Endre Moen HW 3

3.2

$MA(1) \Rightarrow Y_t = \theta Z_{t-1} + Z_t \quad , Z_t \sim N(0,4)$

a) Show $Z_t = \sum_{j=0}^{\infty} (-\theta)^j Y_{t-j}$

$Z_t = Y_t - \theta Z_{t-1} = Y_t - \theta(Y_{t-1} - \theta Z_{t-2})$

$Z_{t-1} = Y_{t-1} - \theta Z_{t-2}$

$t-1$

$= Y_t - \theta Y_{t-1} + \theta^2 Z_{t-2} = Y_t - \theta Y_{t-1} + \theta^2(Y_{t-2} - \theta Z_{t-3})$

$= Y_t - \theta Y_t + \theta^2 Y_t - \theta^3 Z_{t-3}$

$\Rightarrow (-\theta)^0 Y_t + (-\theta)^1 Y_{t-1} + (-\theta)^2 Y_{t-2} + \dots + (-\theta)^j Z_{t-j}$

$= \sum_{j=0}^{m} (-\theta)^j Y_{t-j} + (-\theta)^m Z_{t-m} \xrightarrow[m \to \infty]{} \sum_{j=0}^{\infty} (-\theta)^j Y_{t-j} + o(1)$

b) Show: $Y_t = \sum_{j=0}^{p} (-\theta)^j Y_{t-j} + Z_t + \sum_{j=p+1}^{\infty} (-1)^j \theta^j Y_{t-j}$

a) $\Rightarrow Z_t = \sum_{j=0}^{\infty} (-\theta)^j Y_{t-j} = Y_t + \sum_{j=1}^{\infty} (-\theta)^j Y_{t-j}$

$\Rightarrow Y_t = Z_t - \sum_{j=1}^{\infty} (-\theta)^j Y_{t-j} = \cancel{Z_t + \sum (-\theta)^{j+1} Y_{t-j}}$

$= Z_t + \sum_{j=1}^{\infty} (-1)^{j+1} \theta^j Y_{t-j}$

$= \underbrace{Z_t}_{a} + \underbrace{\sum_{j=1}^{p} (-1)^{j+1} \theta^j Y_{t-j}}_{b} + \underbrace{\sum_{j=p+1}^{\infty} (-1)^{j+1} \theta^j Y_{t-j}}_{c}$

if $c \to 0$ then $Y_t = a + b = AR(p)$

3.3 / Q: Find $\{\pi_j\}$ for MA(1) process:

a) (3.2) $Y_t = \theta Z_{t-1} + Z_t$, $Z_t \sim N(0,$

MA(q): process

$\pi_0 = 1 \iff 1 = \pi_0$

$\pi_1 = -\theta \iff 0 = \pi_1 + \theta_1 \pi_0$

$\pi_2 = -\theta(-\theta) \iff 0 = \pi_2 + \theta_1 \pi_1 + \theta_2 \pi$

$\pi_3 = -\theta(-\theta)^2 \iff 0 = \pi_3 + \theta_1 \pi_2 + \theta_2 \pi_1 + \theta_3 \pi$

$\vdots \qquad\qquad \Downarrow$

$\pi_j = (-\theta)(-\theta)^{j-1} \iff 0 = \pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k}$

$j = 1, 2, \ldots$ $= (-\theta)^j$

MA(1) Polynomial

$\theta(z) = \theta z + 1 \xRightarrow{\text{invertible}} \theta(z) \neq 0 \quad |z| \leq 1$

$\pi(z) = \frac{1}{(z)} = \frac{1}{\theta z + 1}$

MA(1): $0 = \pi_1 + \theta_1 \pi_0 \iff \pi_1 = -\theta \cdot 1$

$$\pi_j = \begin{cases} 1 & j = 0 \\ -\theta & j = 1 \end{cases}$$

MA(q):

$$\pi_j = \begin{cases} 1 & j = 0 \\ \sum_{k=1}^{q} -\theta_k \pi_{j-k} & j > 0 \end{cases}$$

b) <u>Q:</u> Explain $\pi_0 = 1$. Hint use (3.5)

(3.5) $\quad \text{I}: \quad X_t = \theta(B) Z_t$

$\quad\quad\quad \text{II}: \quad Z_t = \pi(B) X_t$

<u>Remark</u>: $I = B^0$ And $\pi^{(1)}(B) = \pi(B) - \pi_0 I$

$$\Rightarrow \pi(B) = \pi_0 I + \pi^{(1)}(B)$$

<u>A:</u>

From I: $X_t = \theta(B) Z_t \Leftrightarrow \dfrac{X_t}{Z_t} = \theta(B)$

$$\overset{\text{use II}}{\Longrightarrow} \frac{X_t}{\pi(B) X_t} = \theta(B) \Leftrightarrow \frac{1}{\pi(B)} = \theta(B)$$

$$\overset{\text{Remark}}{\Longleftrightarrow} \frac{1}{\pi_0 I + \pi^{(1)}(B)} = \theta(B)$$

$$\Longleftrightarrow \quad 1 = \theta(B)\left(\pi_0 I + \pi^{(1)}(B)\right)$$

<u>Note</u> $\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$

$\theta(0) = 1$

$$\overset{B^0}{\Longleftrightarrow} \quad 1 = \theta(B^0)\left(\pi_0 I + \pi^{(1)}(B^0)\right)$$

$$= \theta(B^0)\left(\pi_0 I + \underset{\shortparallel}{\pi(B^0)} - \pi_0 I\right)$$

$$= \theta(B^0)\left(\pi_0 I + \pi_0 I - \pi_0 I\right)$$

$$1 = \theta(B^0)(\pi_0 I)$$

$$1 = \pi_0$$

3.3 c) Q Show representation of MA(q) process as a AR(∞) and can be written

$$(3.7) \qquad X_t = -\pi^{(1)}(B)X_t + Z_t$$

Hint: use (3.6) $\quad \pi(B) = \pi_0 I + \pi^{(1)}(B)$

AR $\qquad \phi(B)X_t = \theta(B)Z_t \qquad$ MA

Causal: $\qquad\qquad$ Invertible

$\phi(z) = 1 - \phi z - .. \phi_p z^p = 0 \qquad \theta(z) = 1 + \theta_1 z + .. \theta_p z^p \neq 0$

$\Rightarrow \quad |z| \leq 1 \quad \Leftarrow$

$\Rightarrow$ MA(∞) representation	AR(∞)
$X_t = \underbrace{\frac{\theta(B)}{\phi(B)}}_{\Psi(B)} Z_t$	$Z_t = \underbrace{\frac{\phi(B)}{\theta(B)}}_{\pi(B)} X_t$
$\Psi(B) = 1 + \sum_{j=1} \Psi_j B^j$	$\pi(B) = 1 + \sum_{j=1}^{\infty} \pi_j B^j$

AR(1)

$X_t = \phi X_{t-1} + Z_t$

AR(∞) of MA(1): $\quad X_t = Z_t + \theta Z_{t-1}$

$$X_t = (1 + \theta B)Z_t$$

And $|\theta| < 1$

$$Z_t = \frac{X_t}{1 - (-\theta B)} \overset{\text{geom.}}{\underset{\text{Series}}{=}} X_t - \theta B X_t + \theta^2 B^2 X_t - \dots$$

$$\Leftrightarrow X_t = -(\theta B X_t - \theta^2 B^2 X_t + \dots) + Z_t$$

$$\Leftrightarrow X_t = Z_t - \sum_{j=1} (-\theta)^j B^j X_t$$

$$\underline{\phi_j = -(-\theta)^j} = (-1)\cdot(-1)^j \theta^j = (-1)^{j+1} \theta^j = \pi_j$$

$MA(2) \to AR(\infty)$

$\Rightarrow \quad X_t = (1 - \theta_1 B - \theta_2 B^2) Z_2$

which can be factorized

$$X_t = (1 - Z_1 B)(1 - Z_2 B) Z_t$$

$$\frac{X_t}{(1 - Z_1 B)(1 - Z_2 B)} = \frac{1}{(1 - Z_1 B)} \frac{X_t}{(1 - Z_2 B)}$$

$$= \underbrace{(1 + Z_1 B + Z_2 B^2 + \dots)}_{k} (X_t + Z_2 B X_t + \dots) + Z_t$$

$$= k (X_t + Z_2 B X_t + \dots) + Z_t$$

Similar for $MA(q)$

3.4/ a) Q: Write ARMA(1,1) in filter-form and give $\phi(z)$, $\theta(z)$

A: $X_t = \phi X_{t-1} + \theta Z_{t-1} + Z_t \qquad t \in \mathbb{Z}$

$X_t - \phi X_{t-1} = \theta Z_{t-1} + Z_t$

$\phi(z) = 1 - \phi z \qquad \theta(z) = 1 + \theta z$

Def filter form $\phi(B) X_t = \theta(B) Z_t$

$\phi(B) = 1 - \phi B \qquad \theta(B) = 1 + \theta B$

b) Q: Identify ξ when $\phi(z) = 1 - \frac{z}{\xi}$

A: $\phi(z) = 0 \Leftrightarrow 1 - \frac{z}{\xi} = 0 \Leftrightarrow \xi = z$ is root

3.4/ c) Q: Write $\psi(z) = \dfrac{\theta(z)}{\phi(z)}$ in terms of (θ, ϕ)

and by direct calculation find $\{\psi_j\}$

(3.9) $\psi(z) = \psi_0 + \psi_1 z + \psi_2 z^2 + \dots$

$$\psi(z) = \frac{1 + \theta_1 z + \theta_2 z^2 + \dots \theta_q z^q}{1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p}$$

ARMA(1,1)

$$\Rightarrow \frac{1 + \theta z}{1 - \phi z}$$

The function $f(z) = \dfrac{1}{\phi(z)} = \dfrac{1}{1 - \phi z}$

has a singul. at $1 - \phi z = 0$ $\quad z = 1/\phi = \xi_1$

Since $|\phi| < 1 \Rightarrow \xi_1 > 1$

$|\phi| < 1$, By "important eq p 48" $\dfrac{1}{\phi(z)} = \sum_{j=0}^{\infty} \phi^j z^j$

$$\Rightarrow \frac{1}{\phi(z)} = \frac{1}{1 - \phi z} = \left(1 + \phi^1 z^1 + \phi^{-2} z^{-2} + \dots\right)$$

$$\Rightarrow \psi(z) = (1 + \theta z)\left(1 + \phi z + (\phi z)^2 + (\phi z)^3 + \dots\right)$$

$$= 1 + \phi z + (\phi z)^2 + (\phi z)^3 + \dots) + (\theta z + \theta \phi z^2 + \theta \phi^2 z^3$$

$$= 1 + (\phi + \theta) z + (\phi + \theta \phi^2) z^2 + (\phi + \theta \phi^3) z^4$$

$$= 1 + (\phi + \theta) z + (1 + \theta \phi) \phi z^2 + (1 + \theta \phi) \phi^2 z^4$$

$$= 1 + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z^j = \psi_j(z)$$

$$\left.\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= \phi + \theta \\ \psi_3 &= (\phi + \theta)\phi \end{aligned}\right\} \psi_j = \begin{cases} 1 & j = 0 \\ (\phi + \theta)\phi^{j-1} & j \geq 1 \end{cases}$$

$$\phi(B) X_t = \theta(B) Z_t$$

$$X_t = \frac{\theta(B)}{\phi(B)} Z_t = \left[1 + (\phi + \theta) \sum_{j=1}^{\infty} (\phi^{j-1} B^j)\right] Z_t$$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

$\Rightarrow$ MA(∞) representation.

d) Q: Find

(3.10) $$\pi(z) = \frac{\phi(z)}{\theta(z)} = \pi_0 + \pi_1 z + \pi_2 z^2 + \ldots$$

$$\pi(z) = \frac{1 - \phi z}{1 - (-\theta z)} \Rightarrow \frac{1}{1 - (-\theta z)} = (1 + (-\theta z) + (-\theta z)^2 + (-\theta z)^3 + \ldots)$$

$$\Rightarrow \pi(z) = (1 - \phi z)(1 - \theta z + \theta^2 z^2 - \theta^3 z^3 + \ldots)$$

$$= 1 - \theta z + \theta^2 z^2 - \theta^3 z^3 + \ldots)$$
$$+ (-\phi z + \phi\theta z^2 - \phi\theta^2 z^3 + \ldots)$$

$$= 1 - (\theta + \phi) z + (\theta^2 + \phi\theta) z^2 - (\theta^3 - \phi\theta^2) z^3 + \ldots$$

$$= 1 + (\theta + \phi) \sum_{j=1}^{\infty} (-1)^j \theta^{j-1} \cdot z^j$$

$$\Longrightarrow \pi_j = \begin{cases} 1 & j = 0 \\ -(\theta + \phi)(-\theta)^{j-1} & j \geq 1 \end{cases}$$

$$Z_t = \frac{\phi(B)}{\theta(B)} X_t \Leftrightarrow X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}$$

$$\Leftrightarrow X_t = (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j} + Z_t$$

$$\Downarrow \phi = 0$$

AR(∞)

$$\theta \sum (-\theta)^{j-1} B^j X_t + Z_t$$

$$\sum (-1)^{j-1} \theta^j B^j X_t + Z_t$$

$$(-1)^{j-1} = (-1)^{j+1}$$

$$= \sum (-1)(-1)^j \theta^j B^j X_t + Z_t$$

From 3.3 c)

$$\boxed{X_t = Z_t - \sum (-\theta)^j B^j X_t}$$

3.4 e) **Q:** Express (3.8) as $AR(\infty)$

$$AR(\infty) \quad X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} \dots + Z_t$$

$$(3.8) \quad X_t = \phi X_{t-1} + \theta Z_{t-1} + Z_t$$

A: $\Leftrightarrow X_t - \phi X_{t-1} = \theta Z_{t-1} + Z_t$

$$\Rightarrow (1 - \phi B) X_t = (1 + \theta B) Z_t$$

$$\Rightarrow \phi(z) X_t = \theta(z) Z_t$$

$$X_t = \frac{\theta(z) Z_t}{\phi(z)} \Rightarrow MA(\infty)$$

$$\frac{\theta(z)}{\theta(z)} = Z_t \qquad \Rightarrow AR(\infty)$$

3.4 d) $\Rightarrow$

$$Z_t = X_t \left(1 - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j\right)$$

$$= X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}$$

$$X_t = \left[(\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}\right] + Z_t$$

$$X_t = (\phi + \theta) X_{t-1} - (\phi + \theta)\theta X_{t-2} + (\phi + \theta)\theta^2 X_{t-3}$$
$$- (\phi + \theta)\theta^3 X_{t-4} + \dots + Z_t$$

3.4/ f) Q: How large must j s.t $|\phi_k| < \frac{1}{2} \cdot 10^{-9}$ $\forall k \geq j$

when $(\phi, \theta) = (0.7, 0.7)$

A: $\phi \cdot |\theta|^{j-1} = 0.7 \cdot 0.7^{j-1} = 0.5 \cdot 10^{-9}$

$= 0.7^{j} = 0.5 \cdot 10^{-9}$

$j \ln 0.7 = -9 \ln(0.5)$

$$\frac{\ln(0.5) + \ln(10^{-9})}{\ln 0.7} = j = \frac{\ln(0.5 \cdot 10^{-9})}{\ln 0.7}$$

$$\| $$

$$\frac{\ln 0.5 + (-9)\ln(10)}{\ln 0.7} \Rightarrow \quad = 60.04$$

$$\underline{\underline{j = 61}}$$

3.5/ a)

Q: Prove marginal dist of Y_t is $N(0, \gamma(0))$

MA(1) $\quad Y_t = \theta Z_{t-1} + Z_t \quad t \in \mathbb{Z}$

independent $\quad Z_t \sim N(0, 4)$

A: $\gamma(0) = cov(Y_{t+h}, Y_t)$

$\gamma(0) = \gamma(0,0) = \gamma(t,t) = cov(Y_t, Y_t)$

$= E((Y_t - \mu_Y(t))(Y_t - \mu_Y(t)))$

$\Rightarrow \mu_Y(t) = E[Y_t] = E(\theta Z_{t-1} + Z_t$

$= \theta E[Z_{t-1}] + E[Z_t] = 0$

$\Rightarrow E(Y_t \cdot Y_t) = E(Y_t^2)$

$= E((\theta Z_{t-1} + Z_t)(\theta Z_{t-1} + Z_t))$

$= E(\theta^2 Z_{t-1}^2 + \theta Z_t Z_{t-1} + \theta Z_{t-1} \cdot Z_t + Z_t^2)$

$= E[\theta^2 Z_{t-1}^2 + Z_t^2] = \sigma^2(\theta^2 + 1)$

$\sigma^2 = 4 \Rightarrow 4(\theta^2 + 1)$

$\gamma(0) = 4(\theta^2 + 1)$

$\underline{Y_t \sim N(0, 4(\theta^2 + 1))}$

b) Q Extend a) to $Y_N = (Y_1 .. Y_n)^T$

$$Y_n = (Y_1 .. Y_n)^T \sim N(0, \Gamma_n)$$

$\Gamma_n \in n \times n$ $\{\gamma(i-j),\ 1 \le i, j \le n\}$

Hint: Use multinomial distribution or transformation formula by adding Z_0

so $Y_n = (Y_1 .. Y_n, Z_0)^T$

Transformation formula: joint pdf $f(x,y)$

consider (u,v), $u = g(x,y)$ $v = h(x,y) \Rightarrow f_{(u,v)}(u,v) = f_{xy}(g^{-1}(u,v)\ h^{-1}(u,v))\ |$

[independence $\Rightarrow \sigma_{xy} = \rho_{xy} = 0$
$\Leftarrow$ not in general] we dont know this from a)

$$\Gamma_2 = \begin{bmatrix} Var(Y_1) & cov(Y_1, Y_2) \\ cov(Y_2, Y_1) & var(Y_2) \end{bmatrix} = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}$$

A: Multinomdal dist:

$$N(\underline{\mu}, \Sigma) \sim f(x) = \frac{1}{(2\pi)^{k/2} \sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

precision matrix

$$\Rightarrow \Gamma_n = \begin{bmatrix} \gamma(0) & \cdots & \gamma(n) \\ \vdots & & \\ \gamma(n) & & \gamma(0) \end{bmatrix}$$

Multinomdal $\Rightarrow \underline{Y} \sim N(0, \Gamma_n)$

3.5/ $S = \{t_1, t_2 .. t_m\}$ $t_{j-1} < t_j$

c) $Y_S = (Y_{t_1} .. Y_{t_m})^T$

Q: Explain $Y_S \sim N(0, \Sigma)$, $\sigma_{ij} = \gamma(t_j - t_i)$

A: from b) $\Gamma_n = \begin{bmatrix} \gamma(0) & .. & \gamma(n) \\ \vdots & \ddots & \\ \gamma(n) & & \gamma(0) \end{bmatrix}$

$\sigma_{ii} = \gamma(t_j - t_i)$ $\forall i \in 1..m$

$= \gamma(0)$

$\sigma_{ij} = \gamma(t_i - t_j) = \gamma(i-j)$ $\forall i < j \in 1..m$

$\Rightarrow \Sigma = \Gamma_{m-1} = \begin{bmatrix} \gamma(0) & \gamma(1-2) & .. & \gamma(1-m) \\ \gamma(2-1) & \ddots & & \\ \vdots & & & \\ \gamma(m-1) & & & \gamma(0) \end{bmatrix}$

and we know $\gamma(-h) = \gamma(h)$

d) Q Indicate how to extend your argument to general MA(q) process

A: From 3.3 c) $MA(q) \rightarrow AR(\infty)$

MA(q) can be written as a

MA(1) process into a $AR(\infty)$ process

times a konstant k

ie $\underbrace{(1 + Z_1 B + Z_2 B^2 + ..)}_{k}(X_t + Z_2 B X_t + ..) + Z_t$

$= k \cdot AR(\infty)$

so MA(q) contrib konstant to MA(1)

3.6/ Q: $\{X_t\}$ is MA(q) process

s.t. $\theta(z)$ has unit root, $z = 1$

$\theta(z=1) = 0$

1) Explain $Var(\bar{X}) = O(\frac{1}{n^2})$

2) Explain $Var(\bar{X}) = O(\frac{1}{n}$ when $z = -1$

Hint: Look at MA(1) $\Rightarrow$ $\theta(B) = (1-B)\theta_1(B)$

if $\theta(z=1) = 0$

$\deg(\theta_1(z)) = q-1$ and $k = 1$

A: By Fundamental Thm of Algebra

a $\deg_q$ $q-1$ has $q-1$ number of

roots in $\mathbb{C}$. So we can write $\theta_1(z)$

as $\theta_1(z) = (z-\xi_1)(z-\xi_2)\ldots(z-\xi_{q-1})$

$\Longrightarrow$ We only need to consider MA(1)

MA(1) $X_t = \theta Z_{t-1} + Z_t$

$\Rightarrow$ $\theta = -1$

$\Rightarrow$ $X_t = -Z_{t-1} + Z_t$ $\Rightarrow$ $\theta(B) = (1-B)\theta_1(B)$

$\Rightarrow$ $\theta(z) = 1 - \theta z$

$\theta(1) = 1 - \theta = 0$

$\theta = 1$

$\gamma(0) = E(Z_t - Z_{t-1})(Z_t - Z_{t-1}))$

$= E(Z_t^2 - Z_{t-1}Z_t - Z_tZ_{t-1} + Z_{t-1}^2)$

$= \sigma^2 + \sigma^2$

3.7/ Q: Is a moving average of a moving average process a moving average process?

Fact

A: $\boxed{\text{In } MA(q) \quad \gamma(p) = 0 \quad \text{if } q < p}$

$MA(1): X_t = \theta Z_{t-1} + Z_t$

$$\overline{X}_3 = \frac{1}{3}(X_1 + X_2 + X_3)$$

$$= \frac{1}{3}(\theta Z_0 + Z_1 + \theta Z_1 + Z_2 + \theta Z_2 + Z_3)$$

$$= \frac{1}{3}(\theta(Z_0 + Z_1 + Z_2) + (Z_1 + Z_2 + Z_3))$$

let $Z_1 + Z_2 + Z_3 = Z$ $\quad Z_i \sim WN(0, \sigma^2)$

$c \cdot Z_i \sim WN(0, c \cdot \sigma^2)$

$$3\overline{X}_t^{(3)} = \theta(Z_{t-3} + Z_{t-2} + Z_{t-1}) + (Z_{t-2} + Z_{t-1} + Z_t)$$

$$Z_t^* = Z_{t-2} + Z_{t-2} + Z_t \sim N(0, 9 \cdot \sigma^2)$$

$$3X_t^{(3)} = \theta_1 Z_{t-3} + \theta_1 Z_{t-2} + \theta_1 Z_{t-1} + Z_t^*$$

Which is $MA(3)$ process because

$$\gamma(4) = Cov(X_{t+4}^{(3)}, X_t) =$$

$$= E(\theta_1(Z_{t+1} + Z_{t+2} + Z_{t+1} + Z_{t+4})(\theta(Z_{t-3} + Z_{t-2} + Z_{t-1} + Z_t)$$

$$= \theta^2 Z_{t+1}(Z_{t-3} + Z_{t-2} + Z_{t-1}) + \theta(Z_{t+1} \cdot Z_t) +$$

$$+ \theta^2 Z_{t+4}(Z_{t-3} + Z_{t-2} + Z_{t-1}) + \theta(Z_{t+1} \cdot Z_t)$$

$= 0$ because $Z_{t+i} \neq Z_{t-j}$ $\quad \forall i \in 1..4$

$\forall j \in 0..3$

Fact

$\Longrightarrow$ Mean of $MA(q)$ process is $MA(kq)$ process for some $k \in \mathbb{Z}^+$

3.8

a) (3.4) $$Y_t = \underbrace{\sum_{j=1}^{p} (-\theta)^j Y_{t-j}}_{AR(p)} + Z_t + \underbrace{\sum_{j=p+1}^{\infty} (-1)^{j+1} \theta^j Y_{t-j}}_{\text{vanish}}$$

We can use (3.4) to construct a predictor Y_{n+1}

Assume that $n \geq p+1$ then

$$\tilde{Y}_{n+1} = \sum_{j=1}^{p} (-\theta)^j Y_{n+1-j}$$

To find the variance of predictor error: we have:

$$Y_{n+1} = Z_{n+1} + \sum_{j=1}^{p} (-\theta)^j Y_{n+1-j} + \sum_{j=p+1}^{\infty} (-\theta)^j Y_{n+1-j}$$

$$\Leftrightarrow \quad Y_{n+1} = Z_{n+1} + \tilde{Y}_{n+1} + \Delta$$

$$\Leftrightarrow \quad Y_{n+1} - \tilde{Y}_{n+1} = Z_{n+1} + \Delta \quad \oplus$$

a) Q: Explain $\| Y_{n+1} - \tilde{Y}_{n+1} \|^2 = \| Z_{n+1} \|^2 + \|\Delta\|$

A: Z_{n+1}, Δ are uncorrelated $\Rightarrow \langle Z_{n+1}, \Delta \rangle = 0$

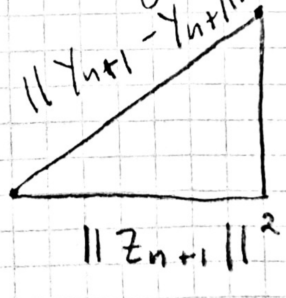
$$\| Z_{n+1} + \Delta \|^2 = \| Z_{n+1} \|^2 + \|\Delta\|^2 + \underbrace{2\,cov(Z_{n+1}, \Delta)}_{= 0 \text{ by indep}}$$
$$= \| Z_{n+1} \|^2 + \|\Delta\|^2$$

by $\oplus$

$$\Rightarrow \| Y_{n+1} - \tilde{Y}_{n+1} \| = \| Z_{n+1} \|^2 + \|\Delta\|^2$$

$$\Rightarrow Var(Z_{n+1}, \Delta) = var(Z_{n+1}) + var(\Delta)$$

information-loss when using $\tilde{Y}_{n+1}$ is used to estimate Y_{n+1}



$\|\Delta\|^2$ - error due to estimator

$\| Z_{n+1} \|^2$ - variance / entropy

AKA - Kullback - Liebler divergence

3.8] b) Q: Calculate $\|\Delta\|^2$ in general and for $p=5$, $\theta=0.7$, $\sigma^2=4$

A: From 3.8 a) $\Delta = \sum_{j=p+1}^{\infty} (-\theta)^j Y_{n+1-j}$

$=$

$$Var(\Delta) = Var\left(\sum_{j=p+1}^{\infty} (-\theta)^j Y_{n+1-j}\right)$$

continuouity of Var(·)

$$= \sum_{j=p+1}^{\infty} Var\left((-\theta)^j, Y_{n+1-j}\right)$$

$$= \sum_{j=p+1}^{\infty} (\theta^j)^2 \cdot Var(Y_{n+1-j})$$

$$= \sum_{j=p+1}^{\infty} \theta^{2j} \cdot \left[\sigma^2(1+\theta^2)\right]$$

$$\Longrightarrow \sum_{j=6}^{\infty} (0.7)^{2j} \cdot \underbrace{(4\cdot(1+0.7^2))}_{4\cdot(1+0.49) = 5.96}$$

$$= \left(\sum_{j=1}^{\infty} (0.7)^{2j} - \sum_{j=1}^{5} (0.7)^{2j}\right)(5.96)$$

$$= \left(\sum_{j=0} (0.49)^2 - \sum_{j=0}^{5} (0.49)^j\right](5.96$$

$$= \frac{1}{1-0.49} - \left[1+0.49+0.49^2 + \_^3 + \_^4 + \_^5\right].$$

$$= [1.96 - (1.93)](5.96) = 0.03 \cdot 5.96$$

$$= \underline{0.1788}$$

$$\Rightarrow \|Y_{n+1} - \hat{Y}_{n+1}\|^2 = 4 + 0.1788 \approx 4$$