Prove It: Mathematics and Certainty

STSC 278

Material Mathematics Paper

Mathematical Symbol Systems: Rigor, Intuition, and Pedagogy

Author:

Emmett Neyman

Professor:

Dr. Stephanie Dick

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Since the ancient Greek mathematicians and before, mathematics has always had a major written component. Not only is it easier to share and distribute math when it lives on the page rather than in one's mind, but it is easier to understand math when you see it and can read it rather than only hear it. Mathematicians throughout history have even argued that the mathematical notation used can influence how one thinks about, learns, and understands math. Few believed in this idea more strongly than the British mathematicians Bertrand Russell and Alfred North Whitehead. Around the beginning of the twentieth century, the mathematical community was deep in a crisis of foundations. The newly discovered paradoxes of axiomatic set theory generated a sense of anxiety among mathematicians. Russell and Whitehead believed they had a solution: a completely axiomatized formal system of logic that could express all mathematical truths without introducing any paradoxes or inconsistencies. And at the heart of their system: a radically new logical notation that lay the groundwork for many modern systems. Although this new symbol system made their groundbreaking book *Principia Mathematica* famously difficult to read, the authors argued that it had multiple benefits over more traditional methods. Mathematical symbol systems also sparked debate early in the nineteenth century when algebra, calculus, and logic were permeating into other fields of mathematics. During this period, geometers argued about the use of algebraic symbol systems in geometry. Also during this time, as religion was slowly becoming disentangled from math, symbol systems were seen as a secular alternative to more intuitive methods backed by religious metaphysics. In this essay I will argue that mathematical symbol systems, and more generally mathematical notations, play a significant role how rigorous and intuitive mathematics is. I will also argue that symbol systems and notations are important factors in how people learn and teach math. I'll finish this paper by observing that over the course of the nineteenth century the benefits and drawbacks of formal symbol systems changed.

One of the first debates concerning mathematical symbol systems emerged in Europe during the beginning of the nineteenth century about the best way to do geometry. This debate was particularly active in France and Italy. On one side of the debate were the so called *analysts* who believed that the best way to do geometry was algebraically, using the power of the calculus. On the other side were the *synthesists* who advocated the use of more physical methods similar to Euclid's geometry. The synthetic camp had two main arguments against the use of analytic methods in geometry. The first was that the calculus lacked solid foundations; it relied to heavily on nebulous metaphysics such as the concept of an infinitesimal. Without rigorous foundations, synthetic geometers argued, how could one be sure that analytic results were correct? Their second argument focused on the intuitiveness of the analytic methods. They argued that when geometry was done using only algebra and calculus, it was stripped of the natural, physical intuition it holds. Only by using constructions, diagrams, and physical models could mathematician take advantage of this intuition and use it to successfully solve problems. The synthetic geometers of the time argued that analysts were not doing real mathematics, they were only moving symbols around on a page. Synthetic geometers believed that solving equations was not in the true spirit of geometry. In summary, the synthetic geometers criticized the analytic methods because they lacked both rigor and intuition. The one advantage analytic methods had over synthetic ones was how easy they were to use and to teach. Because solving equations using algebra and calculus did not require any geometric intuition, all a new mathematician needed to know in order to solve problems were the rules of the algebraic and analytic symbol systems. Although debates over mathematics were not new, this was one of the first debates in which notation and symbol systems played a significant role.

While the debate over the correct way to do geometry was waging in France and Italy, a similar debate over the use of symbol systems was happening in England. At the center of this debate was the role of religion in mathematics. One of the first secular universities in all of England, London University, had just opened its doors and the British logician Augustus De Morgan was soon given a job there as a professor of mathematics. As one of the first non-religious academic institutions in England, the University faced many challenges during its early years. In mathematics specifically, the challenge was to separate the study of mathematics from religion and other theological foundations. While at first this issue may seem to have no relation to the use of mathematical symbol systems, the connection is actually very deep. Proponents of a religion-based mathematical education made an argument very similar to one made by the synthetic geometers. They believed that teaching students logic and algebra, two fields full of symbols and their manipulation, did not fully develop a student's intellect. In other words, teaching someone how to do mathematics via the rules of a symbol system did not properly teach them how to reason or how to think. This view fit nicely into a mathematics backed by religion. It preached that students learn how to understand things in their entirety and be able to reason about them effectively until they are sure of their results. On the other side of the debate were mathematicians such as De Morgan who believed that logic and algebra more closely modeled how humans actually thought about math. These secular mathematicians viewed symbol-based math as a model of cognition that was not rooted in religious beliefs. De Morgan in particular spent a large part of his life searching for valid interpretations for algebraic symbol systems so that they would have more solid and rigorous foundations. He believed that only with solid foundations could symbolic mathematics truly be an accurate model for how mathematicians reason about mathematical problems. He also believed that with solid foundations, symbol systems would no longer be thought of as just manipulating symbols per a set of rules. Rather, they would be thought of as a different way of describing mathematical ideas.

Like the debate over geometry taking place in France and Italy, the debate in England during the beginning of the nineteenth century focused on the various benefits and downsides of using mathematical symbol systems. One of the most similar connections between the two debates stems from how symbols were viewed as a tool for pedagogy. In the debate between analytic and synthetic geometry, synthetic geometers criticized how easy analytic methods were to learn. They believed that geometric insight was an essential tool for the geometer and something that could only be gained by intense and prolonged study. In the debate in England, religiously minded mathematicians also criticized the use of symbol systems as a pedagogical tool. They argued that learning only these rote systems did not adequately teach a student how to use reason and did not instill them with a passion for knowledge. In both cases though, the use of mathematical symbols were viewed as an easy method for solving (but not understanding) real world problems, even by those who advocated other methods. One difference between the two debates concerns the foundations of these symbol systems and their level of rigor. Synthetic geometers looked down on analytic methods for lacking proper, metaphysics-free foundations. However, in England, symbol systems were viewed as a tool to secularize the study of math, a tool that did not rely on any religious foundations. Supporters of the use of algebra and logic praised these systems for providing a non-religious model of human cognition. They believed that the foundations of these systems lay in the meanings ascribed to the symbols. If the given meanings were rigorous then so too would be the symbol systems. In both debates however, opponents of these formal systems attacked these calculus-based systems for lacking concrete foundations.

During the nineteenth century, the use of algebra and calculus increased all throughout the mathematical community. Significant work was done to ensure calculus stood on solid foundations. New mathematics such as set theory was developed in order to try to unify seemingly different fields of math. New, non-Euclidean geometries were discovered that made geometers question what they believed to be true. Altogether, math was a different animal heading into the twentieth century. However, these changes were not all for the better. A creeping sense of anxiety had permeated the mathematical community. This anxiety turned into a full-on foundations crisis at the beginning of the twentieth century. A group of mathematicians known as *logicists* argued to the rest of the community that the solution to this crisis was to base all of mathematics on a consistent, axiomatic logical system. Mathematicians such as Peano and Frege were pioneers in this pursuit, developing new logical systems and translating subsets of math (such as arithmetic) into their new formal systems. However paradoxes in set theory soon stopped them in their tracks. Russell and Whitehead soon took up the project and did so at an even larger scale. They developed a fully axiomatic symbol system with the goal of translating all of modern mathematics into their new system.

It's important to note that this type of symbol system is of a slightly different kind than the ones discussed so far in this paper. So far, we have looked at symbol systems used in algebra, calculus, and naive logics. These systems were all developed alongside the math they represent. Russell and Whitehead's formal system, on the other hand, was created *after* the math it aims to represent. In a sense, Russell and Whitehead's system is a language of metamathematics since it was developed to represent already developed math. Another difference related to this fact is that this system was designed with rigor, logical consistency, and solid foundations as a priority. Symbol systems of algebra and calculus were designed to make new fields of math understandable and usable; rigor and foundations were never priorities. On the other hand, because rigor was placed at such a high priority in Russell and Whitehead's system, the clarity and intuitiveness of the system was sacrificed – an observation noted by the authors themselves. They note, however, that the non-intuitiveness is only temporary. Like any other formal symbol system, if one works in it enough, it becomes more intuitive and easier to understand. When Russell and Whitehead set out to write *Principia Mathematica*, they hoped that their symbol system would soon become a popular choice for doing math, much like the symbol systems of algebra and calculus became in the nineteenth century. It was soon apparent however, that mathematicians were not keen to use this new system. To answer why, we simply have to look at the differences listed above. The symbol system written for *Principia Mathematica* prioritized rigor and consistency over ease of use and intuitiveness. Unlike algebra and calculus, which were praised for how easy they were to learn and use, Russell and Whitehead's system was complicated and non-intuitive which forever cursed it as a pedagogical tool.

After looking at a few different symbol systems used between the early nineteenth century and the early twentieth century, it is clear that the benefits of symbol systems aren't always the same. Earlier systems were popular because of how easy and mechanical they made mathematics even though they sacrificed some rigor and intuition. Later systems were popular for the solid foundations and consistency they ensured even though they might be complicated to learn and to use. Regardless of what benefits these systems have though, each of these systems uniquely affects how mathematicians can do and think about math while using them.

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