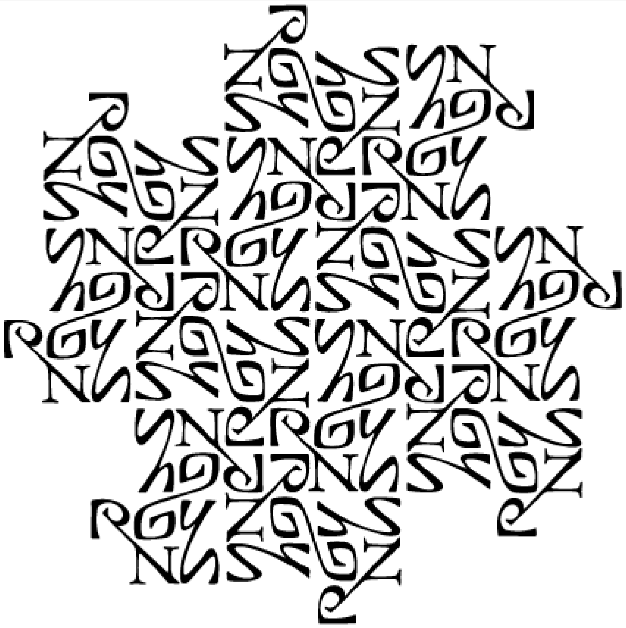
AR(p) + MA(q) = ARMA(p, q)



ARMA(p, q) Model

Definition (ARMA(p, q) Model)

A time series is ARMA(p, q) if it is stationary and satisfies

$$Z_t = \alpha + \underbrace{\phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p}}_{\text{AR part}} + \underbrace{\theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}}_{\text{MA part}} + a_t \qquad (\star)$$

Using the AR and MA operators, we can rewrite $(\star)$ as

$$\phi(B)\dot{Z}_t = \theta(B)a_t$$

The textbook defines the ARMA process with a slightly different parametrization given by

$$Z_t = \alpha + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t \qquad (\star)$$

Causality and Invertibility of ARMA(p, q)

Facts:

- The ARMA(p, q) model given by $\phi(B)Z_t = \theta(B)a_t$ is *causal* if and only if
$$|\phi(z)| \neq 0 \text{ when } |z| \leq 1$$
i.e. all roots including complex roots of $\phi(z)$ lie outside the unit disk.
- The ARMA(p, q) model given by $\phi(B)Z_t = \theta(B)a_t$ is *invertible* if and only if
$$|\theta(z)| \neq 0 \text{ when } |\theta| \leq 1$$
i.e. all roots including complex roots of $\theta(z)$ lie outside the unit disk.

Parameter Redundancy

The process

$$Z_t = a_t \qquad (\star)$$

is equivalent to

$$.5Z_{t-1} = .5a_{t-1} \qquad (\star\star)$$

which is also equivalent to $(\star) - (\star\star)$, i.e.

$$Z_t - .5Z_{t-1} = a_t - .5a_{t-1}$$

Z_t in the last representation is still white noise, but has an ARMA(1,1) representation.

- We remove redundancies in an ARMA model $\phi(B)Z_t = \theta(B)a_t$ by canceling the common factors in $\theta(B)$ and $\phi(B)$.

Example – Removing Redundancy

Example

Consider the process

$$Z_t = .4Z_{t-1} + .45Z_{t-2} + a_t + a_{t-1} + .25a_{t-2}$$

which is equivalent to

$$\underbrace{(1 - .4B - .45B^2)}_{\phi(B)} Z_t = \underbrace{(1 + B + .25B^2)}_{\theta(B)} a_t$$

Is this process really ARMA(2,2)? No! Factoring $\phi(z)$ and $\theta(z)$ gives

$$\phi(z) = 1 - .4z - .45z^2 = (1 + .5z)(1 - .9z)$$
$$\theta(z) = (1 + z + .25z^2) = (1 + .5z)^2$$

Removing the common term $(1 + .5z)$ gives the reduced model

$$Z_t = .9Z_{t-1} + .5a_{t-1} + a_t$$

Example – Causality and Invertibility

Example (cont.)

The reduced model is $\phi(B)Z_t = \theta(B)a_t$ where

$$\phi(z) = 1 - .9z$$
$$\theta(z) = 1 + .5z$$

- There is only one root of $\phi(z)$ which is $z = 10/9$, and $|10/9|$ lies outside the unit circle so this ARMA model is causal.
- There is only one root of $\theta(z)$ which is $z = -2$, and $|-2|$ lies outside the unit circle so this ARMA model is invertible.

ARMA Mathematics

Start with the ARMA equation

$$\phi(B)Z_t = \theta(B)a_t$$

To solve for Z_t, we simply divide! That is

$$Z_t = \underbrace{\frac{\theta(B)}{\phi(B)}}_{\psi(B)} a_t \qquad \text{"MA}(\infty)\text{ representation"}$$

Similarly, solving for a_t gives

$$a_t = \underbrace{\frac{\phi(B)}{\theta(B)}}_{\pi(B)} Z_t \qquad \text{"AR}(\infty)\text{ representation"}$$

The key is to figuring out the ψ_j and π_j in

$$\psi(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j \qquad \psi(B) = 1 + \sum_{j=1}^{\infty} \psi_j B^j$$

Example – MA(∞) Representation

Example (cont.)

Note that

$$\begin{aligned}
\psi(z) = \frac{\theta(z)}{\phi(z)} &= \frac{1 + .5z}{1 - .9z} \\
&= (1 + .5z)(1 + .9z + .9^2z^2 + .9^3z^3 + \cdots) \qquad \text{for } |z| \le 1 \\
&= 1 + (.5 + .9)z + (.5(.9) + .9^2)z^2 + (.5(.9^2) + .9^3)z^3 + \cdots \quad \text{for } |z| \le 1 \\
&= 1 + (.5 + .9)z + (.5 + .9)(.9)z^2 + (.5 + .9)(.9^2)z^3 + \cdots \quad \text{for } |z| \le 1 \\
&= 1 + \underbrace{(.5 + .9)}_{1.4} \sum_{j=1}^{\infty} .9^{j-1} z^j \qquad \text{for } |z| \le 1
\end{aligned}$$

This gives the following MA(∞) representation:

$$Z_t = \frac{\theta(B)}{\phi(B)} a_t = \left(1 + 1.4 \sum_{j=1}^{\infty} .9^{j-1} B^j\right) a_t = a_t + 1.4 \sum_{j=1}^{\infty} .9^{j-1} a_{t-j}$$

Example – AR(∞) Representation

Example (cont.)

Similarly, one can show

$$\begin{aligned}\pi(z) &= \frac{\phi(z)}{\theta(z)} \\ &= \frac{1 - .9z}{1 + .5z} \\ &\vdots \\ &= 1 - 1.4\sum_{j=1}^{\infty}(-.5)^{j-1}z^j \qquad \text{for } |z| \le 1\end{aligned}$$

This gives the following AR(∞) representation:

$$Z_t = 1.4\sum_{j=1}^{\infty}(-.5)^{j-1}a_{t-j} + a_t$$

Homework 3a

Read §4.1 and §4.2.

- **Derive the equation for $\pi(z)$ on slide 10.**
- **Do exercise #3.14(a) only for models (i), (ii), and (iv)**
- **Do exercise #3.14(b,c) only for model (ii)**