Mandatory homework 4 Stat 211 - H21

Teaching assistant: Robert Clay Glastad

March 10, March 17: 0815 - 1000

Version-A-210219 2021-03-17-14-48-15

Due at the end of March 22

Contents

4.1. ARMA(p, q)

PROBLEM 4.1
Consider a causal ARMA(p, q) model

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + \sum_{j=1}^{q} \theta_j Z_{t-j} + Z_t \tag{4.1}$$

where the he autoregressive- and moving average polynomial have no common roots. The model is causal iff all the roots of the autoregressive polynomial $\phi(z)$ have modulus strictly larger than one. In that case

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \tag{4.2}$$

Since the model is causal,

$$\psi_j = \sum_{k=1}^{p\wedge j} \phi_k \psi_{j-k} + \theta_j, \qquad j \geq 0 \tag{4.3}$$

with $\theta_0 = 1$

a) What is meant by invertibility for this model? Formulate a necessary and sufficient condition for the model to be invertible.
b) Formulate (4.1) and (4.2) in backward shift operator notation.
c) Find an analoguous structure to (4.3) for the linear filter $\{\pi_j\}$ when the model is invertible.
d) Explain that (4.3) is a homogenous p-order difference equation with boundary conditions conditions.
e) Suppose that the model is causal. Explain that ψ_j decreases with an exponential rate towards zero. Can you give a precise rate?

4.2. ARMA$(2, 3)$

PROBLEM 4.2
Consider a causal ARMA$(2, 3)$ given by (4.1) where the linear representation satisfies (4.3). Let

$$\boldsymbol{\phi} = (1.7, -0.9)', \qquad \boldsymbol{\theta} = (-1.4, 0.8, 0.1)', \qquad \sigma^2 = 1. \tag{4.4}$$

a) Check that (3) defines a causal and invertible ARMA(p, q) model.
b) Use R and plot $\{\psi_j,\ j = 0, \dots, 50\}$ when the parameteres are given by (4.4).

4.3. THEORETICAL YW FOR ARMA(p, q)

PROBLEM 4.3
Consider a causal ARMA(p, q). Then by (4.2) and (4.3),

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{q-|h|} \theta_{j+h} \psi_j, \qquad h \geq 0. \tag{4.5}$$

a) For an ARMA(2,3), find explicit formulas for $\{\gamma(h),\, h = 0, \dots, 4\}$ expressed by the model parameters and write them in matrix notation.
b) Complete the description with the homogenuous difference equation $\phi(B)\gamma(h) = 0$ for $h \geq 4$.
c) Implement the results in R, compute and plot $\{\gamma(h),\, h = 0, \dots, 50\}$ with parameter values given by (4.4). Check your calculations with help of an R-function.

4.4. The roots of the characteristic polynomial in an AR(p) model determine the coeffecients in the model

Problem 4.4
Let $\phi(B)X_t = Z_t$ be a causal AR(p) model with $\{Z_t\}$ as white noise with variance σ_Z^2. The characteristic polynomial, $\phi(z)$, can factorised in linear factors by the fundamental theorem of algebra

$$\phi(z) = \left(1 - \frac{z}{\xi_1}\right)\left(1 - \frac{z}{\xi_2}\right) \cdots \left(1 - \frac{z}{\xi_p}\right) \tag{4.6}$$

where $\{\xi_1, \dots, \xi_p\}$ are the roots of $\phi(z)$ including multiplisities.
For the calculations you may replace the roots their respectively inverses, so that (4.6) becomes

$$\phi(z) = \left(1 - \alpha_1 z\right)\left(1 - \alpha_2\, z\right) \cdots \left(1 - \alpha_p\, z\right) \tag{4.7}$$

where $\alpha_j \stackrel{\text{def}}{=} \xi_j^{-1}$. For a causal model $|\alpha_j| < 1$ for all j.

a) Let $p = 2$, use (4.7) and show that
$$\phi_1 = \frac{1}{\xi_1} + \frac{1}{\xi_2}, \qquad \phi_2 = -\frac{1}{\xi_1}\frac{1}{\xi_2}.$$
Hint: $\phi(z) = 1 - \phi_1\, z - \phi_2\, z^2$.
b) Explain with help of (4.6) that for any causal model; $|\phi_p| < 1$. *Hint:* $\phi(z) = 1 - \phi_1\, z - \phi_2\, z^2 - \cdots - \phi_p\, z^p$.

4.5. Theoretical Yule Walker for the AR(2) model

Problem 4.5
Let $\{X_t\}$ be a causal AR(2) process with white noise process $\mathrm{WN}(0, 0, \sigma^2)$,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t, \qquad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \tag{4.8}$$

Then by (4.5)

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \delta_{0h}\, \sigma^2, \qquad h \geq 0, \qquad \delta_{0h} = \begin{cases} 1, & h = 0; \\ 0, & \text{outerwise.} \end{cases} \tag{4.9}$$

a) Multiply (4.8) with X_{t-h} for $h = 0, 1, 2$, take the expectation of equations and deduce (4.9) without reference to (4.5).

b) Divide the equations by γ_0 and verify that for this model,

$$\begin{aligned} (1-\phi_2)\rho(1) &= \phi_1 \\ -\phi_1\rho(1) + \rho(2) &= \phi_2 \\ \gamma_0\big(1-\phi_1\rho(1)-\phi_2\rho(2)\big) &= \sigma^2. \end{aligned} \tag{4.10}$$

c) Solve the first two equations above with respect to $\rho(1)$, $\rho(2)$ and then find a formula for γ_0 in terms of model parameters. Argue from (4.10) that the following boundaries on the parameters are necessary for a causal model:

$$\begin{aligned} \phi_2 &= 1, \\ \phi_2 + \phi_1 &= 1, \\ \phi_2 - \phi_1 &= 1. \end{aligned} \tag{4.11}$$

But $\phi_2 = -1$ seems to be difficult to squeeze out of (4.11). However, you can use Problem 4.4.

d) Use R and draw the rectangle defined by (4.11) in a $\phi_1\phi_2$-coordinate system.

e) Show that the roots of the characteristic polynomial is defined by

$$z^2 + \frac{\phi_1}{\phi_2}z - \frac{1}{\phi_2} = 0,$$

and explain that the part of the triangle(4.11) that defines the region with complex roots is inside the triangle and bounded below by horizontal line $\phi_2 = -1$ and above by the curve $\phi_2 = -1/4\phi_1^2$. Add curve to your drawing. Where in the triangle does the model has just one root of multiplisity 2? What about the corners and edges? *Hint: The arguments for the curve can be justified with help of the root formula of a quadratic equation.*

4.6. Asymptotic covariance matrix for $\widehat{\boldsymbol{\phi}}$ in an AR(2) model

Problem 4.6
The asymptotic covariance matrix for the least square estimator of $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^T$ in an AR(p) process with iid noise is $\sigma^2\Gamma_p^{-1}$. Compare the asymptotic variance of $\widehat{\phi}_1$ when the estimated an true model ia and AR(1) with the corresponding asymptotic variance for the estimatior of ϕ_1 in an AR(2) model when $\phi_2 = 0$. Here

$$\Gamma_p = \begin{pmatrix} \gamma_0 & \cdots & \gamma_{p-1} \\ \vdots & \ddots & \vdots \\ \gamma_{p-1} & \cdots & \gamma_0 \end{pmatrix}.$$

What you must do here is to calculate Γ_p^{-1} and then simply as much as possible.

4.7. A programming challenge

Problem 4.7
Write a R-program that calculates the ACVF for an causal AR(p) model from (4.9) given $\boldsymbol{\phi}$ and σ^2 for an arbitray p. Test the program for

$$\boldsymbol{\phi} = (1.5, -0.7, -0.20, 0.05)', \qquad \sigma^2 = 2, \tag{4.12}$$

and plot the first 50 lags of the ACVF and the ACF.

4.8. Complex roots

Problem 4.8
Consider an AR(2) model with 2 complex roots,

$$\phi(z) = \left(1 - \frac{z}{\xi}\right)\left(1 - \frac{z}{\overline{\xi}}\right)$$

where $|\xi| > 1$. Let $\alpha = 1/\xi$ so that

$$\phi(z) = \big(1 - \alpha\, z\big)\,\big(1 - \overline{\alpha} z\big), \qquad \alpha = r\ \exp(i\theta), \qquad \xi = r^{-1}\exp(-i\theta).$$

a) Express ϕ_1 and ϕ_2 as functions of r and τ .

Let $\theta \sim$ uniform$[0, 2\pi)$, $r \sim$ uniform$[0, 1)$ and define $\alpha = r\ \exp(i\theta)$.

b) Find $\mathbb{E}\,\phi_1$, $\mathbb{E}\,|\phi_1|$ and $\mathbb{E}\,\phi_2$.

An AR(2) with complex roots has a ACF with periodic structure. According to theory for homogeneous difference equations.

$$\gamma(h) = r^h\big\{c\ \exp(ih\theta) + \overline{c}\ \exp(-ih\theta)\big\}, \qquad h \geq 0 \tag{4.13}$$

where $c = u + iv$ and $\overline{c} = u - iv$ is its the complex conjugate. This complex constant contains 2 unknowns parametersa that are determined by the initial conditions. Alternatively,

$$\gamma(h) = a\, r^h \cos(\theta h + b), \qquad h \geq 0 \tag{4.14}$$

where (a, b) is fixed by the initial conditions. From (4.14) you see that the ACVF has a periodic structure. If $\theta = 2\pi/m$ for an integer m then the period is exactly m, but the amplitude is damped.

c) Do a drawing as described above and plot the ACF. You can handle that by first computing ϕ_1 , ϕ_2 and then use an R package for the essential job. Alternatively, you can use results from Problem 4.5.
d) Formulate the equations that determine (u, v) in (4.13) or (a, b) in (4.14).

4.9. Durbin Levinson applied on an MA(2)

Problem 4.9
Let

$$X_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t$$

with $\boldsymbol{\theta} = (1.8, 0.9)'$

a) Find the roots for this the model by the R-function **polyroot**. Is the model invertible?
b) Apply Durbin Levinson on this model for $n = 1, \ldots, 50$.
c) Compare the coefficients with the invertible representation $\{\pi_j,\ j \geq 0\}$,

$$\pi_j = \begin{cases} 1, & \text{for } j = 0; \\ -\theta_1, & \text{for } j = 1\ ; \\ -\theta_1\pi_{j-1} - \theta_2\pi_{j-2}, & \text{for } j \geq 2. \end{cases}$$

4.10. Prediction with an MA model that has a unit root.

Problem 4.10
Let $X_t = Z_t - Z_{t-1}$ where $\{Z_t\}$ is white noise with variance $\sigma_Z^2 = 1$. We want to predict X_{n+1} given $\{X_n, \dots, X_1\}$.

a) Is this model invertible?
b) Use the Durbin Levinson algorithm with R and calculate $\{\phi_{nj}, \nu_n, , \ j = 1, \dots, n, \ \}$ for $n = 1, \dots, 50$. Then choose $j = 50$ and calculate ϕ_{n50}, ν_n for $n = 50, \dots, 500$ and plot $\{\phi_{n50}, \, n = 50, \dots, 500\}$ and $\{\nu_n, \, n = 50, \dots, 500\}$ You must compute all the ϕ_{nj}s but you only need to store one of them for each n in addition to ν_n.

We need to understand the numerical results.

c) Show that

$$\|X_{n+1} - \widehat{X}_{n+1}\|^2 = \|Z_{n+1}\|^2 + \| - Z_n - \widehat{X}_{n+1}\|^2$$

so the best we can hope for is to predict $-Z_n$ from $\{X_n, \dots, X_1\}$.

d) Show that $\widehat{X}_2 = -1/2X_1$ and the prediction error variance, $\nu_1 = 3/2$.
e) Show that $\widehat{X}_3 = -2/3X_2 - 1/3X_1$ and the prediction error variance, $\nu_2 = 4/3$. Compare with a).

Based on the two previous points we guess that

$$\widehat{X}_{n+1} = -\sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) X_{n+1-j}, \qquad \nu_n = \frac{n+2}{n+1}. \tag{4.15}$$

f) Prove (4.15) and compare with a).

In the notation of Durbin Levinson,

$$\widehat{X}_{n+1} = \sum_{i=1}^{n} \phi_{nj} X_{n+1-j}. \tag{4.16}$$

g) Explain that in this example,

$$\phi_{nj} = -1 + \frac{1}{n+1} \underset{n}{\longrightarrow} -1.$$

In what sense is this a counterexample and in what sense does it extend the theory.

4.11. Prediction with a noncausal AR(p) process

Problem 4.11
Let $\{X_t\}$ be the solution of a noncausal AR(p) model with white $\{\widetilde{Z}_t\}$ noise process $\sigma_{\widetilde{Z}}^2$. How does the sequence of $\widehat{X}_{n+1}$ predictors looks like for $n \geq p$ and what about the predictor error variance?