Math 2300: Discrete Mathematics
Logic 4 - Nested Quantifiers Textbook Section

Terms, Concepts, and Examples

- More than one quantifier may be necessary to capture the meaning of a statement. For example, "Everybody loves someone."

  Two quantifiers are **nested** if one is within the scope of the other.

  $$\forall x \exists y (x + y = 0)$$

- Here are some examples of translating quantified statements, you should be able to move in either direction. (symbols to words and word to symbols).

  Use integers as the domain.

$\forall x \forall y (x + y = y + x)$	For all integers x and y, $x + y = y + x$.
$\forall x \exists y (x = -y)$	For all integers x, there exists an integer y such that $x = -y$.
$\forall x \forall y ((x > 0 \wedge y < 0) \implies xy < 0)$	For all integers x and y, if x is positive and y is negative, then xy is negative. OR The product of a positive integer and a negative integer is a negative integer.
$\forall x \forall y ((x > 0) \wedge (y > 0)) \implies x + y > 0)$	The sum of two positive integers is always positive.

- Determining truth values of nested quantifiers:

$\forall x \forall y P(x, y)$	$\forall x \exists y P(x, y)$
True when $P(x, y)$ is true for every pair x, y. **False** when there is a pair x, y for which $P(x, y)$ is false.	**True** when for every x there is a y for which $P(x, y)$ is true. **False** when there is an x such that $P(x, y)$ is false for all y.
$\exists x \forall y P(x, y)$	$\exists x \exists y P(x, y)$
True when there is an x such that $P(x, y)$ is true for all y **False** when for every x there is a y for which $P(x, y)$ is false.	**True** when there is a pair x, y for which $P(x, y)$ is true. **False** when $P(x, y)$ is false for every pair x, y.

  Video Example of Nested Quantifiers

- The order of the quantifiers matters when the quantifiers are NOT the same.

  Example: Compare the following sets of statements: Assume $P(x, y)$ is the predicate $x + y = 10$.

  $\forall x \exists y P(x, y)$

  $\exists y \forall x P(x, y)$

  Solution: These statements are NOT the same. $\forall x \exists y P(x, y)$ says for all integers, there exists an integer such that $x + y = 10$. (So for instance, if $x = 7$ y would be 3). This is a True statement.

While $\exists y \forall x P(x, y)$ says, there exists an integer y such that $x + y = 10$ for all integers x. This is a False statement since the integer y changes for each x we pick so we can't find just ONE y that works for any x.

Example: Assume $P(x, y, z)$ is the predicate $x + y = z$.

$\forall x \forall y \exists z P(x, y, z)$

$\exists z \forall x \forall y P(x, y, z)$

Solution: Again these statements are NOT the same. In the first statement you pick any x and y you want and find a z that works for it. In the second statement you find one single z that works for ANY x and y you pick.

Video Example of Quantifier Order

Practice Problems

1. Let $Q(x, y)$ be the statement $xy = 0$ If the domain for both variables consists of all integers, what are the truth values of the following statements?
   (a) $Q(0, 3)$
   (b) $Q(6, 2)$
   (c) $\forall y Q(1, y)$
   (d) $\exists x Q(x, 4)$
   (e) $\forall x \exists y Q(x, y)$
   (f) $\exists x \forall y Q(x, y)$
   (g) $\forall x \forall y Q(x, y)$

2. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y", where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.
   (a) $\exists x \exists y Q(x, y)$
   (b) $\exists x \forall y Q(x, y)$
   (c) $\exists y \forall x Q(x, y)$
   (d) $\forall y \exists x Q(x, y)$
   (e) $\forall x \forall y Q(x, y)$