Math 2300: Discrete Mathematics
Recursion 1 - Sequences and Recurrence Relations Textbook Section

Terms, Concepts, and Examples

- A **sequence** is a mapping from a subset of the integers (usually $\{0, 1, 2, 3, \ldots\}$ or $\{1, 2, 3, \ldots\}$) to a set S. The notation a_n denotes the element from n. a_n is called a term of the sequence.

  Example: Consider the sequence $[a_n]$ where $a_n = \frac{n}{n+1}$. The list of the terms of this sequence, beginning with a_1
  $$a_1, a_2, a_3, a_4, \ldots$$
  starts with
  $$1, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \ldots$$
  Sometimes we give an explicit formula for a sequence (like above). Another way is to provide some initial terms together with a rule for determining subsequent terms.

- A **recurrence relation** for the sequence $[a_n]$ is an equation that express a_n in terms of one of more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a nonnegative integer. The **initial conditions** for such a recurrence relation specify the terms that precede the first term where the recurrence relation takes effect.

  Example: Let $[a_n]$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + na_{n-2} + 1$ for $n = 2, 3, 4, \ldots$ and suppose that $a_0 = 1$ and $a_1 = 2$. Find a_2, a_3 and a_4.
  $$\begin{aligned} a_2 &= a_1 + 2a_0 + 1 \\ &= 2 + 2 * 1 + 1 \\ \therefore a_2 &= 5 \end{aligned}$$
  $$\begin{aligned} a_3 &= a_2 + 3a_1 + 1 \\ &= 5 + 3 * 2 + 1 \\ \therefore a_3 &= 12 \end{aligned}$$
  $$\begin{aligned} a_4 &= a_3 + 4a_2 + 1 \\ &= 12 + 4 * 5 + 1 \\ \therefore a_4 &= 33 \end{aligned}$$

  Video Example of Recurrence Relation

- We say we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

Example: Determine whether the sequence $[a_n]$ where $a_n = 2^n$ for every nonnegative integer n is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Using the equation $a_0 = 2^0 = 1$, $a_1 = 2^1 = 2$, and $a_2 = 2^2 = 4$. Because $a_2 = a_1 - a_0 = 2 * 2 - 1 = 3$ and this is not the value of a_2, we say that $[a_n]$ where $a_n = 2^n$ is not a solution (or closed formula) for the recurrence relation.

Video Example of Checking

- We can solve a recurrence relation using **iteration**.

  Example: Solve the recurrence relation $a_n = a_{n-1} + 3$ when $a_1 = 2$.

  We are looking for a closed formula, so we will successively apply the recurrence relation until we see a pattern.

$$\begin{aligned}
a_2 &= a_1 + 3 = 2 + 3 \\
a_3 &= a_2 + 3 = (2 + 3) + 3 = 2 + 3 \cdot 2 \\
a_4 &= a_3 + 3 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3 \\
&\vdots \\
a_n &= a_{n-1} + 3 = (2 + 3(n-2)) + 3 = 2 + 3(n-1)
\end{aligned}$$

  So our closed formula is $a_n = 2 + 3(n-1)$.

  Video Example of Solving a Recurrence Relation

- A **recursively defined function** has two parts:

  BASIS STEP: Specify the value of the function at zero

  RECURSIVE STEP: Give a rule for finding its value at an integer from its value at smaller integers.

  This is similar to a recurrence relation, but using function notation.

  Example: Consider the **Fibonacci numbers** f_n where $f(0) = 0$, $f(1) = 1$ and for $n \geq 2$

$$f(n) = f(n-1) + f(n-2)$$

  Applying the formula gives

$$\begin{aligned}
f(2) &= f(1) + f(0) = 1 + 0 = 1 \\
f(3) &= f(2) + f(1) = 1 + 1 = 2 \\
f(4) &= f(3) + f(2) = 2 + 1 = 3 \\
f(5) &= f(4) + f(3) = 3 + 2 = 5 \\
f(6) &= f(5) + f(4) = 5 + 3 = 8
\end{aligned}$$

  Thinking of this as a recurrence relation we would write $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. Generating the sequence $[0, 1, 1, 2, 3, 5, 8, \ldots]$.

  Video Example of Recursive Function

Practice Problems

1. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

   (a) $a_n = 6a_{n-1},\ a_0 = 2$

   (b) $a_n = -2a_{n-1},\ a_0 = -1$

   (c) $a_n = a_{n-1}^2,\ a_1 = 2$

   (d) $a_n = na_{n-1} + a_{n-2}^2\ a_0 = -1, a_1 = 0$

   (e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$

2. Show that the sequence $[a_n]$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

   (a) $a_n = 0$

   (b) $a_n = 1$

   (c) $a_n = (-4)^n$

   (d) $a_n = 2(-4)^n + 3$

3. Find the solution to each of these recurrence relations with the given initial conditions. Use forward or backward iteration.

   (a) $a_n = -a_{n-1}, a_0 = 5$

   (b) $a_n = a_{n-1} + 3, a_0 = 1$

   (c) $a_n = 2a_{n-1} - 3, a_0 = -1$

4. Find the next four terms of the recursive function if $f(n)$ is defined recursively by

   (a) $f(n+1) = f(n) + 2, f(0) = 1$

   (b) $f(n+1) = 2^{f(n)}, f(0) = 1$

   (c) $f(n+1) = 3f(n) + 7, f(0) = 3$

   (d) $f(n+1) = f(n) + 3f(n-1), f(0) = -1, f(1) = 2$

   (e) $f(n+1) = f(n)f(n-1), f(0) = f(1) = 1$

5. Give a recursive definition of the sequence $[a_n], n = 1, 2, 3, \dots$ if

   (a) $a_n = 6n$

   (b) $a_n = 4n - 2$

   (c) $a_n = n(n+1)$

   (d) $a_n = 5$