(Uniform) Continuity, (Uniform) Convergence

David Jekel

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The distinctions between continuity, uniform continuity, convergence, and pointwise convergence deserve repeated explanation, since they are important but easily confused.

Let's compare the definitions. In the following, X and Y are metric spaces, and $f : X \to Y$ and $f_n : X \to Y$ are functions.

Continuity

- In continuity, you are only considering one function f (not a sequence of functions).[1]
- Continuity describes how $f(x)$ changes *when you change x*.
- It says that for each x_0, if x is close to x_0, then $f(x)$ is close to $f(x_0)$.
- The definition reads: $\forall x_0 \in X, \forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in X$, $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$.

Uniform Continuity

- Uniform continuity is a stronger version of continuity. As before, you are only considering one function f (not a sequence of functions).
- Uniform continuity describes how $f(x)$ changes *when you change x*.
- If f is uniformly continuous, that means that if x is close to x_0, then $f(x)$ is close to $f(x_0)$. Importantly, it requires that how close $f(x)$ and $f(x_0)$ are only depends on how close x and x_0 are. The same estimate works for all possible values of x and x_0.
- The definition reads: $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x_0 \in X, \forall x \in X$, $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$.

[1] Well, maybe you have a sequence $\{f_n\}$ of continuous functions, but in that case, the definition of continuity applies to each function f_n independently. It only considers one function at a time.

- Note that the only thing that changed relative to the definition of continuity was that "$\forall x_0$" moved later, but this makes all the difference. In the statement of continuity, putting x_0 first allows the value of δ to depend on both ϵ and x_0, but for uniform continuity the value of δ only depends on ϵ. Thus, you can make *one* choice of δ such that $f(x)$ and $f(x_0)$ will be *uniformly* close together for all values of x and x_0 within a distance of δ from each other.

Pointwise Convergence

- To discuss pointwise convergence $f_n \to f$, you need to have a sequence of functions $\{f_n\}$, not just one function.

- Convergence describes how $f_n(x)$ changes *when you change* n (but don't change x).

- $f_n \to f$ pointwise means that for each $x \in X$, if n is large enough, then $f_n(x)$ is close to $f(x)$.

- The definition reads: $\forall x \in X, \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$.

Uniform Convergence

- Uniform convergence is a stronger version of convergence. To discuss pointwise convergence $f_n \to f$, you need to have a sequence of functions $\{f_n\}$, not just one function.

- Uniform convergence describes how $f_n(x)$ changes *when you change* n (but don't change x).

- $f_n \to f$ means tha if n is large enough, then $f_n(x)$ is close to $f(x)$ uniformly for all values of x.

- The definition reads: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall x \in X$, $n \geq N$ implies $d(f_n(x), f(x)) < \epsilon$.

- Note that the only thing that changed relative to the definition of pointwise convergence was that "$\forall x$" moved later, but this makes all the difference. In the statement of convergence, putting x first allows the value of N to depend on both ϵ and x, but for uniform converence the value of N only depends on ϵ. Thus, you can make *one* choice of N such that $f_n(x)$ and $f(x)$ will be *uniformly* close together for all values of x whenever $n \geq N$.

Examples and Theorems

- The following functions $\mathbb{R} \to \mathbb{R}$ are uniformly continuous: x, $\sin x$, $1/(1+x^2)$, $\arctan x$.

- The following functions $\mathbb{R} \to \mathbb{R}$ are *not* uniformly continuous: x^2, $\sin x^2$, any polynomial of degree at least 2, e^x.
- If f_n is continuous for each n and $f_n \to f$ uniformly, then f is continuous.
- If f_n is continuous for each n and $f_n \to f$ pointwise, then f might not be continuous. For example, consider $f_n : \mathbb{R} \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} 0, & x \leq 0 \\ nx, & 0 \leq x \leq 1/n \\ 1, & x \geq 1. \end{cases} \qquad f(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

  Then $f_n \to f$ pointwise but not uniformly.
- If f_n is uniformly continuous and $f_n \to f$ uniformly, then f is uniformly continuous.
- Similarly, if f_n is uniformly continuous and $f_n \to f$ pointwise, then f might not be continuous, or f might be continuous and not uniformly continuous.