CSE 470N B

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28th February 2023

Lecture notes from the 2023 undergraduate course Quantum Computing, given by Professor James D Kiper at Miami University at Benton Hall in the academic year 2022-2023. This course covers introductory quantum computing concepts. Credit for the material in these notes is due to Professor James D Kiper, while the structure is loosely taken from the in-class lectures. The credit for the typesetting is my own.

Disclaimer: This document will inevitably contain some mistakes–both simple typos and legitimate errors. Keep in mind that these are the notes of an undergraduate student in the process of learning the material, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email, in English, at sayahie@miamioh.edu.

For more notes like this, visit my GitHub profile.

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Spring Term: 2023,
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Thu, 26 January 2023, 2:50pm – 4:10pm

Lecture 1: Week 1, Thursday

Definition 1.1

A *bit* is a binary digit that can take on one of two values, 0 or 1.

Definition 1.2

A *qubit* is analogous to a bit in a quantum computer, but can take on a superposition of the values 0 and 1–it can be in a state of 0 and 1 at the same time.

Definition 1.3

The *planetary model of the atom* is a model of the atom in which the electrons orbit the nucleus in a circular orbit. The planetary model of the atom was developed by Niels Bohr in 1913.

The Stern-Gerlach experiment, first successfully performed in 1922, demonstrated that the magnetic field of an electron can be used to separate the electron into two different states, one with a magnetic field pointing up and one with a magnetic field pointing down. Silver atoms with random spatial orientations were sent straight between two magnets, with the atoms hitting a detector on the other side. The detector was able to detect which direction the atoms were moving in, and the results showed that the atoms were split into two groups, one with a magnetic field pointing up and one with a magnetic field pointing down–'the magnetism was quantised'. This was not expected–the initial hypothesis was that the atoms would form a continuous pattern instead of falling onto two points on the detector, as the spatial orientations were random.

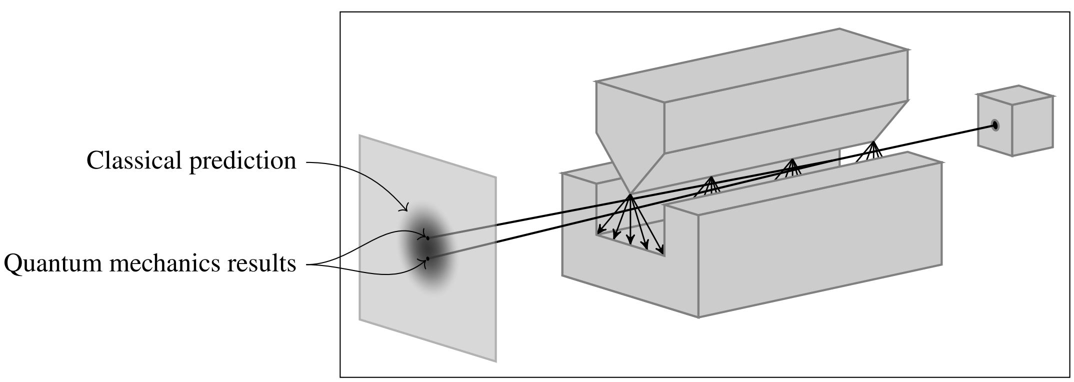

Figure 1: Stern-Gerlach Experiment. Figured designed by Clemens Koppensteiner

Note:-

An electron orbiting in a circular orbit generates a magnetic field.

Particles have some properties, such as 'colour' (with two possible values: black or white), and 'hardness' (with two possible values: soft, hard). We can build detectors that, when given many particles, show a long-run probability of detecting a particle with a certain property. These detectors can be repeated (eg, a colour detector followed by another colour detector) without the probability changing. These detectors demonstrate that the properties are also probabilistically independent (as in, the results are not correlated between a particle's colour, hardness, etc).

Definition 1.4

The *uncertainty principle* states that the probability of measuring a certain property of a particle is inversely proportional to the probability of measuring a different property of the same particle. This is demonstrated in Figure 2. In other words, the more certain we are of measuring one property of a particle, the less certain (or more uncertain) we are of measuring a different property of the same particle.

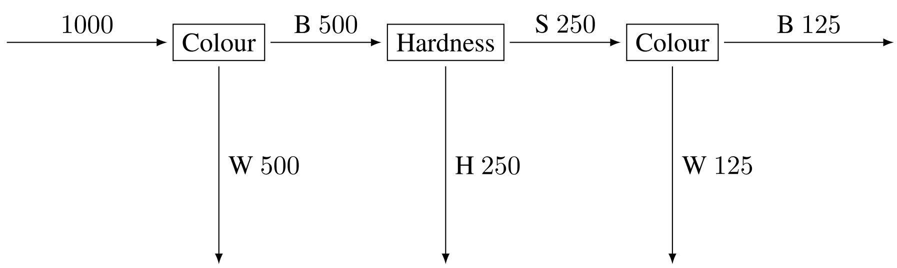

Figure 2: Repeated detectors that detect colour and hardness, demonstrating the uncertainty principle. By measuring the hardness, we became uncertain of the colour; the 250 'black' & 'soft' particles were redetected as 125 black and 125 white.

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Lecture 2: Week 2, Tuesday

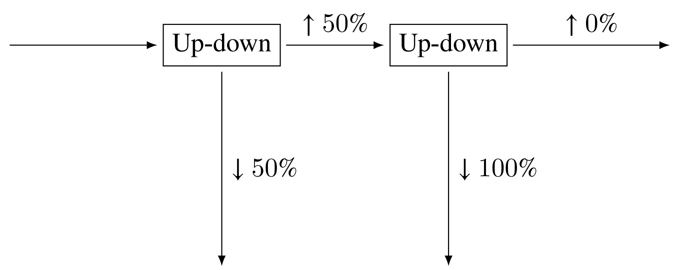

Figure 3: Two repeated detectors of whether a particle's spin is up or down. The same property is being measured; the percentages heading into the second detector are known.

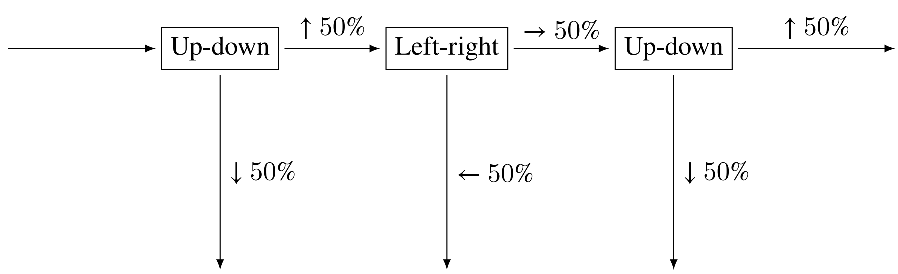

Figure 4: Three repeated detectors, now detecting three different properties of a particle. Now, the percentages for spin up or down are not known; the first and last detectors are probabilistically independent because of the middle detector.

'To talk about an electron that is both spin up and spin right is nonsensical.'

Definition 2.5

Tunneling is a phenomenon in quantum mechanics where an object on the quantum scale can penetrate barriers in a manner that's contradictory to what classical mechanics predicts. In other words, an object can sometimes move through something that should seemingly stop its movement.

Definition 2.6

Quantum decoherence is when the wave function that describes the quantum state of a particle 'collapses' (ie, the quantum state can no longer be predicted or described by the wave function). With decoherence, information about the system is lost into the environment; if a quantum system were perfectly isolated (ie, if nothing could interact with it), it would maintain coherence indefinitely.

With n qubits, a quantum algorithm can search up to 2^n states simultaneously. This is the advantage of quantum computers–modelling complex systems and searching through a large set of possibilities is where quantum computers can be useful.

Thu, 2 February 2023, 2:50pm – 4:10pm

Lecture 3: Week 2, Thursday

'The state of a quantum system corresponds to a vector in a vector space of complex numbers.'

Definition 3.7

A *vector* is a list of numbers. The length (or magnitude; denoted by $||v\rangle|$) of a vector can be found by calculating the square root of the sum of squares of the horizontal and vertical components. Scalar multiplication can be performed by multiplying every value of a vector by the scalar. A *unit vector* is a vector, $\vec{v}$ with a magnitude $|v| = 1$. Vector addition can be performed by adding every element in a vector with the element in the corresponding position in another vector. Vector multiplication (also referred to as finding a dot product, or an inner product; denoted by the product $\langle v|w\rangle$) can be done between two vectors with the same dimensions. If the result of the multiplication is 0, the two vectors are *orthogonal*.

Example. Row ('bra' in Dirac notation): $\langle v| = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$

Column ('ket' in Dirac notation): $|w\rangle = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Magnitude: $|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$

$||w\rangle| = \sqrt{a^2 + b^2}$

Scalar multiplication: $|v\rangle = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$4 \cdot |v\rangle = 4 \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$

Vector addition: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 7 \end{bmatrix}$

Multiplication: $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} = -2 + 6 + 28 = 32$

Note:-

For the purposes of this course, we must be able to find the length of a vector, preform scalar multiplication, perform vector addition, and check for orthogonality.

Definition 3.8

A set of *basis vectors* is a set of vectors that can be combined in a linear combination to make any other vector in the vector space.

Example. Possible basis vectors with two dimensions:

$$\begin{bmatrix} 76.9513 \\ \pi \end{bmatrix} = 76.9513 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \pi \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Possible basis vectors with three dimensions:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note:-

$$|{\rightarrow}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|{\leftarrow}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$|{\nearrow}\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{-\sqrt{3}}{2} \end{bmatrix}$$

$$|{\swarrow}\rangle = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

Tue, 7 February 2023, 2:50pm – 4:10pm

Lecture 4: Week 3, Tuesday

Qubits are represented by unit vectors.

Definition 4.9

A vector is *orthonormal* if it is both a unit vector and orthogonal.

Example. The following are the basis vectors for R^3: $\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix}$

This means that any vector can be written as a linear combination of these basis vectors.

If we were talking about spin, for example, we could use R^2, with the following basis vectors:

$|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}$

$|\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

$|\rightarrow\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

$|\leftarrow\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$

Meaning,

$|\nearrow\rangle = |\uparrow\rangle \cdot |\rightarrow\rangle = \begin{bmatrix}\frac{1}{2}\\\frac{-\sqrt{3}}{2}\end{bmatrix}$

$|\swarrow\rangle = |\downarrow\rangle \cdot |\leftarrow\rangle \begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{bmatrix}$.

Definition 4.10

A *matrix is a rectangular array of numbers.*
The ‘gates’ that are the primary components of quantum computing algorithms correspond to matricies.

Definition 4.11

To *transpose* a matrix means to ‘rotate’ it so that its rows become columns, and its columns become rows.

Example. $M = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
M has 3 rows by 2 columns.

$$M^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

M^T has 2 rows by 3 columns.

If multiplying two matricies, one with dimensions $n \times m$ and the other with dimensions $m \times p$, then the result will have dimensions $n \times p$ (ie, the resultant matrix's dimensions will be number of the first matrix's rows, by the number of the second matrix's columns).

Example. $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 & 3 \\ 4 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 29 & -5 \\ 16 & 37 & -4 \\ 18 & 45 & -3 \end{bmatrix}$

Definition 4.12

An *identity matrix* is a matrix when, multiplied with another matrix, simply yields that other matrix.

Example. $\begin{bmatrix} 14 & 29 & -5 \\ 16 & 37 & -4 \\ 18 & 45 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 29 & -5 \\ 16 & 37 & -4 \\ 18 & 45 & -3 \end{bmatrix}$

Note:-

If a matrix, H, were multiplied by its transpose (ie, $H \cdot H^T$), and if the multiplication were to yield an identity matrix, I, then it is orthogonal. In other words, if $H \cdot H^T = I$, then H is orthogonal. When dealing with complex numbers, this is called a *unitary matrix* instead, rather than an 'orthogonal matrix'.

Definition 4.13

A *tensor product* (represented as $A \otimes B$) can be found by, for each value in the left-hand side, multiplying said value with all of the values on the right-hand side. This operation can be referred to as calculating the *Kronecker product* or *matrix direct product*, when dealing specifically with matricies as opposed to other tensors.
A *tensor* is a generalisation of matricies; where a matrix has two dimensions (rows and columns), a tensor can have any number of dimensions.

Example. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 14 \\ -3 \\ 12 \\ 21 \end{bmatrix}$

The state of a quantum system corresponds to a vector. The state of a quantum system is a tensor product of qubits. A vector is separable if it can be written as a tensor product of two other vecotrs. A vector is entangled if it *cannot* be written as the tensor product of two other vectors.

Tue, 7 February 2023, 2:50pm – 4:10pm

Lecture 5: Week 3, Tuesday

'The state of a quantum system is a vector in a vector space of complex numbers.'

Using the following basis vectors, $|\uparrow\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|\downarrow\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, a quantum state can be represented as $a|0\rangle + b|1\rangle$, where a and b are probability amplitudes (ie, a^2 is the probability of getting $|0\rangle$, b^2 is the probability of getting $|1\rangle$, and $a^2 + b^2 = 1$). To determine if a quantum state is valid, square the values of a and b, and then add them together; if the sum of the squares is not equal to 1, then the state is invalid.

If we 'measure' a qubit in the horizontal direction we are using the basis vectors ($\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$). Some of the values that we're going to repeatedly run into may be familiar from an earlier education in trigonometry.
$\sin(45^\circ) = \frac{1}{\sqrt{2}}$
$\cos(45^\circ) = \frac{1}{\sqrt{2}}$
$\sin(30^\circ) = \frac{1}{2}$ $\cos(60^\circ) = \frac{1}{2}$
$\cos(30^\circ) = \frac{\sqrt{3}}{2}$ $\sin(60^\circ) = \frac{\sqrt{3}}{2}$

Note:-

We cannot distinguish between $|v\rangle$ and $-|v\rangle$.
$|v\rangle = a|0\rangle + b|1\rangle$ has the same probabilities involved as $-|v\rangle = -a|0\rangle - b|1\rangle$ (being a^2 for $|0\rangle$ and b^2 for $|1\rangle$).

If we were to rotate our observation appparatus (or perspective) by Θ°, the new basis vectors would be ($\begin{bmatrix} \cos(\Theta) \\ -\sin(\Theta) \end{bmatrix}$, $\begin{bmatrix} \sin(\Theta) \\ \cos(\Theta) \end{bmatrix}$).

Tue, 14 February 2023, 2:50pm – 4:10pm

Lecture 6: Week 4, Tuesday

Definition 6.14

A *qubit* is a unit vector ('ket') in C^2. When we measure a qubit, we are, in effect, choosing a direction for measurement. Which actually means that we are choosing an orthonormal basis vector.

Note:-

To represent classical bits, we can do so with the equation, $|v\rangle = x \cdot |b_1\rangle + y \cdot |b_2\rangle$ where $x^2 + y^2 = 1$.
We will always write it so that $|b_1\rangle = 0$ and that $|b_2\rangle = 1$, not the other way around; the order in which we write our basis vectors conveys that the first will represent a 0, and that the second will represent a 1.

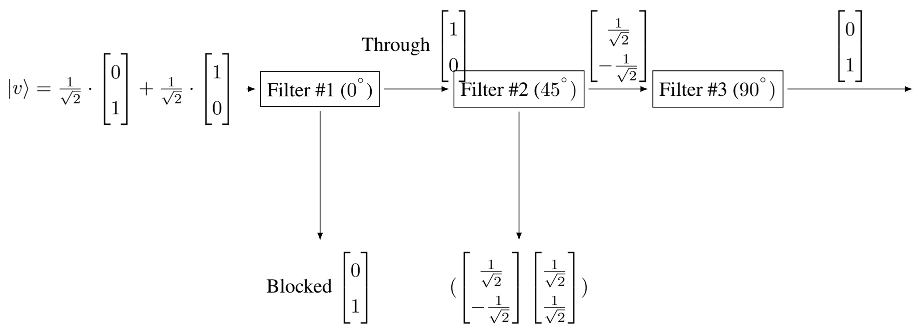

Figure 5: Three repeated filters, blocking photons. The probabilities became a certain outcome.

Definition 6.15

When two waves collide and destroy one another, that is *destructive interference*. When the two waves combine, that is *constructive interference*.

Example. Find a and b when $|v\rangle = a \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $|v\rangle$ is the interaction between $|{\leftarrow}\rangle$ and $|{\rightarrow}\rangle$.

$|v\rangle = \frac{1}{\sqrt{2}} \cdot |{\leftarrow}\rangle + \frac{1}{\sqrt{2}} \cdot |{\rightarrow}\rangle = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Therefore, $a = 1$, $b = 0$. $|{\leftarrow}\rangle$ and $|{\rightarrow}\rangle$, both with their own probabilities, constructively interfered to achieve one certain outcome.

Thu, 23 February 2023, 2:50pm – 4:10pm

Lecture 7: Week 5, Thursday

Note:-

A system of n qubits will be in C^{2^n} space; ie, if you preform tensor multiplication on n qubits (each being in C^2 space–meaning each vector has two elements), you will get C^{2^n} space–meaning the resultant vector will have 2^n elements.

Note:-

If two qubits are *not* entangled, then we can examine each one independently. Additionally, we can represent this state both as a vector in C^4 space, and as the tensor product of two vectors in C^2 space.

Suppose we wanted to perform tensor multiplication with groupings; to do so, we would multiply as we would any other algebraic expression: $|vw\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = (ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle)$, where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Note:-

$|v\rangle \otimes |w\rangle = |v\rangle|w\rangle = |vw\rangle \neq \langle v|w\rangle$.

The standard basis vectors for $C^4 = C^2 \otimes C^2$ space are $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. This is equivalent to $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Tue, 28 February 2023, 2:50pm – 4:10pm

Lecture 8: Week 6, Tuesday

Two qubits are entangled if, when we measure (observe) one, the state of the other qubit changes instantaneously. The state of any quantum system corresponds to a unit vector in a vector space; that vector can be represented as a linear combination of basis vectors. If two qubits are *not* entangled, then we can examine (measure) one qubit without affecting the state of the other—they are independent. Suppose that the basis vectors that we are using for the first qubit are $|a_1\rangle$ and $|a_2\rangle$, and the basis vectors that we are using for the second qubit are $|b_1\rangle$ and $|b_2\rangle$. Then the state of this system of two qubits is represented by a vector in R^2 like $r|a_0b_0\rangle + s|a_0b_1\rangle + t|a_1b_0\rangle + u|a_1b_1\rangle$. Since r, s, t, and u are probability amplitudes, $r^2 + s^2 + t^2 + u^2 = 1$.

Example. Separable (not entangled; unentangled)

$|v\rangle = x_0|a_0\rangle + x_1|a_1\rangle \qquad x_0^2 + x_1^2 = 1$
$|w\rangle = y_0|b_0\rangle + y_1|b_1\rangle \qquad y_0^2 + y_1^2 = 1$

$|v\rangle \otimes |w\rangle = (x_0|a_0\rangle + x_1|a_1\rangle) \otimes (y_0|b_0\rangle + y_1|b_1\rangle) = x_0y_0|a_0b_0\rangle + x_0y_1|a_0b_1\rangle + x_1y_0|a_1b_0\rangle + x_1y_1|a_1b_1\rangle$

The squared probability amplitudes sum up to 1.
$(x_0y_0)^2 + (x_0y_1)^2 + (x_1y_0)^2 + (x_1y_1)^2 = 1$
To demonstrate this we can use the fact that $x_0^2 + x_1^2 = 1$ and $y_0^2 + y_1^2 = 1$.
$x_0^2(y_0^2 + y_1^2) + x_1^2(y_0^2 + y_1^2) = 1$
$x_0^2 + x_1^2 = 1$

If a vector is entangled, then it cannot be represented as the tensor product of two vectors,

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

A quick trick to determine if a vector in C^4 is entangled is to check if the product of the first and last elements is equal to the product of the middle two elements; if it is, then the vector is not entangled—it is separable. If the products are different, then the vector is entangled (ie, if $ac \cdot bd \neq ad \cdot bc$, then the vector is entangled).

Example. Separable (not entangled; unentangled)

$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$

1. What is the probability that the second qubit is $|0\rangle$?
2. Now, let's measure the first qubit. Suppose we get $|0\rangle$. What is the probability that the second qubit is $|0\rangle$?
3. Now, what is the probability that the second qubit is $|0\rangle$?

Solution

1. $(ac)^2 + (bc)^2 = a^2c^2 + b^2c^2 = c^2(a^2 + b^2) = c^2$.
2. $ac|00\rangle + ad|01\rangle$. This is an invalid quantum state; $(ac)^2 + (ad)^2 \neq 1$. We know this because we know that $(ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 = 1$. We can normalise using the length, $\sqrt{(ac)^2 + (ad)^2} = \sqrt{a^2c^2 + a^2d^2} = \sqrt{a^2(c^2 + d^2)} = a$. $\frac{ac|00\rangle}{a} + \frac{ad|01\rangle}{a} = c|00\rangle + d|01\rangle$. This is a valid quantum state.
3. $(c)^2 = c^2$. This is the same as the probability that the second qubit is $|0\rangle$ when we don't measure the first qubit. This means that the first qubit and the second qubit are independent. They are *not* entangled.

Example. Entangled

$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

1. What is the probability that the second qubit is $|0\rangle$?
2. Now, let's measure the first qubit. Suppose we get $|0\rangle$. What is the probability that the second qubit is $|0\rangle$?
3. Now, what is the probability that the second qubit is $|0\rangle$?

Solution

1. $a^2 + c^2$
2. The system state becomes $a|00\rangle + b|01\rangle$. This is not a valid quantum state, because $(a)^2 + (b)^2 + (c)^2 + (d)^2 = 1$. We can normalise using the length, $\sqrt{a^2 + b^2}$. $\frac{a|00\rangle}{\sqrt{a^2+b^2}} + \frac{b|01\rangle}{\sqrt{a^2+b^2}}$. This is a valid quantum state.
3. $\frac{a^2}{a^2+b^2}$. This does not equal $a^2 + c^2$ (the probability that the second qubit is $|0\rangle$ when we don't measure the first qubit). This means that the first qubit and the second qubit are not independent. They are entangled; they both changed.

Hadamard gate: $\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

CNot gate: $\mathrm{CNot} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \\ c \end{bmatrix}$.

Notes