HW 1 Endre Moen

1) $Z \sim N(0, \sigma^2)$

a) $X_t = a + bZ_t + cZ_{t-2}$

<u>Stationary:</u> 1) $E[X_t] = a + bE[Z_t] + cE[Z_{t-2}]$
$= a + 0 + 0$

2) $\gamma_X(t+h, t) = cov(X_{t+h}, X_t) =$
$= E[(X_{t+h} - E[X_{t+h}])(X_t - E[X_t])]$
$= E[(X_{t+h} - a)(X_t - a)]$
$= E[(a + bZ_{t+h} + cZ_{t+h-2} - a)$
$(a + bZ_t + cZ_{t-2} - a)]$
$= E[(bZ_{t+h} + cZ_{t+h-2})(bZ_t + cZ_{t-2})]$
$= E(b^2 Z_{t+h}Z_t + bc Z_{t+h-2}Z_t + bc Z_{t-2}Z_{t+h} + c^2 Z_{t+h-2}Z_{t-2}$
$= E[b^2 Z_{t+h}Z_t] + E[bc(Z_t Z_{t+h-2} + Z_{t-2}Z_{t+h})] + E[c^2 Z_{t+h-2}Z_{t-2}]$
$Var(Z) = E[Z_t^2] - E[Z_t]^2 = E[Z_t^2] - 0 = \sigma$

$$\cancel{t}\ \gamma_X(t+h, t) = \begin{cases} (b^2 + c^2)\sigma^2 & h = 0 \\ bc\sigma^2 & |h| = 2 \\ 0 & \text{else} \end{cases}$$

3) $Var(X_t) = E[X_t^2] - E[X_t]^2$
$E[X_t^2] = Var(X_t) + a^2$

$Var(X_t) = cov(X_t, X_t) = E(bZ_t + cZ_{t-2})(bZ_t + cZ_{t-2})$
$= b^2\sigma^2 + bc\underbrace{E[Z_{t-2}Z_t]}_{=0} + bc\underbrace{E[Z_t Z_{t-2}]}_{=0} + c^2\sigma^2$
$= (b^2 + c^2)\sigma^2$

$E[X_t^2] = (b^2 + c^2)\sigma^2 + a^2 < \infty$

1 b) $X_t = a + b Z_1 \cos(ct) + c Z_2 \sin(ct)$

1) $E[X_t] = a$

2) $\gamma_x(t+h, t) = E[(X_{t+h} - a)(X_t - a)]$

$= E[(b Z_1 \cos(c(t+h)) + c Z_2 \sin(c(t+h)))(b Z_1 \cos(ct) + c Z_2 \sin(ct))]$

$= b^2 \sigma^2 \underbrace{\cos(c(t+h))}_{x} \underbrace{\cos(ct)}_{y} + 0 + 0 + c^2 \sigma^2 \sin(c(t+h)) \sin(ct)$

if $b^2 = c^2 \Rightarrow b^2 \sigma^2 (\cos(x)\cos(y) + \sin(x)\sin y) = \cos(x-y)$

$= \cos(ct + ch - ct)$

if $b^2 \neq c^2 \Rightarrow$ not stationary $\quad \gamma_x \not\perp t \Leftarrow \quad = \cos(ch)$

3) $E[x^2] = Var(X_t) + a^2 = b^2\sigma^2 \cos^2(ct) + c^2\sigma^2 \sin^2(ct) + a^2 < \infty$

$\sin x, \cos x \in [0,1]$

1) c) $X_t = a + b Z_t \cos(ct) + c Z_{t-1} \sin(ct)$

1) $E[X_t] = 0$

2) $\gamma_x(t+h, t) = E((b Z_{t+h} \cos(c(t+h)) + c Z_{t+h-1} \sin(c(t+h)))$

$(b Z_t \cos(ct) + c Z_{t-1} \sin(ct))$

$= b^2 Z_{t+h} Z_t \cos(c(t+h)) \cos(ct) + bc Z_{t+h-1} Z_t \sin(c(t+h)) \cos(ct)$

$+ bc Z_{t+h} Z_{t-1} \cos(c(t+h)) \sin(ct) + c^2 Z_{t+h-1} Z_{t-1} \sin(c(t+h)) \sin(ct)$

1 $h = 0 \Rightarrow b^2 \sigma^2 \cos(ct) + c^2 \sigma^2 \sin^2(ct)$

$\Rightarrow b^2 = c^2 \Rightarrow b^2\sigma^2(\cos^2(ct) + \sin^2(ct)) = b^2\sigma^2$

2 $h = 1 \Rightarrow bc\,\sigma^2 \sin(c(t+1)) \cos(ct)$

3 $h = -1 \Rightarrow bc\,\sigma^2 \cos(c(t-1)) \sin(ct)$

4 else $\Rightarrow 0$

γ_x indep of t if 1 and $b = c$ or 4 $h \neq 0, h \neq \pm 1$

or 3 and 2 $c = \pm k\pi$ or $c = 0$, or $c = \pm 2k\pi$

d) $X_t = a + bZ_0$

1) $E[X] = a$

2) $\gamma_X(t+h, t) = cov(X_{t+h}, X_t) = E[(X_{t+h} - a)(X_t - a)]$

$= E[(bZ_0)(bZ_0)]$

$= b^2 E[Z_0^2] = b^2\sigma^2 \not\ni t$

3) $E[X^2] = var(X_t) + E[X]^2$

$= b^2\sigma^2 + a^2 < \infty$

e) $X_t = Z_0 \cos(ct)$

$E[X_t] = 0$

$\gamma_X = E[Z_0 \cos(c(t+h))(Z_0 \cos(ct))]$

$= E[Z_0^2 \cos(c(t+h)) \cos(ct)]$

$$\begin{cases} \sigma^2 \cos^2(ct) & h = 0 \\ \sigma^2 \cos(c(t+h)) \cos(ct) & h \neq 0 \end{cases}$$

$\cos 0 = 1$

$\cos \pi = -1$

$\cos 2\pi = 1$

Stationary

if $c = \pm 2k\pi$ or $c = \pm(2k-1)\pi$ or $c = \pm\frac{(2k-1)\pi}{2}$

3) $E[X^2] = \sigma^2 \cos^2(ct) < \infty$ as $\cos x \in [0,1]$

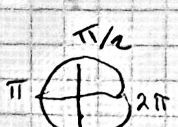
f) 1/ $X_t = Z_t Z_{t-1}$ 1) $E[X_t] = 0$

2) $\gamma_X(t+h, t) = E[(Z_{t+h} Z_{t+h-1})(Z_t Z_{t-1})]$

$\frac{\pi}{2}$:

$h = 1$: $E[Z_{t+1} Z_t Z_t Z_{t-1}] = \sigma^2$

$h = 0$: $E[Z_t Z_{t-1} Z_t Z_{t-1}] = \sigma^2\sigma^2 = \sigma^4$



$h = -1$: $E[Z_{t-1} Z_{t-2} Z_t Z_{t-1}] = \sigma^2$

else $E[\quad] = 0$

$\frac{8\pi}{2}$

3) $E[x^2] = 0 + \sigma^4 < \infty$

X_t - stationary

1.5/ $\{X_t\}$ given $X_t = Z_t + \theta Z_{t-2}$

$\{Z_t\} \sim WN(0,1)$

a) find autocovariance $\gamma_X(h) = cov(X_{t+h}, X_t)$

autocorrelation $\rho_X(h) = \frac{\gamma_X(h)}{\gamma(0)} = cor(X_{t+h}, X_t)$

1) $E(X) = E[Z_t + \theta Z_{t-2}] = 0$

2) $\gamma_X = cov(X_{t+h}, X_t) = E[X_{t+h} \cdot X_t]$

$= E[(Z_{t+h} + \theta Z_{t-2+h}) \cdot (Z_t + \theta Z_{t-2})]$

$= E[Z_{t+h} Z_t] + E[\theta Z_{t-2+h} Z_t] + E[\theta Z_{t+h} Z_{t-2}]$
$+ E[\theta^2 Z_{t-2+h} Z_{t-2}]$

$$\gamma_X(h) = \begin{cases} \sigma^2 + \theta^2\sigma^2 & h=0 \\ \theta\sigma^2 & |h|=2 \\ 0 & h \text{ else} \end{cases}$$

$$\rho_X(h) = \begin{cases} \frac{\sigma^2(1+\theta^2)}{\sigma^2(1+\theta^2)} = 1 & , h=0 \\ \frac{\theta}{1-\theta^2} & , |h|=2 \\ 0 & , \text{else} \end{cases}$$

b) Compute the variance of the sample mean

$X_1 = Z_1 + \theta Z_{-1}$ $X_2 = Z_2 + \theta Z_0$ $X_3 = Z_3 + \theta Z_1$ $X_4 = Z_4 + \theta Z_2$

$E[X_4]$ $\bar{X}_4 = \left(Z_1(1+\theta) + Z_2(1+\theta) + Z_3 + Z_4 + \theta(Z_{-1} + Z_0)\right)/4 = 0$

$Var(X_4) = Var(\bar{X}_4) = \frac{1}{4^2} Var(\quad) = E[X^2] - E^2[X]$

$= \frac{1}{4^2} E[(1+\theta)(Z_1 + Z_2) + Z_3 + Z_4 + \theta(Z_{-1} + Z_0)]^2$

$= \frac{1}{4^2} (1+\theta)^2 [E Z_1^2 + E Z_2^2) + E Z_3^2 + E Z_4^2 + \theta(E Z_{-1}^2 + E Z_0^2)$

$= \frac{1}{4^2} \; 2(1+\theta)^2 + 2 + 2\theta = \frac{1}{4}(1+\theta)^2 + 1 + \theta$

$\theta = 0.8 \Rightarrow \frac{1}{4}(1.8)^2 + 1.8 = 1.26$

c) in b) $\theta = 0.8$, in c) $\theta = 0.6$?

$$Var = E[X^2] - E^2[X]$$
$$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{0}$$

$$Var(\bar{X}_4) = \frac{1}{4^2} Var(X_1 + X_2 + X_4)$$

$$E^2 \bar{X}_4 = 0$$

$$Var(X_1 X_2) = E[X_1 X_2] - E[X_1] E[X_2]$$

$$E[X_1] = E[X_2] = 0$$

$$Var(X_1, X_2) = E[X_1 X_2] = cov(X_1, X_2)$$

$$cov(Z_t, Z_s) = 0 \quad \forall s \neq t$$

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b)
$$Var(\bar{X}_4) = (E[X^2] - E[X^2]) \quad E[X^2] = 0$$

$$= E[X^2] - 0 \quad \text{Fra} \quad E[Z_s \cdot Z_t] = cov(Z_s \cdot Z_t) = 0$$

$$\Rightarrow E\left[\frac{1}{4^2}\left(Z_1(1+\theta) + Z_2(1+\theta) + Z_3 + Z_4 + \theta(Z_{-1} + Z_0)\right)^2\right]$$

$$= \frac{1}{4^2} E\left[\left(Z_1^2(1+\theta)^2 + Z_2^2(1+\theta)^2 + Z_3^2 + Z_4^2 + \theta^2(Z_{-1}^2 + Z_0^2)\right)\right]$$

$$= \frac{1}{16}\left[1 \cdot (1+\theta)^2 + 1 \cdot (1+\theta) + 1 + 1 + \theta^2(1+1)\right)$$

$$Var[Z_i] = E[Z_i^2] - E[Z_i] = E[Z_i^2] = 1$$

$$= \frac{1}{16}\left(2(1+\theta)^2 + 2 + 2\theta^2\right) = \underline{\frac{1}{4}\left((1+\theta)^2 + 1 + \theta^2\right)}$$

$$\theta = 0.8 = \frac{1}{4}(1.8)^2 + 1.8 = 1.26$$

c) $\theta = -0.8 = \frac{1}{4}(0.2^2 + 1 - 0.8) = \underline{0.06}$

1.6/ $\{X_t\}$ AR(1) def in 1.4.5

First-order Autoregressive AR(1)

$X_t = \phi X_{t-1} + Z_t \quad t \in \mathbb{Z}$ $\quad Z_t \sim WN(0, \sigma^2)$, $\phi < 1$

$X_1 = Z_1$ $\quad X_3 = \phi(\phi X_1 + Z_2) + Z_3$

$X_2 = \phi X_1 + Z_2$ $\quad X_4 = \phi(\phi(\phi X_1 + Z_2) + Z_3) + Z_4$

$= \phi^3 X_1 + \phi^2 Z_2 + \phi Z_3 + Z_4$

$\overline{X}_4 = X_1(1 + \phi + \phi^2 + \phi^3) + Z_2(1 + \phi + \phi^2) + Z_3(1+\phi) + Z_4 / 4$

$Var(\overline{X}_4) = \frac{1}{16}\Big[Var(X_1)(1+\phi+\phi^2+\phi^3)^2 + Var(Z_2)(1+\phi+\phi^2)^2 + Var(Z_3)(1+\phi)^2 + Var(Z_4)$

$Var(X_1) = cov(X_1, X_1)$

$= cov(X_t, X_t) = \gamma_X(0) = \frac{\sigma^2}{1-\phi^2}$

$\Rightarrow \frac{1}{16}\Big[\frac{\sigma^2}{1-\phi^2}(1+\phi+\phi^2+\phi^3)^2 + (1+\phi+\phi^2)^2 + (1+\phi)^2 + 1$

$Z_t \sim WN(0,1)$

$\phi = 0.9 \Rightarrow \frac{1}{16}\Big[\frac{1}{1-0.9^2}(11.83) + 7.34 + 3.61 + 1\Big)$

$= 1/16\,[62.3 + 7.34 + 3.61 + 1] = 4.64$

b) $\phi = -0.9$: $\frac{1}{16}[0.17 + 0.828 + 0.01 + 1] = \underline{0.1255}$

1.7/ $\{X_t\}$, $\{Y_t\}$ uncorrelated
ie X_r, Y_t —"— $\forall r, \forall t$

1) $\Rightarrow \{X_t + Y_t\}$ stationary and uncorrelated

2) $\gamma_{X+Y} = \gamma_X + \gamma_Y$

1) $E[X_t + Y_t] = E[X_t] + E[Y_t] = \mu_x + \mu_y$

2) $\gamma_{X+Y}(t+h,t) = cov(X_{t+h} + Y_{t+h}, X_t + Y_t)$

$= E(X_{t+h} + Y_{t+h} - (\mu_x + \mu_y))(X_t + Y_t - (\mu_x + \mu_y))$

$= E[(\underline{X_{t+h}} + Y_{t+h})(\underline{X_t} + Y_t) - \underline{(\mu_x + \mu_y)}(\underline{X_t} + Y_t)$

$- \underline{(\mu_x + \mu_y)}(\underline{X_{t+h}} + Y_{t+h}) + (\mu_x + \mu_y)^2$

$\Rightarrow E[(X_{t+h} \cdot X_t - \mu_x X_t - \mu_x X_{t+h} + \mu_x^2)]$

$= cov(X_{t+h}, X_t) = E[(X_{t+h} - \mu_x)(X_t - \mu_x)]$

$= E[X_{t+h} X_t - \mu_x X_t - \mu_x X_{t+h} + \mu^2$

$\Rightarrow E[X_{t+h} Y_t - \mu_x Y_t - X_{t+h}\mu_y + \mu_x \mu_y$

$0 = cov(X_{t+h}, Y_t) = E[(X_{t+h} - \mu_x)(Y_t - \mu_y)$

$\boxed{X, Y \text{ uncorr}}$ $= E[X_{t+h} Y_t - \mu_x Y_t - X_{t+h}\mu_y + \mu_x \mu_y]$

$0 = cov(X_t, Y_{t+h}) = 0$

$\Rightarrow cov(Y_{t+h}, Y_t)$

$\gamma_{X+Y}(t+h,t) = cov(X_{t+h}, X_t) + cov(Y_{t+h}, t)$

$= \underline{\underline{\gamma_X(h) + \gamma_Y(h)}}$

$\Rightarrow \gamma_{X+Y}$ - stationary and uncorrelated