CAS PY 452 — Quantum Physics II

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Discussion Notes
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Perturbing two harmonic oscillators

Problem statement: Consider a particle in a two-dimensional isotropic quantum harmonic oscillator; that is, consider the following Hamiltonian:

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}m\omega^2\hat{y}^2. \tag{1}$$

Now perturb the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}'$ with,

$$\hat{H}' = \lambda\hat{x}\hat{y} \tag{2}$$

where $\lambda \ll m\omega^2/2$. Find the first and second-order corrections to the energy of the first excited state (which is doubly-degenerate), and find the first-order corrections to the first excited state.

Solution We will use ladder operators to compute inner products; if any of this is unclear, you should look at the discussion notes from Oct. 5. Here are some identities for reference:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}), \tag{3}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}), \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a}). \tag{4}$$

Now that we are dealing with two quantum harmonic oscillators, there are two sets of ladder operators: $\hat{a}_x, \hat{a}_x^\dagger$ and $\hat{a}_y, \hat{a}_y^\dagger$. The first few states are,

$$\begin{aligned}
|\psi_{00}\rangle &= |\psi_0(x)\rangle\,|\psi_0(y)\rangle & (5)\\
|\psi_{10}\rangle &= |\psi_1(x)\rangle\,|\psi_0(y)\rangle & (6)\\
|\psi_{01}\rangle &= |\psi_0(x)\rangle\,|\psi_1(y)\rangle & (7)\\
|\psi_{11}\rangle &= |\psi_1(x)\rangle\,|\psi_1(y)\rangle & (8)\\
|\psi_{20}\rangle &= |\psi_2(x)\rangle\,|\psi_0(y)\rangle & (9)\\
|\psi_{02}\rangle &= |\psi_0(x)\rangle\,|\psi_2(y)\rangle & (10)\\
&\vdots & (11)
\end{aligned}$$

We can write the un-perturbed Hamiltonian $\hat{H} = \hbar\omega(\hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y + 10) = \hbar\omega(\hat{N}_x + \hat{N}_y + 1)$ as a matrix in the basis $|\psi_{00}\rangle, |\psi_{10}\rangle, |\psi_{01}\rangle, |\psi_{11}\rangle, |\psi_{20}\rangle, |\psi_{02}\rangle, |\psi_{21}\rangle, |\psi_{12}\rangle, \ldots$:

$$\hat{H}_0 = \hbar\omega\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\\
\cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\\
\cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots\\
\cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdots\\
\cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdots\\
\cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdots\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdots\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{12}$$

and the perturbation as a matrix:

$$
\begin{aligned}
\hat{H}'_{n_x n_y m_x m_y} &= \langle \psi_{m_x m_y} | \hat{H}' | \psi_{n_x n_y} \rangle = \lambda \langle \psi_{m_x m_y} | \hat{x}\hat{y} | \psi_{n_x n_y} \rangle = \frac{\hbar\lambda}{2m\omega} \langle \psi_{m_x m_y} | (\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_y^\dagger + \hat{a}_y) | \psi_{n_x n_y} \rangle \\
&= \frac{\hbar\lambda}{2m\omega} \langle \psi_{m_x m_y} | \, (\hat{a}_x^\dagger + \hat{a}_x)(\sqrt{n_y+1} \, |\psi_{n_x, n_y+1}\rangle + \sqrt{n_y} \, |\psi_{n_x, n_y-1}\rangle) \\
&= \frac{\hbar\lambda}{2m\omega} \langle \psi_{m_x m_y} | \, (\sqrt{(n_y+1)(n_x+1)} \, |\psi_{n_x+1, n_y+1}\rangle + \sqrt{n_y(n_x+1)} \, |\psi_{n_x+1, n_y-1}\rangle \\
&\quad + \sqrt{(n_y+1)n_x} \, |\psi_{n_x-1, n_y+1}\rangle + \sqrt{n_y n_x} \, |\psi_{n_x-1, n_y-1}\rangle) \\
&= \frac{\hbar\lambda}{2m\omega} (\sqrt{(n_y+1)(n_x+1)} \delta_{m_x, n_x+1} \delta_{m_y, n_y+1} + \sqrt{n_y(n_x+1)} \delta_{m_x, n_x+1} \delta_{m_y, n_y-1} \\
&\quad + \sqrt{(n_y+1)n_x} \delta_{m_x, n_x-1} \delta_{m_y, n_y+1} + \sqrt{n_y n_x} \delta_{m_x, n_x-1} \delta_{m_y, n_y-1})
\end{aligned}
\tag{13}
$$

$$
\implies \hat{H}' = \frac{\hbar\lambda}{2m\omega} \begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & 0 & 1 & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdots \\
\cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdots \\
1 & \cdot & \cdot & \cdot & \sqrt{2} & \sqrt{2} & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\
\cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdots \\
\cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & 2 & \cdot & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \tag{14}
$$

We can see that the first excited states, $|\psi_{10}\rangle, |\psi_{01}\rangle$, have off-diagonal matrix elements within their block of the matrix $\hat{H}'$. This means that when we compute the sum,

$$
\sum_{m_x, m_y \neq 0,1} \frac{\langle \psi_{m_x, m_y} | \hat{H}' | \psi_{n_x, n_y} \rangle}{E_n^{(0)} - E_m^{(0)}} |\psi_{m_x, m_y}\rangle, \tag{15}
$$

there will be a zero in the denominator for the $m_x, m_y = 1, 0$ term without a zero in the numerator. However, if we diagonalize the block $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ by choosing new basis states,

$$
|\psi_-\rangle = \frac{1}{\sqrt{2}} (|\psi_{01}\rangle - |\psi_{10}\rangle), \qquad |\psi_+\rangle = \frac{1}{\sqrt{2}} (|\psi_{01}\rangle + |\psi_{10}\rangle), \tag{16}
$$

we can compute,

$$
\begin{aligned}
\hat{H}'_{\pm, m_x m_y} &= \langle \psi_{m_x m_y} | \hat{H}' | \psi_\pm \rangle = \frac{\hbar\lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | \, (\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_y^\dagger + \hat{a}_y) \, (|\psi_{01}\rangle \pm |\psi_{10}\rangle) \\
&= \frac{\hbar\lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | \, (\hat{a}_x^\dagger + \hat{a}_x) \left(\sqrt{2} \, |\psi_{02}\rangle + |\psi_{00}\rangle \pm (|\psi_{11}\rangle + 0) \right) \\
&= \frac{\hbar\lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | \left(\sqrt{2}(|\psi_{12}\rangle + 0) + (|\psi_{10}\rangle + 0) \pm (\sqrt{2} \, |\psi_{21}\rangle + |\psi_{01}\rangle) \right) \\
&= \frac{\hbar\lambda}{2m\omega} \langle \psi_{m_x m_y} | \, (|\psi_{12}\rangle \pm |\psi_{21}\rangle \mp |\psi_\pm\rangle) = \frac{\hbar\lambda}{2m\omega} \left(\delta_{12, m_x m_y} \pm \delta_{21, m_x m_y} \mp \delta_{\pm, m_x m_y} \right).
\end{aligned}
$$

These new states are still energy eigenstates because a linear combination of eigenvectors with a common eigenvalue is also an eigenvector with the same eigenvalue; for a visualization of these new states in the position representation, see figure 1. In the new basis $|\psi_{00}\rangle, |\psi_-\rangle, |\psi_+\rangle, |\psi_{11}\rangle, |\psi_{20}\rangle, |\psi_{02}\rangle, |\psi_{21}\rangle, |\psi_{12}\rangle, \ldots$, the perturbed Hamiltonian is,

$$\hat{H}' = \frac{\hbar\lambda}{2m\omega}\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & -1 & 0 & \cdot & \cdot & \cdot & -1 & 1 & \cdots \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdots \\ 1 & \cdot & \cdot & \cdot & \sqrt{2} & \sqrt{2} & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & 2 & \cdots \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 2 & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{17}$$

Because we diagonalized the degenerate block by changing bases, we see that when we compute the $+,-$ terms in the sum on line 15, we do not have to deal with a zero in the denominator because the zero in the numerator takes precedence. We have discovered that when we perturb the Hamiltonian, the first excited state splits into two states $|\psi_-\rangle, |\psi_+\rangle$, and we can compute the corrections to the energies and states using the matrix elements from line 17 and 12:

$$E_-^{(1)} = \langle\psi_-|\hat{H}'|\psi_-\rangle = -\frac{\hbar\lambda}{2m\omega} \tag{18}$$

$$E_+^{(1)} = \langle\psi_+|\hat{H}'|\psi_+\rangle = +\frac{\hbar\lambda}{2m\omega} \tag{19}$$

$$E_-^{(2)} = \frac{|\langle\psi_{21}|\hat{H}'|\psi_-\rangle|^2}{\hbar\omega(2-4)} + \frac{|\langle\psi_{12}|\hat{H}'|\psi_-\rangle|^2}{\hbar\omega(2-4)} = \frac{\hbar\lambda^2}{4m^2\omega^3}\left(\frac{|-1|^2}{-2} + \frac{|1|^2}{-2}\right) = -\frac{\hbar\lambda^2}{4m^2\omega^3} \tag{20}$$

$$E_+^{(2)} = \frac{|\langle\psi_{21}|\hat{H}'|\psi_+\rangle|^2}{\hbar\omega(2-4)} + \frac{|\langle\psi_{12}|\hat{H}'|\psi_+\rangle|^2}{\hbar\omega(2-4)} = \frac{\hbar\lambda^2}{4m^2\omega^3}\left(\frac{|1|^2}{-2} + \frac{|1|^2}{-2}\right) = -\frac{\hbar\lambda^2}{4m^2\omega^3} \tag{21}$$

$$|\psi_-^{(1)}\rangle = \frac{\langle\psi_{21}|\hat{H}'|\psi_-\rangle}{\hbar\omega(2-4)}|\psi_{21}\rangle + \frac{\langle\psi_{12}|\hat{H}'|\psi_-\rangle}{\hbar\omega(2-4)}|\psi_{12}\rangle = \frac{\lambda}{2m\omega^2}\left(\frac{-1}{-2}|\psi_{21}\rangle + \frac{1}{-2}|\psi_{12}\rangle\right) = \frac{\lambda}{4m\omega^2}(|\psi_{21}\rangle - |\psi_{12}\rangle) \tag{22}$$

$$|\psi_+^{(1)}\rangle = \frac{\langle\psi_{21}|\hat{H}'|\psi_+\rangle}{\hbar\omega(2-4)}|\psi_{21}\rangle + \frac{\langle\psi_{12}|\hat{H}'|\psi_+\rangle}{\hbar\omega(2-4)}|\psi_{12}\rangle = \frac{\lambda}{2m\omega^2}\left(\frac{1}{-2}|\psi_{21}\rangle + \frac{1}{-2}|\psi_{12}\rangle\right) = -\frac{\lambda}{4m\omega^2}(|\psi_{21}\rangle + |\psi_{12}\rangle) \tag{23}$$

Exact solution By changing coordinates, we can solve this problem exactly! If we use rotated coordinates u, v so that,

$$u = \frac{1}{\sqrt{2}}(x+y), \qquad v = \frac{1}{\sqrt{2}}(x-y), \tag{24}$$

$$x = \frac{1}{\sqrt{2}}(u+v), \qquad y = \frac{1}{\sqrt{2}}(u-v), \tag{25}$$

the perturbed Hamiltonian becomes,

$$\begin{aligned}\hat{H} = \hat{H}_0 + \hat{H}' &= \frac{1}{2m}(\hat{p}_u^2 + \hat{p}_v^2) + \frac{1}{4}m\omega^2\left((u+v)^2 + (u-v)^2\right) + \lambda(u+v)(u-v) \\ &= \frac{1}{2m}(\hat{p}_u^2 + \hat{p}_v^2) + \frac{1}{2}m\omega^2\left[\left(1 + \frac{\lambda}{m\omega^2}\right)u^2 + \left(1 - \frac{\lambda}{m\omega^2}\right)v^2\right],\end{aligned} \tag{26}$$

so we have now have a non-isotropic two-dimensional harmonic oscillator. In the u-direction, $\omega_u = \omega\sqrt{1 + \lambda/(m\omega^2)}$, and in the v-direction $\omega_v = \omega\sqrt{1 - \lambda/(m\omega^2)}$. This means that the exact energies and their power series expansion are,

$$E_{n_u,n_v} = \hbar\omega\left[\sqrt{1 + \frac{\lambda}{m\omega^2}}\left(n_u + \frac{1}{2}\right) + \sqrt{1 - \frac{\lambda}{m\omega^2}}\left(n_v + \frac{1}{2}\right)\right] \tag{27}$$

$$= \hbar\omega(1 + n_u + n_v) + \lambda\frac{\hbar(n_u - n_v)}{2m\omega} - \lambda^2\frac{\hbar(1 + n_u + n_v)}{8m^2\omega^3} + O(\lambda^3), \tag{28}$$

where $(n_u, n_v) = (0,0), (1,0), (0,1), (1,1), \ldots$ are the quantum numbers. If we substitute quantum numbers $(n_u, n_v) = (0,1)$ or $(n_u, n_v) = (1,0)$ into the power series expansion, we see that we successfully recover our perturbative approximations!

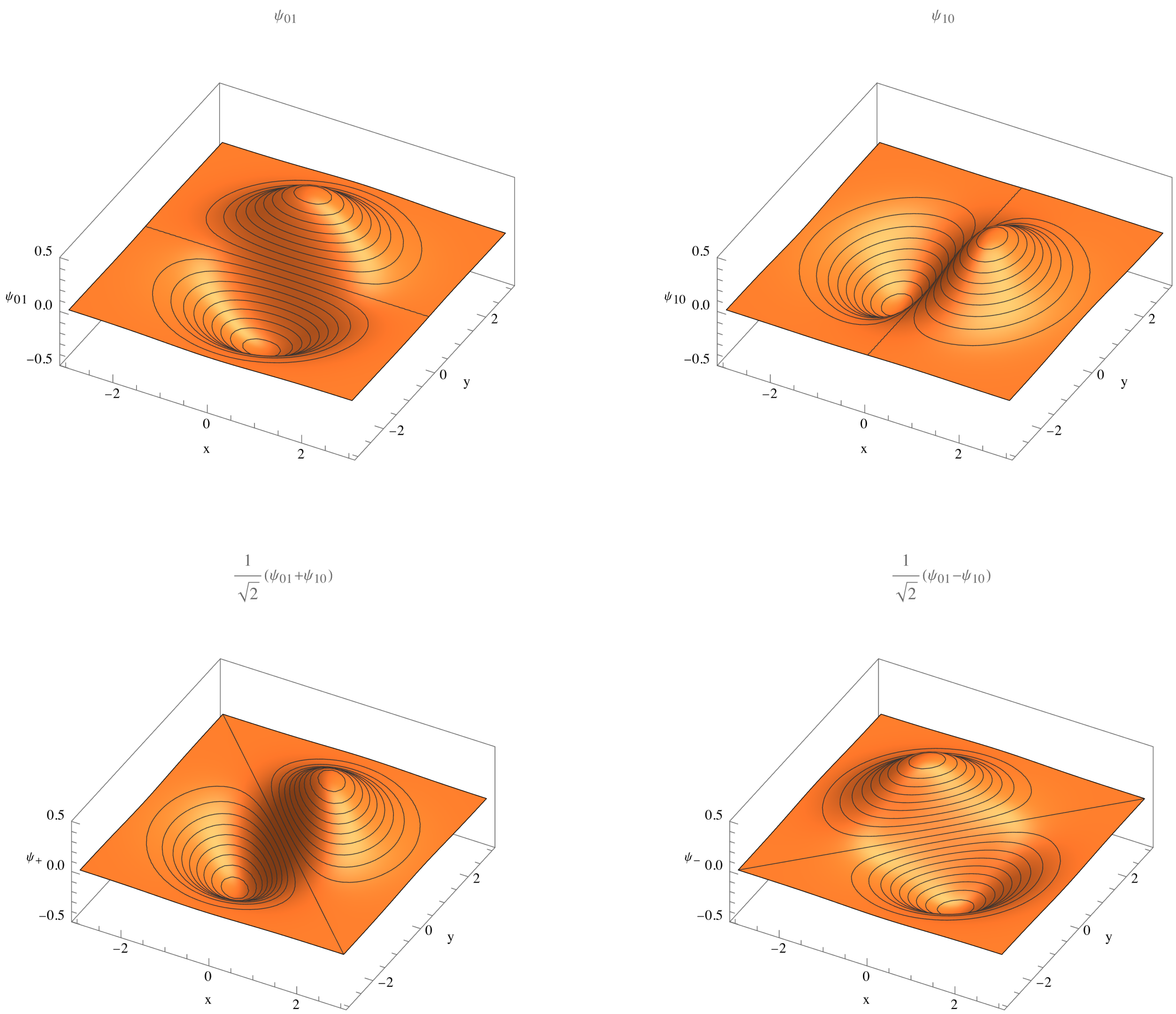

Figure 1: Number states $|\psi_{01}\rangle$, $|\psi_{10}\rangle$ and 'good' states $|\psi_+\rangle$, $|\psi_-\rangle$ in the position representation.