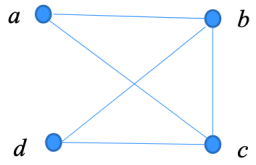
Math 2300: Discrete Mathematics
Graph Theory-Definitions Textbook Section

Terms, Concepts, and Examples

- A **graph** $G = (V, E)$ consists of a nonempty set V of **vertices** and a set E of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**. An edge is said to **connect** its endpoints.

  Example: This graph has vertices $V = \{a, b, c, d\}$ and edge set
  $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.



- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a **simple graph**. Graphs that may have multiple edges connecting the same vertices are called **multigraphs**.

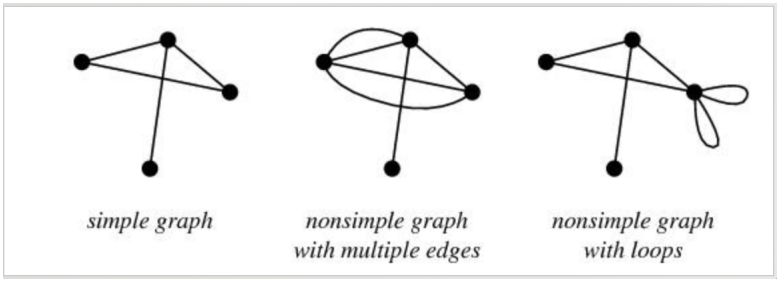
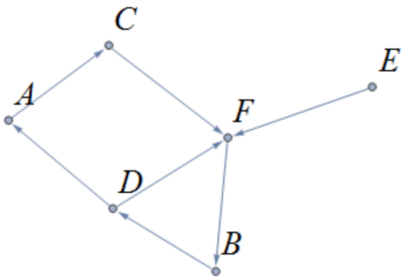


Image from MathWorld

- A **directed graph** (or digraph) (V, E) is an extension of a graph where each edge has a direction. Each directed edge is associated with an ordered pair of vertices. The directed edge with the ordered pair (u, v) is said to start at u and end at v.

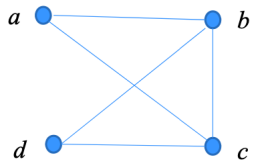
  Example: This graph has vertices $V = \{A, B, C, D, E, F\}$ and edge set
  $E = \{\{A, C\}, \{C, F\}, \{E, F\}, \{F, B\}, \{D, F\}, \{B, D\}, \{D, A\}\}$.



Video Example of Graph Types

- The **degree** of a vertex v in an undirected graph, denoted $d(v)$ is the number of edges in the graph G containing vertex v.

  Example: In our simple undirected graph from above, we see $d(a) = 2, d(b) = 3, d(c) = 3$ and $d(d) = 2$.



  Note that $d(a) + d(b) + d(c) + d(d) = 10$ which is twice the number of edges. This will always be the case.

  Video Example of Degree

- **The Handshaking Theorem**: The sum of the degrees of the vertices of a graph $G = (V, E)$ will be twice the number of edges.

$$\sum_{v \in V} d(v) = 2|E|$$

  Example: How many edges are there in a graph with 10 vertices each of degree six?

  Solution: $\sum_{v \in V} d(v) = 10 * 6 = 60$ which is twice the number of edges. This means there are 30 edges total.

  Video Example of Handshaking Theorem

Practice Problems

1. Draw graph models, stating the type of graph (simple, multigraph or directed) used to represent airlines routes where every day there are three flights from Boston to Newark, two flights from Newark to Boston, four flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, two flights from Newark to Washington, three flights from Washington to Newark, and one flight from Washington to Miami when

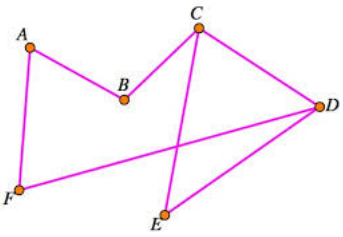
   (a) an edge between vertices representing cities that have a flight between them (in either direction).

   (b) an edge between vertices representing cities for EACH flight that operates between them (in either direction)

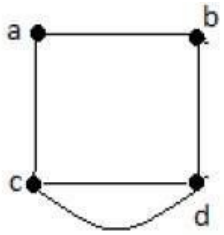
   (c) an edge from a vertex representing a city where a flight starts to the vertex representing the city where it lands.

2. Find the number of vertices, the number of edges, and the degree of each vertex in the given graph.

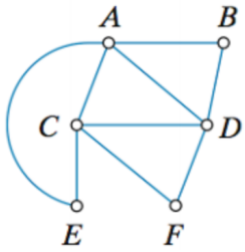
(a)



(c)



(b)



3. Can a simple graph exist with 15 vertices each of degree 5?