Basic concepts and the AR(1) model

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Stationarity

1. A time series $\{X_t,\ t \in \mathbb{Z}\}$ is strictly stationary if for any $h \geq 0$ the distribution of $(X_t, X_{t+1}, \ldots, X_{t+h})$ does not depend on t
2. A time series $\{X_t,\ t \in \mathbb{Z}\}$ is weakly stationary if
   - i) $\mathbb{E}\, X_t^2 < \infty$ for all t.
   - ii) $\mathbb{E}\, X_t \equiv \mu$.
   - iii) $\gamma(t+h, t) = \mathrm{Cov}(X_{t+h}, X_t)$ is independent of t for any h so that $\gamma(h) \stackrel{\text{def}}{=} \gamma(t+h, h)$ is well defined. The function γ is defined on $\mathbb{Z}$.

ACVF and ACF

We see that

i) Weak stationarity does not in general imply strict stationarity.

ii) Strict stationarity plus finite variance implies weak stationarity.

iii) In the Gaussian case these two concepts are equal.

We say that a time series is stationary if it is weakly stationary. If it is stationary, γ is the autocovariance function [ACVF]. and $\rho = \gamma/\gamma(0)$ is autocorrelation function [ACF].

White noise process

DEFINITION 1. $\{Z_t,\, t \in \mathbb{Z}\}$ is a white noise process with variance σ_Z^2 if it is stationary with zero mean and uncorrelated, i.e

$$\gamma_Z(h) = \begin{cases} \sigma_Z^2, & \text{for } h = 0; \\ 0, & \text{otherwise} \end{cases} = \delta_{0,h}\, \sigma_Z^2$$

We write this as $\{Z_t\} = \{Z_t,\, t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma_Z^2)$.

DEFINITION 2. If $\{Z_t,\, t \in \mathbb{Z}\}$ is a white noise process and addition is iid then we write $\{Z_t,\, t \in \mathbb{Z}\} \sim \mathrm{IID}(0, \sigma_Z^2)$.

REMARK 1. We may drop the subindex «Z» and just write $\{Z_t,\, t \in \mathbb{Z}\} \sim \mathrm{IID}(0, \sigma^2)$.

Iteration of the AR(1) model

The model is

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z} \tag{1.1}$$

Suppose that $\{X_t\}$ is a stationary solution of (1.1)

$$\begin{aligned} X_t &= \phi\big(\phi X_{t-2} + Z_{t-1}\big) + Z_t \\ &= \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t \end{aligned} \tag{1.2}$$

We continue this backward substitutions (m steps)

$$\begin{aligned} X_t &= \phi^{m+1} X_{t-m-1} + \phi^m Z_{t-m} + \phi^{m-1} Z_{t-m+1} + \cdots + \phi Z_{t-1} + Z_t \\ &= \phi^{m+1} X_{t-m-1} + \sum_{j=0}^{m} \phi^j Z_{t-j} \end{aligned} \tag{1.3}$$

The backward recursion converges

Let t be fixed.

$$S_m \stackrel{\text{def}}{=} \sum_{j=0}^{m} \phi^j Z_{t-j}, \qquad S \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

$$V_m \stackrel{\text{def}}{=} \sum_{j=0}^{m} |\phi|^j |Z_{t-j}|, \qquad V \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} |\phi|^j |Z_{t-j}|$$

$$\begin{aligned} \mathbb{E}\, V &= \sum_{j=0}^{\infty} |\phi|^j \mathbb{E}\, |Z_{t-j}| \qquad \text{all terms are nonnegative} \\ &\leq \frac{\sigma}{1 - |\phi|} < \infty \end{aligned}$$

Absolutely convergence with probability one

This means that

$$\mathbb{P}(V < \infty) = 1$$

and the sum which defines S converges absolutely with probability 1. Moreover

$$\begin{aligned}
\mathbb{E}\,V^2 &= \mathbb{E}\left(\sum_{j=0}^{\infty} |\phi|^j |Z_{t-j}|\right)\left(\sum_{k=0}^{\infty} |\phi|^k |Z_{t-k}|\right) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\phi|^j |\phi|^k \mathbb{E}\,|Z_{t-j}||Z_{t-k}| \\
&\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\phi|^j |\phi|^k \sigma^2 \\
&= \frac{\sigma^2}{(1-|\phi|)^2} < \infty
\end{aligned}$$

Mean square convergence

By the calculations above we see that

$$\mathbb{E}\left(V - V_m\right)^2 \leq |\phi|^{2m} \mathbb{E}\, V^2 = \mathcal{O}(1)$$

Hence

$$V_m \xrightarrow[n]{L^2} V$$

and

$$S_m \xrightarrow[n]{L^2} S$$

We define

$$X_t \stackrel{\text{def}}{=} S(t) = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

The autocovariance function

As a consequence of the calculations that we already have done,

$$\mathrm{Cov}(X_{t+h}, X_t) = \mathbb{E} \sum_{j=0}^{\infty} \phi^j Z_{t+h-j} \sum_{j=0}^{\infty} \phi^j Z_{t-j}$$

Now, for $h \geq 0$,

$$\sum_{j=0}^{\infty} \phi^j Z_{t+h-j} = \sum_{j=0}^{\infty} \phi^j Z_{t-(j-h)} = \sum_{j=-h}^{\infty} \phi^{j+h} Z_{t-j}$$

$$\begin{aligned}\mathrm{Cov}(X_{t+h}, X_t) &= \mathbb{E} \sum_{j=-h}^{\infty} \phi^{j+h} Z_{t-j} \sum_{k=0}^{\infty} \phi^k Z_{t-k} \\ &= \sum_{k=0}^{\infty} \phi^{2k+h} \sigma^2 = \frac{\sigma^2}{1-\phi^2} \phi^h\end{aligned}$$

A direct verification of the proposed solution of the AR(1) model

The model is

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z}$$

Inserting the proposed solution $X_t \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} \phi_j Z_{t-j}$ on the right hand side gives

Right han side = left hand side

$$
\begin{aligned}
\text{r.h.s.} &= \phi X_{t-1} + Z_t \\
&= \phi \Big\{ \sum_{j=0}^{\infty} \phi^j Z_{t-1-j} \Big\} + Z_t \\
&= \sum_{j=0}^{\infty} \phi^{j+1} Z_{t-1-j} + Z_t \\
&= \sum_{j=0}^{\infty} \phi^{j+1} Z_{t-(j+1)} + Z_t \\
&= \sum_{j=1}^{\infty} \phi^{j} Z_{t-j} + Z_t \\
&= \sum_{j=0}^{\infty} \phi^{j} Z_{t-j} = X_t \\
&= \text{l.h.s.}
\end{aligned}
$$

The ACVF follows from YW

We note that

$$\begin{aligned}\operatorname{Cov}(X_t, Z_t)) &= \operatorname{Cov}(\phi X_{t-1} + Z_t, Z_t) \\ &= \phi \operatorname{Cov}(X_{t-1}, Z_t) + \operatorname{Cov}(Z_t, Z_t)) \\ &= 0 + \sigma^2 \qquad\qquad \text{since } X_{t-1} = \textstyle\sum_{j=0}^{\infty} \phi^j Z_{t-1-j}\end{aligned}$$

and

$$\operatorname{Cov}(X_{t-h}, Z_t)) = 0 \;\text{ for }\; h \geq 1$$

Multiplying and integrating

Multiplying with X_{t-h}

$$X_t = \phi X_{t-1} + Z_t$$

is

$$X_t X_{t-h} = \phi X_{t-1} X_{t-h} + Z_t X_{t-h}$$

We integrate both sides of this equation,

$$\mathbb{E}\, X_t X_{t-h} = \phi\, \mathbb{E}\, X_{t-1} X_{t-h} + \mathbb{E}\, Z_t X_{t-h}$$

which gives Yule-Walker equations for this model

$$\gamma(h) = \phi\, \gamma(h-1) + \delta_{0,h}\, \sigma^2, \quad h \geq 0$$

Solving YW with respect to the autocovariance function

$$\gamma(h) = \phi^h \gamma(0), \qquad h \geq 0, \qquad \gamma(0) = \phi\gamma(1) + \sigma^2$$

Hence

$$\gamma(h) = \frac{\sigma^2}{1-\phi^2}\,\phi^h, \quad h \geq 0$$

Simulation of the AR(1) process

```
#__________number of obs
N<-100
m<-100  #burn in
M<-N+m
#____________parameters
phi<-0.8
sigma<- 1.0
#_____________Gaussian
Z<-rnorm(M,0,sigma)
X<-Z

#_______loop____
for( t in 2:M) X[t] <- phi* X[t-1] + Z[t]

X<-X[(m+1):M]
```

```
#______plot_____
head<-paste("phi=",phi,",   N=",N, sep="")
plot( 1:N, X, col="blue", las=1, type="l"
      , xlim=c(1,N), xlab="t",
      ylab= "X_t", main=head)



#__________acf and pacf_____
gamma<-acf(X)
alpha<-pacf(X)
mu<-mean(X)
sigmax<-sd(X)
D<-density(X)
x<-D\$x
y<-dnorm(x,mu,sigmax)

#________marginal density_______

plot(D, col="blue", main=head)
points(x,y, col="red", type="l")
```

Example AR(1)

EXAMPLE 1. For the AR(1) model with $|\phi| < 1$ and $\{Z_t, t \in \mathbb{Z}\} \sim \mathrm{WN}(0, \sigma_Z^2)$ the solution $\{X_t, t \in \mathbb{Z}\}$ is a stationary time series with zero mean and ACVF

$$\gamma(h) = \frac{\sigma_Z^2}{1-\phi^2}\, \phi^{|h|} = \gamma(0)\phi^{|h|}$$

The correlation funcion

$$\rho(h) = \phi^{|h|}$$

and the partical correlation function is

$$\alpha(h) = \begin{cases} \phi, & \text{for } h = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Calculating PACF for an AR(1)

For calculating PACF we assume that $\{X_t,\, t \in \mathbb{Z}\}$ is a Gaussian process. We can do that since partial correlation depends only on the covariance structure. It does not depend on the mean value and therefore we can assume that it is zero. By definition, for $h > 1$,

$$\alpha(h) = \mathrm{Corr}\big(X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right],\, X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, X_h\right]\big)$$

$$= \frac{\mathrm{Cov}\big(X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right],\, X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, X_h\right]\big)}{\big\|X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right]\big\|\, \big\|X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, \right]\big\|}$$

wherr $\|\cdot\|$ is the standard deviation.

The residual is orthogonal to the subspace

By Pythagoras, since the left hand side is uncorrelated with $\operatorname{span}\{X_h, \ldots, X_1\}$,

$$\operatorname{Cov}\Big(X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right], \mathbb{E}\left[X_1 \,|\, X_2, \ldots, X_h\right]\Big) = 0$$

Since $h \geq 2$,

$$\begin{aligned}
\mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right] &= \mathbb{E}\left[\phi X_h + Z_{h+1} \,|\, X_h, \ldots, X_2\right] \\
&= \mathbb{E}\left[\phi X_h \,|\, X_h, \ldots, X_2\right] + \mathbb{E}\left[Z_{h+1} \,|\, X_h, \ldots, X_2\right] \\
&= \phi X_h
\end{aligned}$$

Inserting previous calculations

We insert these calculations,

$$\alpha(h) = \frac{\operatorname{Cov}\big(X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right],\, X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, X_h\right]\big)}{\|X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right]\| \; \|X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, \right]\|}$$

$$= \frac{\operatorname{Cov}\big(X_{h+1} - \phi X_h,\, X_1\big)}{\|X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right]\| \; \|X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, \right]\|}$$

$$= \frac{\operatorname{Cov}\big(Z_{h+1},\, X_1\big)}{\|X_{h+1} - \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_2\right]\| \; \|X_1 - \mathbb{E}\left[X_1 \,|\, X_2, \ldots, \right]\|}$$

$$= 0$$

Descriptive statistics

Given obs $\{X_1, \ldots, X_n\}$

i) Plot the time series. It is the graph: $\{(t, X_t), t = 1, \ldots, n\}$.

ii) Empirical mean. It is an ordinary mean.

$$\overline{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t$$

iii) Empirical ACVF. it is given by

$$\widehat{\gamma}(h) = \frac{1}{n} \sum_{t}^{n-h} (X_{t+h} - \overline{X}_n)(X_t - \overline{X}_n), \qquad h \geq 0$$

iv) Empirical ACF. it is given by

$$\widehat{\rho}(h) = \frac{\widehat{\gamma}(h)}{\widehat{\gamma}(0)}$$

v) Emprical PACF

$$\widehat{\alpha}(h) = \widehat{\phi}_{hh}, \qquad h \geq 1$$

PACF

The empirical partial autocorrelation function, $\alpha(\cdot)$ is a bit more difficult. It is defined with Gaussian formulas,

$$
\begin{aligned}
\widehat{X}_{h+1} &\stackrel{\text{def}}{=} \mathbb{E}\left[X_{h+1} \,|\, X_h, \ldots, X_1\right] \\
&= \mu + \sum_{j=1}^{h} \phi_{hj} \big(X_{h+1-j} - \mu\big) \\
\alpha(h) &= \phi_{hh} \\
\widehat{\alpha}(h) &= \widehat{\phi}_{hh}
\end{aligned}
$$

where $\{\widehat{\phi}\}$ is estimated under the assumption of stationarity.

PACF II

Let $Y = X_{h+1}$ and $\mathbf{V} = (X_h, \ldots, X_1)^T$. Then

$$\begin{bmatrix} \phi_{h1} \\ \phi_{h2} \\ \vdots \\ \phi_{hh} \end{bmatrix} = \Sigma_{\mathbf{V}}^{-1} \Sigma_{\mathbf{V},Y}$$

and

$$\Sigma_{\mathbf{V}} = \Gamma_h$$
$$\Sigma_{\mathbf{V},Y} = (\gamma_1, \ldots, \gamma_h) = \boldsymbol{\gamma}_h$$

Hence

$$\begin{bmatrix} \phi_{h1} \\ \phi_{h2} \\ \vdots \\ \phi_{hh} \end{bmatrix} = \Gamma_h^{-1} \boldsymbol{\gamma}_h$$

Noteb that $\alpha(h)$ depends on $\rho(1), \ldots, \rho(h)$.

Example $h = 2$

$$\begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

$$= \frac{1}{\gamma^2(0) - \gamma^2(1)} \begin{bmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix}$$

which shows that

$$\alpha(2) = \phi_{22}$$

$$= \frac{\gamma(0)\gamma(2) - \gamma^2(1)}{\gamma^2(0) - \gamma^2(1)}$$

$$= \frac{\rho(0)\rho(2) - \rho^2(1)}{\rho^2(0) - \rho^2(1)}$$

$$= \frac{\rho(2) - \rho^2(1)}{1 - \rho^2(1)}$$