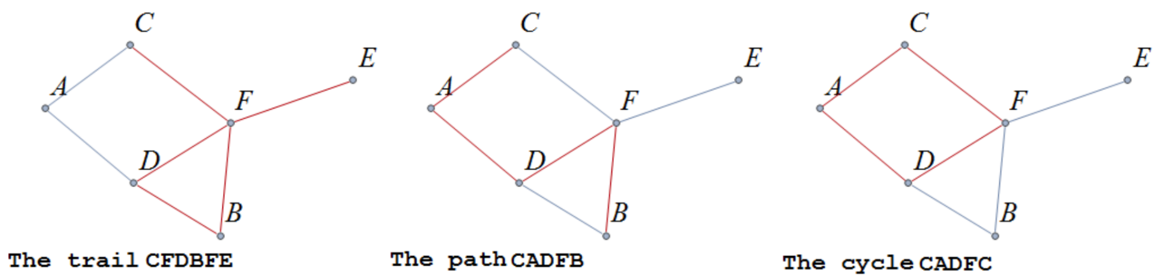
Math 2300: Discrete Mathematics
Graph Theory 2 - Euler and Hamilton Textbook Section

Terms, Concepts, and Examples

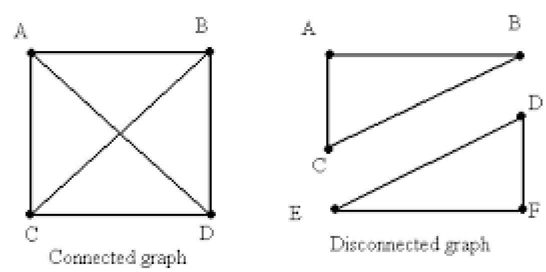
- A **walk** on a graph $G = (V, E)$ is a finite, non-empty, alternating sequence of vertices and edges. A **trail** is a walk that does not repeat an edge, ie. all edges are distinct. A **path** is a trail that does not repeat a vertex. A **circuit** (or **cycle**) is a non-empty trail in which the only repeating vertices are the beginning and ending vertices.

  Example: In the graphs below the first shows a trail $CFDBFE$. It is not a path since the vertex F is repeated. The second shows a path $CADFB$ and the third a cycle $CADFC$.



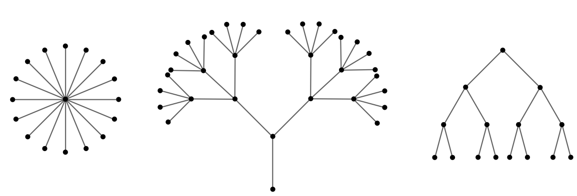
- A graph is **connected** if there is a path from each vertex to every other vertex.

  Example: In the graphs below the first shows a connected graph. The second is not connected since there is no path from vertex A to vertex E. The second graph is said to have two **connected components**.



  Video Example of Connected Graph

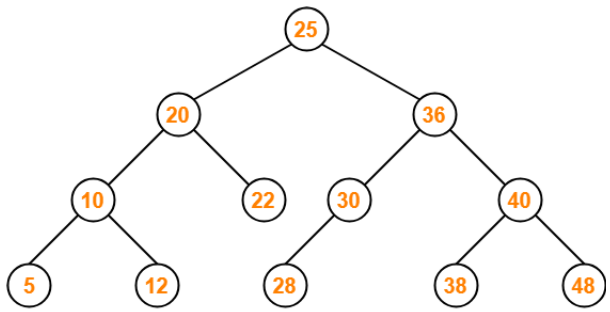
- A **tree** is a connected graph with no circuits. A group of disconnected trees is called a **forest**.



Video Example of Identifying Trees

- **Binary Search Trees** have the property that the node to the left contains a smaller value than the node pointing to it and the node to the right contains a larger value than the node pointing to it.

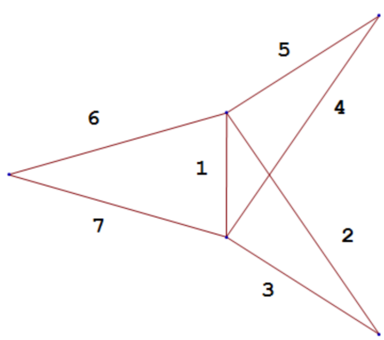
  In this image the tree begins at 25 and is ordered numerically with a node on the right if it is smaller and on the left if it is larger.



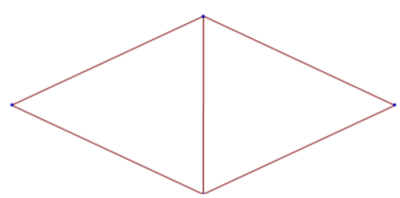
Binary Search Tree

Video Example of Binary Search Tree

- Informally an **Eulerian graph** is one in which there is a closed (beginning and ending with the same vertex) trail that includes all edges. An **Eulerian path** is a path that visits every edge in the graph. An **Eulerian circuit** is a closed (path) trail containing each edge of the graph $G = (V, E)$, exactly once and and returns to the start vertex. A graph with an Eulerian circuit is considered Eulerian or is said to be an Eulerian graph.



An Eulerian graph,with Eulerian circuit 1, 2, ..., 7



A graph that is not Eulerian

Video Example of Euler Path - Bridges of Konigsberg

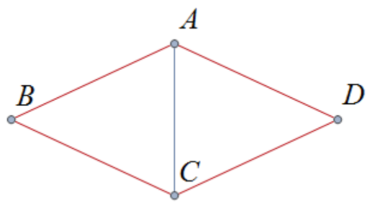
- A connected graph with at least two vertices has an Euler circuit if and only if each vertex of the graph has an even degree.

- A connected graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

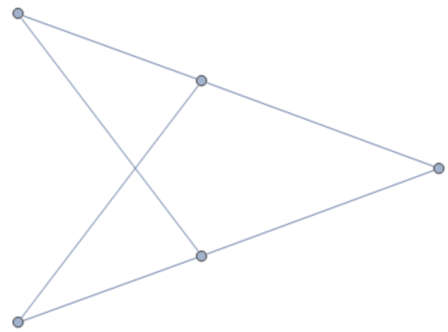
  Video Example Existence of Euler Circuit

- A cycle, in a graph $G = (V, E)$ is said to be a **Hamiltonian cycle** if every vertex, except for the starting and ending vertex in V is visited exactly once. A **Hamiltonian path** is a path that visits each vertex exactly once (but doesn't necessarily start and end in the same place).

  A graph is said to be a **Hamiltonian graph** or Hamiltonian, if it contains a Hamiltonian cycle.



A Hamiltonian graph with Hamiltonian cycle, ABCDA

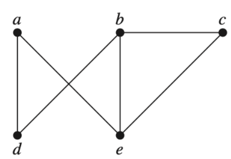


A graph that is not Hamiltonian

Video Example Hamilton Circuits

Practice Problems

1. Does each of these lists of vertices form a path in the graph? Which are circuits?



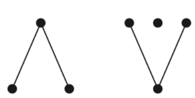
   (a) a, e, b, c, b

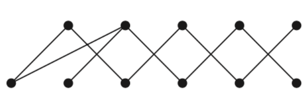
   (b) e, b, a, d, b, e

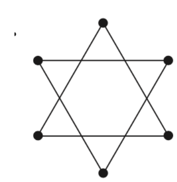
   (c) a, e, a, d, b, c, a

   (d) c, d, b, a, e, c

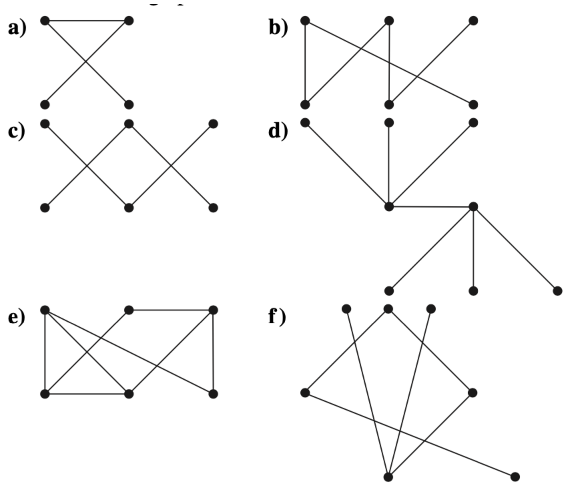
2. Determine whether the given graph is connected. How many connected components does each graph have?

   (a) 

   (b) 

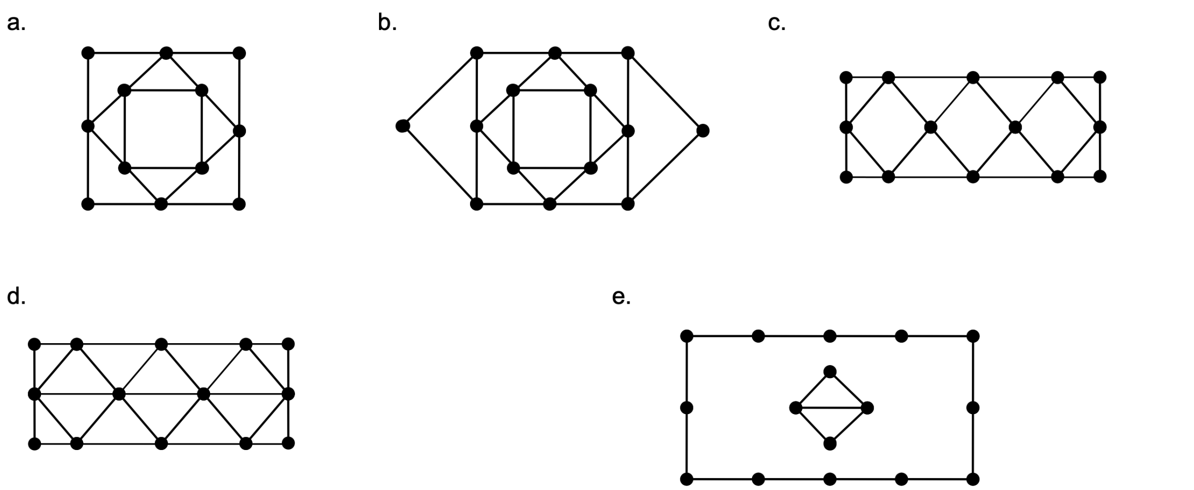
   (c) 

3. Determine whether the graphs are trees, forests, or neither.



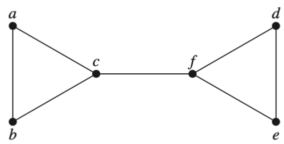
4. Build a binary search tree for the words *pepper, cucumber, broccoli, cauliflower, sweet potato, zucchini,* and *mushroom* using alphabetical order.

5. Determine whether each of the following graphs has an Euler circuit, an Euler path, or neither.

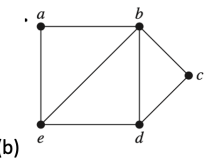


6. A night watchman must walk the streets of the green Hills subdivision. The night watchman needs to walk only once along each block. Draw a graph that models this situation.

7. Determine whether each of the following graphs has a Hamilton circuit, a Hamilton path, or neither.



(a)



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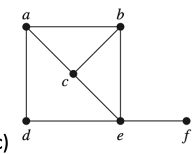
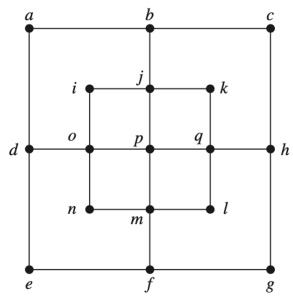
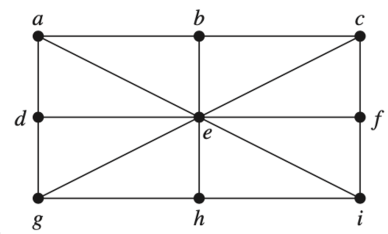


(c)



(d)



(e)