Summable filter and symbolic operators

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Summable filters

1. Introduction

We have seen that even a simple model like the $\mathrm{AR}(p)$ has a solution expressed as an infinite $\mathrm{MA}(\infty)$, and that is an infinite expansion.

Likewise an inveritble $\mathrm{MA}(1)$ process has an infinite expansion as an $\mathrm{AR}(\infty)$. In these slides you will learn to manipulate with such expansions.

Table 1: Model and representation

model	expansion	
AR(p)	MA(∞)	
MA(q)	AR(∞)	
ARMA(p, q)	MA(∞)	

Symbolic shift operator

2. Filter

2.1. Backward shift operator

Definition 1. Let $\{X_t\}$ be a stationary time series. Then the symbolic shift operator is denoted by «B» and $BX_t \stackrel{\text{def}}{=} X_{t-1}$.

Remark 1. We also use $B^j X_t = X_{t-j}$ and $B^{-j} X_t = X_{t+j}$.

Summable filters

DEFINITION 2. A general ℓ^2-filter $\{\psi_j,\, j \in \mathbb{Z}\}$ satisfies $\sum_j \psi_j^2 < \infty$. It is summable if $\sum_j |\psi_j| < \infty$. Here a filter is summable if not specified to be something else.

REMARK 2. A summable filter is a ℓ^1-filter.

DEFINITION 3. Let $\{Y_t\}$ be a stationary time series and ψ a filter. and let ω be an outcome. Let $y_s = Y_s(\omega)$ for $s \in \mathbb{Z}$. Then

$$X_t = \psi(B)Y_t$$

is defined by $X_t(\omega) \stackrel{\text{def}}{=} \sum_j \psi_j Y_{t-j}(\omega) = \sum_j \psi_j y_{t-j} = (\mathbf{y} * \psi)(t)$.

DEFINITION 4. Let ψ be a summable filter. Its genrating function is $\phi(z) = \sum_j z^j$ where z is complex varibale.

REMARK 3. The genearing filter function $\psi(z)$ is at least well defined on the unit circle.

Convolution

2.2. Convolution

Remark 4. The filter operation; going from $\{Y_t\}$ to $\{X_t\}$, is a convolution; $X = \psi * Y$.

Remark 5. The outcome ω could be $\{y_t, t \in \mathbb{Z}\}$.

Conserving stationarity

PROPOSITION 1. Let $\{Y_t\}$ be a stationary time series and ψ a filter. Then $X_t \stackrel{\text{def}}{=} \psi(B)Y_t$ is a well defined and stationary time series with ACVF;

$$\gamma_X(h) = \sum_u \Big(\sum_j \psi_{j+h-u}\psi_j\Big)\gamma_Y(u). \tag{1}$$

Convolution notation

REMARK 6. We see that (1) is close to a doble convolution. Let ψ' be defined as $\psi'_j = \psi_{-j}$,

$$\begin{aligned}\sum_j \psi_{j+h-u}\psi_j &= \sum_j \psi_{h-u+j}\psi'_{-j} \\ &= \sum_j \psi_{h-u-j}\psi'_j \\ &= \psi * \psi'(h-u).\end{aligned}$$

We insert this formula in (1),

$$\begin{aligned}\gamma_X(h) &= \sum_u (\psi * \psi')(h-u)\gamma_Y(u) = (\psi * \psi') * \gamma_Y(h) \\ &= \psi * \psi' * \gamma_Y(h).\end{aligned} \tag{2}$$

Proof

Proof of Proposition 1.

We use that $S_n \stackrel{\text{def}}{=} \sum_{|j| \le n} \psi_j Y_{t-j}$ converges pointwise to X_t and $|S_n| \le U$. where

$$U \stackrel{\text{def}}{=} \sum_j |\psi_j||Y_{t-j}|.$$

Then

$$\mathbb{E}\,U = \sum_j |\psi_j| \mathbb{E}\,|Y_{t-j}| \le \mathbb{E}^{1/2} Y_t^2 \sum_j |\psi_j| < \infty.$$

This means that the infinite sum converges absolutely with probability one. Hence $\{X_t\}$ is is well defined.

Stationarity

The varaiable U dominate ;$|X_t| \leq U$ and U has finite expectation. Therefore

$$\mathbb{E}\, X_t = \sum_j \psi_j \mathbb{E}\, Y_{t-j} = \mu_Y \sum_j \psi_j \qquad \text{independent of } t.$$

What about the variance? Ou dominating variable has finite variance since

$$\begin{aligned}
\mathbb{E}\, U_t^2 &= \sum_j \sum_k |\psi_j||\psi_k| \mathbb{E}\, |Y_{t-j}||Y_{t-k}| \\
&\leq \sum_j \sum_k |\psi_j||\psi_k| \mathbb{E}\, Y_t^2 \\
&= \mathbb{E}\, Y_t^2 \Big(\sum_j |\psi_j| \Big)^2 \\
&< \infty.
\end{aligned}$$

Variance and covariance

We are use again that $S_n(t) = \sum_{|j|\leq n} \psi_j Y_{t-j}$ converges pointwise to X_t and

$$|S_n(t+h)S_n(t)| \leq U(t+h)U(t) \leq 2U_{t+h}^2 + 2U_t^2$$

where the two terms on the right hand side have finite expectation. This means that we have justfied that covariance of X_{t+h} and X_t can be computed by exchanging covariance (integration) with summation.

DCT

By the dominating convergence theorem,

$$\gamma_X(h) = \sum_j \sum_k \psi_j \psi_k \mathrm{Cov}(Y_{t+h-j}, Y_{t-k})$$

$$\gamma_X(h) = \sum_j \sum_k \psi_j \psi_k \gamma_Y(h - j + k)$$

$$\gamma_X(h) = \sum_j \sum_k \psi_j \psi_k \gamma_Y(h - j + k)$$

substitute with $u = h - j + k \Rightarrow j = h - u + k$

$$= \sum_u \sum_k \psi_{k+h-u} \psi_k \gamma_Y(u).$$

$\square$

Convolution of white noise

COROLLARY 1. Let $X_t = \psi(B)Z_t$ with $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Then

$$\gamma_X(h) = \sigma^2 \sum_j \psi_{j+h}\psi_j.$$

PROOF.
In this case $\gamma_Y(u) = \sigma^2 \delta_{0,u}$. $\square$

Note that $\gamma_X = \sigma^2 \psi * \psi'$.

Example

2.3. AR(1)

Example 1. In a causal AR(1) process $\psi_j = \phi^j$ for $j \geq 0$ and zero otherwise. Let $h \geq 0$. Then

$$\begin{aligned}
\gamma_X(h) &= \sigma^2 \sum_j \psi_{j+h}\psi_j \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_{j+h}\psi_j \\
&= \sigma^2 \sum_{j=0}^{\infty} \phi^{h+j+j} \\
&= \sigma^2 \phi^h \sum_{j=0}^{\infty} \phi^{2j} \\
&= \frac{\sigma^2}{1-\phi^2}\phi^h.
\end{aligned}$$

Filters commute

2.4. Filter calculus

PROPOSITION 2. Let $\{Y_t\}$ be a stationary time series and let α, β be filters. Define $U_t = \beta(B)Y_t$ and $X_t = \alpha(B)U_t$. Then

$$X_t = \psi(B)Y_t$$

where $\psi(z) = \alpha(z)\beta(z)$.

REMARK 7. This is a calculus rule. Since $\psi(z) = \beta(z)\alpha(z)$, this means that the filter operations commute.

Proof

Proof of Proposition 2.

The proof shows that an oop

Let

$$U \stackrel{\text{def}}{=} \sum_j |\alpha_j| \sum_k |\beta_k| |Y_{t-j-k}|$$

Then

$$\begin{aligned}\mathbb{E}\, U &\leq \mathbb{E}\, |Y_t| \sum_j |\alpha_j| \sum_k |\beta_k| \\ &= \mathbb{E}\, |Y_t| \left[\sum_j |\alpha_j|\right] \quad \left[\sum_k |\beta_k|\right] < \infty.\end{aligned}$$

Hence we absolutely convergence with probability one.

Unbounded sequence

Let ω be an outcome outside the exceptional set and let $y_s = Y_s(\omega)$. Then

$$\begin{aligned}
&= \sum_j |\alpha_j| \sum_k |\beta_k||y_{t-j-k}| \\
&= \sum_k |\beta_k| \sum_j |\alpha_j||y_{t-j-k}| \\
&= \sum_u \left(\sum_k |\beta_{u+k}||\alpha_k| \right) |y_{t-u}| \\
&= \sum_u (|\beta| * |\widetilde{\alpha}|)(u) |y_{t-u}| \\
&= |\psi|(B)|y_t|.
\end{aligned}$$

$\square$

REMARK 8. Note that in general $\{y_t\}$ is an unbounded sequence.

The inverse of a filter

2.5. A polynomial filter has an inverse

THEOREM 1. Let $\{Y_t\}$ be a stationary time serie, β a filter and $X_t \stackrel{\text{def}}{=} \beta(B)Y_t$. Suppose that β is finite, the polynomial $\beta(z)$ has constant term equal to 1 and none of its roots are on the unit circle.

Then there exists a unique filter α such that $Y_t = \alpha(B)X_t$.

REMARK 9. We sometimes write

$$X_t = \beta(B)Y_t \quad \Rightarrow \quad Y_t = \beta^{-1}(B)X_t.$$

REMARK 10. The filter α is the inverse filter of β.

A polynomial expressed as function of its roots

PROPOSITION 3. Let $\beta(z)$ be a polynomial of grade m with constant term equal to 1. Then

$$\beta(z) = \prod_{i=1}^{m}(1 - \xi_i^{-1} z)$$

where $\{\xi_1, \ldots, \xi_m\}$ are the m roots of the polynomial including multiplisities,

PROOF OF PROPOSITION 3.
By the fundamental theorem in algebra the roots exist. Let

$$g(z) \stackrel{\text{def}}{=} \beta(z) - \prod_{i=1}^{m}(1 - \xi_i^{-1} z).$$

Then g is a polynomial of at most grade m. But g has $m + 1$ roots and therefore $g \equiv 0$. $\square$

Proof Theorem 1

PROOF OF THEOREM 1. Let β be a polynomial of grade m with constant term equal to 1. By Proposition 3

$$\beta(z) = \prod_{i=1}^{m}(1 - \xi_i^{-1} z). \tag{3}$$

$\{\xi_i,\ i = 1, \ldots, m\}$ are the roots of β including multiplisities. By (3), we can factorise β as a product of two polynomial factors; β_1 with all roots outside the unit circle, say r; β_2 with all roots inside the unit circle.

$$\beta(z) = \beta_1(z)\beta_2(z) = \left(\prod_{i=1}^{r}(1 - \xi_i^{-1} z\right) \left(\prod_{i=r+1}^{m}(1 - \xi_i^{-1} z)\right).$$

Consider the root ξ_i. If $|\xi_i| > 1$, then

$$\frac{1}{1 - \xi_i^{-1} z} = \sum_{n=0}^{\infty} \left(\frac{z}{\xi_i}\right)^n, \qquad |z| < |\xi|$$

since the right hand is a convergent geometric series under the given condition.

Roots inside the unit circle

If $|\xi_i| < 1$, then

$$\frac{1}{1-\xi_i^{-1}z} = \frac{1}{(\xi_i^{-1}z)(\xi z^{-1}-1)}$$
$$= \frac{1}{(-\xi_i^{-1}z)(1-\xi z^{-1})}$$
$$= \left(\frac{-\xi}{z}\right)\left(\frac{1}{1-\xi z^{-1}}\right).$$

Therefore

$$\frac{1}{1-\xi_i^{-1}z} = \left(\frac{-\xi_i}{z}\right)\sum_{n=0}^{\infty}\left(\frac{\xi_i}{z}\right)^n$$
$$= -\sum_{n=1}^{\infty}\left(\frac{\xi_i}{z}\right)^n$$
$$= -\sum_{n=1}^{\infty}\xi_i^n z^{-n}, \qquad |z| > |\xi_i|.$$

The two factors

Hence

$$\alpha_1(z) \stackrel{\text{def}}{=} \frac{1}{\beta_1(z)} = \prod_{i=1}^{r} \Big(\sum_{n=0}^{\infty} \Big(\frac{z}{\xi_i}\Big)^n \Big), \qquad |z| < \min_{i \le r} |\xi_i|,$$

(4)

$$\alpha_2(z) \stackrel{\text{def}}{=} \frac{1}{\beta_2(z)} = (-1)^{m-r} \prod_{i=r+1}^{m} \Big(\sum_{n=1}^{\infty} \Big(\frac{\xi_i}{z}\Big)^n \Big), \qquad |z| > \max_{i \ge r+1} |\xi_i|,$$

and

$$\alpha(z) \stackrel{\text{def}}{=} \frac{1}{\beta(z)} = \alpha_1(z)\, \alpha_2(z).$$

Simple trick

We have used a quite simple method but it is uttermost importance. Let

$$\delta = \min\big(\max(1 - |\xi_i|, |\xi_i| - 1)\big).$$

Then

$$\alpha_1(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad |z| \in (1-\delta, 1+\delta),$$

$$\alpha_2(z) = \sum_{n=1}^{\infty} b_n z^{-n}, \qquad |z| \in (1-\delta, 1+\delta)$$

where the coefficients $\{a_n\}$ and $\{b_n\}$ are defined in terms of the roots (4) .

Finally

We can write

$$\alpha(z) \stackrel{\text{def}}{=} \frac{1}{\beta(z)} = \sum_n c_n z^n, \qquad |z| \in (1-\delta, 1+\delta)$$

and $|c_n| = \mathcal{O}((1-\delta)^n)$.

Now,

$$\alpha(B)X_t = \alpha(B)\beta(B)Y_t = \Big(\alpha(B)\beta(B)\Big)Y_t = \psi(B)Y_t = Y_t$$

by Proposition 2.

$\square$

Example $m = 2$

2.6. The AR(2) characteristic polynomial has two roots

For $m = 2$ the two roots must be one of the following possibilities:

i) 1 multiple real root: ξ.

ii) 2 different real roots: ξ_1 and ξ_2.

iii) One complex root and its conjugate: ξ and $\overline{\xi}$

2 complex roots

Suppose that we have two complex roots that are stritly inside the unit circle. Then

$$\begin{aligned}\alpha(z) &= \Big(-\sum_{j=1}^{\infty} \xi^j z^{-n}\Big)\Big(-\sum_{k=1}^{\infty} \overline{\xi}^k z^{-k}\Big) \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} z^{-(j+k)} \xi^n \overline{\xi}^k \\ &= \sum_{n=2}^{\infty} \Big(\sum_{k=1}^{n-1} \xi^{n-k} \overline{\xi}^k\Big) z^{-n} \\ &= \sum_{n=2}^{\infty} c_n z^{-n}.\end{aligned}$$

Remark 11. The filter is real.

Exponential rate

The coefficients are

$$c_n \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \xi^{n-k}\overline{\xi}^{\,k} = (a * \overline{a})(n) - \xi^n - \overline{\xi}^{\,n}, \qquad a_n \stackrel{\text{def}}{=} \xi^n \;.cr$$

The sequence $\{c_n\}$ decreases with an exponential rate towards zero. We have

$$|c_n| \leq (n+1)|\xi|^n \leq C(|\xi| + \epsilon)^n.$$

In o-notation

$$c_n = \mathcal{O}(n|\xi|^n) = o((|\xi| + \epsilon)^n)$$

where we had to pay with a small epsilon in order to get rid of the polynomial factor n.

Generating functions

2.7. Generating autocovariane function

Definition 5. Let γ be a covariance function that is summable. Then the generating function for γ is

$$G(z) = \sum_h \gamma(h) z^h, \qquad z \in \mathbb{C}$$

which is at least well defined $|z| = 1$.

If $\{X_t\}$ is stationary with summable covariance function we write G_X for its generating function. For a filter ψ we have

$$\psi(z) = \sum_j \psi_j z^j.$$

MA(1)

EXAMPLE 2. [MA(1)]
For an MA(1),

$$\gamma(h) = \sigma^2\Big[(1+\theta^2)\mathbb{1}(h=0) + \theta\mathbb{1}(h=1) + \theta\mathbb{1}(h=-1)\Big],$$

and

$$G(z) = \sigma^2\big[1+\theta^2+\theta z+\theta z^{-1}\big].$$

Now

$$(1+\theta z)(1+\theta z^{-1}) = 1+\theta z+\theta z^{-1}+\theta^2$$

and

$$G(z) = \sigma^2(1+\theta z)(1+\theta z^{-1}) = \sigma^2\theta(z)\theta(z^{-1}).$$

AR(1)

EXAMPLE 3. [AR(1)]

$$\gamma(h) = \frac{\sigma^2}{1-\phi^2}\Big[\mathbb{1}(h=0) + \sum_{n=1}^{\infty}\phi^h \mathbb{1}(h=n) + \sum_{n=1}^{\infty}\phi^h \mathbb{1}(h=n),\Big]$$

$$\begin{aligned}
G(z) &= \frac{\sigma^2}{1-\phi^2}\Big[1 + \sum_{n=1}^{\infty}\phi^h z^n + \sum_{n=1}^{\infty}\phi^h z^{-n}\Big] \\
&= \frac{\sigma^2}{1-\phi^2}\Big[1 + \frac{\phi z}{1-\phi z} + \frac{\phi z^{-1}}{1-\phi z^{-1}}\Big] \\
&= \frac{\sigma^2}{1-\phi^2}\Big[\frac{(1-\phi z)(1-\phi z^{-1}) + \phi z(1-\phi z^{-1}) + \phi z^{-1}(1-\phi z)}{(1-\phi z)(1-\phi z^{-1})}\Big] \\
&= \frac{\sigma^2}{1-\phi^2}\Big[\frac{1-\theta^2}{(1-\phi z)(1-\phi z^{-1})}\Big] \\
&= \frac{\sigma^2}{(1-\phi z)(1-\phi z^{-1})} = \frac{\sigma^2}{\phi(z)\phi(z^{-1})}.
\end{aligned}$$

From a convolution to a product

2.8. The autocovariance generating function formula

PROPOSITION 4. Let $\{Y_t\}$ be a stationary time series with summable covariance function and let $X_t = \psi(B)Y_t$. Then

$$G_X(z) = \psi(z)\psi(z^{-1})G_Y(h). \tag{5}$$

PROOF OF PROPOSITION 4. The simple proof use (2);

$$\gamma_X = \psi * \psi' * \gamma_Y$$

which immediately gives (5) since $\psi'(z) = \psi(z^{-1})$.

We can also obtain the result from Propopsition 1 by brute force.

Brute force

$$
\begin{aligned}
G_X(z) &= \sum_h \gamma_X(h) z^h \\
&= \sum_h \left(\sum_u \left(\sum_j \psi_{j+h-u}\psi_j \right) \gamma_Y(u) \right) z^h \\
&= \sum_h \sum_u \sum_j \psi_{j+h-u}\psi_j \gamma_Y(u) z^h \\
&= \sum_h \sum_u \sum_j \psi_{j+h-u} z^{j+h-u} \psi_j z^{-j} \gamma_Y(u) z^u \\
&= \sum_u \sum_j \left(\sum_h \psi_{j+h-u} z^{j-u+h} \right) \psi_j z^{-j} \gamma_Y(u) z^u \\
&= \sum_u \sum_j \left(\sum_h \psi_h z^h \right) \psi_j z^{-j} \gamma_Y(u) z^u \\
&= \sum_u \gamma_Y(u) z^u \psi(z) \sum_j \psi_j z^{-j} \\
&= G_Y(u)\psi(z)\psi(z^{-1}). \qquad \square
\end{aligned}
$$

Linear process

COROLLARY 2. Let $\{X_t\}$ be an MA(q) process. Then

$$G(z) = \sigma^2\theta(z)\theta(z^{-1}).$$

PROOF.
In this case $\gamma_Y(h) = \sigma^2\delta_{o,h}$. □

COROLLARY 3. Let $\{X_t\}$ be a general linear process, $X_t = \sum_j a_j Z_{t-j}$ with summable coeffficients. Then

$$G(z) = \sigma^2 a(z)a(z^{-1}).$$

PROOF.
In this case $\gamma_Y(h) = \sigma^2\delta_{o,h}$. □

The ARMA(p, q) **model**

3. ARMA(p, q)

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q} + Z_t, \quad t \in \mathbb{Z}. \tag{6}$$

The model

i) From a theoretical point of view it is model with $\{Z_t\}$ as input. Then it is an anlogoue of a SDE; stochastic differential equation. In that perspective a solution of the equation (model) is any stationary process $\{X_t\}$ that satisfies the model equation.

ii) Another perspectve is that $\{X_t\}$ is given. Then the problem is to find a white noise process $\{Z_t\}$ and an appropriate ARMA(p, q) model such that (6) holds.

iii) We will try to distinguish between a model and a process.

iv) The model may also cover nonstationary processes like unit types if the time index starts at zero instead of minus infinity.

The parts

$$X_t = \underbrace{\phi_1 X_{t-1} + \cdots + \phi_p X_{t-p}}_{\text{autoregressive part}} \quad + \quad \underbrace{\theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}}_{\text{moving average part}} \quad + \underbrace{Z_t}_{\text{residual}} \quad .$$

A regression model

This could be seen as a regression model and the full name is an autoregressive moving average of order (p, q).

i) It is autorregression since the regressors are pervious versions of the dependent variable. The dependent variable is the present observation.
ii) Previous innovations are also included as regressors. These regressors are not directly observable. But they should not have an own life.
iii) If $q = 0$, then the model reduces to an $\mathrm{AR}(p)$ model.
iv) If $p = 0$, then the model is a $\mathrm{MA}(q)$ model.
v) If both p and q are zero, then it is a white noise process.
vi) It is a Gaussian $\mathrm{ARMA}(p, q)$ model if the residuals are Gaussian.
vii) It is Markovian if the residuals are iid.
viii) It has a state space representation.

Other regression models for time series

We also have time series regression models of more standard type;

$$Y_t = \beta_0 + \sum_{j=1}^{p} \beta_j X_{t-j} + Z_t, \qquad t \geq 0$$

with $\{(X_t, Y_t)\}$ as a possible stationary vector time series. The big difference is that the dependent variable occurs only on the left hand side. If lags of the Y-procress are also included as regressors in this model and at same time we add a linear model for X_t, then we get a vector autoregressive model; VARMA(p, q).

Associated Polynomials

The associated characteristic polynomial for the model is

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p.$$

The moving average polynomial is

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q.$$

AR(p) examples

EXAMPLE 4. [AR(1)]

$$X_t = \phi X_{t-1} + Z_t.$$

EXAMPLE 5. [AR(2)]

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t.$$

MA(q) examples

EXAMPLE 6. [MA(1)]

$$X_t = \theta Z_{t-1} + Z_t.$$

EXAMPLE 7. [MA(2)]

$$X_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t.$$

ARMA(p, q) examples

EXAMPLE 8. [ARMA$(1, 1)$]

$$X_t = \phi X_{t-1} + \theta Z_{t-1} + Z_t.$$

EXAMPLE 9. [ARMA$(2, 1)$]

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta Z_{t-1} + Z_t.$$

EXAMPLE 10. [ARMA$(2, 2)$]

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t.$$

Filter form

3.1. Filter form of the ARMA(p, q) model

We reorganize the model equation

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

which is

$$X_t - \phi_1 B X_t - \cdots - \phi_p B^p X_t = Z_t + \theta_1 B Z_t + \theta_2 B^2 Z_t + \cdots + \theta_q B^q Z_t.$$

In terms of the the polynomials,

$$\phi(B) X_t = \theta(B) Z_t.$$

Existence of theorem

3.2. EXISTENCE AND UNIQUENESS THEOREM FOR AN ARMA(p,q) MODEL

THEOREM 2. Let

$$\phi(B)X_t = \theta(B)Z_t \tag{7}$$

be an ARMA(p,q) model with input $\{Z_t\}$ that is a WN$(0,\sigma_Z^2)$. Assume that the polynomials ϕ and θ has no common roots. The polynomial ϕ and θ have grade p and q, respectively. Then the model as a unique stationary solution $\{X_t\}$ iff the characteristic polynomial has no roots on the unit circle. The solution is

$$X_t = \psi(B)Z_t, \qquad \psi(z) \overset{\text{def}}{=} \frac{\theta(z)}{\phi(z)}. \tag{8}$$

Model vs process

DEFINITION 6. Let $\{X_t\}$ be a stationary process. If $\{(X_t - \mu)\}$ satisfies (6) for a μ, some (p, q) and for a noise process $\{Z_t\}$, then we say that $\{X_t\}$ is an ARMA process.

REMARK 12.

i) Note that we say nothing uniqueness here. We will later show that we do not have uniqueness here in the sense that an ARMA process will satisfy several ARMA models.

ii) It is important to recognize the difference between an ARMA model and an ARMA process.

Proof of Theorem 2.
Suppose that ϕ has no roots on the unit circle. Then by Proposition **??**, $\{X_t\}$ is well defined and solves (10). By the same Proposition this stationary solution in unique given that ϕ has no roots on the unit circle. It is a bit harder to show that the root condition ia also necesessary. $\square$

The noise process expressed in terms of the observable time series

3.3. THE π-FILTER IN AN ARMA(p, q) MODEL

COROLLARY 4. Suppose that the conditions in Theorem 2 that ensure a stationary solution hold. If in addition θ has no roots unit circle, then

$$Z(t) = \Pi(B)X_t, \qquad \Pi(z) \overset{\text{def}}{=} \frac{\phi(z)}{\theta(z)}. \tag{9}$$

PROOF.
Apply Theorem 1. $\square$

Autocovariance generating formula for an $\mathrm{ARMA}(p,q)$ process

3.4. The G function for an $\mathrm{ARMA}(p,q)$ process

Theorem 3. Suppose that the conditions in Theorem 2 then the solution $\{X_t\}$ has generating function

$$G(z) = \sigma^2 \frac{\theta(z)\theta(z^{-1})}{\phi(z).\phi(z^{-1})}$$

Proof.
Apply Theorem 2 and by Proposition 4. $\square$

Causal Filters

3.5. CAUSALITY AND INVERTIBILITY FOR AN ARMA(p, q) MODEL

DEFINITION 7. The filter ψ is causal if $\psi_j \equiv 0$ on $(-\infty, -1]$.

If $X_t = \psi(B)Y_t$ and ψ is causal, then

$$X_t = \sum_{j=0}^{\infty} \psi_j Y_{t-j}$$

Causality and invertibility

Definition 8. An ARMA(p, q) model is causal if ϕ and θ has no common roots and ϕ has all roots strictly outside the unit circle.

Definition 9. An ARMA(p, q) model is invertible if ϕ and θ has no common roots and θ has all roots strictly outside the unit circle.

The linear representation for a causal ARMA model is computable in an efficient way

3.6. The covariance structure for a causal ARMA(p,q) model

Theorem 4. Let

$$\phi(B)X_t = \theta(B)Z_t \tag{10}$$

be a causal and invertible ARMA(p,q) model with $\{X_t\}$ as the unique solution. Then

$$\psi_j = \begin{cases} 1, & j = 0; \\ \sum_{k\geq 1} \phi_k \psi_{j-k} + \theta_j & j \geq 1, \end{cases} \qquad \pi_j = \begin{cases} 1, & j = 0; \\ \sum_{k\geq 1} -\theta_k \pi_{j-k} - \phi_j & j \geq 1. \end{cases}$$

Proof of Theorem 4.

We insert

$$X_{t-k} = \sum_{j=0}^{\infty} \psi_j Z_{t-k-j} = \sum_{j=0}^{\infty} \psi_j Z_{t-(j+k)} = \sum_{j=0}^{\infty} \psi_{j-k} Z_{t-j}$$

in (10) ,

$$\sum_{j=0}^{\infty} \psi_j Z_{t-j} - \sum_{k=1}^{p} \phi_k \sum_{j=0}^{\infty} \psi_{j-k} Z_{t-j} = \sum_{j=0}^{\infty} \theta_j Z_{t-j}. \tag{11}$$

where $\theta_0 = 1$, $\theta_j \stackrel{\text{def}}{=} 0$ for $j \notin [0, q]$ and $\psi_j \equiv 0$ for $j < 0$.

The second term of the left hand of (11) is rewritten and we get the same summation for all three parts that the equation consists of;

$$\sum_{j=0}^{\infty} \psi_j Z_{t-j} - \sum_{j=0}^{\infty} \sum_{k=1}^{p} \phi_k \psi_{j-k} Z_{t-j} = \sum_{j=0}^{\infty} \theta_j Z_{t-j}. \tag{12}$$

Identical zero

Finally, we take all parts with a common summation on the left hand side of the equality sign in (12), and then we see that

$$\sum_{j=0}^{\infty} \Bigg\{ \underbrace{\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} - \theta_j}_{\text{has to be } \equiv 0} \Bigg\} Z_{t-j} = 0.$$

ARMA(1, 1)

EXAMPLE 11. [ARMA(1, 1)]
Let $\{X_t\}$ be given by a causal and invertible ARMA(1, 1) model,

$$X_t = \phi X_{t-1} + \theta Z_{t-1} + Z_t.$$

Then

$$\psi_j = \begin{cases} 1, & \text{, for } j = 0 \text{ ;} \\ \phi + \theta & \text{, for } j = 1; \\ \phi \psi_{j-1} & \text{, for } j \geq 2. \end{cases}$$

This gives

$$\psi_j = \phi^{j-1}\psi_1, \quad j \geq 1, \qquad \psi_1 = \phi + \theta,$$

$$= \begin{cases} 1, & \text{, for } j = 1; \\ \phi^{j-1}(\phi + \theta) & \text{, for } j \geq 1. \end{cases}$$

Hence

$$X_t = (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j} + Z_t.$$

The invertibility part

We also see that

$$\pi_j = \begin{cases} 1, & \text{, for } j = 0; \\ -\theta - \phi & \text{, for } j = 1; \\ -\theta\pi_{j-1} & \text{, for } j \geq 2. \end{cases}$$

This gives

$$\pi_j = (-\theta)^{j-1}\pi_1, \quad j \geq 1, \qquad \pi_1 = -\theta - \phi$$

$$= \begin{cases} 1, & \text{, for } j = 1; \\ \theta^{j-1}(-1)^j(\phi + \theta) & \text{, for } j \geq 1, \end{cases}$$

$$Z_t = (\phi + \theta)\sum_{j=1}^{\infty}(-1)^j\theta^{j-1}X_{t-j} + X_t,$$

and

$$X_t = (\phi + \theta)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j} + Z_t.$$