HW 4 Endre Moen

4.1 | ARMA(p, q)

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + \sum_{j=1}^{q} \theta_j Z_{t-j} + Z_t \quad (4.1)$$

AR(p) causal $\Rightarrow$ $$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \text{(MA}(\infty)) \quad (4.2)$$

And —— " —— $$\psi_j = \sum_{k=1}^{p \wedge j} \phi_k \psi_{j-k} + \theta_j \quad j \geq 0 \quad (4.3)$$

with $\theta_0 = 1$

Q: a) What is meant with invertibility for this model
Formulate necessary and sufficient cond. for invertible:

A: 1) sufficient nonsingularity: $\gamma(0) > 0 \wedge \gamma(h) \xrightarrow[h \to \infty]{} 0$

sufficient $\theta \in (-1, 1) \wedge \theta + \phi > 0$

2) Invertibility: X_t invertible if there exists constants $\{\pi_j\}$ s.t. $\sum_0^{\infty} |\pi_j| < \infty$ And (i)

$$Z_t = \sum_0^{\infty} \pi_j X_{t-j} \quad \text{And} \quad (ii)$$

$$\theta(z) = 1 + \theta z + \ldots + \theta_q z^q \neq 0 \quad \forall \ |z| \leq 1 \quad (iii)$$

Necessary: $\{(i), (ii), (iii)\}$

b) (4.1)

$$X_t = \sum_{k=1}^{p} \varphi (B^k) X_t + \sum_{j=1}^{q} \theta (B^j) Z_t + Z_t$$

(4.2) $$X_t = \sum_{j=0}^{\infty} \psi (B^j) Z_t$$

c) Q: Find equiv formula for (4.3) when model invertible using $\{\pi_j\}$

A: ① $$\pi_j = -\phi_j - \sum_{k=1}^{q} \theta_k \pi_{j-k} \qquad (3.1.8)$$

pure MA(q) $-\ \psi(B) = \dfrac{\theta(B)}{\phi(B)}$ $\qquad \pi(B) = \dfrac{\phi(B)}{\theta(B)}$ pure AR(p)

As AR(∞) representation

② $$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

③ $$\pi(B) = \sum_{j=0}^{\infty} \phi^j B^j$$

④ $$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j$$ And ⑤ $$\pi(z) = \frac{\phi(z)}{\theta(z)}$$

4.1 d) Q: Explain that (4.3) is a homogenious p-order difference equation with boundary conditions.

A: Eg: ARMA(2,1) $\Rightarrow p = 2$

$$\Rightarrow \psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$$

$$\Rightarrow \psi_0 = 1 \,, \quad \psi_1 = \phi_1 + \theta_1$$

$$\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 \,, \quad \psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1$$

$$\dots \quad \psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2}$$

$$\Rightarrow \left.\begin{array}{l} \psi_0 = 1 \\ \psi_1 = \theta_1 + \phi_1 \end{array}\right] \text{initial conditions}$$

$$\underbrace{\psi_j = \phi_1 \psi_{j-1} - \phi_2 \psi_{j-2} = 0 \,, \quad j \geq 1}_{\text{homogenious difference eq.}}$$

polynomial homogenious eq.

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 \,, \quad \text{roots } z_1, z_2$$

z_1, z_2 can be 1) identically real
2) different, real
3) complax - conjugate pair

$\Rightarrow$ general solution $\psi_j = c_1 z_1^{-j} + c_2 z_2^{-j}$

$\Rightarrow$ 1) identically real : $\psi_j = z_1^{-j}(c_1 + c_2 j)$
with initial condition: $\psi_0 = c_1$,
$\psi_1 = z_1^{-1}(c_1 + c_2)$

2) different, real : $\psi_j = c_1 z_1^{-j} + c_2 z_2^{-j}$

initial condition $\psi_0 = c_1 + c_2$

$\psi_1 = c_1 z_1^{-1} + c_2 z_2^{-1}$

3) complex pair : $\neq re^{i\theta}$ $r = e^{Re(z)}$, $\theta = Im(z)$, $\theta = Im(\bar{z})$

$$\psi_j = e^{Re(z)}\left(c_1 \cos(Im(z)) + c_2 \sin(Im(z))\right)$$

with initial condition $\psi_0 = c_1 + c_2$

$\psi_1 = c_1 Re(z) + Im(z) c_2$

$$\Rightarrow \psi_j = \theta_j + \sum_{k=1}^{p \wedge j} \phi_k \psi_{j-k}$$

is a homogenius p-order difference equation

As ψ_j is expressed at the p'th order difference - the term ψ_{j-p}

$$\left[\psi_j = \theta_j + \dots + \psi_{j-p} \right]$$

ss

$\psi^{(p)}$ - p'th derivation

equivelence of differential eqs.

And $\theta_j = 0 \quad \forall j$

And

$p \wedge j = \min(p, j)$ as $\phi_k = 0 \quad \forall k > p$

And Initial condition $\{\psi_0 \dots \psi_{p-1}\}$

4.1 e) Q: Suppose that the model is causal. Explain that ψ_j decreases with exponential rate towards zero.

Can you give a precise rate.

A: 1) General solution

$$C_1 e^{r_1 t_0} + C_2 e^{r_2 t_0}$$

2/ $$C_1 e^{r_1 t_0} + C_2 t e^{r_1 t_0}$$

3/ $$r e^{i\theta}$$

Since the time-series needs to be stationary $\Rightarrow$ 1) it cannot be exponentially growing

or 2) oscillating

$\Rightarrow$ hence its exponentially decreasing.

4.2 ARMA(2,3)

$\phi = (1.7, -0.9) \quad \theta = (-1.4, 0.8, 0.1) \quad \sigma^2 = 1$

a)

Q: Find roots

A: $X_t = \sum^{p} \phi_k X_{t-k} + \sum^{q} \theta_j Z_{t-j} + Z_t$

$\Leftrightarrow X_t - \sum^{p} \phi_k X_{t-k} = \sum^{q} \theta_j Z_{t-j} + Z_t$

$X_t - 1.7 X_{t-1} + 0.9 X_{t-2} = -1.4 Z_{t-1} + 0.8 Z_{t-2} + 0.1 Z_{t-3} + Z_t$

AR $\phi(z) = 1 - 1.7z + 0.9z^2$

MA $\theta(z) = 1 - 1.4z + 0.8z^2 + 0.1z^3$

$\phi: \quad +1.7 \pm \sqrt{1.7^2 - 4 \cdot 1 \cdot 0.9}/2 = 1.7 \pm \sqrt{-0.7}/2$

$z_1 = 1.7 + \sqrt{-0.7}/2 \qquad z_2 = 1.7 - \sqrt{-0.7}/2$

$|z_1| = |z_2| = \sqrt{(1.7 + \sqrt{-0.7}/2)^2} = \frac{\sqrt{10}}{3} > 1$

MA: θ $\quad \theta(z) = 1 - 1.4z + 0.8z^2 + 0.1z^3$

$z_1 = -9.6 \qquad z_2 = 0.8 + i\,0.65$

$z_3 = 0.8 - i\,0.65$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$|z_2| = |z_3| = 1.02 > 1$

$\Rightarrow \phi$ - is causal (AR(2))

$\Rightarrow \theta$ - is invertible (MA(3))

4.3 / ARMA(p, q)

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{q} \theta_{j+h} \psi_j \qquad h \geq 0$$

a) Q: ARMA(2, 3), find formula for $\gamma(h)$, $h = 0..3$ expressed by model parameter, in matrix form:

$\boxed{A:}$

$$\gamma(h) - \sum^{2} \phi_k \gamma(h-k) = \sigma^2 \sum_{j=0}^{3} \theta_{j+h} \psi_j \quad , \quad \gamma(-h) = \gamma(h)$$

$h=0$ $\quad \gamma(0) - \gamma(1)\phi_1 - \phi_2 \gamma(2) = \sigma^2(1 + \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3) = \sigma^2 S_0$

$h=1$ $\quad \gamma(1) - \gamma(0)\phi_1 - \phi_2\gamma(1) = \sigma^2(\theta_1 + \theta_2\psi_1 + \theta_3\psi_2) \quad \sigma^2 S_1$

$h=2$ $\quad \gamma(2) - \gamma(1)\phi_1 - \phi_2\gamma(0) = \sigma^2(\theta_2 + \theta_3\psi_1) \quad = \sigma^2 S_2$

$h=3$ $\quad \gamma(3) - \gamma(2)\phi_1 - \phi_2\gamma(1) = \sigma^2(\theta_3) \quad = \sigma^2 S_3$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & 0 \\ -\phi_1 & 1 & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 \\ 0 & -\phi_2 & -\phi_1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{bmatrix}}_{B} = \sigma^2 \underbrace{\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}}_{C}$$

4.3 b) Q: Complete the description with homogeniouse difference equation $\phi(B)\gamma(h)=0$, $h \geq 4$

A: $\phi(B)\gamma(h) = \gamma(h) - \phi_1 \gamma(h-1) - \phi_2 \gamma(h-2) = 0$ $\quad \forall h \geq 4$

$$\Rightarrow \begin{bmatrix} \begin{array}{cccccc} & & A & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & -\phi_2 & -\phi_1 & 1 \\ & & & -\phi_2 & -\phi_1 & 1 \\ & & & & & \\ & & & -\phi_2 & -\phi_1 & 1 \end{array} \end{bmatrix} \begin{bmatrix} B \\ \\ \gamma(4) \\ \gamma(5) \\ \vdots \\ \gamma(n) \end{bmatrix} \begin{array}{c} = \\ \\ = \\ \\ \\ \\ \end{array} \begin{bmatrix} C \\ \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}$$

4.4) $\phi(B)X_t = Z_t$ causal AR(p) $Z_t \sim \sigma_t^2$

char poly: $\phi(z) = \left(1-\frac{z}{\xi_1}\right)\left(1-\frac{z}{\xi_2}\right) \cdot \left(1-\frac{z}{\xi_n}\right)$

$\alpha_n = \left(\frac{1}{\xi_n}\right)$, causal $\Rightarrow |\alpha_n| < 1$

$\Rightarrow \phi(z) = (1-\alpha_1 z_1) \dots (1-\alpha_n z_n)$

a) $p=2$, show: $\phi_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2}$, $\phi_2 = -\frac{1}{\xi_1}\frac{1}{\xi_2}$

Hint: $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$

A: $1 - \left(\frac{1}{\xi_1} + \frac{1}{\xi_2}\right) z - \left(-\frac{1}{\xi_1}\frac{1}{\xi_2}\right) z^2 = 0$

$1 - (\alpha_1 + \alpha_2) z - (-\alpha_1 \alpha_2) z^2 = 0$

$1 - (\alpha_1 + \alpha_2) z + \alpha_1 \alpha_2 z^2$

$(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 + \alpha_2)^2 - 4 \cdot \alpha_1 \alpha_2} \, / 2$

$= (\alpha_1 + \alpha_2 \pm \sqrt{\alpha_1^2 + \alpha^2 + 2\alpha_1\alpha_2 - 4\alpha_1\alpha_2} \, / 2$

$= (\alpha_1 + \alpha_2) \pm \sqrt{\alpha_1^2 + \alpha_2^2 - 2\alpha_1\alpha_2} \, / 2$

$(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 - \alpha_2)^2} \, / 2$

$= \alpha_1 + \alpha_2 \pm (\alpha_1 - \alpha_2) / 2 = 0$

$\alpha_1 + \alpha_2 + \alpha_1 - \alpha_2 / 2$ | $\alpha_1 + \alpha_2 - (\alpha_1 - \alpha_2) / 2$

$2\alpha_1 / 2$ | $2\alpha_2 / 2$

$\alpha_1 = 0$ | $\alpha_2 = 0$

4.4) b) Q: Explain with help of (4.6) that for any causal model; $|\phi_p| < 1$

Hint: $\phi(z) = 1 - \phi z - \phi_2 z^2 - \dots - \phi_p z^p$

A: $1 - \alpha_n z = 0 \iff \alpha_1 z = 1$

$z = \xi_n$

causal $\Rightarrow |z| > 1$.

$\Rightarrow \phi_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2} < 2 \quad \phi_2 = \frac{1}{\xi_1} \frac{1}{\xi_2} < 1$

$\Rightarrow \phi_p = \frac{1}{\xi_1} \cdot \frac{1}{\xi_2} \cdot \dots \cdot \frac{1}{\xi_p} < 1$

4.5/ X_t causal AR(2), $Z_t \sim WN(0, \sigma^2)$

(4.8) $$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

by (4.5),

(4.9) $$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \delta_{0h} \sigma^2 \qquad h \geq 0$$

$$\delta_{0h} = \begin{cases} 1 & h=0 \\ 0 & \text{else} \end{cases}$$

a) Q: Multiply (4.8) with X_{t+h}, $h \in 0,1,2$

take the expectation of eq and deduce (4.9) without ref to (4.5)

A: $$X_t \cdot X_{t-h} = \phi_1 X_{t-1} X_{t-h} + \phi_2 X_{t-2} X_{t-h} + Z_t X_{t-h}$$

$$E(X_t X_{t-h}) = \phi_1 E(X_{t-1} X_{t-h}) + \phi_2 E(X_{t-2} X_{t-h}) + \delta_{0h} \sigma^2$$

$h=0$ $$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2)$$

$h=1$ $$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$$

$h=2$ $$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0)$$

$$\frac{\gamma(k)}{\gamma(0)} = \phi_1 \frac{\gamma(h-1)}{\gamma(0)} + \phi_2 \frac{\gamma(h-2)}{\gamma(0)}$$

$$(\Rightarrow) \quad \rho_h = \phi_1 \rho_{h-1} + \phi_2 \rho_{h-2}$$

b) Q: divide by $\gamma(0)$ and verify

A] I $h=0$: $\rho_0 = \phi_1 \rho_1 + \phi_2 \rho_2 + \frac{\sigma^2}{\gamma(0)}$

II $h=1$ $\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1$, $\rho_0 = 1$

$\Leftrightarrow \rho_1(1-\phi_2) = \phi_1 \Leftrightarrow \rho_1 = \frac{\phi_1}{1-\phi_2}$

III $h=2$ $\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$

$\rho_2 = \phi_1\left(\frac{\phi_1}{1-\phi_2}\right) + \phi_2 \Leftrightarrow \rho_2 = \frac{\phi_1^2}{1-\phi_2} + \frac{\phi_2(1-\phi_2)}{1-\phi_2}$

$\Leftrightarrow \rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2}$

II $\Rightarrow \phi_1 = \rho_1(1-\phi_2)$

III $\Rightarrow \phi_2 = \rho_2 - \phi_1 \rho_1$

I $\Rightarrow \frac{\sigma^2}{\gamma(0)} = 1 - \phi_1 \rho_1 - \phi_2 \rho_2 \Leftrightarrow \sigma^2 = (1 - \phi_1 \rho_1 - \phi_2 \rho_2)\gamma(0)$

c) Q: Solve the first two eq above with respect to ρ_1 + ρ_2 and then find a formula for γ_0 in terms of model parameters. Argue from (4.10) that the following boundaries on the parameters are necessary for a causal model:

$$\begin{cases} \phi_2 = 1 & (1) \\ \phi_2 + \phi_1 = 1 & (2) \\ \phi_2 - \phi_1 = 1 & (3) \end{cases}$$

A] I $\quad \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2$

II $\quad \gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$

III $\quad \gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0)$

II $\quad \gamma(1) = \dfrac{\phi_1 \gamma(0)}{1-\phi_2}$

III $\quad \gamma(2) = \phi_1 \left(\dfrac{\phi_1 \gamma(0)}{1-\phi_2}\right) + \phi_2 \gamma(0) = \left[\dfrac{\phi_1^2 + (1-\phi_2)\phi_2}{1-\phi_2}\right] \gamma(0)$

$$\overset{\text{II, III}}{\Longrightarrow} \text{I} \quad \gamma(0) = \phi_1 \left[\frac{\phi_1 \gamma(0)}{1-\phi_2}\right] + \phi_2 \left[\frac{\phi_1^2 + (1-\phi_2)\phi_2}{1-\phi_2}\right] \gamma(0) + \sigma^2$$

$$\gamma(0) \left[\frac{1-\phi_2}{1-\phi_2} - \frac{\phi_1^2}{1-\phi_2} - \frac{\phi_2 \phi_1^2 + (1-\phi_2)\phi_2^2}{1-\phi_2}\right] = \sigma^2$$

$$\gamma(0) = \frac{(1-\phi_2)\,\sigma^2}{1 - \phi_2 - \phi_1^2 - \phi_2 \phi_1^2 - \phi_2^2 + \phi_2^3}$$

$$(1-\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right] = (1-\phi_2)\left[1 - 2\phi_2 + \phi_2^2 - \phi_1^2\right]$$

$$= 1 - \phi_2 - 2\phi_2 - 2\phi_2^2 + \phi_2^2 + \phi_2^3 - \phi_1^2 - \phi_1^2 \phi_2$$

$$= 1 - \phi_2 - \phi_2^2 + \phi_2^3 - \phi_1^2 - \phi_1^2 \phi_2$$

$$\Rightarrow \gamma(0) = \frac{(1-\phi_2)\,\sigma^2}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]}$$

III $\gamma(1) = \frac{\phi_1}{1-\phi_2}\gamma(0)$

$$= \left(\frac{\phi_1}{1-\phi_2}\right)\left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma^2}{(1-\phi_2)^2-\phi_1^2}$$

II $\rho_2 = \phi_2 + \frac{\phi_1^2}{1-\phi_2}$ I $\rho_1 = \frac{\phi_1}{1-\phi_2}$

Fact $|\rho_n| \leq 1$

$\overset{\text{fact}}{\Longrightarrow} \; -1 \leq \frac{\phi_1}{1-\phi_2} < 1 \iff -(1-\phi_2) < \phi_1 < 1-\phi_2$

$+$ I

$\phi_2 - 1 < \phi_1 < 1-\phi_2$

$\Rightarrow -1 < \phi_1 - \phi_2$

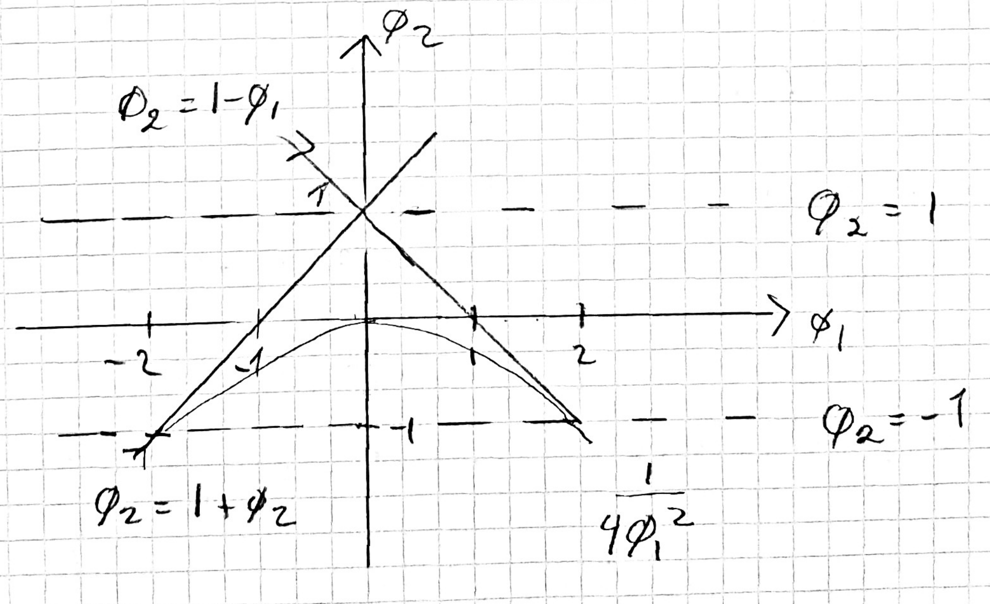
$\phi_1 + \phi_2 < 1$ which is 2

$\hookrightarrow 1 > -(\phi_1 - \phi_2) \iff 1 > \phi_2 - \phi_1$ which is 3

$|\phi_2| = \left|\frac{1}{\xi_1}\frac{1}{\xi_2}\right| \overset{\text{causal}}{\Longrightarrow} |\xi_i| > 1 \; \forall i \Rightarrow |\phi_2| < 1$

So ϕ_2 is bounded above by $\phi_2 = 1$

below by $\phi_2 = -1$



4.5 e) Q: Show that the roots of the characteristic polynomial is def by:

$$z^2 + \frac{\phi_1}{\phi_2} z - \frac{1}{\phi_2} = 0$$

and explain that the part of the triangle (4.11) that defines the region with complex roots inside the triangle and bounded below by horizontal line $\phi_2 = -1$ and above by $\phi_2 = 4\phi_1^2$. Add curv to drawing. Where does the model have roots of multiplicity 1? What about corners/edges?

A:

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\phi(z)$ is standard characteristic func.

It can be rewritten: $\lambda = z^{-1}$

$$z^{-2} - \phi_1 z^{-1} - \phi_2 = \lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

$|z| > 1 \iff |\lambda| < 1$

causal

$$\Rightarrow \quad -(-\phi_1) \pm \sqrt{\phi_1^2 + 4\phi_2} \,/\, 2 \cdot (-\phi_2)$$

$\Rightarrow$ real roots $\Rightarrow \phi_1^2 + 4\phi_2 \geq 0$

$$\phi_1^2 \geq -4\phi_2$$

$$-\frac{\phi_1^2}{4} \geq \phi_2$$

1 root, multiplicity 2: $-\frac{\phi_1^2}{4} = \phi_2$

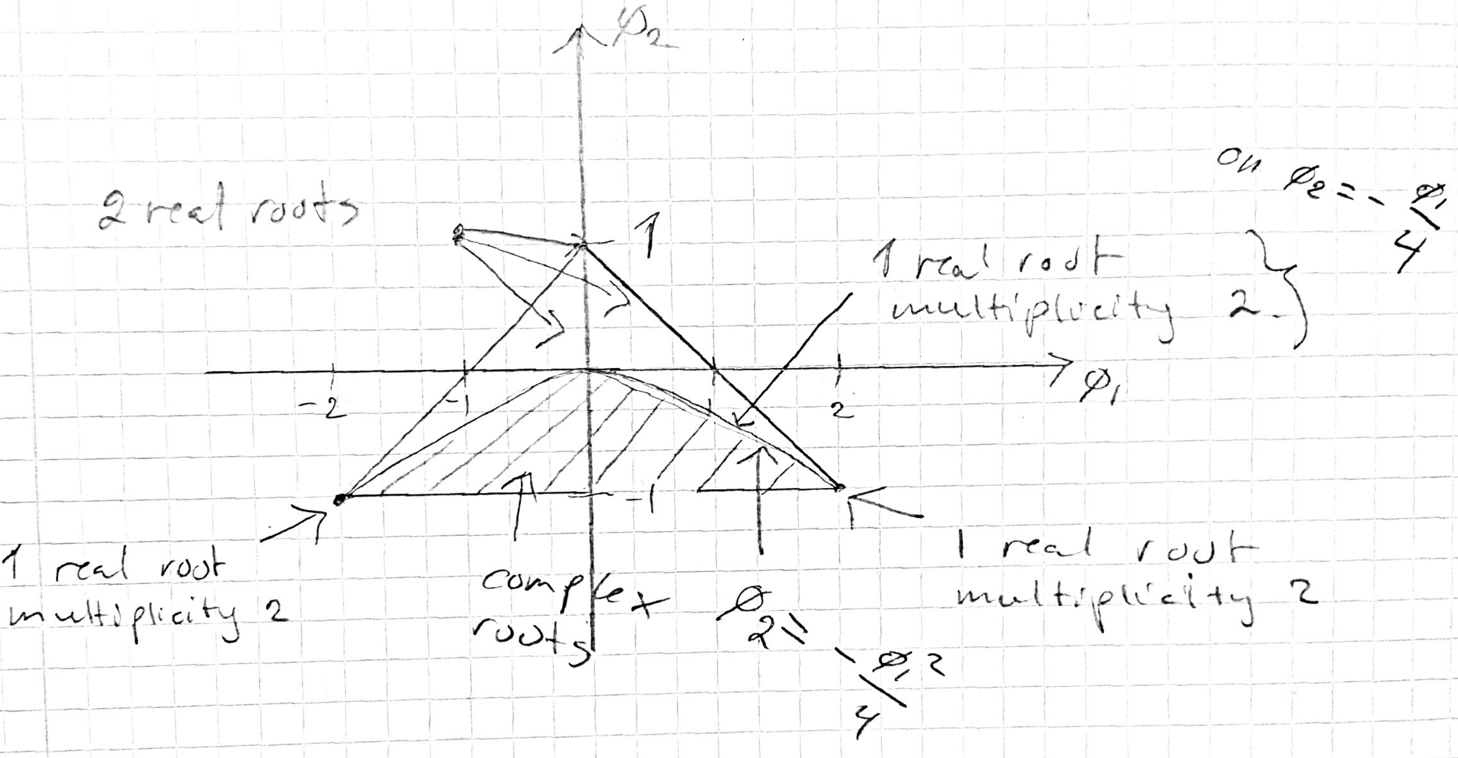
$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0 \quad / \cdot \frac{1}{\phi_2}$$

$$\iff \frac{1}{\phi_2} - \frac{\phi_1}{\phi_2} z - z^2 = 0 \quad / \cdot -1$$

$$\iff z^2 + \frac{\phi_1}{\phi_2} z - \frac{1}{\phi_2} = 0$$

$a = 1$ $\quad b$ " $\quad c = -\frac{1}{\phi_1}$

$$\Rightarrow z = -\frac{\varphi_1}{\varphi_2} \pm \sqrt{\left(\frac{\varphi_1}{\varphi_2}\right)^2 - 4\cdot\left(-\frac{1}{\varphi_2}\right)} \Big/ 2\cdot 1$$



4.6/ Q: Asymptotic cov. matrix for least squares estimator of $\phi = (\phi_1, \dots \phi_p)^T$ in AR(p) process with iid $\sigma^2 \Gamma_p^{-1}$. Compare the asymptotic variance of $\hat{\phi}_1$ when the estimated true model is AR(1) with corresponding asymptotic variance for the estimator of ϕ_1 in an AR(2) model when $\phi_2 = 0$, Here:

$$\Gamma_p = \begin{bmatrix} \gamma_0 & \dots & \gamma_{p-1} \\ \vdots & \ddots & \\ \gamma_{p-1} & & \gamma_0 \end{bmatrix}$$

Calculate Γ_p^{-1} and simplify.

$\boxed{A}$ From 4.5 c): AR(1)

$$\gamma(0) = \phi_1 \gamma(1) + \sigma^2$$

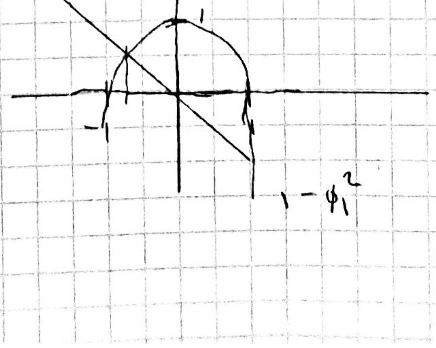
$$\gamma(1) = \phi_1 \gamma(0)$$

$$\Rightarrow \gamma(0) = \phi_1(\phi_1 \gamma(0)) + \sigma^2 \iff \gamma(0)(1 - \phi_1^2) = \sigma^2$$

$$\gamma(0) = \frac{\sigma^2}{1-\phi^2}$$

$$\Gamma_1 = \frac{\sigma^2}{1-\phi_1^2} \Rightarrow \Gamma_1^{-1} = \frac{1-\phi_1^2}{\sigma^2}$$

$$\Rightarrow \sigma^2 \Gamma^{-1} = 1 - \phi_1^2$$



from 4.59 AR(2) : $\gamma(0) = \dfrac{(1-\varphi_2)}{(1+\varphi_2)} \dfrac{\sigma^2}{(1-\varphi_2)^2 - \varphi_1^2}$

$$\gamma(1) = \frac{\phi_1}{1-\phi_2}\gamma(0)$$

$$\Gamma_2 = \begin{bmatrix} \dfrac{(1-\varphi_2)}{(1+\varphi_2)} \dfrac{\sigma^2}{(1-\varphi_2)^2 - \varphi_1^2} & \dfrac{\phi_1}{1-\phi_2}\gamma(0) \\ \dfrac{\phi_1}{1-\phi_2}\gamma(0) & \dfrac{1-\phi_2}{(1+\phi_2)} \dfrac{\sigma^2}{(1-\phi_2)^2 - \phi_1^2} \end{bmatrix}$$

$\dfrac{1}{ad-bc}$

$$\Gamma_2^{-1} = \begin{bmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{bmatrix} \cdot \frac{1}{\gamma(0)^2 - (-\gamma(1) \cdot -\gamma(1))}$$

$$\gamma(0)^2 - (-\gamma(1) \cdot -\gamma(1)) = \gamma(0)^2 - \gamma(1)^2$$

$\rightarrow$ complicated

use $\phi_2 = 0$

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2} \qquad \gamma(1) = \phi_1 \gamma(0)$$

$$\Gamma_2^{-1} = \begin{bmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{bmatrix} \cdot \frac{1}{\gamma(0)^2 - \gamma(1)^2}$$

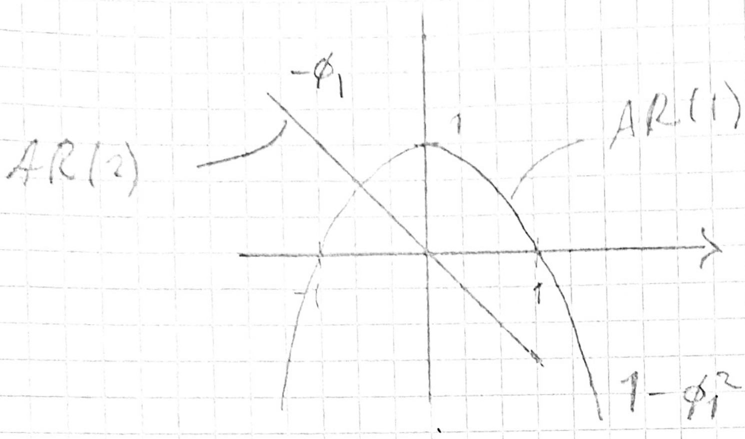
$$\gamma(0)^2 = \frac{\sigma^4}{(1-\phi_1^2)^2} \qquad \gamma(1)^2 = \frac{\phi_1^2 \sigma^4}{(1-\phi_1^2)^2}$$

$$\Rightarrow \gamma(0)^2 - \gamma(1)^2 = \frac{\sigma^4(1-\phi_1^2)}{(1-\phi_1^2)^2} = \frac{\sigma^4}{1-\phi_1^2}$$

$$(\gamma(0)^2 - \gamma(1)^2)^{-1} = \frac{1-\phi_1^2}{\sigma^4}$$

$$\Gamma_2^{-1} = \begin{bmatrix} \dfrac{\sigma^2}{1-\phi_1^2} \cdot \dfrac{1-\phi_1^2}{\sigma^4} & \dfrac{-\phi_1 \cdot \sigma^2}{1-\phi_1^2} \cdot \dfrac{1-\phi_1^2}{\sigma^4} \\ \dfrac{-\phi_1 \sigma^2}{1-\phi_1^2} \dfrac{1-\phi_1^2}{\sigma^4} & \dfrac{\sigma^2}{1-\phi_1^2} \dfrac{1-\phi_1^2}{\sigma^4} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{\sigma^2} & \dfrac{-\phi_1}{\sigma^2} \\ \dfrac{-\phi_1}{\sigma^2} & \dfrac{1}{\sigma^2} \end{bmatrix} \Rightarrow \sigma^2 \Gamma_2^{-1} = \begin{bmatrix} 1 & -\phi_1 \\ -\phi_1 & 1 \end{bmatrix}$$



$1-\phi_1^2 = -\phi_1 \iff 1+\phi_1-\phi_1^2$

$0 = 1+\phi_1-\phi_1^2 \Rightarrow -1 \pm \sqrt{1+4}/-2 \Rightarrow -(-1+\sqrt{5})/2 = -0.62$

$\hookrightarrow -(-1-\sqrt{5})/2 = 1.62$

Since $|\phi_1| < 1 \Rightarrow$

AR(1) $\overset{\text{model}}{\vee}$ has less variance if $\phi_1 \in (-1, -0.62)$

AR(2) —— " —— $\phi_1 \in (-0.62, 1)$

even if process is AR(1)

4.8 Consider AR(2) with complex root:

$$\phi(z) = \left(1 - \frac{z}{\xi_1}\right)\left(1 - \frac{z}{\bar{\xi}_1}\right)$$

where $|\xi| > 1$, let $\alpha = \frac{1}{\xi}$ so that

$$\phi(z) = (1 - \alpha z)(1 - \bar{\alpha} z),$$

$$\alpha = re^{i\theta} = |z| e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$

$$\xi = \frac{1}{r} e^{-i\theta} = |z|^{-1} e^{-i\theta} = |z|^{-1}(\cos\theta - i\sin\theta)$$

a) Q: express ϕ_1, ϕ_2 as function of r, T

A:

$$\phi(z) = (1 - \alpha z)(1 - \bar{\alpha} z) = 1 - \phi_1 z - \phi_2 z^2$$

$$1 - \bar{\alpha} z - \alpha z + \alpha\bar{\alpha} z^2 = 1 - \phi_1 z - \phi_2 z^2$$

$$\alpha = re^{i\theta} \qquad \qquad 1 - (\bar{\alpha} + \alpha) z + \alpha\bar{\alpha} z^2$$

$$\bar{\alpha} = re^{-i\theta}$$

$$\Rightarrow \alpha + \bar{\alpha} = r(e^{i\theta} + e^{-i\theta}) = r($$

$$= r(\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta})$$

$$= r\, 2\cos\theta$$

$$\bar{\alpha}\alpha = re^{i\theta} \cdot re^{-i\theta} = r^2 (e^{i\theta - i\theta}) = r^2$$

$$\Rightarrow 1 - (\bar{\alpha} + \alpha) z + \alpha\bar{\alpha} z^2$$

$$= 1 - 2r\cos\theta\, z + r^2 z^2 = 1 - \phi_1 z - \phi_2 z^2$$

$$\Rightarrow \phi_2 = -r^2, \quad \phi_1 = 2r\cos\theta$$

4.8/ b) let $\theta \sim U[0, 2\pi)$, $r \sim U[0,1)$

$$\alpha = r \cdot e^{i\theta}$$

Q: find $E Q_1$, $E Q_2$, $E|Q_1|$

A: $\phi_2 = -r^2$ $\qquad \phi_1 = 2r\cos\theta$

$$E(\phi_2) = -E(r^2)$$

$E(x^2)$ for $U[a,b]$ is $\dfrac{b^3 - a^3}{3b - 3a}$

$\Rightarrow$ $U[0,1]$ is $\dfrac{1-0}{3-0} = \dfrac{1}{3}$

$$-E(r^2) = \underline{\underline{-\frac{1}{3}}}$$

$E(\phi_1)$: $\quad f_\theta : \dfrac{1}{b-a} = \dfrac{1}{2\pi}$

$$\Rightarrow E(\phi_1) = E(2r\cos\theta) = 2E(r)E(\cos\theta)$$

$$= 2\int_0^1 \int_0^{2\pi} r\cos\theta \cdot \frac{1}{2\pi} \, d\theta \, dr$$

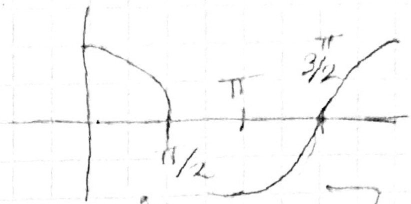
$$= \frac{2}{2\pi}\int_0^1 r\left[\sin\theta\Big|_0^{2\pi} dr = \frac{1}{\pi}\int_0^1 r\cdot(0-0)\,dr = 0\right.$$

$\cos\theta'$
$= -\sin\theta$

$$\Rightarrow E(\phi_1) = \underline{\underline{0}}$$

$\sin\theta'$
$= \cos\theta$

$E(|\phi_1|) =$



$$\frac{2}{2\pi}\int_0^1 r\left[\int_0^{\pi/2}\phi_1\,d\theta - \int_{\pi/2}^{3\pi/2}\theta_1\,d\theta + \int_{3\pi/2}^{2\pi}\theta_1\,d\theta\right]dr$$

$$= \frac{1}{\pi}\int_0^1 r\left[\sin\theta\Big|_0^{\pi/2} - \sin\theta\Big|_{\pi/2}^{3\pi/2} + \sin\theta\Big|_{3\pi/2}^{2\pi}\right]dr$$

$$\frac{1}{\pi}\int_0^1 r\ (1-0) - (-1-1) + (0-(-1))\,]\,dr$$

$$1+1+1+1=4$$

$$\frac{4}{\pi}\int_0^1 r\,dr = \frac{4}{\pi}\left(\frac{1}{2}r^2\Big|_0^1 = \frac{4}{\pi}\cdot\frac{1}{2}\cdot(1-0) = \frac{2}{\pi}\right.$$

4.8 d) Given AR(2) with complex roots has ACF with periodic structure. According to theory for homogenous difference eq.

$$\gamma(h) = r^h \{ c \exp(ih\theta) + \bar{c} \exp(-ih\theta) \}, \quad h \geq 0$$

$c = u + iv$, $\bar{c} = u - iv$

$c, \bar{c}$ determined by initial condition, or

(4.14)
$$\gamma(h) = a r^h \cos(\theta h + b), \quad h \geq 0$$

when (a, b) initial condition.

From (4.14) you see that ACVF has a periodic structure. If $\theta = \frac{2\pi}{m}$, $m \in \mathbb{Z}$, then the period is m, and amplitude damped.

d) Q: formulate the eq to give (u, v) in (4.13) or (a, b) in (4.14)

A: Note $\cos(x+y) = \cos x \cos y - \sin x \sin y$

$$\gamma(0) = a r^0 \cos(\theta \cdot 0 + b) = a \cos(b)$$

$$\gamma(1) = a r \cos(\theta + b) = ar(\cos\theta \cos b - \sin\theta \sin b)$$

$$\gamma(2) = a r^2 \cos(2\theta + b) = ar^2(\cos 2\theta \cos b - \sin 2\theta \sin b)$$

$$\rho_0 = 1 \Rightarrow \frac{a \cos b}{\gamma(0)} = 1$$

$$\rho_1 = \frac{ar(\cos\theta \cos b - \sin\theta \sin b)}{a \cos b} = r\cos\theta - \frac{r \sin\theta \sin b}{\cos b}$$

$$\rho_2 = \frac{ar^2(\cos 2\theta \cos b - \sin 2\theta \sin b)}{a \cos b}$$

$$= r^2 \cos 2\theta - \frac{r^2 \sin 2\theta \sin b}{\cos b}$$

4.9/ a) A: Yes, $\varepsilon_1 = 1.05$ $\varepsilon_2 = 1.05$

$|\varepsilon_i| > 1 \Rightarrow$ invertible

$$X_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t$$

$$\underline{\theta} = (1.8, 0.9), \sigma^2 = 1$$

b) Q: Apply Durbin-Levinson for n=1..50

$$\gamma(0) = \sigma^2(1 + \theta_1^2 + \theta_2^2) = 5.05$$

$$\gamma(1) = \frac{\theta_1 + \theta_1\theta_2}{\gamma(0)} = 0.67$$

$$\gamma(2) = \frac{\theta_2}{\gamma(0)} = 0.18$$

$$\gamma(h) = 0 \qquad h \geq 3$$

$n = 0$, $V_0 = \gamma(0)$

$n = 1$, $\phi_{11} = \frac{\gamma(1)}{\gamma(0)} = \frac{\gamma(1)}{V_0} = 0.13$

$$V_1 = V_0(1 - \phi_{11}^2) = 4.95$$

$n = 2$ $\phi_{22} = (\gamma(2) - \phi_{11}\gamma(1)) / V_1 = 0.02$

$\phi_{21} = (\phi_{11} - \phi_{22}\phi_{11}) = 0.13$

$V_2 = V_1(1 - \phi_{22}^2) = 4.96$

$n = 3$ $\phi_{33} = (\gamma(3) - \phi_{21}\gamma(2) - \phi_{22}\gamma(1)) / V_2 = -0.01$

$\phi_{31} = (\phi_{21} - \phi_{33}\phi_{22}) = 0.13$

$\phi_{32} = (\phi_{22} - \phi_{33}\phi_{21}) = 0.02$

$V_3 = V_2(1 - \phi_{23}^2) = 4.96$

4.9/ c) Q: Compare coeff with invertible rep:

$$\pi_j = \begin{cases} 1 & j=0 \\ -\theta_1 & j=1 \\ -\theta_1 \pi_{j-1} - \theta_2 \pi_{j-2} & j \geq 2 \end{cases}$$

$\pi_0 = 1$, $\pi_1 = -1.8$

$\pi_2 = -1.8\,\pi_1 - \theta_2 \pi_0$

$= 1.8^2 - 0.9 = 2.34$

$\pi_3 = -\theta_1 \pi_2 - \theta_2 \pi_1 = -2.6$

$\hat{X}_3 = \phi_{21} X_2 + \phi_{22} X_1$

$\hat{X}_4 = \phi_{31} X_3 + \phi_{32} X_2 + \phi_{33} X_1$

4.11) X_t - solution non-causal AR(p)
$\tilde{Z}_t \sim WN(0, \sigma^2)$

Q: How does the seq. of $\hat{X}_{n+1}$ predictors look like for $n \geq p$, what about the predictor error variance?

A: X_t - non-causal $\Rightarrow \hat{X}_{n+1}$ depends on all future values.

4.10 / a) is the model invertible?

$X_t = Z_t - Z_{t-1}$ ⟹ Random walk

⟹ non-stationary

$\theta = -1$

c) Show $\| X_{n+1} - \hat{X}_{n+1} \|^2 = \| Z_{n+1} \|^2 + \| -Z_n - \hat{X}_{n+1} \|^2$

so the best we can do is pred $-Z_n$ $\{X_n \dots X_n\}$

$$\| X_{n+1} - \hat{X}_{n+1} \|^2 = \| \theta Z_n + Z_{n+1} - \hat{X}_{n+1} \|^2$$

$$\leq \| Z_{n+1} \|^2 + \| \theta Z_n - \hat{X}_{n+1} \|^2$$

$$= \| Z_{n+1} \|^2 + \| -Z_n - \hat{X}_{n+1} \|^2$$

$\hat{X}_{n+1} = \phi_{n1} X_n + \dots + \phi_{nn} X_1$, Z_{n+1} independent of $X_1, \dots X_n$, and Z_n,

Z_i iid and Z_n dependent on X_n