

## Perturbing two harmonic oscillators

**Problem statement:** Consider a particle in a two-dimensional isotropic quantum harmonic oscillator; that is, consider the following Hamiltonian:

$$\hat{H}_0 = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}m\omega^2\hat{y}^2. \quad (1)$$

Now perturb the Hamiltonian  $\hat{H} = \hat{H}_0 + \hat{H}'$  with,

$$\hat{H}' = \lambda\hat{x}\hat{y} \quad (2)$$

where  $\lambda \ll m\omega^2$ . Find the first and second-order corrections to the energy of the first excited state (which is doubly-degenerate), and find the first-order corrections to the first excited state.

**Solution** We will use ladder operators to compute inner products; if any of this is unclear, you should look at the discussion notes from Oct. 5. Here are some identities for reference:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right), \quad (3)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a}). \quad (4)$$

Now that we are dealing with two quantum harmonic oscillators, there are two sets of ladder operators:  $\hat{a}_x, \hat{a}_x^\dagger$  and  $\hat{a}_y, \hat{a}_y^\dagger$ . The first few states are,

$$|\psi_{00}\rangle = |\psi_0(x)\rangle |\psi_0(y)\rangle \quad (5)$$

$$|\psi_{10}\rangle = |\psi_1(x)\rangle |\psi_0(y)\rangle \quad (6)$$

$$|\psi_{01}\rangle = |\psi_0(x)\rangle |\psi_1(y)\rangle \quad (7)$$

$$|\psi_{11}\rangle = |\psi_1(x)\rangle |\psi_1(y)\rangle \quad (8)$$

$$|\psi_{20}\rangle = |\psi_2(x)\rangle |\psi_0(y)\rangle \quad (9)$$

$$|\psi_{02}\rangle = |\psi_0(x)\rangle |\psi_2(y)\rangle \quad (10)$$

$$\vdots \quad (11)$$

We can write the un-perturbed Hamiltonian  $\hat{H} = \hbar\omega(\hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y + 10) = \hbar\omega(\hat{N}_x + \hat{N}_y + 10)$  as a matrix in the basis  $|\psi_{00}\rangle, |\psi_{10}\rangle, |\psi_{01}\rangle, |\psi_{11}\rangle, |\psi_{20}\rangle, |\psi_{02}\rangle, |\psi_{21}\rangle, |\psi_{12}\rangle, \dots$ :

$$\hat{H}_0 = \hbar\omega \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (12)$$

and the perturbation as a matrix:

$$\begin{aligned}
\hat{H}'_{n_x n_y m_x m_y} &= \langle \psi_{m_x m_y} | \hat{H}' | \psi_{n_x n_y} \rangle = \lambda \langle \psi_{m_x m_y} | \hat{x} \hat{y} | \psi_{n_x n_y} \rangle = \frac{\hbar \lambda}{2m\omega} \langle \psi_{m_x m_y} | (\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_y^\dagger + \hat{a}_y) | \psi_{n_x n_y} \rangle \\
&= \frac{\hbar \lambda}{2m\omega} \langle \psi_{m_x m_y} | (\hat{a}_x^\dagger + \hat{a}_x)(\sqrt{n_y + 1} |\psi_{n_x, n_y+1}\rangle + \sqrt{n_y} |\psi_{n_x, n_y-1}\rangle) \\
&= \frac{\hbar \lambda}{2m\omega} \langle \psi_{m_x m_y} | (\sqrt{(n_y + 1)(n_x + 1)} |\psi_{n_x+1, n_y+1}\rangle + \sqrt{n_y(n_x + 1)} |\psi_{n_x+1, n_y-1}\rangle \\
&\quad + \sqrt{(n_y + 1)n_x} |\psi_{n_x-1, n_y+1}\rangle + \sqrt{n_y n_x} |\psi_{n_x-1, n_y-1}\rangle) \\
&= \frac{\hbar \lambda}{2m\omega} (\sqrt{(n_y + 1)(n_x + 1)} \delta_{m_x, n_x+1} \delta_{m_y, n_y+1} + \sqrt{n_y(n_x + 1)} \delta_{m_x, n_x+1} \delta_{m_y, n_y-1} \\
&\quad + \sqrt{(n_y + 1)n_x} \delta_{m_x, n_x-1} \delta_{m_y, n_y+1} + \sqrt{n_y n_x} \delta_{m_x, n_x-1} \delta_{m_y, n_y-1})
\end{aligned} \tag{13}$$

$$\Rightarrow \hat{H}' = \frac{\hbar \lambda}{2m\omega} \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdots \\ \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdots \\ 1 & \cdot & \cdot & \cdot & \sqrt{2} & \sqrt{2} & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdots \\ \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & 2 & \cdot & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{14}$$

We can see that the first excited states,  $|\psi_{10}\rangle, |\psi_{01}\rangle$ , have off-diagonal matrix elements within their block of the matrix  $\hat{H}'$ . This means that when we compute the sum,

$$\sum_{m_x, m_y \neq 0, 1} \frac{\langle \psi_{m_x, m_y} | \hat{H}' | \psi_{n_x, n_y} \rangle}{E_n^{(0)} - E_m^{(0)}} |\psi_{m_x, m_y}\rangle, \tag{15}$$

there will be a zero in the denominator for the  $m_x, m_y = 1, 0$  term without a zero in the numerator. However, if we diagonalize the block  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  by choosing new basis states,

$$|\psi_{-}\rangle = \frac{1}{\sqrt{2}} (|\psi_{01}\rangle - |\psi_{10}\rangle), \quad |\psi_{+}\rangle = \frac{1}{\sqrt{2}} (|\psi_{01}\rangle + |\psi_{10}\rangle), \tag{16}$$

we can compute,

$$\begin{aligned}
\hat{H}'_{\pm, m_x m_y} &= \langle \psi_{m_x m_y} | \hat{H}' | \psi_{\pm} \rangle = \frac{\hbar \lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | (\hat{a}_x^\dagger + \hat{a}_x)(\hat{a}_y^\dagger + \hat{a}_y) (|\psi_{01}\rangle \pm |\psi_{10}\rangle) \\
&= \frac{\hbar \lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | (\hat{a}_x^\dagger + \hat{a}_x) (\sqrt{2} |\psi_{02}\rangle + |\psi_{00}\rangle \pm (|\psi_{11}\rangle + 0)) \\
&= \frac{\hbar \lambda}{2\sqrt{2}m\omega} \langle \psi_{m_x m_y} | (\sqrt{2} (|\psi_{12}\rangle + 0) + (|\psi_{10}\rangle + 0) \pm (\sqrt{2} |\psi_{21}\rangle + |\psi_{01}\rangle)) \\
&= \frac{\hbar \lambda}{2m\omega} \langle \psi_{m_x m_y} | (|\psi_{12}\rangle \pm |\psi_{21}\rangle \mp |\psi_{\pm}\rangle) = \frac{\hbar \lambda}{2m\omega} (\delta_{12, m_x m_y} \pm \delta_{21, m_x m_y} \mp \delta_{\pm, m_x m_y}).
\end{aligned}$$

These new states are still energy eigenstates because a linear combination of eigenvectors with a common eigenvalue is also an eigenvector with the same eigenvalue; for a visualization of these new states in the position representation,

see figure 1. In the new basis  $|\psi_{00}\rangle, |\psi_{-}\rangle, |\psi_{+}\rangle, |\psi_{11}\rangle, |\psi_{20}\rangle, |\psi_{02}\rangle, |\psi_{21}\rangle, |\psi_{12}\rangle, \dots$ , the perturbed Hamiltonian is,

$$\hat{H}' = \frac{\hbar\lambda}{2m\omega} \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & -1 & 0 & \cdot & \cdot & \cdot & -1 & 1 & \dots \\ \cdot & 0 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \dots \\ 1 & \cdot & \cdot & \cdot & \sqrt{2} & \sqrt{2} & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & 2 & \dots \\ \cdot & 1 & 1 & \cdot & \cdot & \cdot & 2 & \cdot & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (17)$$

Because we diagonalized the degenerate block by changing bases, we see that when we compute the  $+$ ,  $-$  terms in the sum on line 15, we do not have to deal with a zero in the denominator because the zero in the numerator takes precedence. We have discovered that when we perturb the Hamiltonian, the first excited state splits into two states  $|\psi_{-}\rangle, |\psi_{+}\rangle$ , and we can compute the corrections to the energies and states using the matrix elements from line 17 and 12:

$$E_{-}^{(1)} = \langle \psi_{-} | \hat{H}' | \psi_{-} \rangle = -\frac{\hbar\lambda}{2m\omega} \quad (18)$$

$$E_{+}^{(1)} = \langle \psi_{+} | \hat{H}' | \psi_{+} \rangle = +\frac{\hbar\lambda}{2m\omega} \quad (19)$$

$$E_{-}^{(2)} = \frac{|\langle \psi_{21} | \hat{H}' | \psi_{-} \rangle|^2}{\hbar\omega(2-4)} + \frac{|\langle \psi_{12} | \hat{H}' | \psi_{-} \rangle|^2}{\hbar\omega(2-4)} = \frac{\hbar\lambda^2}{4m^2\omega^3} \left( \frac{|-1|^2}{-2} + \frac{|1|^2}{-2} \right) = -\frac{\hbar\lambda^2}{4m^2\omega^3} \quad (20)$$

$$E_{+}^{(2)} = \frac{|\langle \psi_{21} | \hat{H}' | \psi_{+} \rangle|^2}{\hbar\omega(2-4)} + \frac{|\langle \psi_{12} | \hat{H}' | \psi_{+} \rangle|^2}{\hbar\omega(2-4)} = \frac{\hbar\lambda^2}{4m^2\omega^3} \left( \frac{|1|^2}{-2} + \frac{|1|^2}{-2} \right) = -\frac{\hbar\lambda^2}{4m^2\omega^3} \quad (21)$$

$$|\psi_{-}^{(1)}\rangle = \frac{\langle \psi_{21} | \hat{H}' | \psi_{-} \rangle}{\hbar\omega(2-4)} |\psi_{21}\rangle + \frac{\langle \psi_{12} | \hat{H}' | \psi_{-} \rangle}{\hbar\omega(2-4)} |\psi_{12}\rangle = \frac{\lambda}{2m\omega^2} \left( \frac{-1}{-2} |\psi_{21}\rangle + \frac{1}{-2} |\psi_{12}\rangle \right) = \frac{\lambda}{4m\omega^2} (|\psi_{21}\rangle - |\psi_{12}\rangle) \quad (22)$$

$$|\psi_{+}^{(1)}\rangle = \frac{\langle \psi_{21} | \hat{H}' | \psi_{+} \rangle}{\hbar\omega(2-4)} |\psi_{21}\rangle + \frac{\langle \psi_{12} | \hat{H}' | \psi_{+} \rangle}{\hbar\omega(2-4)} |\psi_{12}\rangle = \frac{\lambda}{2m\omega^2} \left( \frac{1}{-2} |\psi_{21}\rangle + \frac{1}{-2} |\psi_{12}\rangle \right) = -\frac{\lambda}{4m\omega^2} (|\psi_{21}\rangle + |\psi_{12}\rangle) \quad (23)$$

From these results, the second-order corrections energies are,

$$E_{-} = 2\hbar\omega - \frac{\hbar\lambda}{2m\omega} - \frac{\hbar\lambda^2}{4m^2\omega^3} + O\left(\left(\frac{\lambda}{m\omega^2}\right)^3\right), \quad (24)$$

$$E_{+} = 2\hbar\omega + \frac{\hbar\lambda}{2m\omega} - \frac{\hbar\lambda^2}{4m^2\omega^3} + O\left(\left(\frac{\lambda}{m\omega^2}\right)^3\right). \quad (25)$$

To find the first-order corrected states, we must normalize the states:

$$|\psi_{-}'\rangle = \frac{1}{\sqrt{1 + \frac{1}{2} \left(\frac{\lambda}{2m\omega^2}\right)^2}} \left( \frac{1}{\sqrt{2}} (|\psi_{01}\rangle - |\psi_{10}\rangle) + \frac{\lambda}{4m\omega^2} (|\psi_{21}\rangle - |\psi_{12}\rangle) + O\left(\left(\frac{\lambda}{m\omega^2}\right)^2\right) \right), \quad (26)$$

$$|\psi_{+}'\rangle = \frac{1}{\sqrt{1 + \frac{1}{2} \left(\frac{\lambda}{2m\omega^2}\right)^2}} \left( \frac{1}{\sqrt{2}} (|\psi_{01}\rangle - |\psi_{10}\rangle) - \frac{\lambda}{4m\omega^2} (|\psi_{21}\rangle + |\psi_{12}\rangle) + O\left(\left(\frac{\lambda}{m\omega^2}\right)^2\right) \right) \quad (27)$$

**Exact solution** By changing coordinates, we can solve this problem exactly! If we use rotated coordinates  $u, v$  so that,

$$u = \frac{1}{\sqrt{2}}(x + y), \quad v = \frac{1}{\sqrt{2}}(x - y), \quad (28)$$

$$x = \frac{1}{\sqrt{2}}(u + v), \quad y = \frac{1}{\sqrt{2}}(u - v), \quad (29)$$

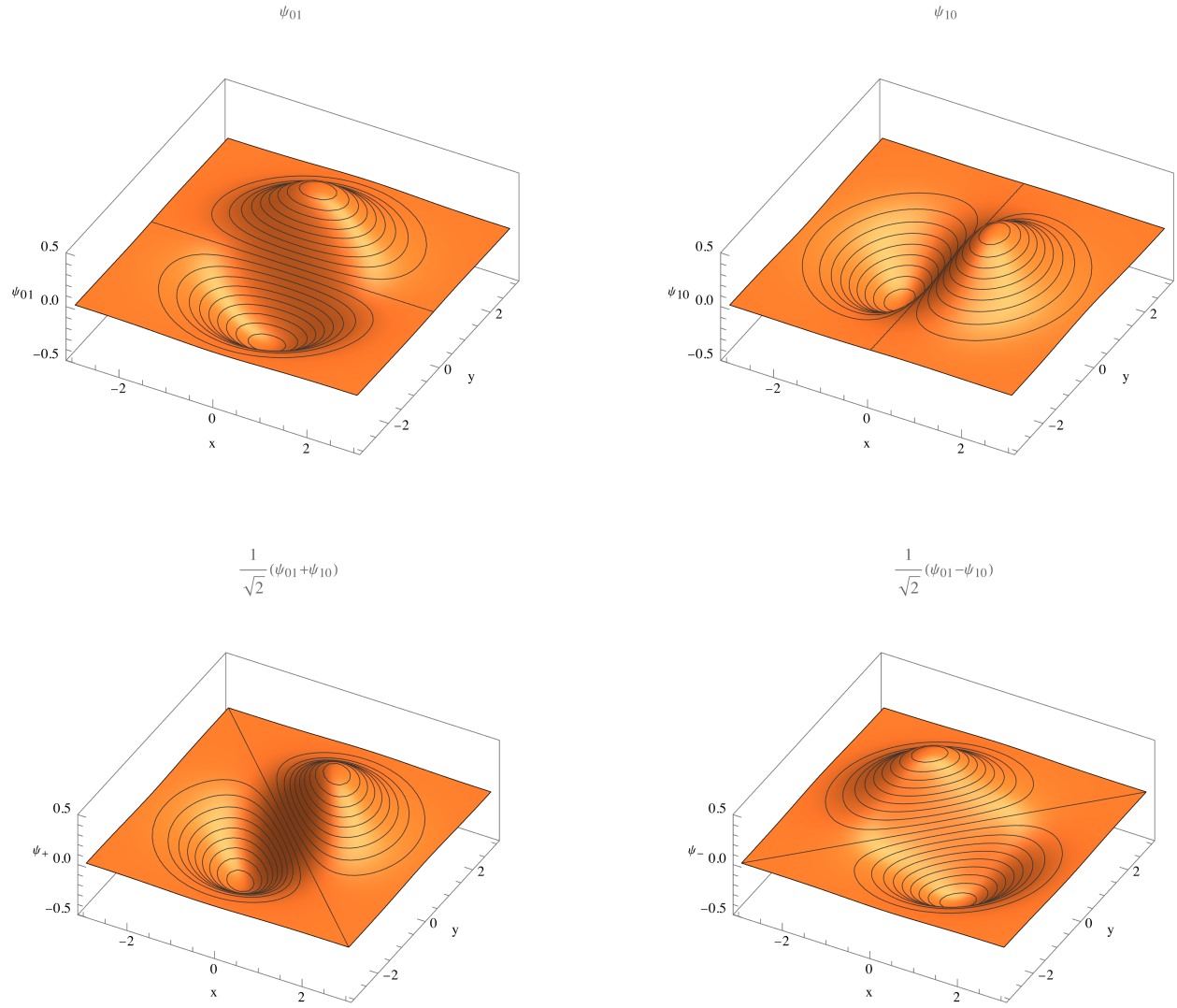


Figure 1: Number states  $|\psi_{01}\rangle, |\psi_{10}\rangle$  and ‘good’ states  $|\psi_{+}\rangle, |\psi_{-}\rangle$  in the position representation.

the perturbed Hamiltonian becomes,

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}' = \frac{1}{2m}(\hat{p}_u^2 + \hat{p}_v^2) + \frac{1}{4}m\omega^2((u+v)^2 + (u-v)^2) + \lambda(u+v)(u-v) \\ &= \frac{1}{2m}(\hat{p}_u^2 + \hat{p}_v^2) + \frac{1}{2}m\omega^2 \left[ \left(1 + \frac{\lambda}{m\omega^2}\right) u^2 + \left(1 - \frac{\lambda}{m\omega^2}\right) v^2 \right],\end{aligned}\tag{30}$$

so we have now have a non-isotropic two-dimensional harmonic oscillator. In the  $u$ -direction,  $\omega_u = \omega\sqrt{1 + \lambda/(m\omega^2)}$ , and in the  $v$ -direction  $\omega_v = \omega\sqrt{1 - \lambda/(m\omega^2)}$ . This means that the exact energies and their power series expansion are,

$$E_{n_u, n_v} = \hbar\omega \left[ \sqrt{1 + \frac{\lambda}{m\omega^2}} \left(n_u + \frac{1}{2}\right) + \sqrt{1 - \frac{\lambda}{m\omega^2}} \left(n_v + \frac{1}{2}\right) \right]\tag{31}$$

$$= \hbar\omega (1 + n_u + n_v) + \lambda \frac{\hbar(n_u - n_v)}{2m\omega} - \lambda^2 \frac{\hbar(1 + n_u + n_v)}{8m^2\omega^3} + O(\lambda^3),\tag{32}$$

where  $(n_u, n_v) = (0, 0), (1, 0), (0, 1), (1, 1), \dots$  are the quantum numbers. If we substitute quantum numbers  $(n_u, n_v) = (0, 1)$  or  $(n_u, n_v) = (1, 0)$  into the power series expansion, we see that we successfully recover our perturbative approximations!