HW 2 Endic Moen

2.1/ a) $X_t = B_0 + B_1 t + Z_t$, $Z_t \sim N(0,1)$

$E[X_t] = B_0 + B_1 t + \underbrace{E[Z_t]}_{=0}$ - <u>depends on t</u>

weakly stationary: 1) $\mu_X(t)$ indep of t
2) $\gamma_X(t+h, t)$ indep of t

b) Prove $\nabla X_t = X_t - X_{t-1}$, $\nabla X_t = B_0 + B_1 t + Z_t - B_0 - B_1(t-1) - Z_{t-1}$

1) $= B_1(t-t+1) + Z_t - Z_{t-1} = B_1 + Z_t - Z_{t-1}$, $E[\Delta X_t] = B_1$

2) $\gamma_{X'}(t+h, t) = E[(X'_{t+h} - E[X'_{t+h}])(X'_t - E[X'_t])]$

$= cov(\nabla X_{t+h}, \nabla X_t) = cov(B_1 + Z_{t+h} - Z_{t-1+h}, B_1 + Z_t - Z_{t-1})$

Rules
$cov(X, a) = 0$
$cov(X, X) = var(X)$
$cov(aX, bY) = ab\, cov(Y, X)$
$cov(X+a, Y+b) = cov(X, Y)$

$= cov(Z_{t+h}, Z_t) - cov(Z_{t-1+h}, Z_t) - cov(Z_{t+h}, Z_{t-1})$
$+ cov(Z_{t-1+h}, Z_{t-1})$

$h=0 \Rightarrow \sigma^2 - 0 - 0 + \sigma^2 = 2\sigma^2$
$h=1 \Rightarrow 0 - \sigma^2 - 0 + 0 = -\sigma^2$
$h=-1 \Rightarrow 0 - 0 - \sigma^2 + 0 = -\sigma^2$
$|h|>1 \Rightarrow = 0$

1) & 2) $\Rightarrow$ stationary

2.1 c) $Z_t = Y_t \sim N(\mu_Y, \sigma^2)$, $\gamma_Y(h)$

$$X_t = B_0 + B_1 t + Y_t$$

$$\nabla X_t = B_1 + Y_t - Y_{t-1}$$

$$E[\nabla X_t] = B_1$$

$$\gamma_{\nabla X}(t+h, t) = cov(Y_{t+h-1}, Y_{t-1})$$

$$= cov(B_1 + Y_{t+h} - Y_{t+h-1}, B_1 + Y_t - Y_{t-1})$$

$$= cov(Y_{t+h}, Y_t) - cov(Y_{t+h}, Y_{t-1}) - cov(Y_{t+h-1}, Y_t) + cov(Y_{t+h-1}, Y_{t-1})$$

Def $cov(Y_{t+h}, Y_t) = \gamma_Y(h)$

$$= \gamma(h) - cov(Y_{t+h}, Y_{t-1}) - cov(Y_{t+h-1}, Y_t) + cov(Y_{t+h-1}, Y_{t-1})$$

$$cov(Y_{t+h-1}, Y_{t-1}) = cov(Y_{t+h}, Y_t) = \gamma(h)$$

$t = t+1$

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$\Rightarrow (t+h, t-1)$

2 $t = t+1$

$\Rightarrow (t + \boxed{1 + h}, t) = h+1$

3 $t + \boxed{h-1}, t = h-1$

4 $t+h-1, t-1 = c(Y_{t+h}, Y_t) = \gamma(h)$

$t = t+1$

$$\Rightarrow \gamma(h) - \gamma(h+1) - \gamma(h-1) + \gamma(h)$$

$$= 2\gamma(h) - \gamma(h+1) - \gamma(h-1)$$

$$\Rightarrow \gamma(h) = \begin{cases} 2\sigma^2 & h=0 \\ \sigma^2 & h = \pm 1 \\ 0 & h > 1 \end{cases}$$

2.2 f) γ_u $\quad u_t = \mu + Z_t + \theta Z_{t-1} \qquad Z \sim N(0, \sigma_z^2)$

$E[u_t] = \mu$

$\mathrm{cov}(u_{t+h}, u_t) = \mathrm{cov}(Z_{t+h}; Z_t) + \mathrm{cov}(Z_{t+h}, \theta Z_{t-1})$
$+ \mathrm{cov}(\theta Z_{t-1+h}, Z_t) + \mathrm{cov}(\theta Z_{t-1+h}, \theta Z_{t-1})$

$= \mathrm{cov}(\quad) + \theta\,\mathrm{cov}(\quad)$
$+ \theta\,\mathrm{cov}(\quad) + \theta^2\,\mathrm{cov}(\quad)$

$h=0 \Rightarrow \sigma^2 + \sigma^2\theta^2 = \sigma^2(1+\theta^2)$

$h = \pm 1 \Rightarrow \theta \cdot \sigma^2$

$h > 1 \Rightarrow 0$

γ_u - is defined for $h \in [-1, 1]$

$Ma(q)$ is a moving average process of order q

$x_t = Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$

$Ma(1)$ has memory of $t, t-1$

which fits with γ_u - U_t is therefore a $Ma(1)$ process

2.2) g) $\rho(h) = \dfrac{\gamma(h)}{\gamma(0)}$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1+\theta^2)} = \frac{\theta}{(1+\theta^2)}$$

$$\gamma(0) = \sigma^2(1+\theta^2)$$

I $\sigma^2 = \dfrac{\gamma(0)}{1+\theta^2}$

II $\rho(1)\cdot(1+\theta^2) = \theta$

III $\rho(1) + \rho(1)\theta^2 - \theta = 0$

$\Downarrow$

$\theta^2\rho - \theta + \rho = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\theta = \frac{1 \pm \sqrt{1^2 - 4\rho\rho}}{2\rho}$$

$$\boxed{\hat{\theta} = \frac{1 \pm \sqrt{1 - 4\rho(1)^2}}{2\rho(1)} \qquad \hat{\sigma}^2 = \frac{\gamma(0)}{1+\hat{\theta}^2}}$$

62, 83.-84, 85
MA(1) - 84

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2.3 1) $cov(X_1, X_3 | X_2)$

$$X_t = Z_t + \theta Z_{t-1} \quad - MA(1)$$

MA(1)
$\Rightarrow \rho(h) = 0 \quad h \geq 2$

2) partial correlation for MA(1) process

1) $$\rho_{X_1 X_3 \cdot X_2} = \frac{\sigma_{X_1 X_3 \cdot X_2}}{\sqrt{\sigma_{X_1 X_1 \cdot X_2}} \cdot \sqrt{\sigma_{X_3 X_3 \cdot X_2}}}$$

2) PACF - PArtical Correlation Function - MA(1)

$$\alpha(h) = \phi_{hh} = \frac{-(-\theta)^h}{(1+\theta^2 + \dots + \theta^{2h})}$$

$$cov(X_1, X_3) = \begin{bmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{bmatrix}$$

$$\rho_{13 \cdot 2} = \frac{\rho_{13} - \rho_{21}\rho_{23}}{(1-\rho_{21}^2)^{1/2}(1-\rho_{23}^2)^{1/3}}$$

$(t+h-1, t-1)$

$cov(X_{t+h}, X_t) = cov(Z_{t+h}, Z_t) + \theta\, cov(Z_{t+h-1}, Z_t) + \theta(Z_{t+h}, Z_{t-1}) + \theta^2 cov(Z_{t+h-1}, Z_{t-1})$

$(Z_{t+h} + \theta Z_{t+h-1})$ $\Rightarrow h = 0 \Rightarrow \sigma^2 + \theta^2\sigma^2 = \sigma^2(1+\theta^2)$

$(Z_t + \theta Z_{t-1})$ $\quad h = \pm 1 \Rightarrow \sigma^2\theta$

$\gamma(0) = \sigma^2(1+\theta^2) \qquad \gamma(\pm 1) = \sigma^2\theta$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\gamma(h)}{\sigma^2(1+\theta^2)} \Rightarrow \rho(1) = \frac{\sigma^2\theta}{\sigma^2(1+\theta^2)} = \frac{\theta}{(1+\theta^2)}$$

$$\phi_{11} = \rho_1, \quad \phi_{22} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\left(\frac{\theta}{1+\theta^2}\right)^2}{1-\left(\frac{\theta}{1+\theta^2}\right)^2} \frac{(1+\theta^2)^2}{(1+\theta^2)^2} = \frac{\theta^2}{(1+\theta^2)^2 - \theta^2}$$

$(1+\theta^2)(1+\theta^2) = 1 + 2\theta^2 + \theta^4$

$$= \frac{\theta^2}{1 + 2\theta^2 + \theta^4 - \theta^2} = \frac{\theta^2}{1+\theta^2+\theta^4}$$

2.4 a) Explain $\hat{\gamma} = \frac{1}{n}\sum_{t=-\infty}^{\infty} Y_{t+h} Y_t$ $\quad h \in \mathbb{Z}$

$$\hat{\gamma}(h) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-h} (X_{t+h} - \overline{X}_n)(X_t - \overline{X}_n), & h \in [0, n-1] \\ \hat{\gamma}(-h) & h \in [-(n-1), 0] \\ 0 & \text{else} \end{cases}$$

Symmetri

When $Y_t = \begin{cases} X_t - \overline{X}_n & Y_{t+h} = X_{t+h} - \overline{X}_n \quad t \in [1, n] \\ 0 & t \notin [1, n] \end{cases}$

1) let $n \to \infty \Rightarrow \hat{\gamma} = \frac{1}{n}\sum_{t=1}^{\infty} Y_{t+h} Y_t$

2) $t = 0 \Rightarrow \gamma(0) = 0$ by def

3) Symmetri: when $n \to \infty$, $t \in [-1, -n] \Rightarrow \hat{\gamma} = \hat{\gamma}(-h)$
$t \in [-1, -n]$ and $s \in [1, n] \Rightarrow \gamma(-(-t)) = \gamma(s)$

$$\Rightarrow \hat{\gamma} = \sum_{t=-1}^{-\infty} \hat{\gamma}(-h) + 0 + \sum_{t=1}^{\infty} \hat{\gamma}(h) = \sum_{t=-\infty}^{\infty} \hat{\gamma}(h) = \frac{1}{n}\sum_{t=-\infty}^{\infty} Y_{t+h} Y_t$$

b) $a \in \mathbb{R}^m$ show

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i \hat{\gamma}(i-j) a_j = n^{-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{t} a_i Y_{t+i-j} Y_t a_j$$

let $s = t - j$ $\quad = n^{-1}\sum\sum\sum a_i Y_{s+i} Y_{s+j} a_j$

RHS of eq
$$\begin{cases} t > 0 & \mathrm{cov}(Y_{s+i}, Y_{s+j}) = \begin{cases} (\gamma_{s+i})^2 & i = j \\ 0 & i \neq j \end{cases} \\ t = 0 & \gamma(0) = 0 \\ t < 0 & \mathrm{cov}(Y_{-(s+i)}, Y_{-(s+j)}) = \begin{cases} (\gamma_{-(s+i)})^2 \\ 0 \end{cases} \end{cases}$$

LHS: Use $s = t - j$

$\Rightarrow \mathrm{cov}(Y_{s+i}, Y_{s+j}) = \mathrm{cov}(Y_{t+i-j}, Y_t) \Rightarrow h = i - j$

$\Rightarrow Y_{t-j+i}, Y_{t-j+j} \Rightarrow Y_{t+i-j}, Y_t$ $\qquad \gamma(i-j) = \text{LHS}$

2.5 a) Show

$$\overline{X}_n - \mu = \left(\sum_{j=0}^{q} \theta_j\right)\overline{Z}_n + \sum_{j=0}^{q} \theta_j\left(\tfrac{1}{n}U_j - \tfrac{1}{n}V_n(j)\right),$$

$$X_t = u + \sum_{j=0}^{q} \theta_j Z_{t-1}$$

$$\overline{X}_t = \frac{1}{n}\sum_{t=0}^{n}\left(u + \sum_{j=0}^{q}\theta_j Z_{t-1}\right)$$

$$= \frac{n\mu}{\mu} + \frac{1}{n}\sum_{t=0}^{n}\sum_{j=0}^{q}\theta_j Z_{t-1}$$

$$\overline{X}_t - \mu = \frac{1}{n}\sum_{j=0}^{q}\theta_j \sum_{t=0}^{n} Z_{t-1}$$

$$\sum_{t=0}^{n} Z_{t-1} = \sum_{t=0}^{n} Z_t + \sum_{k=1-j}^{0} Z_{1-k} - \sum_{k=n-j+1}^{n} Z_{n-k}$$

$$\sum_{k=1-j}^{0} Z_{1-k} \quad - \sum_{k=n-j+1}^{n} Z_{n-k}$$

$U(j)$ $\quad j=5 \Rightarrow \sum_{k=-4}^{0} Z_{1-k} \Rightarrow Z_5 + Z_4 \ldots + Z_1$

$V(i)$ $\quad j=5 \Rightarrow \sum_{k=n-5+1}^{n} Z_{n-k} = Z_{n-(n-4)} + Z_{n-(n-\ldots}$

$$\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{Z_4} + \qquad \overset{\shortparallel}{Z_5}$$

$$\hookrightarrow \overline{X}_t - u = \sum_{j=0}^{q}\theta_j \overline{Z}_t + \frac{1}{n}\left(\underbrace{\sum_{k=1-j}^{n} Z_{1-k}}_{=\,U_j} - \underbrace{\sum_{k=n-j+1}^{n} Z_{n-k}}_{=\,V_j(n)}\right)$$

2.5 b) Prove $P\left(\frac{1}{n}|V_1(n)| > \varepsilon\right) \leq \frac{q}{\varepsilon^2 n^2}$

Chebyshev ineq

$$P(|X-\mu| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

$$= P(|X-\mu| > k\sigma) \leq \frac{1}{k^2}$$

$$= P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$$\Rightarrow P(|V_j(n)| > \varepsilon \cdot n) \leq \frac{\sigma^2}{\varepsilon^2 n^2}$$

$\sum_{j=1}^{q} Z_n \sim N(0, q)$ since $\{Z_1 \ldots Z_q\}$ iid

$$\Rightarrow P\left(\frac{1}{n}|V_1(n)| > \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2 n^2} = \frac{J \cdot \sigma^2}{\varepsilon^2 n^2} \leq \frac{q \cdot T}{\varepsilon^2 n^2}$$

Prove $\underbrace{\sum_{j=1}^{n} P\left(\frac{1}{n}|V_j(n)| > \varepsilon\right)}_{M} < \infty$, $M_n = P\left(\frac{1}{n}|V_1(n)| > \varepsilon\right)$

$$\Rightarrow \sum_{j=1}^{n} M_n < \sum_{j=1}^{n} \frac{q}{\varepsilon^2 n^2} = n \cdot \frac{q}{\varepsilon^2 n^2} < \infty$$

Borel-Cantelli $\{A_n, n \geq\}$ Seq of events

If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{1}_{A_n} \xrightarrow[n\to\infty]{a.s} 0$

$$\lim_{n\to\infty} \frac{q}{\varepsilon^2 n} = 0 \underset{\text{Cantelli}}{\overset{\text{borel}}{\Rightarrow}} \sum_{i=1}^{\infty} \mathbb{1}_{M_n} \xrightarrow[n\to\infty]{a.s} 0$$

2.5 c) CLT: $Y_n \ni \sqrt{n}(\overline{X_n} - \mu)$

show $Y_n = (\sum_{j=0}^{q} \theta_j)(\sqrt{n}\,\overline{Z_n} + W_n)$

$$W_n = \sum_{j=0}^{q} \theta_j (n^{-1/2} U_j - n^{-1/2} V_n(j))$$

prar $n^{-1/2} U_j \xrightarrow{p} 0 \qquad n^{-1/2} V_n(j) \xrightarrow{p} 0$

s.t $\sqrt{n}(\overline{X_n} - \mu) = Y_n + W_n$ $\qquad W_n = o_p(1)$

Cramer-Slutsky: if $Y_n \xrightarrow{p} X$

i) $X_n = Y_n + Z_n$ And $Z_n \xrightarrow{p} 0 \Rightarrow X_n \xrightarrow{d} 0$

ii $X_n = Y_n Z_n$ And $Z_n \xrightarrow{p} 1 \Rightarrow X_n \xrightarrow{d} Y$

Poof

$E(U_j^2) = E[Z_0^2 + \dots Z_j^2] = j \leq q \qquad j = 1, \dots q$

From b) we know: $\underbrace{P(\frac{1}{n}|V_n(j)| > \varepsilon)}_{M_n} \underset{\text{chebyshev}}{\leq} \frac{q}{\varepsilon^2 n^2}$ and $\sum_{j=0}^{q} M_n \leq \frac{q}{\varepsilon^2 \cdot n} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow \frac{1}{\sqrt{n}} V_n(j) < \frac{q}{n^2 \varepsilon^2} \Leftrightarrow V_n(j) = \frac{\sqrt{n}\, q}{n^2 \varepsilon^2} = \frac{q}{\sqrt{n} \cdot n\, \varepsilon^2} \xrightarrow[n \to \infty]{} 0$

$\Rightarrow \sum_{j=0}^{q} \frac{1}{\sqrt{n}} P(\frac{1}{n}|V_n(j)| > \varepsilon) \leq \frac{q}{\varepsilon^2 n^2} \Leftrightarrow P(|V_n(j)| > \varepsilon) \leq \frac{n \cdot \sqrt{n}\, q}{n^2 \varepsilon^2} = \frac{q}{\sqrt{n}\, \varepsilon^2} \xrightarrow[n \to \infty]{} 0$

$P(\frac{1}{n}|U_j| \geq \varepsilon) = P(|U_j| \geq \varepsilon n) \leq \frac{q}{\varepsilon^2 \cdot n^2} \to 0$

$\Rightarrow \sum \frac{1}{\sqrt{n}} P(\frac{1}{n}|U_j| > \varepsilon) \leq \frac{q}{\varepsilon^2 n^2} \Leftrightarrow \sum P(|U_j| > \varepsilon n) \leq \frac{n \cdot \sqrt{n} \cdot q}{\varepsilon^2 n^2} \to 0$

$\Rightarrow W_n = \sum \theta_j (\frac{1}{\sqrt{n}} U_j - \frac{1}{\sqrt{n}} V_j) \xrightarrow[n \to 0]{} \sum \theta \cdot (0 - 0) = 0$

$\Rightarrow W_n \xrightarrow[n \to \infty]{} 0$

$$\Rightarrow \sqrt{n}\,(\bar{x}_n - u) = Y_n \quad \text{to} \quad \text{når} \quad n \to \infty$$

$$= \left(\sum_{j=0}^{q} \theta_j\right)\left(\sqrt{n}\,\bar{Z}_n\right)$$

$$\sqrt{n}\,\bar{Z}_n \sim N(0, \sigma^2)$$

$$E(cX) = c\,E(x) = 0$$

$$Var(cx) = c^2\,Var(x) = c^2\sigma^2$$

$$\Rightarrow Y_n \sim N\left(0, \left(\sum_{j=0}^{q} \theta_i\right)^2 \cdot \sigma^2\right)$$

$Z_n \sim N(0,1) \Rightarrow \sigma^2 = 1$

$$= N\left(0, \sum_{j=0}^{q} \theta_i\right)$$