Mandatory homework 2 Stat 211 - H21

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Due at the end of February 22

Problem 2.1

Consider a process consisting of a linear trend with an additive noise term consisting of independent random variables Z_t with zero means and variances σ^2, that is,

$$X_t = \beta_0 + \beta_1 t + Z_t$$

where β_0, β_1 are fixed constants.

a) Prove that X_t is non-stationary.
b) Prove that the first difference series $\nabla X_t = X_t - X_{t-1}$ is stationary by finding its mean and autocovariance function.
c) Repeat part (b) if Z_t is replaced by a general stationary process, say Y_t, with mean function μ_Y and autocovariance function γ_Y.

Problem 2.2

a) Install the R-package **astsa**, load the package and the dataset varve. You may use the following code:

```
install.packages("astsa")
library(astsa)
data(varve)
```

Plot the glacial varve data.

The time series exhibits some nonstationarity that can be improved by transforming to logarithms and some additional nonstationarity that can be corrected by differencing the logarithms.

b) Argue that the glacial varves series, say X_t, exhibits heteroscedasticity by computing the sample variance over the first half and the second half of the data. Argue that the transformation $Y_t = \log X_t$ stabilizes the variance over the series. Plot the histograms of X_t and Y_t to see whether the approximation to normality is improved by transforming the data.
c) Plot the series Y_t.
d) Examine the sample ACF of Y_t and comment.
e) Compute the difference $U_t = Y_t - Y_{t-1}$, examine its time plot and sample ACF, and argue that differencing the logged varve data produces a reasonably stationary series. Can you think of a practical interpretation for U_t?
f) Based on the sample ACF of the differenced transformed series computed in (d), argue that the model in (1) below might be reasonable. Assume

$$U_t = \mu + Z_t + \theta Z_{t-1}$$

is stationary when the inputs Z_t are assumed independent with mean 0 and variance σ_Z^2. Show that

$$\gamma_U(h) = \begin{cases} (1+\theta^2), & \text{if } h = 0; \\ \theta\sigma_Z^2, & \text{if } h = 1; \\ 0, & \text{otherwise.} \end{cases}$$

g) Based on part (f), use $\widehat{\rho}_U(1)$ and the estimate of the variance of U_t, $\widehat{\gamma}_U(1)$), to derive estimates of θ and σ_Z^2. This is an application of the method of moments from classical statistics, where estimators of the parameters are derived by equating sample moments to theoretical moments. You can calculate the empirical autocorrelation and autocovariance functions using the acf function, by using the following syntax.

```
acf(u, type = "correlation")
acf(u, type = "covariance")
```

PROBLEM 2.3

Find the partial correlation between X_1 and X_3 given X_2 when $\{X_t\}$ is a MA(1) time series;

$$X_t = Z_t + \theta Z_{t-1}.$$

For the MA(1) we know that $\rho(h) \equiv 0$ for $h \geq 2$. What about the partial correlation for an MA(1) process?

PROBLEM 2.4

The autocovariance function is nonnegative definite and that is a fundamental property. The empirical autovariance function is given by

$$\widehat{\gamma}(h) = \begin{cases} n^{-1}\sum_{t=1}^{n-h}(X_{t+h} - \overline{X}_n)(X_t - \overline{X}_n), & \text{for } h = 0, \dots, n-1; \\ \widehat{\gamma}(-h), & \text{for } -(n-1) \leq h < 0; \\ 0, & \text{otherwise.} \end{cases}$$

In this problem you are asked to prove that $\widehat{\gamma}$ is nonnegative definite, i.e.,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i \widehat{\gamma}(i-j) a_j \geq 0 \qquad (1)$$

for any $\mathbf{a} \in \mathbb{R}^m$ for any $m \geq 1$. Let

$$Y_t \stackrel{\text{def}}{=} \begin{cases} (X_t - \overline{X}_n), & \text{for } 1 \leq t \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

a) Explain that

$$\widehat{\gamma}(h) = n^{-1} \sum_{t=-\infty}^{\infty} Y_{t+h} Y_t, \qquad h \in \mathbb{Z}.$$

b) Let $\mathbf{a} \in \mathbb{R}^m$ be given. Show that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} a_i \widehat{\gamma}(i-j) a_j = n^{-1} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{t} a_i Y_{t+i-j} Y_t \, a_j \tag{2}$$

and do the substitution $s = t - j$. Then you are very close to get a nonnegative number.

PROBLEM 2.5

Let $\{Z_t,\, t \in \mathbb{Z}\}$ be iid with finite expectation and variance σ_Z^2.

$$X_t = \mu + \sum_{j=0}^{q} \theta_j Z_{t-j}$$

with q finite.
Then

$$\overline{X}_n \xrightarrow[n]{\text{a.s.}} \mu,$$
$$\sqrt{n}\big(\overline{X}_n - \mu\big) \xrightarrow[n]{d} \mathcal{N}\Big(0, \sigma^2 \big[\sum \theta_j\big]^2\Big).$$

Can you do the details here? Let

$$U_j \stackrel{\text{def}}{=} \sum_{k=1-j}^{0} Z_{1-k}, \qquad V_j(n) = \sum_{k=n-j+1}^{n} Z_{n-k} \;\text{ for }\; j = 1, \ldots, q \;\text{ and }\; \overline{Z}_n = n^{-1} \sum_{t=1}^{n} Z_t.$$

a) Show that

$$\overline{X}_n - \mu = \big(\sum_{j=0}^{q} \theta_j\big) \overline{Z}_n + \sum_{j=0}^{q} \theta_j \big(n^{-1} U_j - n^{-1} V_n(j)\big). \tag{3}$$

b) Prove that

$$\mathbb{P}(n^{-1} |V_j(n)| > \epsilon) \leq \frac{q}{\epsilon^2 n^2}, \tag{4}$$

and

$$\sum_{j=1}^{n} \mathbb{P}(n^{-1} |V_j(n)| > \epsilon) < \infty.$$

Conclude with reference to Borel Cantelli that $\overline{X}_n \xrightarrow[n]{\text{a.s.}} \mu$.

For the CLT, let

$$Y_n \stackrel{\text{def}}{=} \sqrt{n}(\overline{X}_n - \mu).$$

c) Show that

$$\begin{aligned} Y_n &= \Big(\sum_{j=0}^{q} \theta_j\Big)\sqrt{n}\overline{Z}_n + W_n, \\ W_n &\stackrel{\text{def}}{=} \sum_{j=0}^{q} \theta_j \big(n^{-1/2}U_j - n^{-1/2}V_n(j)\big), \end{aligned} \tag{5}$$

and prove that

$$n^{-1/2}U_j \xrightarrow[n]{\mathbb{P}} 0, \qquad n^{-1/2}V_n(j) \xrightarrow[n]{\mathbb{P}} 0, \qquad j = 1, \ldots, q$$

such that

$$\sqrt{n}(\overline{X}_n - \mu) = Y_n + W_n, \qquad W_n = o_{\mathbb{P}}(1).$$

Use Cramér - Slutsky and write «☐»