Math 2300: Discrete Mathematics
Number Theory 2 - Integer Representations Textbook Section

Terms, Concepts, and Examples

- **Integer Representations** - Let b be an integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

  where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$.

  Base 10 Example:

$$\begin{aligned} 394,256 &= 3 \cdot 100,000 + 9 \cdot 10,000 + 4 \cdot 1,000 + 2 \cdot 100 + 5 \cdot 10 + 6 \cdot 1 \\ &= 3 \cdot 10^5 + 9 \cdot 10^5 + 4 \cdot 10^3 + 2 \cdot 10^2 + 5 \cdot 10^1 + 6 \cdot 10^0 \end{aligned}$$

  Video on Using the Theorem

- **Base Conversion Algorithm** - An algorithm for constructing the base b expansion of an integer n. First, divide n by b to obtain a quotient and remainder, that is,

$$n = bq_0 + a_0 \qquad 0 \leq a_0 < b$$

  the remainder, a_0, is the rightmost digit in the base b expansion of n. Next, divide q_0 by b to obtain

$$q_0 = bq_1 + a_1 \qquad 0 \leq a_1 < b$$

  We see that a_1 is the second digit from the right in the base b expansion of n. Continue this process, successively dividing the quotients by b, obtaining additional base b digits as the remainders. This process terminates when we obtain a quotient equal to zero. It produces the base b digits of n from the right to the left.

  Video on Base Conversion

  1. **Binary** - uses base 2 which means powers of 2 (i.e. 1, 2, 4, 8, 16, 32)

     Examples:

$$\begin{aligned} 46 &= 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 0 \cdot 1 \\ &= 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + 0 \cdot 1 \\ &= (10\ 1110)_2 \end{aligned}$$

$$\begin{aligned} (101\ 1011)_2 &= 1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 64 + 0 \cdot 32 + 1 \cdot 16 + 1 \cdot 8 + 0 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 \\ &= 91 \end{aligned}$$

2. **Octal** - uses base 8 which means powers of 8 (i.e. 1, 8, 64, 512)
   Examples:

$$\begin{aligned} 46 &= 5 \cdot 8 + 6 \cdot 1 \\ &= 5 \cdot 8^1 + 6 \cdot 1 \\ &= (56)_8 \end{aligned}$$

$$\begin{aligned} (725)_8 &= 7 \cdot 8^2 + 2 \cdot 8^1 + 5 \cdot 8^0 \\ &= 7 \cdot 64 + 2 \cdot 8 + 5 \cdot 1 \\ &= 469 \end{aligned}$$

3. **Hexadecimal** - uses base 16 which means powers of 16 (i.e. 1, 16, 256, 4096)
   Since we only have 10 digits, hexadecimal needs to use letters as well:
   0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.
   Examples:

$$\begin{aligned} 46 &= 2 \cdot 16 + 14 \cdot 1 \\ &= 5 \cdot 16^1 + 14 \cdot 1 \\ &= (5E)_8 \end{aligned}$$

$$\begin{aligned} (B3A)_{16} &= 11 \cdot 16^2 + 3 \cdot 16^1 + 10 \cdot 16^0 \\ &= 11 * 256 + 3 * 16 + 10 \\ &= 2874 \end{aligned}$$

Video on Binary and Octal Expansions

Video on Hexadecimal Expansion

Video on Hexadecimal and Binary Expansion

- Conversion between binary, octal, and hexadecimal expansions - Each octal digit corresponds to a block of three binary digits and each hexadecimal digit corresponds to a block of four binary digits.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Video on Conversion

Practice Problems

1. Find the base 9 expansion of the following decimal numbers.
   (a) $(236)_{10}$
   (b) $(1485)_{10}$
   (c) $(230956)_{10}$

2. Convert the decimal expansion of these integers to a binary expansion, an octal expansion, and a hexadecimal expansion.
   (a) 231
   (b) 4532
   (c) 97644

3. Convert the binary expansion of each of these integers to a decimal expansion.
   (a) $(1\ 1111)_2$
   (b) $(10\ 0000\ 0001)_2$
   (c) $(1\ 0101\ 0101)_2$
   (d) $(110\ 1001\ 0001\ 0000)_2$

4. Convert the octal expansion of each of these integers to a decimal expansion.
   (a) $(123)_8$
   (b) $(41)_8$
   (c) $(7325)_8$
   (d) $(101)_8$

5. Convert the octal expansion of each of these integers to a binary expansion.
   (a) $(572)_8$
   (b) $(1604)_8$
   (c) $(423)_8$
   (d) $(2417)_8$

6. Convert the hexadecimal expansion of each of these integers to a binary expansion.
   (a) $(80E)_{16}$
   (b) $(135AB)_{16}$
   (c) $(ABBA)_{16}$
   (d) $(DEFACED)_{16}$

7. Convert the binary expansion of each of these integers to an octal expansion and a hexadecimal expansion.
   (a) $(1111\ 0111)_2$
   (b) $(1010\ 1010\ 1010)_2$
   (c) $(111\ 0111\ 0111\ 0111)_2$
   (d) $(101\ 0101\ 0101\ 0101)_2$