Replica Symmetry Breaking in Ecological Models

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1 Lotka-Volterra Dyanmics (following Guy Bunin)

1.1 Setup

The Lotka-Volterra model describes the dynamics of the populations N_i of species $i = 1, \dots, S$ with the following differential equations:

$$\frac{dN_i}{dt} = \frac{r_i}{K_i} N_i \left(K_i - N_i - \sum_{j \in \{1,\dots,S\} \setminus \{i\}} \alpha_{ij} N_j, \right) \tag{1}$$

where r_i is a natural growth rate of species i, K_i is the natural carrying capacity of species i, and α_{ij} describe the interactions between species i and j. In this analysis, we will assume that these parameters are drawn from normal distributions. We will take,

$$K_i = K + \sigma_K Z_i^{(K)}, \qquad Z_i^{(K)} \sim N(0,1), \qquad \langle Z_i^{(K)} Z_j^{(K)} \rangle = \delta_{ij} \tag{2}$$

$$\alpha_{ij} = \frac{\mu_\alpha}{S} + \frac{\sigma_\alpha}{\sqrt{S}} Z_{ij}^{(\alpha)}, \qquad Z_{ij}^{(\alpha)} \sim N(0,1), \tag{3}$$

$$\left\langle Z_{ij}^{(\alpha)} Z_{ji}^{(\alpha)} \right\rangle = \gamma(1-\delta_{ij}), \qquad \left\langle Z_{ij}^{(\alpha)} Z_{ij}^{(\alpha)} \right\rangle = 1, \qquad \left\langle Z_{ij}^{(\alpha)} Z_{kl}^{(\alpha)} \right\rangle = 0, \text{ (else)}. \tag{4}$$

This last condition with all these δs might seem a bit weird; succinctly, it says $\mathrm{Cor}[\alpha_{ij}, \alpha_{ji}] = \gamma$ for $i \neq j$ and $\mathrm{Cor}[\alpha_{ij}, \alpha_{kl}] = 0$ for $i \neq j, l$ and $j \neq i, k$.

1.2 Cavity solution

In order to analyze the behavior of this model for large S, we will assume that the species' populations are replica-symmetric, meaning that averaging N_i over all species $i = 1, \ldots, S$ in one instantiation of the model and averaging over fluctuations due to $Z_i^{(K)}$ and $Z_{ij}^{(\alpha)}$ in multiple instantiations will yield the same results; this is the assumption of replica symmetry. In order to obtain self-consistency equations for the mean species' populations, we perturb the system by adding another species $i = 0$. The steady-state conditions are then,

$$\frac{dN_i}{dt} = 0 = \overline{N_i}\left(K_i - N_i - \sum_{j\in\{1,\ldots,S\}\setminus\{i\}} \alpha_{ij}\overline{N_j} - \alpha_{i0}\overline{N_0}\right) \tag{5}$$

$$\frac{dN_0}{dt} = 0 = \overline{N_0}\left(K_0 - N_0 - \sum_{j=1}^{S} \alpha_{0j}\overline{N_j}\right). \tag{6}$$

A line over a variable denotes that it is the steady-state quantity. For species $i = 1, \ldots, S$, we can treat the addition of species $i = 0$ as a perturbation to the carrying capacities: $K_i \to K_i - \alpha_{i0}\overline{N}_0$, so we model a linear response,

$$\overline{N}_j = \overline{N}_{j\setminus 0} - \sum_{k=1}^{S} \chi_{jk}\alpha_{k0}\overline{N}_0, \tag{7}$$

where we have defined the susceptibility:

$$\chi_{jk} = \frac{\partial \overline{N}_j}{\partial K_k}. \tag{8}$$

Substituting this into the steady-state condition for species $i = 0$,

$$0 = \overline{N}_0\left(K_0 - \overline{N}_0 - \sum_{j=1}^{S}\alpha_{0j}\overline{N}_{j\setminus 0} + \sum_{j,k=1}^{S}\alpha_{0j}\chi_{jk}\alpha_{k0}\overline{N}_0\right). \tag{9}$$

The last term is self-averaging (i.e., has variance of order $O(S^{-1})$) with mean,

$$\begin{aligned}
\left\langle \sum_{j,k=1}^{S}\alpha_{0j}\chi_{jk}\alpha_{k0}\overline{N}_0 \right\rangle &= \overline{N}_0 \sum_{j,k=1}^{S}\chi_{jk}\langle\alpha_{0j}\alpha_{k0}\rangle = \overline{N}_0\sum_{j,k=1}^{S}\chi_{jk}\left\langle (\frac{\mu_\alpha}{S} + \frac{\sigma_\alpha}{\sqrt{S}}Z_{0j}^{(\alpha)})(\frac{\mu_\alpha}{S} + \frac{\sigma_\alpha}{\sqrt{S}}Z_{k0}^{(\alpha)}) \right\rangle \\
&= \overline{N}_0\sum_{j,k=1}^{S}\chi_{jk}\left(\frac{\mu_\alpha^2}{S^2} + \frac{\sigma_\alpha^2}{S}\left\langle Z_{0j}^{(\alpha)}Z_{k0}^{(\alpha)}\right\rangle\right) = \overline{N}_0\sum_{j,k=1}^{S}\chi_{jk}\left(\frac{\mu_\alpha^2}{S^2} + \frac{\sigma_\alpha^2}{S}\gamma\delta_{jk}\right) \\
&= \overline{N}_0\sigma_\alpha^2\gamma\chi + O(S^{-1/2}),
\end{aligned} \tag{10}$$

where we define, $\chi = \frac{1}{S}\sum_{i=1}^{S}\chi_{ii}$. The second term has mean,

$$\left\langle \sum_{j=1}^{S}\left(\frac{\mu_\alpha}{S} + \frac{\sigma_\alpha}{\sqrt{S}}Z_{0j}^{(\alpha)}\right)\overline{N}_{j\setminus 0} \right\rangle = \frac{\mu_\alpha}{S}S\langle N\rangle + \frac{\sigma_\alpha}{\sqrt{S}}\sum_{j=1}^{S}\left\langle Z_{0j}^{(\alpha)}\overline{N}_{j\setminus 0}\right\rangle = \mu_\alpha\langle N\rangle, \tag{11}$$

where we have used $\langle Z_{0j}^{(\alpha)}\overline{N}_{j\setminus 0}\rangle = \langle Z_{0j}^{(\alpha)}\rangle\langle\overline{N}_{j\setminus 0}\rangle$. The second moment of the second term is,

$$\begin{aligned}\left\langle\left(\sum_{j=1}^{S}\left(\frac{\mu_\alpha}{S}+\frac{\sigma_\alpha}{\sqrt{S}}Z_{0j}^{(\alpha)}\right)\overline{N}_{j\setminus 0}\right)^2\right\rangle &= \left\langle\sum_{j,k=1}^{S}\left(\frac{\mu_\alpha^2}{S^2}+\frac{\mu_\alpha}{S}\frac{\sigma_\alpha}{\sqrt{S}}(Z_{0k}^{(\alpha)}+Z_{0j}^{(\alpha)})+\frac{\sigma_\alpha^2}{S}Z_{0j}^{(\alpha)}Z_{0k}^{(\alpha)}\right)\overline{N}_{k\setminus 0}\overline{N}_{j\setminus 0}\right\rangle \\ &= \frac{\mu_\alpha^2}{S^2}\sum_{j\neq k}\langle\overline{N}_{k\setminus 0}\rangle\langle\overline{N}_{j\setminus 0}\rangle+\frac{\mu_\alpha^2}{S^2}\sum_{j=1}^{S}\langle\overline{N}_{j\setminus 0}^2\rangle+0+\frac{\sigma_\alpha^2}{S}\sum_{j,k=1}^{S}\langle\overline{N}_{k\setminus 0}\overline{N}_{j\setminus 0}\rangle\gamma\delta_{jk} \\ &= \mu_\alpha^2\langle N\rangle^2+(\mu_\alpha^2+\gamma\sigma_\alpha^2)q, \end{aligned} \tag{12}$$

where,

$$q = \frac{1}{S}\sum_{i=1}^{S}\overline{N}_{i\setminus 0}^2. \tag{13}$$

If we model the second term as a normal random variable because it is a sum of independently-distributed normal random variables, we have,

$$0 = \overline{N}_0\left(K-\overline{N}_0-\mu_\alpha\langle N\rangle+\sqrt{\sigma_K^2+q(\mu_\alpha^2+\gamma\sigma_\alpha^2)}Z+\overline{N}_0\sigma_\alpha^2\gamma\chi\right) \tag{14}$$

$$= \overline{N}_0\left(g-\overline{N}_0+\sigma_g Z+\overline{N}_0\sigma_\alpha^2\gamma\chi\right), \tag{15}$$

where $Z\sim N(0,1)$, and,

$$g = K-\mu_\alpha\langle N\rangle, \qquad \sigma_g^2 = \sigma_K^2+q(\mu_\alpha^2+\gamma\sigma_\alpha^2). \tag{16}$$

Solving for $\overline{N}_0$ and keeping only physically-sensible solutions give,

$$\boxed{\overline{N}_0 = \max\left\{0, \frac{g+\sigma_g Z}{1-\gamma\sigma_\alpha^2\chi}\right\}}. \tag{17}$$

1.3 Ramp function-transformed normal distribution

In these computations, we regularly work with normal distributions that are transformed by the 'ramp' function: $\mathrm{ramp}(x)=\max\{0,x\}=x\Theta(x)$. If Z is a standard normal random variable, the PDF of $\mathrm{ramp}(\sigma Z+\mu)$ is,

$$p_{\mathrm{ramp}(\sigma Z+\mu)}(z) = \delta(z)\Phi(-\mu/\sigma)+\frac{1}{\sqrt{2\pi}\sigma}e^{-(z-\mu)^2/2\sigma^2}\Theta(z), \tag{18}$$

where,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-z^2/2}dz = \frac{1}{2}\left(1+\mathrm{erf}(x/\sqrt{2})\right), \tag{19}$$

is the standard normal CDF. The jth ($j \geq 1$) moment is then,

$$W_j(\mu,\sigma) = \langle \mathrm{ramp}(\sigma Z + \mu)^j \rangle = 0 + \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty dz z^j e^{-(z-\mu)^2/2\sigma^2} = \sigma^j \int_{-\mu/\sigma}^\infty \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} (z + \mu/\sigma)^j, \tag{20}$$

$$= \frac{2^{-3/2}}{\sqrt{\pi}} (\sqrt{2}\sigma)^j \left[j \frac{\mu}{\sigma} \Gamma\left(\frac{j}{2}\right) {}_1F_1\left(\frac{1-j}{2}; \frac{3}{2}; -\frac{\mu^2}{2\sigma^2}\right) + \sqrt{2}\Gamma\left(\frac{j+1}{2}\right) {}_1F_1\left(-\frac{j}{2}; \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right)\right], \tag{21}$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind. Observe that $W_j(\mu/\alpha, \sigma/\alpha) = \alpha^{-j} W_j(\mu,\sigma)$. Additionally,

$$W_0(x,1) = 1 \tag{22}$$

$$W_1(x,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + x\Phi(x), \tag{23}$$

$$W_2(x,1) = \frac{1}{\sqrt{2\pi}} x e^{-x^2/2} + (1+x^2)\Phi(x). \tag{24}$$

(Note: w_0 from other papers is just Φ; all other w_j match up for $j \geq 1$.) It follows from integration by parts,

$$W_2(x,1) = \Phi(x) + xW_1(x,1). \tag{25}$$

For a random variable $\Theta(\sigma Z + \mu)$, the PDF is,

$$p_{\Theta(\sigma Z+\mu)}(z) = \frac{1}{2}(1 + \mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right))\delta(z-1) + \frac{1}{2}\mathrm{erfc}\left(\frac{\mu}{\sigma\sqrt{2}}\right)\delta(z), \tag{26}$$

so the jth moment ($j \geq 1$) is,

$$\langle \Theta(\sigma Z + \mu)^j \rangle = 0 + \frac{1}{2}(1 + \mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right))1^j = \frac{1}{2}(1 + \mathrm{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)) = \Phi(\mu/\sigma). \tag{27}$$

1.4 Self-consistency equations

The fraction of surviving species is,

$$\phi = \langle \Theta(\overline{N}_0) \rangle = \langle \Theta(g + \sigma_g Z) \rangle = \Phi(\Delta), \tag{28}$$

where $\Delta = g/\sigma_g$. Taking a derivative of the cavity solution for $\overline{N}_0$ gives,

$$\frac{\partial \overline{N}_0}{\partial K} = \frac{\partial}{\partial K} \frac{g + \sigma_g Z}{1 - \gamma\sigma_\alpha^2\chi} \Theta\left(\frac{g + \sigma_g Z}{1 - \gamma\sigma_\alpha^2\chi}\right) = \frac{\Theta(\overline{N}_0)}{1 - \gamma\sigma_\alpha^2\chi} + \delta\text{-term} \implies \left\langle \frac{\partial \overline{N}_0}{\partial K} \right\rangle = \frac{\phi}{1 - \gamma\sigma_\alpha^2\chi}$$
$$\implies \chi = \frac{\phi}{1 - \chi\gamma\sigma_\alpha^2}. \tag{29}$$

Additionally,

$$\langle N\rangle = \langle \overline{N}_0\rangle = \frac{\sigma_g}{1-\gamma\sigma_\alpha^2\chi}W_1(\Delta,1) = \frac{\sigma_g}{1-\gamma\sigma_\alpha^2\chi}\left(\frac{e^{-\Delta^2/2}}{\sqrt{2\pi}} + \Delta\Phi(\Delta)\right), \tag{30}$$

$$q = \langle \overline{N}_0^2\rangle = \left(\frac{\sigma_g}{1-\gamma\sigma_\alpha^2\chi}\right)^2 W_2(\Delta,1) = \left(\frac{\sigma_g}{1-\gamma\sigma_\alpha^2\chi}\right)^2\left[\frac{\Delta e^{-\Delta^2/2}}{\sqrt{2\pi}} + (1+\Delta^2)\Phi(\Delta)\right]. \tag{31}$$

These constitute the self-consistency equations for the Lotka-Volterra model. Here is a summary of the results:

$$\boxed{\phi = \Phi(\Delta)} \tag{32}$$

$$\boxed{\chi = \frac{\phi}{1-\chi\gamma\sigma_\alpha^2}} \tag{33}$$

$$\boxed{\langle N\rangle = \frac{\sigma}{1-\gamma\sigma_\alpha^2\chi}W_1(\Delta,1)} \tag{34}$$

$$\boxed{q = \left(\frac{\sigma_g}{1-\gamma\sigma_\alpha^2\chi}\right)^2 W_2(\Delta,1)} \tag{35}$$

$$\boxed{g = K - \mu_\alpha\langle N\rangle} \tag{36}$$

$$\boxed{\sigma_g^2 = \sigma_K^2 + q\left(\mu_\alpha^2 + \gamma\sigma_\alpha^2\right)} \tag{37}$$

$$\boxed{\Delta = g/\sigma_g} \tag{38}$$

Using the integration-by-parts identity (25), these quantities can be related explicitly as,

$$(\sigma_g\chi/\phi)^{-2}q = \phi + \Delta(\sigma_g\chi/\phi)^{-1}\langle N\rangle \implies \phi q = \sigma_g^2\chi^2 + g\chi\langle N\rangle \tag{39}$$

$$\implies \phi q = [\sigma_K^2 + q(\mu_\alpha^2 + \gamma\sigma_\alpha^2)]\chi^2 + (K - \mu_\alpha\langle N\rangle)\chi\langle N\rangle \tag{40}$$

1.5 Stability analysis

Looking at the steady-state condition for the $i = 0$ species and incorporating that the last term is self-averaging,

$$0 = \overline{N}_0\left(K_0 - \overline{N}_0 - \sum_{j=1}^{S}\alpha_{0j}\overline{N}_{j\setminus 0} + \overline{N}_0\sigma_\alpha^2\gamma\chi\right) \tag{41}$$

$$= \overline{N}_0\left(g - \frac{\sigma_\alpha}{\sqrt{S}}\sum_{j=1}^{S}Z_{0j}^{(\alpha)}\overline{N}_{j\setminus 0} + \overline{N}_0(\sigma_\alpha^2\gamma\chi - 1) + \sigma_K\delta K_0\right). \tag{42}$$

For surviving species, we can solve to find,

$$\overline{N}_0^+ = \frac{1}{1-\sigma_\alpha^2\gamma\chi}\left(g - \frac{\sigma_\alpha}{\sqrt{S}}\sum_{i,\overline{N}_i>0}Z_{0i}^{(\alpha)}\overline{N}_{i\setminus 0}^+ + \sigma_K\delta K_0\right) \tag{43}$$

Next, we perturb the surviving species $\overline{N}_0^+ \to \overline{N}_0^+ + \varepsilon\eta_i$, where η_i is unit normal random variable which is independent of all other sources of randomness and ε is a small variable controlling the strength of the perturbation. After applying the perturbation and differentiating with respect to ε, we obtain,

$$\frac{d\overline{N}_0^+}{d\varepsilon} = \frac{\sigma_\alpha/\sqrt{S}}{\sigma_\alpha^2\gamma\chi - 1} \sum_{i,\overline{N}_i>0} Z_{0i}^{(\alpha)} \left(\frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} + \eta_i \right), \tag{44}$$

$$\left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 = \frac{\sigma_\alpha^2 S^{-1}}{\left(\sigma_\alpha^2\gamma\chi - 1\right)^2} \sum_{i,j,\overline{N}_i>0,\overline{N}_j>0} Z_{0i}^{(\alpha)} Z_{0j}^{(\alpha)} \left(\frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} + \eta_i \right) \left(\frac{d\overline{N}_{j\backslash 0}^+}{d\varepsilon} + \eta_j \right). \tag{45}$$

Averaging over all sources of randomness yields,

$$\begin{aligned}
\left\langle \left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 \right\rangle &= \frac{\sigma_\alpha^2 S^{-1}}{\left(\sigma_\alpha^2\gamma\chi - 1\right)^2} \sum_{i,j,\overline{N}_i>0,\overline{N}_j>0} \left\langle Z_{0i}^{(\alpha)} Z_{0j}^{(\alpha)} \right\rangle \left(\left\langle \frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} \frac{d\overline{N}_{j\backslash 0}^+}{d\varepsilon} \right\rangle + \langle \eta_i \rangle \left\langle \frac{d\overline{N}_{j\backslash 0}^+}{d\varepsilon} \right\rangle + \langle \eta_j \rangle \left\langle \frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} \right\rangle + \langle \eta_i \eta_j \rangle \right) \\
&= \frac{\sigma_\alpha^2 S^{-1}}{\left(\sigma_\alpha^2\gamma\chi - 1\right)^2} \sum_{i,j,\overline{N}_i>0,\overline{N}_j>0} \delta_{ij} \left(\left\langle \frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} \frac{d\overline{N}_{j\backslash 0}^+}{d\varepsilon} \right\rangle + \delta_{ij} \right) \\
&= \frac{\sigma_\alpha^2 \phi}{\left(\sigma_\alpha^2\gamma\chi - 1\right)^2} \left(\left\langle \left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 \right\rangle + 1 \right).
\end{aligned} \tag{46}$$

Solving for $\left\langle \left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 \right\rangle$ gives,

$$\boxed{\left\langle \left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 \right\rangle = \frac{\sigma_\alpha^2 \phi}{(1 - \chi\gamma\sigma_\alpha^2)^2 - \sigma_\alpha^2 \phi}}. \tag{47}$$

This quantity diverges, meaning that, in a typical ecosystem, at least one species becomes unstable upon perturbation when,

$$\boxed{(1 - \chi^\star\gamma(\sigma_\alpha^\star)^2)^2 - (\sigma_\alpha^\star)^2\phi^\star = 0}. \tag{48}$$

We may solve this equation for $\chi^\star$ to find, $\chi^\star = \frac{1 \pm \sigma_\alpha^\star\sqrt{\phi^\star}}{\gamma(\sigma_\alpha^\star)^2}$. Substituting this result into $1 - \chi\gamma\sigma_\alpha^2 - \phi/\chi = 0$ from the self-consistency equation for the susceptibility and keeping only the physically-sensible solution yields,

$$1 + \frac{\gamma}{1 \pm (\sigma_\alpha^\star\sqrt{\phi^\star})^{-1}} = 0 \implies \boxed{\frac{1}{\sigma_\alpha^\star} = \sqrt{\phi^\star}\,(1+\gamma)}. \tag{49}$$

2 Asymmetric Consumer-Resource Model (aCRM)

2.1 Setup

The Asymmetric Consumer-Resource Model (aCRM) describes the dynamics of the populations N_i of species $i \in \{1, \ldots, S\}$ and abundances R_α of resources $\alpha \in \{1, \ldots, M\}$ with the following coupled differential equations:

$$\frac{dN_i}{dt} = N_i \left(\sum_{\alpha=1}^{M} c_{i\alpha} R_\alpha - m_i \right), \tag{50}$$

$$\frac{dR_\alpha}{dt} = R_\alpha \left(K_\alpha - R_\alpha \right) - \sum_{j=1}^{S} N_j e_{j\alpha} R_\alpha. \tag{51}$$

Describe model parameters. We will take the model parameters to be sampled from a normal distribution:

$$K_\alpha = K + \sigma_K \delta K_\alpha, \qquad \delta K_\alpha \sim N(0,1), \qquad \langle \delta K_\alpha \delta K_\beta \rangle = \delta_{\alpha\beta} \tag{52}$$

$$m_i = m + \sigma_m \delta m, \qquad \delta m_i \sim N(0,1), \qquad \langle \delta m_i \delta m_j \rangle = \delta_{ij} \tag{53}$$

$$c_{i\alpha} = \frac{\mu_c}{M} + \frac{\sigma_c}{\sqrt{M}} d_{i\alpha}, \qquad d_{i\alpha} \sim N(0,1), \qquad \langle d_{i\alpha} d_{j\beta} \rangle = \delta_{ij} \delta_{\alpha\beta} \tag{54}$$

$$e_{i\alpha} = \frac{\mu_e}{M} + \frac{\sigma_e}{\sqrt{M}} \left(\rho d_{i\alpha} + \sqrt{1-\rho^2} x_{i\alpha} \right), \qquad x_{i\alpha} \sim N(0,1), \qquad \langle x_{i\alpha} x_{j\beta} \rangle = \delta_{ij} \delta_{\alpha\beta}, \qquad \langle x_{i\alpha} d_{j\beta} \rangle = 0 \tag{55}$$

Here, $0 < \rho \leq 1$ is a mixture parameter; additionally, let $\gamma = M/S$. If we introduce the averages over one instantiation of the model,

$$\langle R \rangle = \frac{1}{M} \sum_{\alpha=1}^{M} R_\alpha, \qquad \langle N \rangle = \frac{1}{S} \sum_{i=1}^{S} N_i, \tag{56}$$

the aCRM differential equations become,

$$\frac{dN_i}{dt} = N_i \left(g + \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} d_{i\alpha} R_\alpha - \sigma_m \delta m_i \right) \tag{57}$$

$$\frac{dR_\alpha}{dt} = R_\alpha \left(\kappa - R_\alpha - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} N_i \left(\rho d_{i\alpha} + \sqrt{1-\rho^2} x_{i\alpha} \right) + \sigma_K \delta K_\alpha \right), \tag{58}$$

where,

$$g \equiv \mu_c \langle R \rangle - m, \tag{59}$$

$$\kappa \equiv K - \mu_e \gamma^{-1} \langle N \rangle. \tag{60}$$

2.2 Cavity solution

In order to analyze the behavior of the ecological model, we will assume that species and resources are replica-symmetric, meaning that averages over all species and/or resources in a single instantiation of the model are equivalent to ensemble averages over fluctuations in model parameters $(\delta K_\alpha, \delta m, d_{i\alpha}, x_{i\alpha})$ throughout many instantiations of the model. In order to produce self-consistency equations, we will use the cavity method in which we perturb the ecosystem by introducing an additional species $i = 0$ and resource $\alpha = 0$. The aCRM differential equations for species $i = 1, \ldots, S$ become,

$$\frac{dN_i}{dt} = N_i \left(\mu_c \langle R \rangle - [m - \sigma_c M^{-1/2} d_{i0} R_0] + \sigma_c M^{-1/2} \sum_{\alpha=1}^{M} d_{i\alpha} R_\alpha - \sigma_m \delta m_i \right) \tag{61}$$

$$\frac{dR_\alpha}{dt} = R_\alpha \left(\left[K - \sigma_e M^{-1/2} N_0 \left(\rho d_{0\alpha} + \sqrt{1-\rho^2} x_{0\alpha} \right) \right] - \mu_e \gamma^{-1} \langle N \rangle - R_\alpha \right.$$
$$\left. - \sigma_e M^{-1/2} \sum_{i=1}^{S} N_i \left(\rho d_{i\alpha} + \sqrt{1-\rho^2} x_{i\alpha} \right) + \sigma_K \delta K_\alpha \right), \tag{62}$$

and for species $i = 0$ and resource $\alpha = 0$, the differential equations become,

$$\frac{dN_0}{dt} = N_0 \left(g + M^{-1/2} \sigma_c \sum_{\alpha=1}^{M} d_{0\alpha} R_\alpha + M^{-1/2} \sigma_c d_{00} R_0 - \sigma_m \delta m_0 \right) \tag{63}$$

$$\frac{dR_0}{dt} = R_0 \left(\kappa - R_0 - M^{-1/2} \sigma_e \sum_{i=1}^{S} N_j \left(\rho d_{i0} + \sqrt{1-\rho^2} x_{i0} \right) - M^{-1/2} \sigma_e N_0 \left(\rho d_{00} + \sqrt{1-\rho^2} x_{00} \right) + \sigma_K \delta K_\alpha \right). \tag{64}$$

We will analyze the steady-state behavior of this perturbed system, relative to the unperturbed system. A variable with a line on top represents a steady-state value, and a variable like $\overline{N}_{i\backslash 0}$ represents the steady-state quantity without the perturbation by species $i = 0$ and resource $\alpha = 0$. Looking at the perturbed steady-state equations (eqs., 61, 62), we see that we can treat the presence of the additional species and resource as a perturbations to model parameters: $m_i \to m_i - \sigma_c M^{-1/2} d_{i0} \overline{R}_0$ and $K_\alpha \to K_\alpha - \sigma_e M^{-1/2} \overline{N}_0 \left(\rho d_{0\alpha} + \sqrt{1-\rho^2} x_{0\alpha} \right)$. If M is sufficiently large, we can model the perturbation to the original species and resources with linear response:

$$\overline{N}_i = \overline{N}_{i\backslash 0} - \frac{\sigma_e}{\sqrt{M}} \sum_{\beta=1}^{M} \chi_{i\beta}^{(N)} \left(\rho d_{0\beta} + \sqrt{1-\rho^2} x_{0\beta} \right) \overline{N}_0 - \frac{\sigma_c}{\sqrt{M}} \sum_{j=1}^{S} \nu_{ij}^{(N)} d_{j0} \overline{R}_0, \tag{65}$$

$$\overline{R}_\alpha = \overline{R}_{\alpha\backslash 0} - \frac{\sigma_e}{\sqrt{M}} \sum_{\beta=1}^{M} \chi_{\alpha\beta}^{(R)} \left(\rho d_{0\beta} + \sqrt{1-\rho^2} x_{0\beta} \right) \overline{N}_0 - \frac{\sigma_c}{\sqrt{M}} \sum_{j=1}^{S} \nu_{\alpha j}^{(R)} d_{j0} \overline{R}_0, \tag{66}$$

where we have defined the susceptibilities,

$$\chi_{i\beta}^{(N)} \equiv \frac{\partial \overline{N}_i}{\partial K_\beta}, \qquad \chi_{\alpha\beta}^{(R)} \equiv \frac{\partial \overline{R}_\alpha}{\partial K_\beta}, \tag{67}$$

$$\nu_{ij}^{(N)} \equiv \frac{\partial \overline{N}_i}{\partial m_j}, \qquad \nu_{\alpha j}^{(R)} \equiv \frac{\partial \overline{R}_\alpha}{\partial m_j}. \tag{68}$$

2.2.1 Deriving the self-consistency equations for species populations

Substituting the linear response approximation for resources into the aCRM steady-state equation for the additional species give,

$$0 = \overline{N}_0 \left(g + \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} d_{0\alpha} \overline{R}_{\alpha\setminus 0} - \frac{\sigma_c \sigma_e}{M} \sum_{\alpha,\beta=1}^{M} \chi_{\alpha\beta}^{(R)} d_{0\alpha} \left(\rho d_{0\beta} + \sqrt{1-\rho^2} x_{0\beta} \right) \overline{N}_0 \right.$$
$$\left. - \frac{\sigma_c^2}{M} \sum_{\alpha=1}^{M} \sum_{j=1}^{S} \nu_{\alpha j}^{(R)} d_{0\alpha} d_{j0} \overline{N}_0 + \frac{\sigma_c}{\sqrt{M}} d_{00} \overline{R}_0 - \sigma_m \delta m_0 \right). \tag{69}$$

The mean of the third term (involving $\chi_{\alpha\beta}^{(R)}$) is (excluding pre-factors),

$$\left\langle \frac{\sigma_c \sigma_e}{M} \sum_{\alpha,\beta=1}^{M} \chi_{\alpha\beta}^{(R)} d_{0\alpha} \left(\rho d_{0\beta} + \sqrt{1-\rho^2} x_{0\beta} \right) \overline{N}_0 \right\rangle = \overline{N}_0 \frac{\sigma_c \sigma_e}{M} \sum_{\alpha,\beta=1}^{M} \left\langle \chi_{\alpha\beta}^{(R)} \right\rangle \left(\rho \left\langle d_{0\alpha} d_{0\beta} \right\rangle + \left\langle d_{0\alpha} \right\rangle \left\langle x_{0\beta} \right\rangle \sqrt{1-\rho^2} \right)$$
$$= \overline{N}_0 \frac{\sigma_c \sigma_e}{M} \sum_{\alpha,\beta=1}^{M} \left\langle \chi_{\alpha\beta}^{(R)} \right\rangle \left(\rho \delta_{\alpha\beta} + 0 \times 0 \sqrt{1-\rho^2} \right) = \overline{N}_0 \sigma_c \sigma_e \rho \chi, \tag{70}$$

where we have defined $\chi = M^{-1} \sum_{\alpha=1}^{M} \left\langle \chi_{\alpha\alpha}^{(R)} \right\rangle$ and used that $d_{0\alpha}$ and $x_{0\beta}$ are uncorrelated. The variance of this term is of order $O(M^{-1})$, which can be verified by expanding out the second moment. The mean of the fourth term is zero because $d_{0\alpha}$ and d_{j0} are uncorrelated when $\alpha \geq 1, j \geq 1$; the variance of the fourth term is of order $O(M^{-1})$. Keeping only terms of order $O(M^{-1})$,

$$0 = \overline{N}_0 \left(g - \sigma_c \sigma_e \rho \chi \overline{N}_0 + \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} d_{0\alpha} \overline{R}_{\alpha\setminus 0} - \sigma_m \delta m_0 \right) + O(M^{-1/2}). \tag{71}$$

The last two terms above are a sum of many independent random variables, so, by the central limit theorem, we can model these terms as a sum of normal random variables. The mean of these terms is,

$$\left\langle \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} d_{0\alpha} \overline{R}_{\alpha\setminus 0} - \sigma_m \delta m_0 \right\rangle = \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} \left\langle d_{0\alpha} \right\rangle \left\langle \overline{R}_{\alpha\setminus 0} \right\rangle - \sigma_m \left\langle \delta m_0 \right\rangle = \frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} 0 \times \left\langle \overline{R}_{\alpha\setminus 0} \right\rangle - \sigma_m \times 0 = 0. \tag{72}$$

The variance of these terms is,

$$\sigma_g^2 \equiv \mathrm{Var}\left[\frac{\sigma_c}{\sqrt{M}} \sum_{\alpha=1}^{M} d_{0\alpha} \overline{R}_{\alpha\setminus 0} - \sigma_m \delta m_0 \right] = \frac{\sigma_c^2}{M} \sum_{\alpha=1}^{M} \left(\left\langle \overline{R}_{\alpha\setminus 0} \right\rangle^2 \mathrm{Var}\left[d_{0\alpha} \right] + \mathrm{Var}\left[\overline{R}_{\alpha\setminus 0} \right] \mathrm{Var}\left[d_{0\alpha} \right] \right) + \sigma_m^2 \mathrm{Var}\left[\delta m_0 \right]$$
$$= \frac{\sigma_c^2}{M} \sum_{\alpha=1}^{M} \left(\left\langle \overline{R}_{\alpha\setminus 0} \right\rangle^2 + \left\langle \overline{R}_{\alpha\setminus 0}^2 \right\rangle - \left\langle \overline{R}_{\alpha\setminus 0} \right\rangle^2 \right) + \sigma_m^2 = \frac{\sigma_c^2}{M} \sum_{\alpha=1}^{M} \left\langle \overline{R}_{\alpha\setminus 0}^2 \right\rangle + \sigma_m^2 = \sigma_c^2 q_R + \sigma_m^2, \tag{73}$$

where we have defined,

$$q_R \equiv \frac{1}{M}\sum_{\alpha=1}^{M}\left\langle \overline{R}_{\alpha\backslash 0}^2 \right\rangle . \tag{74}$$

Let $Z_N \sim N(0,1)$ be a unit normal random variable so that the large-M limit approximate steady-state condition for the perturbing species becomes,

$$0 = \overline{N}_0 \left(g - \sigma_c\sigma_e\rho\chi\overline{N}_0 + \sigma_g Z_N\right) . \tag{75}$$

Solving for $\overline{N}_0$ and discarding non-physical solutions,

$$\overline{N}_0 = \max\left\{0, \frac{g+\sigma_g Z_N}{\sigma_c\sigma_e\rho\chi}\right\} . \tag{76}$$

2.2.2 Deriving the self-consistency equations for resource abundances

Now, we repeat this process to find a self-consistency equation for the resources. We substitute the linear response approximation for species into the aCRM steady-state equation for the additional resource:

$$\begin{aligned}
0 = \overline{R}_0 \Bigg(& \kappa - \overline{R}_0 - \frac{\sigma_e}{\sqrt{M}}\sum_{i=1}^{S}\overline{N}_{i\backslash 0}\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right) \\
& + \frac{\sigma_e^2}{M}\sum_{i=1}^{S}\sum_{\beta=1}^{M}\chi_{i\beta}^{(N)}\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right)\left(\rho d_{0\beta} + \sqrt{1-\rho^2}x_{0\beta}\right)\overline{N}_0 \\
& + \frac{\sigma_e\sigma_c}{M}\sum_{i,j=1}^{S}\nu_{ij}^{(N)}\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right)d_{j0}\overline{R}_0 - \frac{\sigma_e}{\sqrt{M}}\overline{N}_0\left(\rho d_{00} + \sqrt{1-\rho^2}x_{00}\right) + \sigma_K\delta K_\alpha \Bigg) .
\end{aligned} \tag{77}$$

First, we observe that the fourth term (involving $\chi_{i\beta}^{(N)}$) has zero mean and variance of order $O(M^{-1})$; this can be seen by recalling that $d_{i0}, d_{0\beta}, x_{i0}, x_{0\beta}$ are all independent for $i, \alpha, \beta \geq 1$. Similarly, we see that the variance of the fifth term (involving $\nu_{ij}^{(N)}$) is of order $O(M^{-1})$. We will ignore fluctuations of order $O(M^{-1})$. The mean of the fifth term is,

$$\begin{aligned}
\left\langle \frac{\sigma_e\sigma_c}{M}\sum_{i,j=1}^{S}\nu_{ij}^{(N)}\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right)d_{j0}\overline{R}_0 \right\rangle &= \overline{R}_0\frac{\sigma_e\sigma_c}{M}\sum_{i,j=1}^{S}\left\langle \nu_{ij}^{(N)}\right\rangle \left(\rho\left\langle d_{i0}d_{j0}\right\rangle + \sqrt{1-\rho^2}\left\langle x_{i0}\right\rangle\left\langle d_{j0}\right\rangle\right) \\
&= \overline{R}_0\sigma_e\sigma_c\frac{S}{M}\frac{1}{S}\sum_{i,j=1}^{S}\left\langle \nu_{ij}^{(N)}\right\rangle\left(\rho\delta_{ij} + \sqrt{1-\rho^2}\times 0\times 0\right) = \rho\sigma_e\sigma_c\gamma^{-1}\nu\overline{R}_0,
\end{aligned} \tag{78}$$

where we have defined,

$$\nu \equiv \frac{1}{S}\sum_{i=1}^{S}\left\langle \nu_{ii}^{(N)}\right\rangle . \tag{79}$$

Keeping only terms of order $O(M^{-1})$,

$$0 = \overline{R}_0 \left(\kappa - \overline{R}_0 - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} \overline{N}_{i\setminus 0} \left(\rho d_{i0} + \sqrt{1-\rho^2} x_{i0} \right) + \rho \sigma_e \sigma_c \gamma^{-1} \nu \overline{R}_0 + \sigma_K \delta K_0 \right) + O(M^{-1/2}). \tag{80}$$

Now, we model the third and last terms as a sum of a large number of independent random variables, meaning we can apply the central limit theorem and model the sum of these terms as a normal random variables. Its mean is,

$$\begin{aligned} \left\langle \sigma_K \delta K_0 - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} \overline{N}_{i\setminus 0} \left(\rho d_{i0} + \sqrt{1-\rho^2} x_{i0} \right) \right\rangle &= \sigma_K \langle \delta K_0 \rangle - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} \langle \overline{N}_{i\setminus 0} \rangle \left(\rho \langle d_{i0} \rangle + \sqrt{1-\rho^2} \langle x_{i0} \rangle \right) \\ &= \sigma_K \times 0 - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} \langle \overline{N}_{i\setminus 0} \rangle \left(\rho \times 0 + \sqrt{1-\rho^2} \times 0 \right) = 0. \end{aligned} \tag{81}$$

The variance is,

$$\begin{aligned} \sigma_\kappa^2 &\equiv \mathrm{Var}\left[\sigma_K \delta K_0 - \frac{\sigma_e}{\sqrt{M}} \sum_{i=1}^{S} \overline{N}_{i\setminus 0} \left(\rho d_{i0} + \sqrt{1-\rho^2} x_{i0} \right) \right] = \sigma_K^2 \mathrm{Var}\left[\delta K_0\right] + \frac{\sigma_e^2}{M} \sum_{i=1}^{S} \mathrm{Var}\left[\overline{N}_{i\setminus 0} \left(\rho d_{i0} + \sqrt{1-\rho^2} x_{i0} \right) \right] \\ &= \sigma_K^2 + \frac{\sigma_e^2}{M} \sum_{i=1}^{S} \left(\langle \overline{N}_{i\setminus 0} \rangle^2 \left(\rho^2 \mathrm{Var}\left[d_{i0}\right] + (1-\rho^2) \mathrm{Var}\left[x_{i0}\right] \right) + 0 \times \mathrm{Var}\left[\overline{N}_{i\setminus 0}\right] + \mathrm{Var}\left[\overline{N}_{i\setminus 0}\right] \left(\rho^2 \mathrm{Var}\left[d_{i0}\right] + (1-\rho^2) \mathrm{Var}\left[x_{i0}\right] \right) \right) \\ &= \sigma_K^2 + \frac{\sigma_e^2}{M} \sum_{i=1}^{S} \left(\langle \overline{N}_{i\setminus 0} \rangle^2 + \mathrm{Var}\left[\overline{N}_{i\setminus 0}\right] \right) = \sigma_K^2 + \sigma_e^2 \frac{S}{M} \frac{1}{S} \sum_{i=1}^{S} \left(\langle \overline{N}_{i\setminus 0} \rangle^2 + \left\langle \overline{N}_{i\setminus 0}^2 \right\rangle - \langle \overline{N}_{i\setminus 0} \rangle^2 \right) = \sigma_K^2 + \gamma^{-1} \sigma_e^2 q_N, \end{aligned} \tag{82}$$

where,

$$q_N \equiv \frac{1}{S} \sum_{i=1}^{S} \left\langle N_{i\setminus 1}^2 \right\rangle. \tag{83}$$

The approximate steady-state condition for the added resource then becomes,

$$0 = \overline{R}_0 \left(\kappa - \overline{R}_0 + \sigma_\kappa Z_R + \rho \sigma_e \sigma_c \gamma^{-1} \nu \overline{R}_0 \right). \tag{84}$$

Sovling for $\overline{R}_0$ and discarding nonphysical solutions gives,

$$\overline{R}_0 = \max\left\{ 0, \frac{\kappa + \sigma_\kappa Z_R}{1 - \rho \sigma_e \sigma_c \gamma^{-1} \nu} \right\}. \tag{85}$$

2.2.3 Final self-consistency equations

Some essential quantities of interest are the expected fraction of surviving species ϕ_N and fraction of non-depleted resources ϕ_R. These quantities are computed using the moments calculated in section 1.3 and equations 76 and 84:

$$\phi_N = \left\langle \Theta(\overline{N}_0) \right\rangle = \Phi(\Delta_g), \tag{86}$$

$$\phi_R = \left\langle \Theta(\overline{R}_0) \right\rangle = \Phi(\Delta_\kappa), \tag{87}$$

where $\Delta_\kappa \equiv \kappa/\sigma_\kappa$ and $\Delta_g \equiv g/\sigma_g$. Next, we can differentiate our expressions for $\overline{N}_0$ and $\overline{R}_0$ to get,

$$\begin{aligned}
\frac{\partial \overline{N}_0}{\partial m} &= \frac{\partial}{\partial m}\frac{g+\sigma_g Z_N}{\sigma_c\sigma_e\rho\chi}\Theta\left(\overline{N}_0\right) = -\frac{1}{\sigma_c\sigma_e\rho\chi}\Theta(\overline{N}_0) + \left[\overline{N}_0\delta(\overline{N}_0)\text{-term}\right] \\
\implies \left\langle \frac{\partial \overline{N}_0}{\partial m} \right\rangle &= \nu = -\frac{\phi_N}{\sigma_c\sigma_e\rho\chi} \qquad (88) \\
\frac{\partial \overline{R}_0}{\partial K} &= \frac{\partial}{\partial K}\frac{\kappa+\sigma_\kappa Z_R}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\Theta(\overline{R}_0) = \frac{1}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\Theta(\overline{R}_0) + \left[\overline{R}_0\delta(\overline{R}_0)\text{-term}\right] \\
\implies \left\langle \frac{\partial \overline{R}_0}{\partial K} \right\rangle &= \chi = \frac{\phi_R}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}. \qquad (89)
\end{aligned}$$

We can solve these two equations for χ, ν to obtain the relations,

$$\rho\sigma_c\sigma_e\nu = \left(\gamma^{-1} - \phi_R/\phi_N\right)^{-1}, \qquad \chi = \phi_R - \gamma^{-1}\phi_N. \tag{90}$$

Next, we use equations 76 and 84 and invoke our assumption of replica symmetry to find,

$$\langle N \rangle = \left\langle \overline{N}_0 \right\rangle = \frac{\sigma_g}{\sigma_c\sigma_e\rho\chi}W_1(\Delta_g, 1) = \frac{\sigma_g}{\sigma_c\sigma_e\rho\chi}\left(\frac{e^{-\Delta_g^2/2}}{\sqrt{2\pi}} + \Delta_g\Phi(\Delta_g)\right), \tag{91}$$

$$\langle R \rangle = \left\langle \overline{R}_0 \right\rangle = \frac{\sigma_\kappa}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}W_1(\Delta_\kappa, 1) = \frac{\sigma_\kappa}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\left(\frac{e^{-\Delta_\kappa^2/2}}{\sqrt{2\pi}} + \Delta_\kappa\Phi(\Delta_\kappa)\right), \tag{92}$$

$$q_N = \left\langle \overline{N}_0^2 \right\rangle = \left(\frac{\sigma_g}{\sigma_c\sigma_e\rho\chi}\right)^2 W_2(\Delta_g, 1) = \left(\frac{\sigma_g}{\sigma_c\sigma_e\rho\chi}\right)^2 \left(\frac{\Delta e^{-\Delta^2/2}}{\sqrt{2\pi}} + (1+\Delta^2)\Phi(\Delta)\right), \tag{93}$$

$$q_R = \left\langle \overline{R}_0^2 \right\rangle = \left(\frac{\sigma_\kappa}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\right)^2 W_2(\Delta_\kappa, 1) = \left(\frac{\sigma_\kappa}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\right)^2 \left(\frac{\Delta e^{-\Delta^2/2}}{\sqrt{2\pi}} + (1+\Delta^2)\Phi(\Delta)\right). \tag{94}$$

Recall the substitutions we made earlier:

$$\gamma = M/S \tag{95}$$

$$g \equiv \mu_c \langle R \rangle - m \,, \tag{96}$$

$$\kappa \equiv K - \mu_e \gamma^{-1} \langle N \rangle \,, \tag{97}$$

$$\sigma_g \equiv \sqrt{\sigma_m^2 + \sigma_c^2 q_R} \,, \tag{98}$$

$$\sigma_\kappa \equiv \sqrt{\sigma_K^2 + \gamma^{-1} \sigma_e^2 q_N} \,, \tag{99}$$

$$\Delta_\kappa \equiv \kappa / \sigma_\kappa \,, \tag{100}$$

$$\Delta_N \equiv g / \sigma_g \,. \tag{101}$$

We can solve for the variables $\phi_N, \phi_R, \nu, \chi, \langle N \rangle, \langle R \rangle, q_N, q_R$ using the self-consistency equations 86, 87, 88, 89, 91, 92, 93, 94. Analytically, this is intractable, so it is solved using non-linear least squares.

2.3 Stability analysis of the replica-symmetric solution

In order to analyze the stability of solutions, we will assume that a steady-state replica-symmetric solution is achieved. Then, we will perturb the solution by a small amount and analyze how the resource abundances and species populations change; if the solution diverges, we can conclude the replica symmetry ansatz is broken.

We perturb non-depleted resource abundances by $\varepsilon\eta_\alpha^{(R)}$ and surviving species populations by $\varepsilon\eta_i^{(N)}$, where $\eta_\alpha^{(R)}$ and $\eta_i^{(N)}$ are independent unit normal random variables and $\varepsilon > 0$ is small. From equations 71 and 80, for surviving species and non-depleted resources,

$$\overline{N}_0^+ = \frac{1}{\sigma_c\sigma_e\rho\chi}\left(g + \frac{\sigma_c}{\sqrt{M}}\sum_{\alpha,\overline{R}_\alpha>0} d_{0\alpha}\overline{R}_{\alpha\backslash 0}^+ - \sigma_m\delta m_0\right), \tag{102}$$

$$\overline{R}_0^+ = \frac{1}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\left(\kappa - \frac{\sigma_e}{\sqrt{M}}\sum_{i,\overline{N}_i>0}\overline{N}_{i\backslash 0}^+\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right) + \sigma_K\delta K_0\right). \tag{103}$$

Applying the perturbation gives,

$$\overline{N}_0^+ = \frac{1}{\sigma_c\sigma_e\rho\chi}\left(g + \frac{\sigma_c}{\sqrt{M}}\sum_{\alpha,\overline{R}_\alpha>0} d_{0\alpha}\left(\overline{R}_{\alpha\backslash 0}^+ + \varepsilon\eta_\alpha^{(R)}\right) - \sigma_m\delta m_0\right), \tag{104}$$

$$\overline{R}_0^+ = \frac{1}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\left(\kappa - \frac{\sigma_e}{\sqrt{M}}\sum_{i,\overline{N}_i>0}\left(\overline{N}_{i\backslash 0}^+ + \varepsilon\eta_i^{(N)}\right)\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right) + \sigma_K\delta K_0\right). \tag{105}$$

Differentiating with respect to ε yields,

$$\frac{d\overline{N}_0^+}{d\varepsilon} = \frac{1}{\sigma_e\rho\chi\sqrt{M}}\sum_{\alpha,\overline{R}_\alpha>0} d_{0\alpha}\left(\frac{d\overline{R}_{\alpha\backslash 0}^+}{d\varepsilon} + \eta_\alpha^{(R)}\right), \tag{106}$$

$$\frac{d\overline{R}_0^+}{d\varepsilon} = \frac{\sigma_e/\sqrt{M}}{1-\rho\sigma_e\sigma_c\gamma^{-1}\nu}\sum_{i,\overline{N}_i>0}\left(\overline{N}_{i\backslash 0}^+ + \varepsilon\eta_i^{(N)}\right)\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right). \tag{107}$$

We then square these quantities:

$$\left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2 = \frac{1/M}{(\sigma_e\rho\chi)^2}\sum_{\alpha,\beta,\overline{R}_\alpha>0,\overline{R}_\beta>0} d_{0\alpha}d_{0\beta}\left(\frac{d\overline{R}_{\alpha\backslash 0}^+}{d\varepsilon} + \eta_\alpha^{(R)}\right)\left(\frac{d\overline{R}_{\beta\backslash 0}^+}{d\varepsilon} + \eta_\beta^{(R)}\right), \tag{108}$$

$$\begin{aligned}\left[\frac{d\overline{R}_0^+}{d\varepsilon}\right]^2 = \frac{\sigma_e^2/M}{(1-\rho\sigma_e\sigma_c\gamma^{-1}\nu)^2}\sum_{i,j,\overline{N}_i>0,\overline{N}_j>0}&\left(\frac{d\overline{N}_{i\backslash 0}^+}{d\varepsilon} + \eta_i^{(N)}\right)\left(\frac{d\overline{N}_{j\backslash 0}^+}{d\varepsilon} + \eta_j^{(N)}\right) \\ &\times\left(\rho d_{i0} + \sqrt{1-\rho^2}x_{i0}\right)\left(\rho d_{j0} + \sqrt{1-\rho^2}x_{j0}\right).\end{aligned} \tag{109}$$

Averaging over all sources of randomness and using $\left\langle\left[\frac{dN_0}{d\varepsilon}\right]^2\right\rangle = S^{-1}\sum_{i=1}^S\left[\frac{dN_{i\backslash 0}}{d\varepsilon}\right]^2$ and $\left\langle\left[\frac{dR_0}{d\varepsilon}\right]^2\right\rangle =$

$M^{-1}\sum_{\alpha=1}^{M}\left[\frac{dR_{\alpha\setminus 0}}{d\varepsilon}\right]^2$, which follows from the replica symmetry ansatz, gives us:

$$\begin{aligned}\left\langle\left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2\right\rangle &= \frac{1/M}{(\sigma_e\rho\chi)^2}\sum_{\alpha,\beta,\overline{R}_\alpha>0,\overline{R}_\beta>0}\langle d_{0\alpha}d_{0\beta}\rangle\left(\left\langle\frac{d\overline{R}^+_{\alpha\setminus 0}}{d\varepsilon}\frac{d\overline{R}^+_{\beta\setminus 0}}{d\varepsilon}\right\rangle+\left\langle\frac{d\overline{R}^+_{\alpha\setminus 0}}{d\varepsilon}\right\rangle\left\langle\eta_\beta^{(R)}\right\rangle\right. \\ &\qquad\left.+\left\langle\eta_\alpha^{(R)}\right\rangle\left\langle\frac{d\overline{R}^+_{\beta\setminus 0}}{d\varepsilon}\right\rangle+\left\langle\eta_\alpha^{(R)}\eta_\beta^{(R)}\right\rangle\right) \qquad (110)\\ &= \frac{1/M}{(\sigma_e\rho\chi)^2}\sum_{\alpha,\beta,\overline{R}_\alpha,\overline{R}_\beta>0}\delta_{\alpha\beta}\left(\left\langle\frac{d\overline{R}_\alpha\setminus}{d\varepsilon}\frac{d\overline{R}_{\beta\setminus 0}}{d\varepsilon}\right\rangle+\delta_{\alpha\beta}\right)\\ &= \frac{\phi_R}{(\sigma_e\rho\chi)^2}\left(\left\langle\left[\frac{d\overline{R}_0^+}{d\varepsilon}\right]^2\right\rangle+1\right), \qquad (111)\end{aligned}$$

$$\begin{aligned}\left\langle\left[\frac{d\overline{R}_0^+}{d\varepsilon}\right]^2\right\rangle &= \frac{\sigma_e^2/M}{(1-\rho\sigma_e\sigma_c\gamma^{-1}\nu)^2}\sum_{i,j,\overline{N}_i>0,\overline{N}_j>0}\left(\left\langle\frac{d\overline{N}^+_{i\setminus 0}}{d\varepsilon}\frac{d\overline{N}^+_{j\setminus 0}}{d\varepsilon}\right\rangle+\left\langle\frac{d\overline{N}^+_{i\setminus 0}}{d\varepsilon}\right\rangle\left\langle\eta_j^{(N)}\right\rangle\right. \qquad (112)\\ &\quad\left.+\left\langle\eta_i^{(N)}\right\rangle\left\langle\frac{d\overline{N}^+_{j\setminus 0}}{d\varepsilon}\right\rangle+\left\langle\eta_i^{(N)}\eta_j^{(N)}\right\rangle\right)\left(\rho^2\langle d_{i0}d_{j0}\rangle+\rho\sqrt{1-\rho^2}\left(\langle d_{i0}\rangle\langle x_{j0}\rangle+\langle x_{i0}\rangle\langle d_{j0}\rangle\right)+(1-\rho^2)x_{i0}x_{j0}\right)\\ &= \frac{\sigma_e^2(S/M)/S}{(1-\rho\sigma_e\sigma_c\gamma^{-1}\nu)^2}\sum_{i,j,\overline{N}_i>0,\overline{N}_j>0}\left(\left\langle\frac{d\overline{N}^+_{i\setminus 0}}{d\varepsilon}\frac{d\overline{N}^+_{j\setminus 0}}{d\varepsilon}\right\rangle+\delta_{ij}\right)\left(\rho^2\delta_{ij}+(1-\rho^2)\delta_{ij}\right)\\ &= \frac{\sigma_e^2\gamma^{-1}\phi_N}{(1-\rho\sigma_e\sigma_c\gamma^{-1}\nu)^2}\left(\left\langle\left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2\right\rangle+1\right). \qquad (113)\end{aligned}$$

We can solve this system of equations to obtain:

$$\boxed{\left\langle\left[\frac{d\overline{N}_0^+}{d\varepsilon}\right]^2\right\rangle=\frac{\phi_R\left((1-\nu\rho\sigma_c\sigma_e\gamma^{-1})^2\sigma_e^{-2}+\gamma^{-1}\phi_N\right)}{\left[\rho\chi\left(1-\nu\rho\sigma_c\sigma_e\gamma^{-1}\right)\right]^2-\gamma^{-1}\phi_N\phi_R}}, \qquad (114)$$

$$\boxed{\left\langle\left[\frac{d\overline{R}_0^+}{d\varepsilon}\right]^2\right\rangle=\frac{\gamma^{-1}\phi_N\left(\phi_R+(\rho\sigma_e\chi)^2\right)}{\left[\rho\chi\left(1-\nu\rho\sigma_c\sigma_e\gamma^{-1}\right)\right]^2-\gamma^{-1}\phi_N\phi_R}}. \qquad (115)$$

By analyzing when these quantities diverge, we can conclude that the replica symmetry ansatz is broken when all parameters satisfy the self-consistency equations (86, 87, 88, 89, 91, 92, 93, 94) and:

$$0=\left[\rho^\star\chi^\star\left(1-\nu^\star\rho^\star\sigma_c\sigma_e\gamma^{-1}\right)\right]^2-\gamma^{-1}\phi_N^\star\phi_R^\star. \qquad (116)$$

Using the relations from line 90, we can re-write this as,

$$\phi_R\left((\rho^\star)^2\phi_R^\star-\gamma^{-1}\phi_N^\star\right)=0\implies\boxed{\rho^\star=\sqrt{\gamma^{-1}\frac{\phi_N^\star}{\phi_R^\star}}}=\sqrt{\frac{\#\text{ survivng species}}{\#\text{ non-depleted resources}}}. \qquad (117)$$