.

We have to make sure that our proof for $\text{Con}(T) \to \text{Con}(\text{PA})$ is done solely within PA so we can apply Gödel's Second Incompleteness Theorem later on. Note that i. is a direct result of ii. since the first axiom of T says that formulas θ, a model of T, $\mathfrak{A}$ will satisfy induction for limited formulas θ^*. So, we now need to prove ii. We will do so by induction on θ^*.

Assume that $\theta(\boldsymbol{y}) = \exists x \psi(x; \boldsymbol{y})$. This means that $\theta^*(\boldsymbol{y}; \boldsymbol{z}) = \exists x < z_1 \psi^*(x, \boldsymbol{y}; z_2, ..., z_r)$. We see from this that $\mathfrak{I} \models \theta(\boldsymbol{a})$ iff $\exists b \in I, \boldsymbol{j}$ such that $\mathfrak{A} \models \psi^*(b, \boldsymbol{a}; c(\boldsymbol{j}))$. We see that this only happens when, for some $\boldsymbol{k}'$, $\mathfrak{A} \models \psi^*(b, \boldsymbol{a}; c(\boldsymbol{k}'))$. By the third axiom of T, we see that this only happens when $\mathfrak{A} \models \psi^*(b, \boldsymbol{a}; c(\boldsymbol{k}))$. Thus, we have proved ii. and by extension, that $\text{Con}(T) \to \text{Con}(\text{PA})$.

Combinatorial Principle of Theorem 1 $\rightarrow$ Con(T): This is where the paper gets really dense and difficult. The authors build up the final result from many different lemmas and intermediate propositions concerning partitions of finite graphs and Finite Ramsey Theory. Finally, the authors link the results of these lemmas regarding Finite Ramsey Theory to the axioms of T to show that if Theorem 1 holds, then T is consistent. Again, the authors are careful to work solely inside PA so that Gödel's Theorem can be applied. I'll quickly go over each of the lemmas the authors utilize to prove the claim that if Theorem 1 holds, then T is consistent.

1. Lemma: Given partitions P_0 and P_1 of $[M]^e$, into r_0 and r_1 pieces respectively, there is a partition P of $[M]^e$ into $r_0 \cdot r_1$ pieces such that for $H \subseteq M$, H is homogeneous for P iff H is homogeneous for both P_0 and P_1.

2. Lemma: A set $H \subseteq M$ is homogeneous for a partition P of $[M]^e$ iff every subset of H of size $e+1$ is homogeneous for P.

3. Here, the authors introduce the notation $\sqrt{r}$ to represent the first $s \in \mathbb{N}$ such that $s^2 \geq r$. We note that $\forall r \geq 7,\ r \geq 1 + 2\sqrt{r}$. Lemma: Given $P : [M]^e \rightarrow r$, there is a $P' : [M]^{e+1} \rightarrow (1 + 2\sqrt{r})$ such that $\forall H \subseteq M$ of cardinality $> e+1$, H is homogeneous for P iff H is homogeneous for P'.

4. Lemma: Suppose we are given n partitions $P_0, P_1, ..., P_{n-1}$ where $P_i : [M]^{e_i} \rightarrow r_i$ for $i < n$. Let $e = \max_i e_i$ and $r = \prod_i \max(r_i, 7)$. There is a partition $P : [M]^e \rightarrow r$ such that $\forall H \subseteq M$ of cardinality $> e$, H is homogeneous for P iff H is homogeneous for all P_i, $i < n$.

Here, the authors use the following four lemmas to create a statement of combinatorics carefully constructed to imply the consistency of T.
Proposition 1: For all e, k, r, there is an M such that for any family $\langle P_\xi; \xi < 2^M \rangle$ of partitions $P_\xi : [M]^e \rightarrow r$, there is an X of cardinality $\geq k$ such that:

ii. if $a, b \in X$ and $a < b$, then $a^2 < b$

iii. if $a \in X$ and $\xi < 2^a$, then $X \sim (a+1)$ is homogeneous for P_ξ.

Notice that we skipped i. and went right to ii. This is intentional since ii. and iii. of Proposition 1 correlate to the second and third axioms of T respectively. The authors go on to show that Proposition 1 implies the consistency of T.

Outline of Proof: Given a finite set $S \subseteq T$, let $c_0, c_1, ..., c_k$ be all the constant symbols appearing in S. The authors use Proposition 1 to show that $\langle \omega,\ +,\ \times,\ x_0, ..., x_{k-1} \rangle$ is a model of S where $x_0, ..., x_{k-1}$ are the first k elements of the set X from Proposition 1. The authors note that the first axiom of T is clearly satisfied by our model. They also note that ii. of Proposition 1 handles the second axiom of T. So, all that remains is to show that the third axiom of T is satisfied. Here, the authors show that, for any limited formula in S, you can construct a partition from $[M]^{e'}$ to 2 where e' depends only on the limited formula. Using the fourth lemma from above, the authors combine all the partitions for each limited formula in S into one partition $P_\xi : [M]^e \to r$. Then we can pick an M that is large enough such that $\exists X \subseteq M$ such that the second and third axioms of T are satisfied by choosing $x_0, ..., x_{k-1}$ as described above.

Now all that remains to prove Theorem 2 is to show that Theorem 1 implies this proposition. Here, the authors introduce some notation before proving three more lemmas. Let $g^{(x)}$ be g composed with itself x times. Let $f_0(x) = x + 2$ and let $f_{n+1}(x) = (f_n)^{(x)}(2)$. The authors make a note that this function is so fast growing that it is not primitive recursive.

5. Lemma: For every p there is a $Q : [M]^1 \to p + 1$, where r depends only on m, such that if X is homogeneous for Q and of cardinality ≥ 2, then $\min(X) \geq p$.

6. Lemma: For each m there s a partition $R : [M]^2 \to r$ such that if $X \subseteq M$ is relatively large and homogeneous for R and of cardinality > 2, then $\forall x, y \in X,\ x < y \to f_m(x) < y$.

7. Lemma: Let $P : [M]^e \to s$, where $e \geq 2$, and m be given. There exists a $P^* : [M]^e \to s'$, where s' depends only on m, e, s, such that if there is a relatively large $Y \subseteq M$ homogeneous for P^* of cardinality $> e$, then there is an $X \subseteq M$ such that X is homogeneous for P and has cardinality $\geq e+1$ and $f_m(\min(X))$.

Finally the authors pose the following statement.
Proposition 2: The combinatorial principle of Theorem 1 implies Proposition 1.
Outline of Proof: Given e, r, k, we must find an M with the properties described in Proposition 1. The authors show that, given any M and family of partitions as described in Proposition 1, you can create a new partition $S : [M]^{2e+1} \to 2$. Then, using Q and R as described in lemmas 5 and 6, we can combine Q, R, and S into one partition using the lemma 4. Then, using the lemma 7, we can construct a partition $P^* : [M]^{2e+1} \to s'$. The authors then use this partition along with the fast growing function defined earlier in the paper to show that an M can be found that satisfies the properties of Proposition 1.

The proof for Proposition 2 finishes the proof for Theorem 2.