Mandatory homework 3 Stat 211 - H21

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Contents

In this homework, you are supposed to learn how to simulate and estimate AR/MA models in R, and how to use ACF and PACF to identify models.

3.1. Simulation of an AR(1)

Problem 3.1

Consider an AR(1) process, i.e.

$$X_t = \phi X_{t-1} + Z_t, \qquad t \in \mathbb{Z} \tag{3.1}$$

where $\{Z_t\}$ is iid with $Z_t \sim \mathcal{N}(0,4)$.

a) Simulate the process with $\phi = 0.7$ for $t = 1, \ldots, n$ with $n = 10000$ using the following code:

```
set.seed(1234)
x <- arima.sim(
  model = list(ar = 0.7),
  n = 10000,
  innov = rnorm(10000, sd = 2)
)
```

Remember that the autocorrelation function [ACF] of an AR(1) is given by

$$\rho(h) = \phi^h, \qquad h \geq 1$$

with the partial autocorrelation is

$$\alpha(h) = \begin{cases} \phi, & \text{for } h = 1; \\ 0, & \text{for } h > 1. \end{cases}$$

a) Plot the sample autocorrelation function $\widehat{\rho}$ and sample partial autocorrelation function $\widehat{\alpha}$ of the simulated process. How are these consistent with an AR(1) process? Can you see the relation between ϕ_a nd $\widehat{\rho}(1)$?
b) c) Make a plot comparing the sample autocorrelation with the theoretical autocorrelation. Do the same for the partial autocorrelation. d) Estimate an AR(1) and an AR(2) model for the simulated data using the arima function in R. Compare the two models. Is $\widehat{\phi}_2$ signficantly different from zero? Compare the quality of ϕ_1 for the two models in terms of bias and standard error.

```
?arima
arima(x, order = ...)
```

3.2. Simulation of an MA(1) process

Problem 3.2
Consider an MA(1) process, i.e.

$$Y_t = \theta\, Z_{t-1} + Z_t, \qquad t \in \mathbb{Z} \tag{3.2}$$

where $\{Z_t\}$ is iid with $Z_t \sim \mathcal{N}(0,4)$.

a) Show that

$$Z_t = \sum_{j=0}^{\infty} (-\theta)^j\; Y_{t-j}. \tag{3.3}$$

b) Use (3.3) and explain that

$$Y_t = \sum_{j=1}^{p} (-\theta)^j\; Y_{t-j} + Z_t + \sum_{j=p+1}^{\infty} (-1)^{j+1}\theta^j\, Y_{t-j}. \tag{3.4}$$

This indicates that an MA(1) can be approximated by an AR(p), for p large enough if the rightmost term of (3.4) is neglible.

c) Simulate the process with $\theta = 0.7$ for $t = 1, \ldots, n$ with $n = 10000$ by adapting the code suggested in (3.1).

d) Plot the sample ACF and PACF for the simulated data. How are these consistent with an MA(1) process?

e) Fit an MA(a) nd an AR(5) model to the simulated data. Compare the respective AR coefficients with the correspodning powers of θ in (2). How well does it fit? Will an AR(10) improve the result?

3.3. Filter calculus for an invertible MA(q) process

Problem 3.3
In the following we look at a general invertible MA(q) process

$$X_t = \sum_{j=1}^{q} \theta_j Z_{t-j} + Z_t.$$

In filter notation the MA(q) process , $\{X_t\}$, can be written as

$$\begin{aligned} X_t &= \theta(B) Z_t, \\ Z_t &= \pi(B) X_t \qquad \text{whenever } \theta(z) \text{ has no unit roots.} \end{aligned} \tag{3.5}$$

If the process is invertible then $\pi_j \equiv 0$ for $j < 0$. The filter form of ARMA(p,q) models are quite useful for manipulating these models.

a) Find $\{\pi_j\}$ for the model (3.2) when you assume invertibility. *Hint: Look at (3.4) and find in the textbook or on slides a recursive formula for this filter in terms of the θ_js.*

Let $\mathbb{I} = B^0$, the backward shift operator that leave a variable unchanged, and define $\pi^{(1)}(B) = \pi(B) - \pi_0 \mathbb{I}$ so that

$$\pi(B) = \pi_0\mathbb{I} + \pi^{(1)}(B). \tag{3.6}$$

b) Explain that $\pi_0 = 1$. *Hint: Use both equations in (3.5).*
c) Show that by (3.6) $\mathrm{AR}(\infty)$ representation of the $\mathrm{MA}(q)$ process can be written as

$$X_t = -\pi^{(1)}(B)X_t + Z_t. \tag{3.7}$$

Identify the autoregressive parameters, $\{\phi_j\}$, in this notation,

3.4. ARMA(1, 1)

PROBLEM 3.4
If we combine an $\mathrm{AR}(1)$ and $\mathrm{MA}(1)$ we get an $\mathrm{ARMA}(1,1)$,

$$X_t = \phi X_{t-1} + \theta Z_{t-1} + Z_t, \quad t \in \mathbb{Z}. \tag{3.8}$$

Assume that the model is causal and invertible.
This example is discussed in the textbook and in the slides.

a) Write (3.8) in filter form where you specify $\phi(z)$ and $\theta(z)$.
b) Identify ξ when you write $\phi(z) = 1 - z/\xi$.
c) Write $\psi(z) = \theta(z)/\phi(z)$ in terms of (ϕ, θ) and by direct calculation find $\{\psi_j\}$,

$$\psi(z) = \frac{\theta(z)}{\phi(z)} = \psi_0 + \psi_1 z + \psi_2 z^2 + \cdots. \tag{3.9}$$

d) Likewise, find

$$\pi(z) = \frac{\phi(z)}{\theta(z)} = \pi_0 + \pi_1 z + \pi_2 z^2 + \cdots. \tag{3.10}$$

e) Express the solution of (3.8) as an $\mathrm{AR}(\infty)$,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_k X_{t-k} + \cdots \quad + Z_t.$$

Hint: (3.7)
f) How large must j such that $|\phi_k| < 0.5 \times 10^{-9}$ for all $k \geq j$ when $(\phi, \theta) = (0.7, 0.7)$.
g) Use $(\phi, \theta) = (0.7, 0.7)$ and compute $\{\phi_j, \quad j = 1, \ldots, 50\}$ with R. Plot these coefficients in a reasonable way.

3.5. Gaussian white noise in linear models give Gaussian time series

PROBLEM 3.5
Suppose that $\{Z_t\}$ is iid and normal distributed for the $\mathrm{MA}(1)$ process defined by (3.2).

a) Prove that the marginal distribution of Y_t is $\mathcal{N}(0, \gamma(0))$.

b) Extend a) to $\mathbf{Y}_n = (Y_1, \ldots, Y_n)^T \sim \mathcal{N}(0, \Gamma_n)$ where Γ_n is the $n \times n$ matrix; $\{\gamma(i-j), 1 \le i, j \le n\}$. *Hint: You can rely on basic theory for the multinormal distribution or you can used the transformation formula by adding Z_0 to the Y-vector.*

Let $S = \{t_1, t_2, \ldots, t_m\}$ with $t_{j-1} < t_j$ and $\mathbf{Y}_S = (Y_{t_1}, \ldots, Y_{t_m})^T$.

c) Explain that $\mathbf{Y}_S \sim \mathcal{N}\big(0, \Sigma\big)$ with $\sigma_{ij} = \gamma(t_j - t_i)$.

The results also hold for a general $\mathrm{MA}(q)$ process.

d) Indicate how to extend your arguments to a general $\mathrm{MA}(q)$ process.

3.6. Degenerate variance

Problem 3.6
Let $\{X_t\}$ be a $\mathrm{MA}(q)$ process such that $\theta(z)$ has one unit root that is equal $+1$. Explain that $\mathrm{Var}(\overline{X}_n) = \mathcal{O}(n^{-2})$. Is the same true for one unit root that is different from $+1$? Look at the example with $q = 1$. *Hint: For the general case, you can use that $\theta(B) = (1-B)\theta_1(B)$ if $\theta(z)$ has $+1$ as a root, where the polynomial $\theta_1(z)$ has degree $q-1$ and constant term equal to 1.*

3.7. Iterated operations

Problem 3.7
Is a moving average of moving average process a moving average process?

3.8. Prediction of an MA(1) process

Problem 3.8
We can use (3.4) to construct a predictor for Y_{n+1}.
Assume that $n \ge p+1$. Then

$$\widetilde{Y}_{n+1} = \sum_{j=1}^{p} (-\theta)^j Y_{n+1-j}$$

This is not the optimal predictor, but for p large enough it is pretty close. We what to find the variance of the predictor error. We have

$$\begin{aligned} Y_{n+1} &= Z_{n+1} + \sum_{j=1}^{p} (-\theta)^j Y_{n+1-j} + \sum_{j=p+1}^{\infty} (-\theta)^j Y_{n+1-j} \\ &= Z_{n+1} + \widetilde{Y}_{n+1} + \Delta, \ \text{say} \end{aligned}$$

so that

$$Y_{n+1} - \widetilde{Y}_{n+1} = Z_{n+1} + \Delta.$$

Use $\|\cdot\|^2$ for the variance.

a) Explain that

$$\|Y_{n+1} - \widetilde{Y}_{n+1}\|^2 = \|Z_{n+1}\|^2 + \|\Delta\|^2$$

b) Calulate $\|\Delta\|^2$ in general and for $p = 5$, $\theta = 0.7$ and $\sigma^2 = 4$.