CAS PY 452 — Quantum Physics II

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Discussion Notes
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WKB approximation for the triangle potential

Problem: Consider the 1D quantum system described by the Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \kappa|\hat{x}|. \tag{1}$$

Write the appropriate Bohr-Sommerfeld quantization relation and use the WKB method to approximate the allowed energies of the system.

Solution: This is a potential with no 'vertical walls', so we use the following quantization rule:

$$\int_{x_1}^{x_2} p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar, \qquad n = 1, 2, 3, \ldots \tag{2}$$

where $p(x)$ satisfies $E = \frac{p(x)^2}{2m} + \kappa|x|$, and x_1, x_2 are the classical turning points (i.e., $p(x_1) = p(x_2) = 0$). Solving, we have,

$$p(x) = \sqrt{2m(E - \kappa|x|)} \tag{3}$$

$$p(x) = 0 \implies E - \kappa|x| = 0 \implies x_1 = -\frac{E}{\kappa}, \quad x_2 = +\frac{E}{\kappa}. \tag{4}$$

To find the quantization condition, we are left to compute the integral:

$$\begin{aligned}\sqrt{2m}\int_{-E/\kappa}^{E/\kappa}\sqrt{E - \kappa|x|}dx &= 2\sqrt{2m}\int_0^{E/\kappa}\sqrt{E - \kappa x}dx = 2\sqrt{2mE^3/\kappa}\int_0^1\sqrt{1-u}du = 2\sqrt{2mE^3/\kappa}\int_0^1\sqrt{v}dv \\ &= \frac{4}{3}\sqrt{2mE^3/\kappa} = \left(n - \frac{1}{2}\right)\pi\hbar \implies E_n = \left(\frac{9\pi^2\kappa^2\hbar^2}{128m}\right)^{1/3}(2n-1)^{2/3}, \qquad n = 1, 2, 3, \ldots.\end{aligned} \tag{5}$$

Further notes and comparison with the exact solution: If $\left(9\pi^2\kappa^2\hbar^2/128m\right)^{1/3} = 0.8853 \times (\kappa^2\hbar^2/m)^{1/3}$ is small (i.e. $E^3 \ll \kappa^2\hbar^2/m$), then the spacing between energy levels is very small, and the states may seem to form a continuum. Note that this limit $E^3 \ll \kappa^2\hbar^2/m$ is similar to the small $\hbar$ limit. In this continuum, within a small range δE, the number of states is $\frac{dn}{dE}\delta E = (\frac{d}{dn}E_n)^{-1}\delta E = \left(\frac{6m}{\pi^2\kappa^2\hbar^2}\right)^{1/3}(2n-1)^{1/3}\delta E$. This quantity is the so-called 'density of states.'

In the attached *Mathematica* notebook, we solve the system exactly for $x > 0$ and $x < 0$. Then, we require continuity of the wave-function and its derivative at $x = 0$ in order to obtain a quantization condition for the energies. In figure 1, the WKB approximate energies are plotted. In figure 2, the relative errors $|E_{\text{WKB}} - E_{\text{exact}}|/|E_{\text{exact}}|$ are plotted.

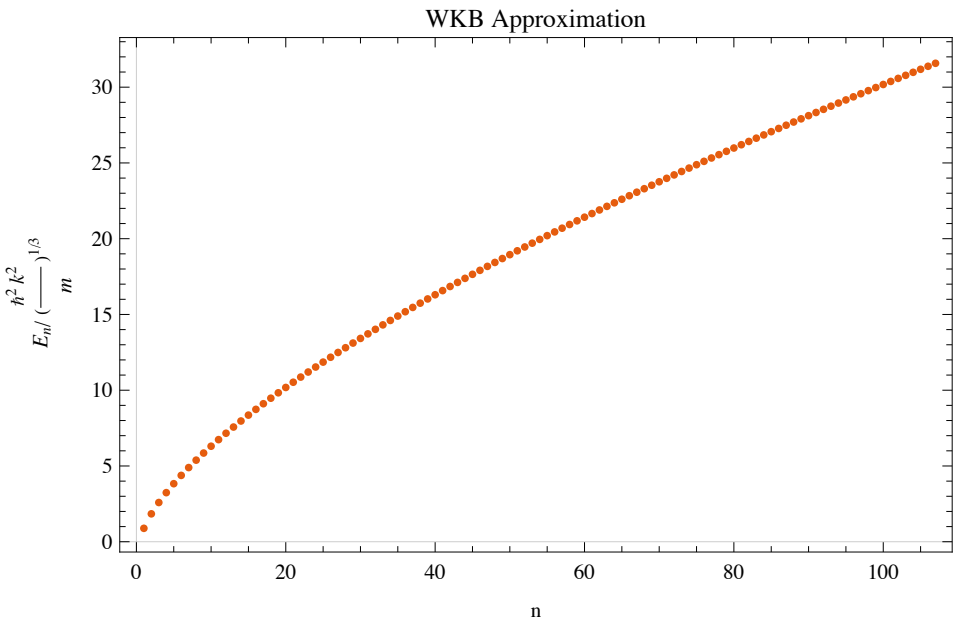

Figure 1: Plot of WKB-approximated energies E_n for various quantum numbers.

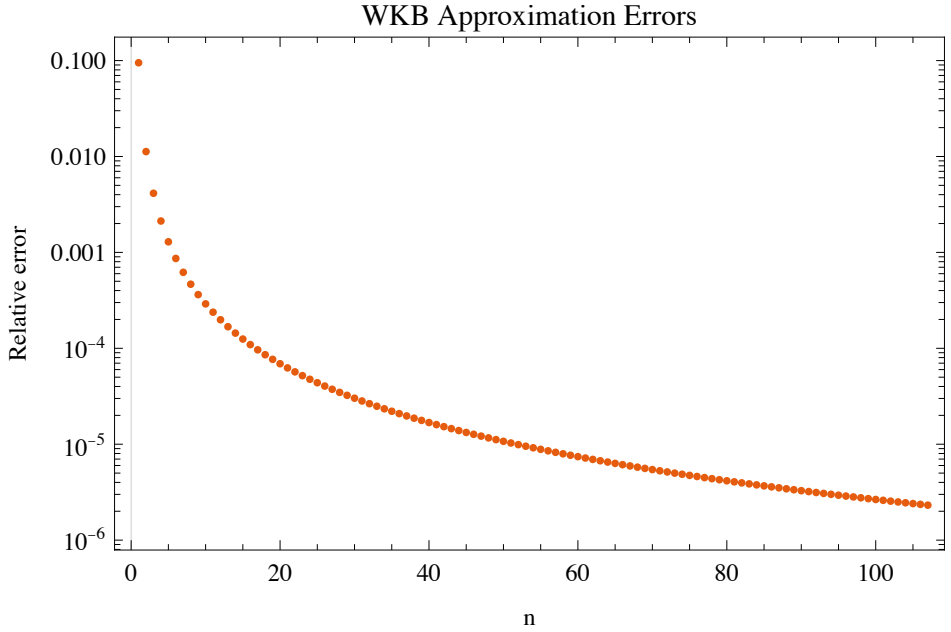

Figure 2: Log-scale plot of WKB relative errors.