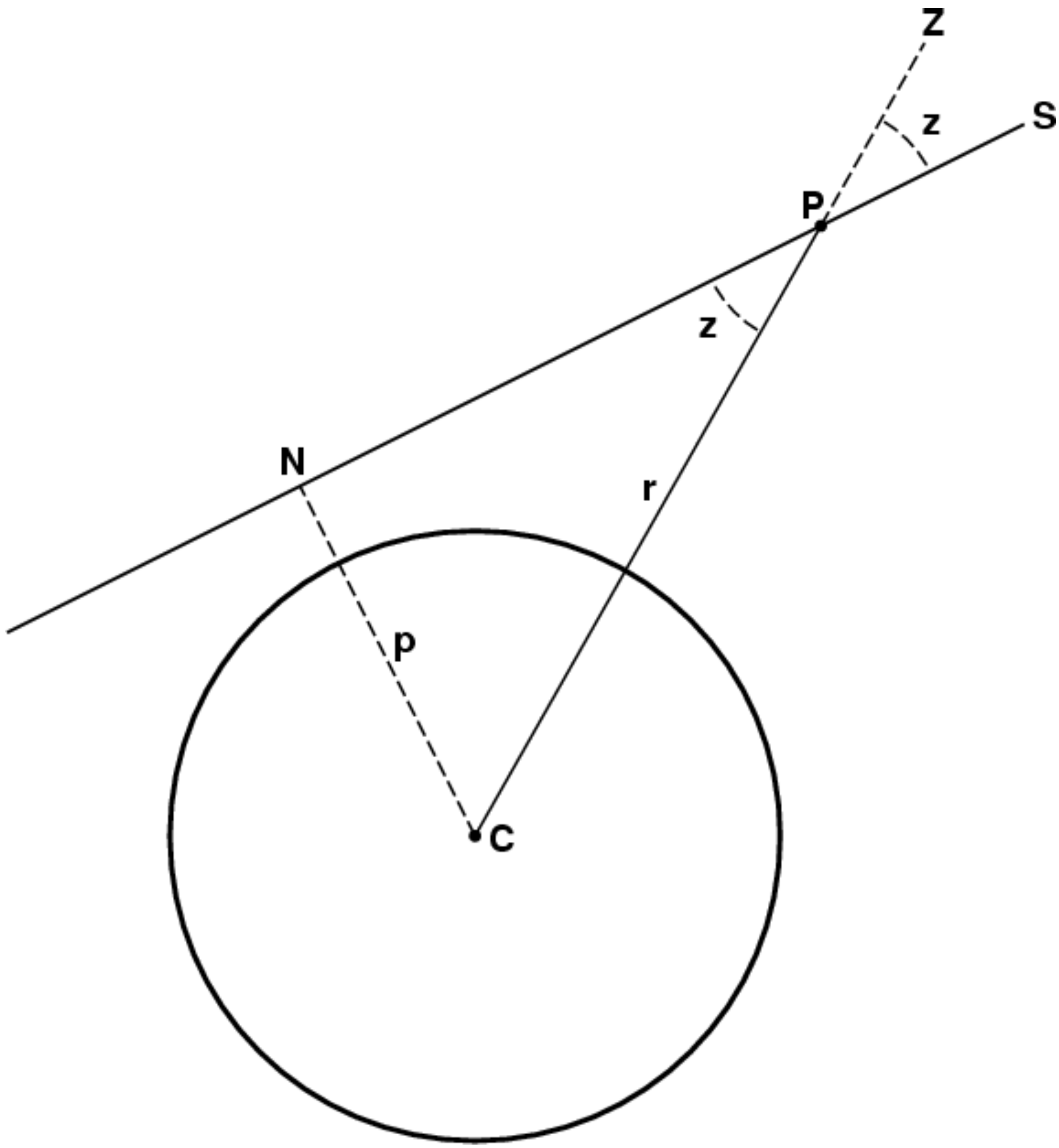
The Refractive Invariant

Introduction

To understand atmospheric refraction quantitatively, you have to understand how Snel's Law works in a spherical atmosphere. This requires only the most elementary trigonometry; just the definitions of sines and cosines are enough to let you follow the argument.

Spherical atmosphere without refraction

To begin with, consider just the geometry of a spherical atmosphere, without refraction. Here's a figure to illustrate this: the big circle centered at C represents the Earth's surface, and the heavy line NPS represents a ray of light coming from a star or the Sun, say, off to the right somewhere in the direction of S. (The plane of the figure is the plane determined by the incoming straight ray NPS and the center of the Earth, C.)



The line CPZ is the local vertical passing through some arbitrary point P on the ray, so that PZ points in the direction of the zenith at P. You can see that the angle ZPS is the local zenith distance, z, of the ray at P. And, because CPZ and NPS are straight lines, the angle CPN is also the zenith distance z.

Now draw the perpendicular from C to the ray NPS, and assume that N is in fact the foot of this perpendicular, which is drawn as a dashed line in the diagram. (That makes N the nearest point to C on the ray.) The angle CPN is now an angle of the right triangle CPN.

Bearing in mind that the angle CPN is also the zenith distance z of the ray at P, we recall from elementary trigonometry that the ratio of the length of the perpendicular CN (of length p) to the hypotenuse of the triangle, CP (of length r), is just the sine of the opposite angle; that is,

$$p/r = \sin z \ .$$

(That's just the definition of the sine of an angle: the opposite side divided by the hypotenuse. You're probably not used to seeing a right triangle turned upside down like this.)

If we clear this of fractions by multiplying both sides by the radial distance, r, from C to P, we have

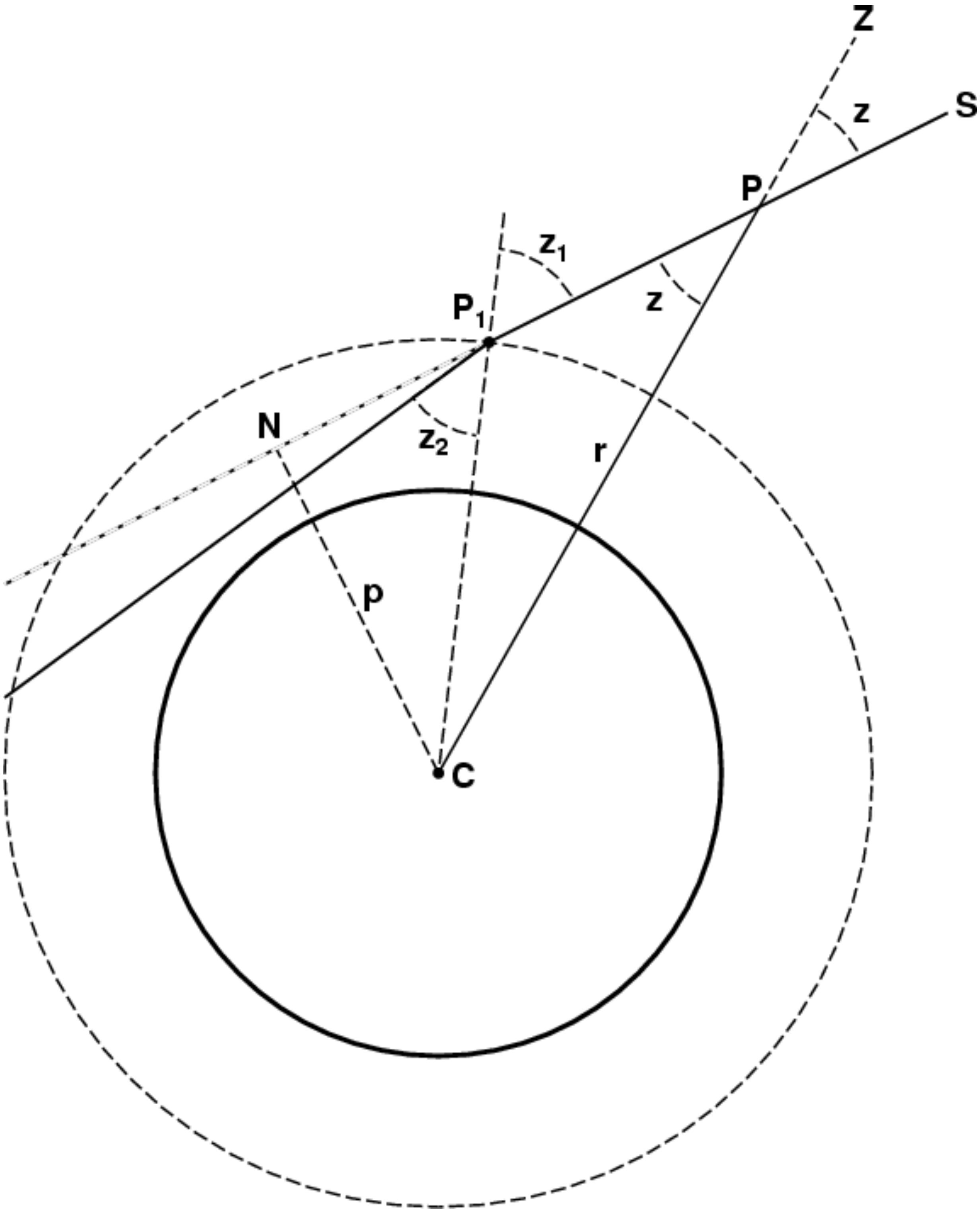
$$p = r \sin z .$$

Notice that this equality holds for every point P along the ray. That is, if P slides back and forth along the (fixed) ray SN, the zenith angle z at P always changes so that the changes in sin z exactly compensate for the changes in radial distance r: their product p remains *invariant*.

We haven't said anything about refraction yet. This is just basic geometry so far. We might call (r sin z) the geometric invariant, to distinguish it from the refractive invariant we will find later.

A refracting interface

Now, let's add a refracting atmosphere to our diagram. We assume it has a horizontal surface — i.e., a sphere concentric with the Earth. So let the dashed circle added in the figure below represent this surface, of radius r_1.



The original ray meets this surface at P_1, where it is refracted instead of continuing along the original path (now shown dotted) through N. Because the local vertical at P_1 has the direction CP_1, which is different from CP, the local zenith distance of the incoming ray is z_1 at P_1.

For simplicity in calculating the refraction at P_1, let us assume [Cassini's oversimplified model] for the moment: the atmosphere is homogeneous below the refracting surface, and there is vacuum above it. That is, the surface is the "top of the atmosphere" very literally. Then the refractive index on the incident (upper) side of the surface $n_1 = 1.000$ exactly, and the refractive index on the lower side is that of air — call it n_2 for convenience.

Now the refractive index of air, n_2, is bigger than unity, so Snel's Law tells us that the angle of refraction z_2 must be smaller than the angle of incidence z_1, because the law of refraction at P_1 is just

$$n_1 \sin z_1 = n_2 \sin z_2,$$

or

$$\sin z_1 = n_2 \sin z_2$$

in our particular case, where $n_1 = 1$ exactly.

Now, multiply this last equation by the value of r at P_1, which is r_1:

$$r_1 \sin z_1 = n_2 r_1 \sin z_2 .$$

From the argument at the end of the previous section, we see that the left side of this equation is just the perpendicular length, p. (The value of the product r sin z is p at *every* point along the original ray; in particular, it is p at the point P_1, so that $r_1 \sin z_1 = p$.) Therefore, the right side is also equal to p.

This is a remarkable result: even after refraction, the product (n r sin z) is still equal to the length of the original perpendicular, p. That's because the invariance of the product (n sin z) across the refracting surface allows us to replace (n sin z) on the lower side with its value on the upper side, where n was unity.

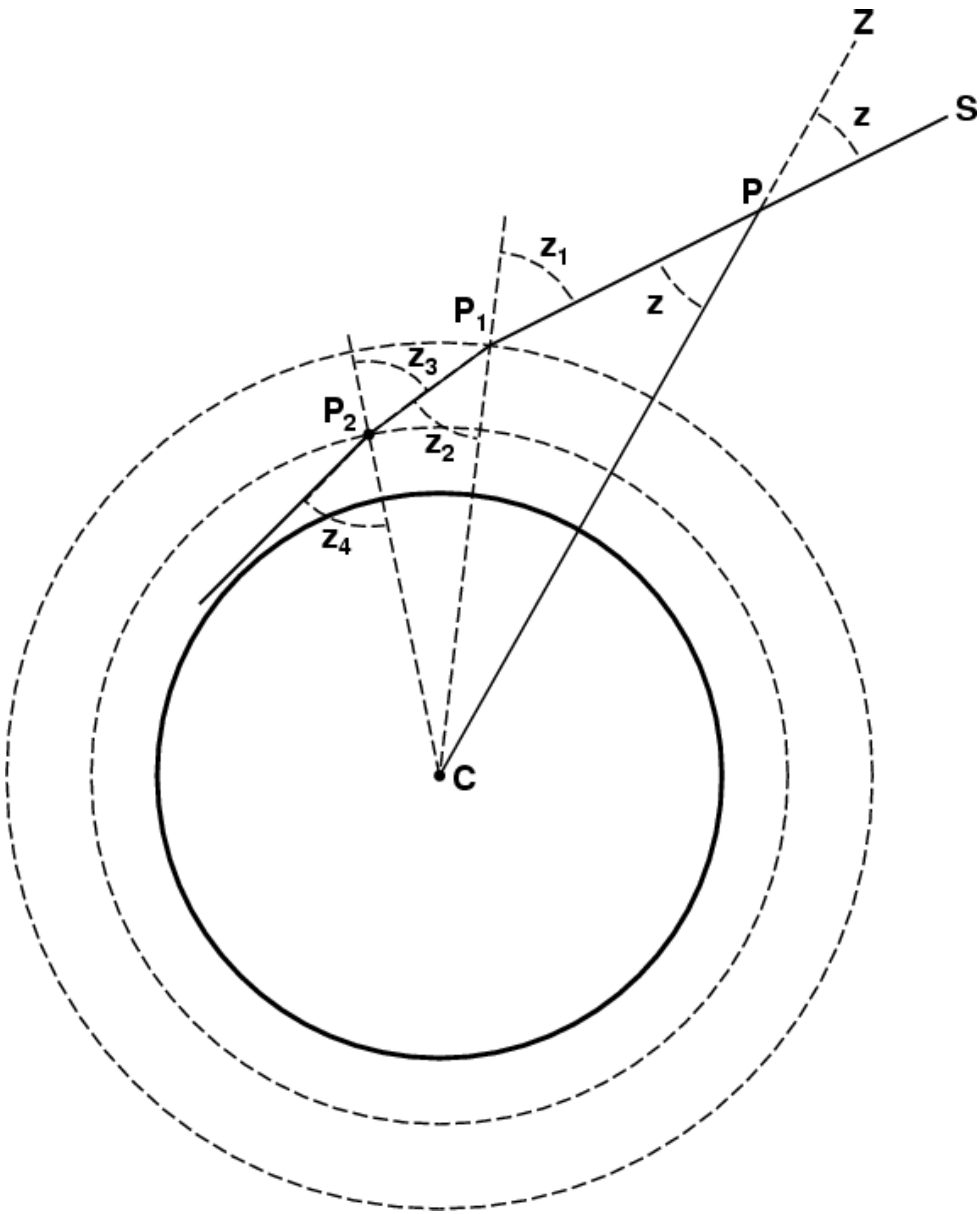
Furthermore, the geometric invariant applies to the ray after refraction: the product (r sin z) is the same everywhere below P_1. And, as the refractive index is also constant throughout the lower layer, the product (n r sin z) is also constant there, and equal to its value p at P_1.

Because (n r sin z) is the same everywhere along the refracted ray, and remains invariant even after refraction, we call it the *refractive invariant*.

Two refractions

Obviously, Cassini's homogeneous atmosphere is not very realistic. We really need to deal with an atmosphere made of many different layers. Suppose, as a start, that we add a second refracting surface below the one considered above. Then the once-refracted ray is refracted a second time when it meets this second interface, at a point we can call P_2.

As mentioned above, the geometric invariant applies to the (straight) refracted ray between the first and second surfaces: the value of (r sin z) at every point along the segment P_1P_2 is constant, and equal to the length of the perpendicular to this straight ray segment from the center, C.



In particular, when the ray meets the second refracting surface at P_2, the value of (r sin z) there is the same as its value at P_1. This value, of course, is the length of the perpendicular from C to the once-refracted ray P_1P_2 ; I refrain from adding this line to the diagram, which is already rather cluttered. Let us call this length p_1.

But we know (from the result of the previous section) that n_1p_1 is just p, the length of the original perpendicular CN in vacuum.

Then, applying Snel's Law again at P_2 gives

$$n_1 \sin z_3 = n_2 \sin z_4 ,$$

where z_3 in the angle of incidence at P_2 (from the medium with refractive index n_1), and z_4 is the angle of refraction into the lowest layer, with refractive index n_2. Consequently, multiplying by r_2, we have

$$n_1 \, r_2 \sin z_3 = n_2 \, r_2 \sin z_4 \, .$$

But the geometric invariant along a straight line allows us to replace $r_2 \sin z_3$ with the corresponding value of this product at P_1, namely $p_1 = r_1 \sin z_2$. So the left side of this equation becomes $n_1 \, r_1 \sin z_2$. And, as we already showed that this equals $r_1 \sin z_1$ or p, we see that this *refractive invariant* is still preserved after a second refraction; the right side of the last centered equation is just the value of (n r sin z) after the second refraction.

And so *ad infinitum* . . .

Clearly, this process can be repeated as many times as we like. No matter how many layers there are, the refractive invariant (n r sin z) remains equal to p in every layer.

We can now extend this process to infinitely many, infinitely thin layers — in other words, to a continuous variation of the refractive index n with height in the atmosphere. The result will always be the same: the refractive invariant is always equal to p, everywhere along the ray:

$$n \, r \sin z = p \, .$$

This fundamental equality is the basis for ray-tracing through any atmospheric model. For the model gives n as a function of r; then the local slope of the ray can be found from sin z.

Those readers who know calculus will see that the path of the ray can be traced if its local slope is known everywhere; differentiating the refractive invariant equation above provide a differential equation from which everything we need can be calculated. But this goes beyond the scope of the present page.

Concluding remarks

What else is the invariant good for?

In addition to providing the basic equation that allows ray-tracing in the atmosphere, the refractive invariant can be used to inspect the properties of rays, without detailed calculations. Here are a few examples; let p be the value of the invariant for a ray.

Shape of a ray path

First of all, because (sin z) is symmetrical about z = 90°, rays that have a maximum or (more often) a minimum are symmetrical about this extremum. This symmetry is discussed on another page; and there will be more to say about these extrema below.

Second, if (nr) is a monotonically increasing function of r, (sin z) = p/nr (and therefore z itself) must decrease monotonically with r; so any ray initially above the horizon approaches the local zenith as we trace it outwards, away from the observer. This means the ray can be nearly horizontal only in the lowest layers of the atmosphere; in the upper atmosphere, the local zenith distance is smaller, and the ray crosses the upper layers more obliquely than the lower ones. So the ray path is less in an upper layer than in a lower one of the same geometric thickness. This is basically why the upper atmosphere has only a very weak effect on refraction: it can never be traversed at grazing incidence.

Comparing the paths of different rays

Bear in mind that p has the same value at all points along a given ray. However, another ray, coming from a different direction in space, will have a different slope from the first at each height above the surface, and so a different value of p.

Let's compare two rays passing through the observer's eye. At the eye, they have different slopes, and so must have different values of the invariant. As the factor (nr) is the same for both rays at any given height, the ray with the larger p must also have a larger value of (sin z): the ray with the steeper slope at the eye remains steeper than the other ray at every other height as well, just as if the rays were straight. So, if both rays are above the *astronomical horizon* at the observer, the rays diverge as they recede from the eye. That means the rays cannot cross anywhere but at the eye; the image seen is necessarily erect. This result is important for mirages.

Conditions for mirages

To have a mirage, we need to have two (or more) paths from the same object point to the eye. That is, two different rays at the eye, traced back to the object, must intersect there. But we have just shown that two rays above the horizon at the observer's eye *cannot* cross, if (nr) is monotonic; so they cannot produce a mirage. Then how are mirages possible?

We can only get a mirage if a ray is *horizontal* somewhere along the line of sight, so that r does not change monotonically along the ray. If (nr) is a monotonic function of r, the ray is horizontal at its *minimum* height above the Earth. Then we see a ray that arrives at the eye from *below* our astronomical horizon. This is the case for the inferior mirage: the inverted image is always below the astronomical horizon.

Another possibility is that the ray is horizontal at a *maximum* height. We can see such rays slightly above (though still very near) the astronomical horizon. But for this case, we require that (nr) is *not* a monotonic function of r; it must have a local maximum. Rays near this maximum can be trapped in a duct. From within the duct, we can see inverted images of distant objects that are also in the duct; this is the classical superior mirage.

Of course, we cannot see celestial objects in the zone at the astronomical horizon that is blocked by the duct. Consequently, mirages of celestial objects — which includes most green flashes — are seen only *below* the astronomical horizon.

The theorem that inverted images require rays to be horizontal somewhere between the object and the observer, so that only erect images are normally seen above the astronomical horizon (in a horizontally-layered atmosphere) was implied by J. B. Biot in his 1810 **monograph** on mirages, and explicitly stated in H. W. Brandes's 1814 **abridged translation** of it into German. It is also explicitly stated by Rudolf Meyer in **his 1935 review** of refraction and mirage phenomena: “The necessary and sufficient condition for a mirage is the horizontal direction of the ray at some place between beginning and end.” The divergence of rays from the observer above the horizon is very nicely proved by Meyer on pp. 776–777 of his **Handbuch der Geophysik** article, using essentially the same argument presented here. Meyer states explicitly on p. 800 that inverted images of heavenly bodies are therefore visible only below the horizon.

Because a ray must be horizontal between the observer and the miraged object, only objects beyond the *apparent horizon* are seen miraged (inverted). (This rule is violated in the common inferior mirages seen on paved roads, because the paving is not perfectly level; all the foregoing discussion assumes a perfectly stratified atmosphere, and this assumption is violated in the surface layer next to a hot road.)

Where a ray is horizontal, the local value of z is 90°, and sin z = 1. At this point, which must be either a maximum or a minimum in height along the ray, n r = p. This condition makes it easy to find where the ray turns back into layers it has traversed before, if we know its slope at any point along the ray.

Dip of the horizon

In particular, a ray must be horizontal at the apparent horizon. Knowing the refractivity profile and the height of the eye, we can then calculate the *dip* of the horizon directly without doing any ray-tracing; for (n r sin z) at the eye must equal (n r) at the horizon. As the dip is the negative of the altitude, which is the complement of the zenith distance, we have

$$\cos(\mathrm{dip}) = n_{hor}\, r_{hor} / n_{eye}\, r_{eye}\ .$$

This relation suggests that the quantity (n r) as a function of height would be very informative; and indeed, a plot of **nr** vs. height is the dip diagram, which is discussed in detail **elsewhere**.

Ducts

The **nr** product is also the key to understanding ducts, which are regions bounded by two layers with the *same* value of **nr**. Because **nr** is the same at the top and bottom of a duct, any ray passing through these two levels has the same value of sin z (and therefore the same value of z) at both of them. A ray that is horizontal within the duct has a *larger* value of **nr** than those at its boundaries, and so cannot reach them; for that would require it to have sin z greater than unity at the duct boundary, to keep (nr sin z) constant. Such rays are trapped within the duct — hence the name.

A ray that is horizontal at the edge of a duct is forced to circle the Earth at constant height indefinitely, assuming a horizontally-uniform atmosphere. These circulating rays are discussed on another page.

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