**CAS PY 452 — Quantum Physics II**

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Discussion Notes
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# WKB approximation for the triangle potential

**Problem:** Consider the 1D quantum system described by the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \kappa|\hat{x}|. \tag{1}$$

Write the appropriate Bohr-Sommerfeld quantization relation, and use the WKB method to approximate the allowed energies of the system.

**Solution:** This is a potential with no 'vertical walls', so we use the following quantization rule:

$$\int_{x_1}^{x_2} p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar, \qquad n = 1, 2, 3, \ldots \tag{2}$$

where $p(x)$ satisfies $E = \frac{p(x)^2}{2m} + \kappa|x|$, and $x_1, x_2$ are the classical turning points (i.e., $p(x_1) = p(x_2) = 0$). Solving, we have,

$$p(x) = \sqrt{2m(E - \kappa|x|)} \tag{3}$$

$$p(x) = 0 \implies E - \kappa|x| = 0 \implies x_1 = -\frac{E}{\kappa}, \quad x_2 = +\frac{E}{\kappa}. \tag{4}$$

To find the quantization condition, we are left to compute the integral:

$$\begin{aligned}\sqrt{2m}\int_{-E/\kappa}^{E/\kappa}\sqrt{E - \kappa|x|}dx &= 2\sqrt{2m}\int_0^{E/\kappa}\sqrt{E - \kappa x}dx = 2\sqrt{2mE^3/\kappa}\int_0^1\sqrt{1-u}du = 2\sqrt{2mE^3/\kappa}\int_0^1\sqrt{v}dv \\ &= \frac{4}{3}\sqrt{2mE^3/\kappa} = \left(n - \frac{1}{2}\right)\pi\hbar \implies E_n = \left(\frac{9\pi^2\kappa^2\hbar^2}{128m}\right)^{1/3}(2n-1)^{2/3}, \qquad n = 1, 2, 3, \ldots.\end{aligned} \tag{5}$$

**Further notes and comparison with the exact solution:** If $\left(9\pi^2\kappa^2\hbar^2/128m\right)^{1/3} = 0.8853 \times (\kappa^2\hbar^2/m)^{1/3}$ is small (i.e. $E^3 \gg \kappa^2\hbar^2/m$), then the spacing between energy levels is very small, and the states may seem to form a continuum. Note that this limit $E^3 \gg \kappa^2\hbar^2/m$ is similar to the small $\hbar$ limit. In this continuum, within a small range $\delta E$, the number of states is $\frac{dn}{dE}\delta E = (\frac{d}{dn}E_n)^{-1}\delta E = \left(\frac{6m}{\pi^2\kappa^2\hbar^2}\right)^{1/3}(2n-1)^{1/3}\delta E$. This quantity is the so-called 'density of states.'

In the attached *Mathematica* notebook, we solve the system exactly for $x > 0$ and $x < 0$. Then, we require continuity of the wave-function and its derivative at $x = 0$ in order to obtain a quantization condition for the energies. In figure 1, the WKB approximate energies are plotted. In figure 2, the relative errors $|E_{\text{WKB}} - E_{\text{exact}}|/|E_{\text{exact}}|$ are plotted; we can see that the WKB approximation becomes more accurate as $n$ increases/at higher energies. This is because the wavefunction is more oscillatory at higher energies, and the WKB approximation is derived from the assumption that the potential varies slowly compared to the wavefunction.

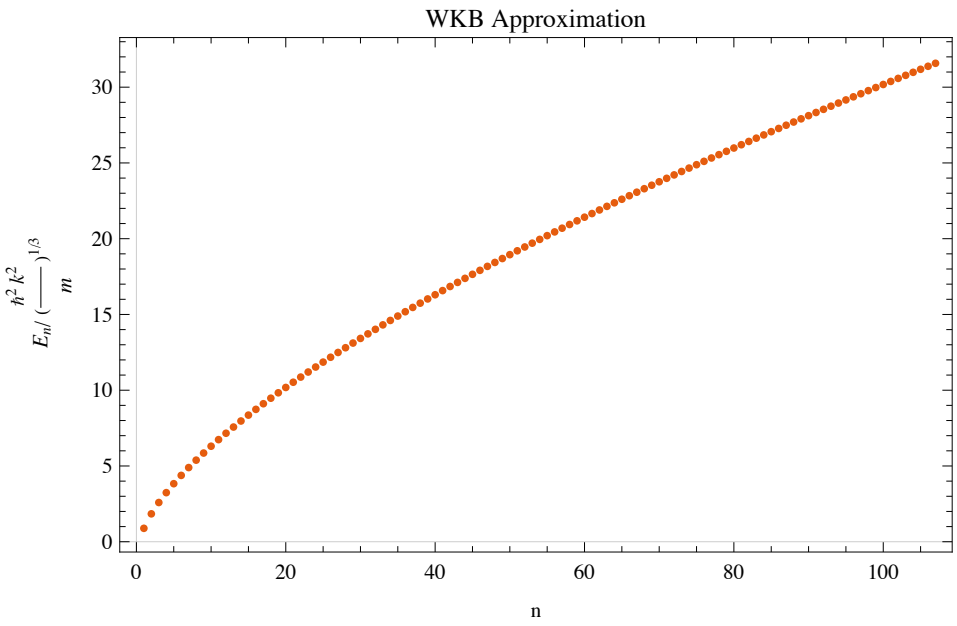


Figure 1: Plot of WKB-approximated energies $E_n$ for various quantum numbers.

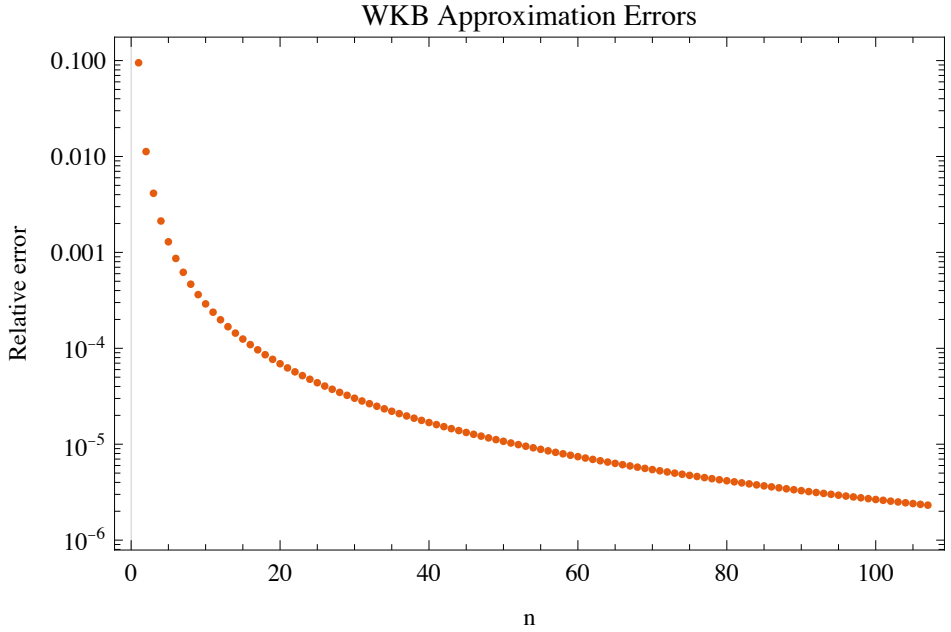


Figure 2: Log-scale plot of WKB relative errors.