Spatial Upscaling Of Stochastic Soil Moisture Model

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- Clarify and Generalize the Stochastic Soil Moisture Equation.
- Derive the random process differential equation of Run-off, EP and leakage.
- Spatial Upscaling of Stochastic Soil Moisture Model.

$$f(s, t + dt)ds = \underbrace{[1 - p_{rain}]f(s + \Delta s, t)d(s + \Delta s)}_{no-rain}$$

$$+ \underbrace{p_{rain}\int_0^s f(z + \Delta z, t)p_{i|z}(s - z)d(z + \Delta z)ds}_{rain}$$

$$\text{(1)}$$

$$p_{rain} = \int_{t}^{t+dt} \lambda(x)dx$$
$$= \lambda(t)dt + o(dt)$$

(2)

No Rain Condition

For the no-rain condition, the soil moisture loss during time $(t, t + dt)$ is Δs, which is the function of the soil moisture s. It could be denoted as follows:

$$\Delta s = \int_t^{t+dt} \rho[s(t)]dt \tag{3}$$

Since $s(t + dt) = s$, take the second order Taylor expansion of Δs, we have:

$$\Delta s = \rho(s)dt + o(dt) \tag{4}$$

Thus:

$$
\begin{aligned}
& f(s + \Delta s, t)d(s + \Delta s) \\
= & f(s + \rho(s)dt + o(dt), t)d(s + \rho(s)dt + o(dt)) \\
= & [f(s,t) + \frac{\partial f(s,t)}{\partial s}\rho(s)dt + o(dt)](1 + \frac{d\rho(s)}{ds}dt)ds \\
= & [f(s,t) + \frac{\partial f(s,t)}{\partial s}\rho(s)dt + f(s,t)\frac{d\rho(s)}{ds}dt + o(dt)]ds
\end{aligned}
\tag{5}
$$

Rainfall Condition

The rainfall condition in equation (1) is a typical convolution form that ensures the soil moisture to be within $(s, s + ds)$ at time $t + dt$. Δz denotes the soil moisture loss during time $(t, t + dt)$, and $z(t + dt) = z$. Thus, as derived above:

$$\Delta z = \int_t^{t+dt} \rho[z(t)]dt = \rho(z)dt + o(dt) \qquad (6)$$

so,

$$
\begin{aligned}
&dt \int_0^s f(z + \Delta z, t) p_{i|z}(s - z) d(z + \Delta z) \\
=&dt \int_0^s f(z, t) p_{i|z - \rho(z)dt - o(dt)}[s - z + \rho(z)dt + o(dt)]dz \\
=&dt \int_0^s f(z, t)\{p_{i|z}(s - z) - \frac{\partial p_{i|z}(s - z)}{\partial z}[\rho(z)dt + o(dt)]\}dz \\
=&dt \int_0^s f(z, t) p_{i|z}(s - z)dz + o(dt)
\end{aligned}
$$

$$(7)$$

Combine (1),(2),(5),(7), we have:

$$
\begin{aligned}
&f(s, t + dt)ds \\
=&f(s,t)ds + \rho(s)\frac{\partial f(s,t)}{\partial s}dtds + f(s,t)\frac{d\rho(s)}{ds}dtds - \lambda(t)f(s,t)dtds \\
&+ \lambda(t)dt \int_0^s f(z,t)p_{i|z}(s-z)dzds + o(dt)
\end{aligned}
\tag{8}
$$

Subtract $f(s,t)ds$ and divide by ds in both sides of equation (11), $lim(dt) \to 0$, we have:

$$
\frac{\partial f(s,t)}{\partial t} = \frac{\partial[\rho(s)f(s,t)]}{\partial s} - \lambda(t)f(s,t) + \lambda(t)\int_0^s f(z,t)p_{i|z}(s-z)dz
\tag{9}
$$

Equation(9) is the general form of soil moisture stochastic differential equation.

Specification

To be more specific, we notice that f(s,t) is discontinuous at the point $(0, t)$ and $(1, t)$. A Dirac function is introduced to describe f(s,t):

$$f(s, t) = g(s, t) + [\delta(s) + \delta(1 - s)](1 - F) \qquad (10)$$

where

$$F \equiv \int_{0^+}^{1^-} g(z, t)dz \qquad (11)$$

which denotes the probability that the soil moisture is within $(0^+, 1^-)$ at time t . $\delta(x)$ is the Dirac function:

$$\delta(x) \equiv \begin{cases} 0 & x \neq 0; \\ \infty & x = 0 \end{cases} \qquad (12)$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (13)$$

we define:

$$p_0(t) \equiv \int_0^{0^+} f(z, t)dz$$
$$= 1 - \int_{0^+}^1 f(z, t)dz \tag{14}$$

Consider that:

$$\int_0^{0^+} f(z, t)p_{i|z}(s - z)dz$$
$$= \int_0^{0^+} f(z, t)[p_{i|0}(s) + \frac{\partial p_{i|z}(s - z)}{\partial z}z + o(z)]dz \tag{15}$$
$$= p_{i|0}(s) \int_0^{0^+} f(z, t)dz$$
$$= p_{i|0}(s)p_0(t)$$

and

$$\int_{0^+}^{s} f(z,t)p_{i|z}(s-z)dz$$

$$= \int_{0}^{s} g(z,t)p_{i|z}(s-z)dz - \int_{0}^{0^+} g(z,t)p_{i|z}(s-z)dz \qquad (16)$$

$$= \int_{0}^{s} g(z,t)p_{i|z}(s-z)dz$$

we have:

$$\frac{\partial f(s,t)}{\partial t} = \frac{\partial \rho(s)f(s,t)}{\partial s} - \lambda(t)f(s,t) + \lambda(t)\int_{0}^{s} f(z,t)p_{i|z}(s-z)dz$$

$$= \frac{\partial \rho(s)f(s,t)}{\partial s} - \lambda(t)f(s,t) + \lambda(t)\int_{0}^{s} g(z,t)p_{i|z}(s-z)$$

$$+ \lambda(t)p_0(t)p_{i|0}(s)$$

$$(17)$$

For the situation that $s = 0$, we construct the integral form forward stochastic function:

$$p_0(t + dt)$$

$$= [1 - \lambda(t)dt]p_0(t) + [1 - \lambda(t)dt] \int_{0^+}^{\rho^{-1}(\frac{\Delta s_{s=0^+}}{dt})} p(u, t)du + o(dt)$$

$$= [1 - \lambda(t)dt]p_0(t) + \int_0^{0^+} p(u, t)du + o(dt)$$

$$= [1 - \lambda(t)dt]p_0(t) + o(dt)$$

(18)

Thus:

$$\frac{dp_0(t)}{dt} = -\lambda(t)p_0(t) \tag{19}$$

$$p_0(t) = p_0(0)e^{-\lambda(t)t} \tag{20}$$

Combine equation(17) and (20), we have the specific form of soil moisture stochastic differential function:

$$\frac{\partial f(s,t)}{\partial t} = \frac{\partial[\rho(s)f(s,t)]}{\partial s} - \lambda(t)f(s,t) + \lambda(t)\int_0^s g(z,t)p_{i|z}(s-z)dz$$
$$+ \lambda(t)p_0(0)e^{-\lambda(t)t}p_{i|0}(s)$$

$$(21)$$

The probability that there to be a normalized run-off between $(r, r + dr)$ during period $(t, t + dt)$, which is denoted as $p(r, t)drdt$, could be expressed as follows:

$$p(r, t)drdt = \int_0^1 f(z, t)[f_p(r + 1 - z, t)drdt]dz \qquad (22)$$

where $f_p(x, t)dxdt$ denotes the probability that the there to be normalized rainfall depth within $(x, x + dx)$ during $(t, t + dt)$.

Stochastic Analysis of Run-off

According to assumption, the rainfall opportunity λ and the normalized single rainfall depth r are independent random variables, thus:

$$f_p(r + 1 - z, t)drdt = p_{rain}p_{r\_depth}(r + 1 - z)dr \qquad (23)$$

where

$$p_{rain} = \lambda(t)dt + o(dt) \qquad (24)$$

Combine equation(22),(23),(24),erase the higher order term, we have:

$$p(r, t) = \lambda(t) \int_0^1 f(z, t)f_{r\_depth}(r + 1 - z)dz \qquad (25)$$

It could be interpreted as that for a single time step, the probability density for there being a run-off generated of depth r equals to $\lambda(t) \int_0^1 f(z, t)f_{r\_depth}(r + 1 - z)dz$.

$$\underbrace{\frac{\partial f(s,t)}{\partial t}}_{0} = \underbrace{\frac{\partial[\rho(s)f(s,t)]}{\partial s} - \lambda(t)f(s,t) + \lambda(t)\int_0^s g(z,t)p_{i|z}(s-z)dz}_{intergration}$$

$$+ \underbrace{\lambda(t)p_0(0)e^{-\lambda(t)t}p_{i|0}(s)}_{0}$$

- Basic physical process abstraction
- Time abstraction (in time domain upscaling) or space heterogeneity abstraction (in space domain upscaling)

stochastic description $+$ law of large numbers.

$$\lim_{n \to \infty} P\{|\sum_{i=1}^{n} E_i - n\mu| < \epsilon\} = 1 \tag{26}$$

Consider the general forward differential equation:

$$f(s, t + dt)ds = \underbrace{[1 - p_{rain}]f(s + \Delta s, t)d(s + \Delta s)}_{no-rain}$$

$$+ \underbrace{p_{rain} \int_0^s f(z + \Delta z, t)p_{i|z}(s - z)d(z + \Delta z)ds}_{rain}$$

We assume:

- Evapotranspiration and Leakage occur uniformly in the study region.
- There is a distribution pattern of soil water replenishment function.

- According to assumption 1, the spatial upscaling of no-rain condition is a linear process which requires no adaptation.
- Due to the heterogeneity of the soil, we should parameterize the soil replenishment process, which refers to $p_{i|z}(s - z)$.

Determination of $p_{i|z}(s - z)$

$p_{i|z}(s - z)$ denotes the probability density that the soil replenishment equals to $(s - z)$ on the condition that the priori soil moisture equals to z.

In the original point scale stochastic soil moisture model, this probability density equals to the probability density that the normalized rainfall depth equals to $(s - z)$, as long as $s < 1$, or not smaller than $1 - z$, when $s = 1$.

In the spatial upscaling condition, z represents the average soil moisture level of the whole region. Given z, the more heterogeneous the region is, the more it is likely to generate run-off with the same rainfall input.

We adopt the idea of Catchment Storage Capacity Curve from Xinanjiang Model to represent the effects caused by heterogeneity.

$$R = \begin{cases} P + z - 1 + [1 - \frac{P+a}{1+b}]^{1+b} & a + P \leq (1+b); \\ P + z - 1 & a + P \geq (1+b) \end{cases} \quad (27)$$

where:

$$a = (1+b)[1 - (1 - z^{\frac{1}{1+b}})] \quad (28)$$

Given:

$$R = \begin{cases} P + z - 1 + [1 - \frac{P+a}{1+b}]^{1+b} & a + P \leq (1+b); \\ P + z - 1 & a + P \geq (1+b) \end{cases} \quad (29)$$

we have:

$$I|z = \begin{cases} 1 - z - [1 - \frac{P+a}{1+b}]^{1+b} & a + P \leq (1+b); \\ 1 - z & a + P \geq (1+b) \end{cases} \quad (30)$$

thus:

$$p_{i|z}(s-z) = \begin{cases} p([1 - \frac{P+a}{1+b}]^{1+b} = 1 - s) & a + P \leq (1+b); \\ p(s=1) & a + P \geq (1+b) \end{cases} \quad (31)$$

Since:

$$p(s=1) = 0 \tag{32}$$

we have the simplified form:

$$p_{i|z}(s-z) = p_{rain\_depth}\{(1+b)[(1-z^{\frac{1}{1+b}}) - (1-s)^{\frac{1}{1+b}}]\} \tag{33}$$

and

$$p_{i|0}(s) = p_{rain\_depth}\{(1+b)[1 - (1-s)^{\frac{1}{1+b}}]\} \tag{34}$$

thus we reached the spatical upscaled stochastic soil moisture equation:

$$\begin{aligned}
\frac{\partial f(s,t)}{\partial t} &= \frac{\partial[\rho(s)f(s,t)]}{\partial s} - \lambda(t)f(s,t) \\
&+ \lambda(t)\int_0^s g(z,t)p_{rain\_depth}\{(1+b)[(1-z^{\frac{1}{1+b}}) - (1-s)^{\frac{1}{1+b}}]\}dz \\
&+ \lambda(t)p_0(0)e^{-\lambda(t)t}p_{rain\_depth}\{(1+b)[1 - (1-s)^{\frac{1}{1+b}}]\}
\end{aligned} \tag{35}$$

Strategy

Since the stochastic soil moisture model adopts the same
hydrologic procedure model as Xinanjiang Model, we first calibrate
the Xinanjiang Model in the study region, then apply the calibrated
parameters in the stochastic soil moisture model to study the
hydrologic pattern of the region.