Math 2300: Discrete Mathematics

Logic 3 - Predicates and Quantifiers Textbook Section

Terms, Concepts, and Examples

The propositions and operators we have defined thus far are not enough. For example:
He is vice president.
$x + y > 10$
The truth value of the statements depends on the value of the "variables" (He, x, and y).
Even including variables is not enough! For example:
Every computer connected to the GGC network is functioning properly.

- A **predicate** is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for variables. We denote predicates as $P(x)$.

- The **domain** of a predicate is the set of all values that may be substituted in place of the variable.

  Example: Use the predicate $P(x)$: "x is a prime number" where the domain (or universe) is all positive integers. The statements $P(2), P(3), P(4), P(5), P(6), \ldots$ are all propositions with different truth values.

  $P(2)$: "2 is a prime number" is a True statement. $P(3)$: "3 is a prime number" is also a True statement. $P(4)$: "4 is a prime number" is a False statement.

  Video Example of Predicates

- **Quanitifiers** are words that refer to quantities such as *some*, or *all* and tell for how many elements a given predicate is true.

  Predicates let us make statements about groups of objects. To do this we use a special quantified expression

- The **Universal Quantifier** $\forall$ is read as "for all".

  Let $Q(x)$ be the predicate and D the domain of x. A universal statement is of the form $\forall x Q(x)$. This statement is true if and only if $Q(x)$ is true for all x in the domain. The statement is false if for at least one value of x exists such that $Q(x)$ is false. We call this one particular x a counterexample.

- The **Existential Quantifier** $\exists$ is read as "there exists".

  Let $Q(x)$ be the predicate and D the domain of x. An existential statement is of the form $\exists x Q(x)$. This statement is true if and only if $Q(x)$ is true for at least one x in the domain. The statement is false if for every value of x in the domain $Q(x)$ is false.

  Example: Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

  1. $\forall x(C(x) \implies F(x))$
  2. $\forall x(C(x) \wedge F(x))$
  3. $\exists x(C(x) \implies F(x))$
  4. $\exists x(C(x) \wedge F(x))$

Solution: $\forall x(C(x) \implies F(x))$ literally translated says "For all people, if a person is a comedian, then the person is funny." More simply, "All comedians are funny."

$\forall x(C(x) \wedge F(x))$ literally translated says "For all people, a person is a comedian and a person is funny." More simply, "Every person is a funny comedian."

$\exists x(C(x) \implies F(x))$ says "There exists a person such that if they are a comedian, then they are funny."

$\exists x(C(x) \wedge F(x))$ literally translated says "There exists a person such that they are a comedian and they are funny." More simply, "Some people are funny comedians."

Video Example on Translating with Quantifiers - Sentences to Symbols

- The table gives a summary of quantified statements.

Statement	When true?	When false?
$\forall x P(x)$	$P(x)$ true for all x	There is an x where $P(x)$ is false
$\exists x P(x)$	There is some x for which $P(x)$ is true	$P(x)$ is false for all x

Example: Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are these truth values?

1. $P(0)$
2. $P(-1)$
3. $P(1)$
4. $\exists x P(x)$
5. $P(2)$
6. $\forall x P(x)$

Solution: $P(0)$ is true since $0 = 0^2$.

$P(1)$ is true since $1 = 1^2$

$P(2)$ is false since $2 \neq 2^2$

$P(-1)$ is false since $-1 \neq (-1)^2$

$\exists x P(x)$ is true since $x = x^2$ for at least one integer (for example 0 and 1 above).

$\forall x P(x)$ is false since $x \neq x^2$ for at least one integer (for example 2 or -1 above).

Video Example on Translating with Quantifiers - Symbols to Sentences

Practice Problems

1. Let $P(x)$ be the predicate $x + 2 > 5$. What are the truth values of $P(2)$, $P(3)$, and $P(5)$?

2. (a) Write down a true universal statement.
   (b) Write down a false universal statement.
   (c) Write down a true existential statement.
   (d) Write down a false existential statement.

3. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x), D(x), F(x)$, quantifiers, and logical operators. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs and ferrets, there is a student in your class who has this animal as a pet.

4. Let $O(x)$ be "x is odd." Let $L(x)$ be "$x < 10$." Let $G(x)$ be "$x > 9$." Let the domain be all integers. What is the truth value of the following statements?

(a) $\exists x \, O(x)$

(b) $\forall x \, [L(x) \implies O(x)]$

(c) $\forall x \, [L(x) \implies \neg \, G(x)]$

(d) $\exists x \, [L(x) \wedge G(x)]$

(e) $\forall x \, [L(x) \vee G(x)]$