5.3 - Higher-Order Taylor Method

Consider solving the initial-value problem for ordinary differential equation:

$$(*) \quad y'(t) = f\big(t,\ y\big),\ a \le t \le b, \quad y(a) = \alpha.$$

Let $y(t)$ be the unique solution of the initial-value problem. In Section 5.2, Euler's Method, a numerical method, is introduced to computes a set $\{y_k\}_{k=0}^{N}$ where $y_k \approx y(t_k)$ and $a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b.$ Recall that Euler's Method is derived using a Taylor polynomial of degree 1. Will a numerical approximation be better if a Taylor polynomial of degree 2 or degree 3 is used? In this section, we derive a numerical method using Taylor Method of order n where $n = 1, 2, \ldots.$

Assume that the initial-value problem is well-posed. Let $h = \dfrac{b-a}{N},$ where N is a positive integer. Define $t_k = t_{k-1} + h$ or $t_k = a + k\,h.$

1. **Taylor Method of order n** (**n-th Order of Taylor Method**):

We first express the solution $y(t_{i+1})$ to the initial-value problem (*) by its nth Taylor polynomial and corresponding remainder centered at t_i:

$$y(t_{i+1}) = y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2!}h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!}h^n + \frac{y^{(n+1)}(t_i^*)}{(n+1)!}h^{n+1}, \text{ where } t_i \le t_i^* \le t_{i+1}$$

$$= y(t_i) + f\big(t_i,\ y(t_i)\big)h + \frac{f'\big(t_i,\ y(t_i)\big)}{2!}h^2 + \ldots + \frac{f^{(n-1)}\big(t_i,\ y(t_i)\big)}{n!}h^n + \frac{f^{(n)}\big(t_i^*,\ y(t_i^*)\big)}{(n+1)!}h^{n+1}$$

$$= y(t_i) + h\left(\underbrace{f\big(t_i,\ y(t_i)\big) + \frac{f'\big(t_i,\ y(t_i)\big)}{2!}h + \ldots + \frac{f^{(n-1)}\big(t_i,\ y(t_i)\big)}{n!}h^{n-1}}_{T_n(t_i,\ y(t_i))} + \underbrace{\frac{f^{(n)}\big(t_i^*,\ y(t_i^*)\big)}{(n+1)!}h^n}_{R_n(t_i)}\right).$$

Define

$$T_n\big(t,\ y(t)\big) = f\big(t,\ y(t)\big) + \frac{f'\big(t,\ y(t)\big)}{2!}h + \ldots + \frac{f^{(n-1)}\big(t,\ y(t)\big)}{n!}h^{n-1}$$

Taylor Method of order n (nth order of Taylor Method):

$$y_0 = \alpha$$
$$y_{i+1} = y_i + hT_n\big(t_i,\ y_i\big),\ i = 0, 1, 2, \ldots, N-1,$$

Notes:

a. $T_1\big(t,\ y(t)\big) = f(t, y(t)).$ So, Euler's Method is the Taylor Method of order 1.

b. $T_n\big(t,\ y(t)\big) = T_{n-1}\big(t,\ y(t)\big) + \dfrac{f^{(n-1)}\big(t,\ y(t)\big)}{n!}h^{n-1}.$ If h is small, for large $n,\ T_n\big(t,\ y(t)\big) \approx T_{n-1}\big(t,\ y(t)\big)$, that means $T_n\big(t,\ y(t)\big)$ will not improve much the approximation.

Example *Use Taylor Method of order 2, 3 and 4 to approximate the solution of the initial-value problem.*

$$y'(t) = y - t^2 + 1,\ 0 \le t \le 2,\ y(0) = 0.5.$$

$$T_2(t_i,\ y_i) = f(t_i,\ y_i) + \frac{f'(t_i,\ y_i)}{2!}h$$

$$T_3(t_i,\ y_i) = T_2(t_i,\ y_i) + \frac{f''(t_i,\ y_i)}{3!}h^2$$

$$T_4(t,\ y(t)) = T_3(t_i,\ y_i) + \frac{f'''(t_i,\ y_i)}{4!}h^3$$

$$f(t,\ y(t)) = y - t^2 + 1,$$

$$f'(t,\ y(t)) = y' - 2t = y - t^2 + 1 - 2t = y - t^2 - 2t + 1$$

$$f''(t,\ y(t)) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$$

$$f'''(t,\ y(t)) = y' - 2t - 2 = y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1$$

Taylor Method of order 2:

$$y_0 = 0.5$$

$$y_{i+1} = y_i + h\left(y_i - t_i^2 + 1 + \frac{h}{2}(y_i - t_i^2 - 2t_i + 1)\right)$$

Taylor Method of order 3:

$$y_0 = 0.5$$

$$y_{i+1} = y_i + h\left(y_i - t_i^2 + 1 + \frac{h}{2}(y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6}(y_i - t_i^2 - 2t_i - 1)\right)$$

Taylor Method of order 4:

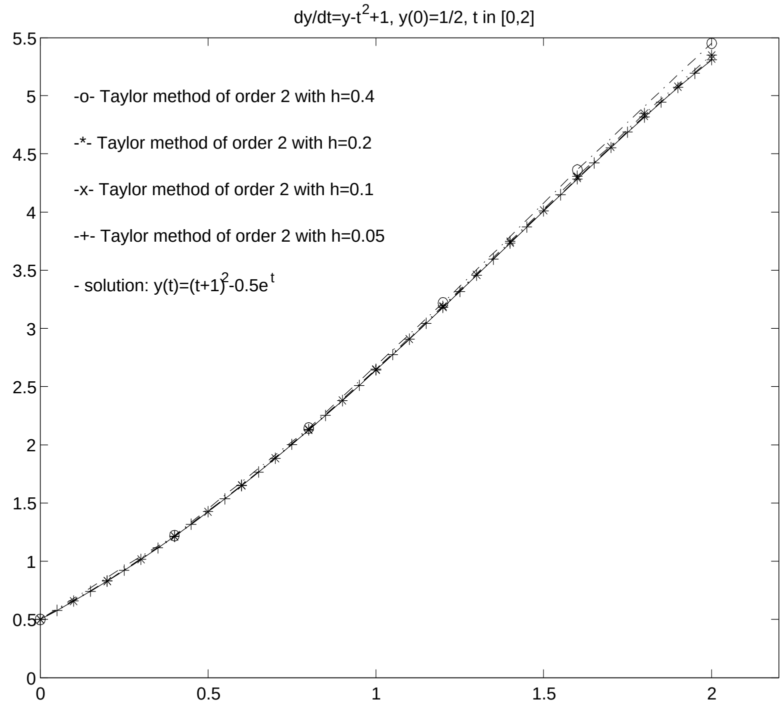
$$y_0 = 0.5$$

$$y_{i+1} = y_i + h\left(y_i - t_i^2 + 1 + \frac{h}{2}(y_i - t_i^2 - 2t_i + 1) + \frac{h^2}{6}(y_i - t_i^2 - 2t_i - 1) + \frac{h^3}{24}(y_i - t_i^2 - 2t_i - 1)\right)$$

Taylor Method of order 2: $h = 0.4$, $h = 0.2$, $h = 0.1$, and $h = 0.05$.

In MatLab:

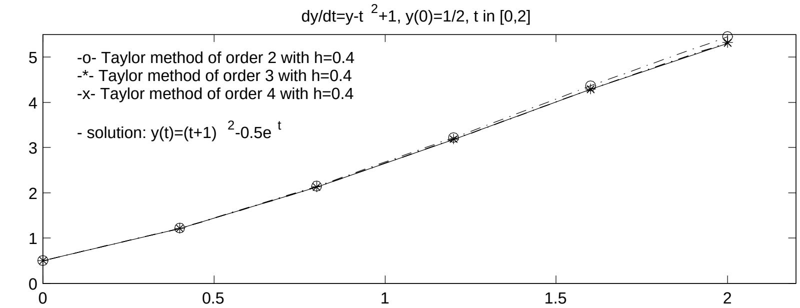
```
>>fun=@(t,y) y-t.^2+1;
>>fun1=@(t,y) y-t.^2-2*t+1;
>>[tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.4);
>>plot(tv,yv,'k-o')
>>hold on
>>[tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.2);
>>plot(tv,yv,'r-*')
>>[tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.1);
>>plot(tv,yv,'b-x')
>>[tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.05);
>>plot(tv,yv,'g-+')
>>title('dy/dt=y-t^2+1, y(0)=1/2, t in [0,2]')
>>hold off
```

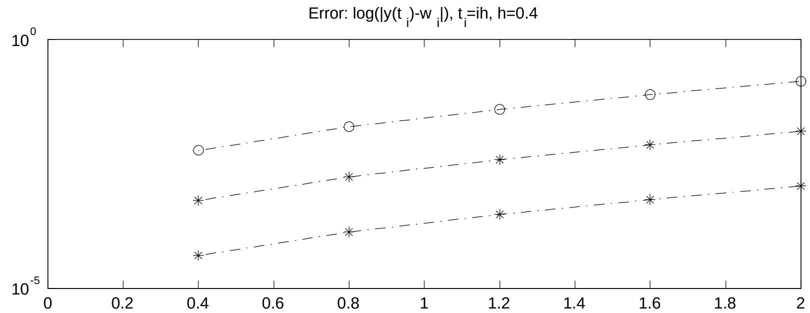


Let $h = 0.4$, comparing Taylor Method of order of 2, 3, and 4.

In MatLab:

```
>>fun=@(t,y) y-t.^2+1;
>>fun1=@(t,y) y-t.^2-2*t+1;
>>fun2=@(t,y) y-t.^2-2*t-1;
(Note that fun3=fun2)
>>[tv yv,n]=tayfun2(fun,fun1,0,2,0.5,0.4);
>>plot(tv,yv,'k-o')
>>hold on
>>[tv yv,n]=tayfun3(fun,fun1,fun2,0,2,0.5,0.4);
>>plot(tv,yv,'r-*')
>>[tv yv,n]=tayfun4(fun,fun1,fun2,fun2,0,2,0.5,0.4);
>>plot(tv,yv,'b-x')
>>title('dy/dt=y-t^2+1, y(0)=1/2, t in [0,2]')
>>hold off
```





2. Local Truncation Error:

Definition *(local truncation error) Consider* ***the difference method****:*

$$y_0 = \alpha$$

$$y_{i+1} = y_i + h\,\phi\big(t_i,\ y_i\big),\ \ i = 0,1,2,\ldots,N-1,$$

The local truncation error *of this difference method is defined as:*

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - \Big(y(t_i) + h\,\phi\big(t_i,\ y(t_i)\big)\Big)}{h}$$

$$= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi\big(t_i,\ y(t_i)\big),\ \ \textit{for } i = 0,1,\ldots,N-1.$$

Note that $\tau_{i+1}(h)$ is a function of h for $i = 0,1,\ldots,N-1$.

When $\phi\big(t_i,\ y(t_i)\big) = T_n\big(t_i,\ y_i\big)$, the local truncation error can be derived as follows.

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h} - \phi\big(t_i,\ y(t_i)\big),\ \text{ for } i = 0,1,\ldots,N-1.$$

$$= \frac{1}{h}\left(y(t_i) + y'(t_i)h + \frac{y''(t_i)}{2!}h^2 + \ldots + \frac{y^{(n)}(t_i)}{n!}h^n + \frac{y^{(n+1)}(t_i^*)}{(n+1)!}h^{n+1} - y(t_i)\right) - T_n\big(t_i,\ y(t_i)\big)$$

$$= \frac{y^{(n+1)}(t_i^*)}{(n+1)!}h^n,\ \text{ where } t_i < t_i^* < t_{i+1}.$$

If $|y^{(n+1)}(t)| \le M$ for t in $\big[a,\ b\big]$, then

$$|\tau_{i+1}(h)| \le \frac{Mh^n}{(n+1)!} \text{ for all } i = 0,1,\ldots,N.$$

Example *Find the local truncation error of Euler's Method.*

Euler's Method: $n = 1$.

$$|\tau_{i+1}(h)| \leq \frac{Mh}{2} \text{ for all } i = 0, 1, \ldots, N.$$

If $|y''(t)| \leq M$ for all t in $\left[a,\ b\right]$, then

$$|\tau_{i+1}(h)| \leq \frac{1}{2}Mh = O(h).$$

Example *Consider the initial-value problem.*

$$y'(t) = y - t^2 + 1,\ 0 \leq t \leq 2,\ y(0) = 0.5.$$

Suppose we know that $y(t) = (t+1)^2 - 0.5e^t$.

(1) For Euler Method:

(i) Find an upper bound (as small as possible) of the true error $|y(t_{i+1}) - y_{i+1}|$ *for* $i = 0, 1, \ldots$ *in term of h.*

(ii) Find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ *for* $i = 0, 1, \ldots$ *in term of h.*

(iii) Compare above error bounds when $h = 0.1$.

(iv) Find h such that $|y(t_{i+1}) - y_{i+1}| \leq 0.01$.

(v) Find h such that $|\tau_{i+1}(h)| \leq 0.01$.

(2) For Taylor Method of Order 2, 3 and 4:

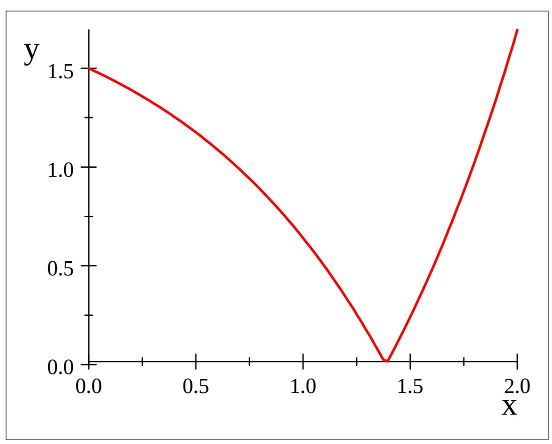
(i) Find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ *for* $i = 0, 1, \ldots$ *in terms of h.*

(ii) Compare above error bounds when $h = 0.1$.

(iii) Find h such that $|\tau_{i+1}(h)| \leq 0.0001$ *for each method.*

$$y(t) = (t+1)^2 - 0.5e^t,\ y'(t) = 2(t+1) - 0.5e^t,\quad y''(t) = 2 - 0.5e^t,\quad y'''(t) = -0.5e^t$$

$$\left| y''(t) \right| = |2 - 0.5e^t| \leq 2 + \frac{1}{2}e^2 = 5.69452805 = M,\quad L = 1$$

Or from the graph of $\left| y''(t) \right| = |2 - 0.5e^t|$, 

,

we have $M = |y''(2)| = |2 - 0.5e^2| = 1.69452805$. Let $M = 1.7$.

(1) For Euler Method:

(i)

$$|y_{i+1} - y(t_{i+1})| \le \frac{Mh}{2L}\left(e^{L(t_{i+1}-a)} - 1\right) \le \frac{(1.7)h}{2(1)}\left(e^{(1)(2-0)} - 1\right) = 0.85(e^2 - 1)h$$

(ii)

$$|\tau_{i+1}(h)| \le \frac{Mh}{2} = \frac{(1.7)h}{2} = 0.85h$$

(iii) For $h = 0.1$,

$|y_{i+1} - y(t_{i+1})| \le 0.85(e^2 - 1)(0.1) = 0.543\,069\,768$ and

$|\tau_{i+1}(h)| \le 0.85(0.1) = 0.085$

$|\tau_{i+1}(h)|$ is smaller than $|y_{i+1} - y(t_{i+1})|$.

(iv) $|y(t_{i+1}) - y_{i+1}| \le 0.85(e^2 - 1)h \le 0.01$, $h \le \dfrac{0.01}{0.85(e^2 - 1)} = 1.841\,384\,03 \times 10^{-3}$, let $h = 0.0018$.

(v) $|\tau_{i+1}(h)| \le 0.85h \le 0.01$, $h \le \dfrac{0.01}{0.85} = 1.176\,470\,59 \times 10^{-2}$, let $h = 0.01176$.

(2) For Taylor Method of order 2, 3, and 4,

(i) $n = 2$, $|\tau_{i+1}(h)| \le \dfrac{Mh^2}{3!} = \dfrac{(1.7)h^2}{6}$

$n = 3$, $|\tau_{i+1}(h)| \le \dfrac{Mh^3}{4!} = \dfrac{(1.7)h^3}{24}$

$n = 4$, $|\tau_{i+1}(h)| \le \dfrac{Mh^4}{5!} = \dfrac{(1.7)h^4}{120}$

(ii) Let $h = 0.1$.

$n = 2$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)(0.1)^2}{6} = 2.833\,333\,33 \times 10^{-3}$

$n = 3$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)(0.1)^3}{4!} = 7.083\,333\,33 \times 10^{-5}$

$n = 4$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)(0.1)^4}{5!} = 1.416\,666\,67 \times 10^{-6}$

$|\tau_{i+1}(h)|$ is getting smaller as n is getting larger.

(iii) $n = 2$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)h^2}{6} \le 0.0001$, $h \le \sqrt{\dfrac{0.0001(6)}{1.7}} = 1.878\,672\,87 \times 10^{-2}$, let $h = 0.018$

$n = 3$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)h^3}{24} \le 0.0001$, $h \le \sqrt[3]{\dfrac{0.0001(24)}{1.7}} = 0.112\,181\,379$, let $h = 0.1$.

$n = 4$, $|\tau_{i+1}(h)| \le \dfrac{(1.7)h^4}{120} \le 0.0001$, $h \le \sqrt[4]{\dfrac{0.0001(120)}{1.7}} = 0.289\,856\,525$, let $h = 0.28$.

Exercises:

1. For each of the initial-value problems,

(1) identify the function $f(t,y)$;

(2) compute $\dfrac{df(t,y(t))}{dt}$;

(3) apply the second order Taylor Method with $N = 2$ without using MatLab program tayfun2.m; and

(4) apply the second order Taylor Method with $h = 0.2$ using MatLab program tayfun2.m.

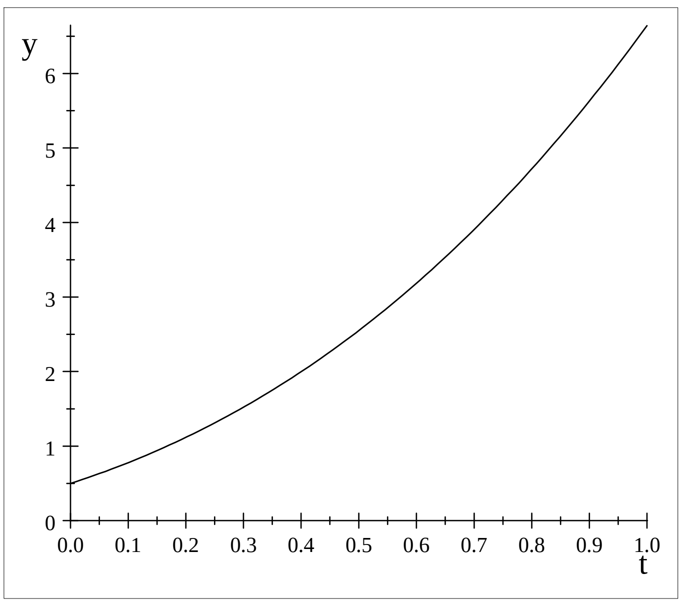
a. $y' = \dfrac{e^t}{y}$, $0 \le t \le 1$, $y(0) = 1$

b. $y' + ty = ty^2$, $0 \le t \le 2$, $y(0) = 0.5$

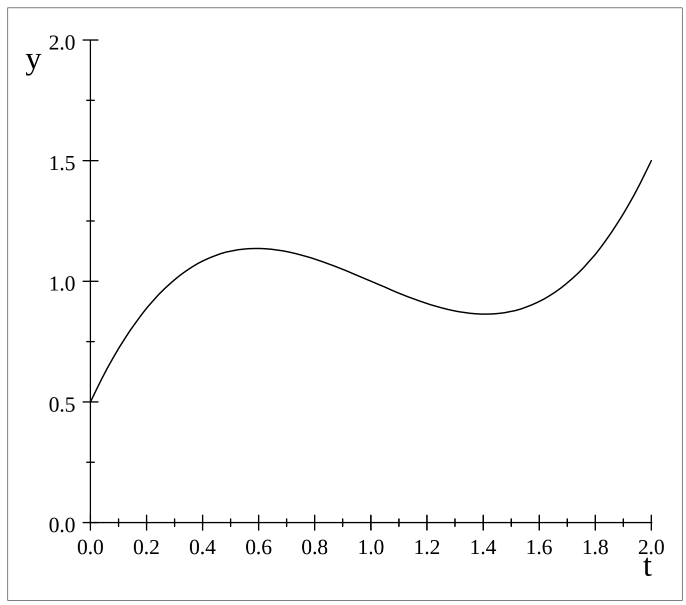
c. $y' = te^{t+y} - 1,\ 0 \le t \le 2,\ y(0) = -1$
d. $y' = e^{2t} + (1 + \frac{5}{2}e^t)y + y^2,\ 0 \le t \le 1,\ y(0) = -1$

2. For each of the initial-value problems,
(1) identify the function $f(t,y)$;
(2) compute $\dfrac{df(t,y(t))}{dt}$, $\dfrac{d^2f(t,y(t))}{dt^2}$ and $\dfrac{d^3f(t,y(t))}{dt^3}$;
(3) apply the Taylor Method of order 4 with $N = 2$ without using MatLab program tayfun4.m; and
(4) apply the Taylor Method of order 4 with $h = 0.1$ using MatLab program tayfun4.m.
a. $y' + 2y^2 = t^2 - 1,\ 0 \le t \le 1,\ y(0) = 0$
b. $y' = \sin(t) - y,\ \pi \le t \le 2\pi,\ y(\pi) = 1$
c. $y' + \dfrac{4y}{t} = t^4,\ 1 \le t \le 2,\ y(1) = 1$
d. $y' = y - t,\ 0 \le t \le 1,\ y(0) = 2$

3. Let $y(t)$ be the solution of the initial-value problem: $y' = f(t,y),\ a \le t \le b,\ y(a) = \alpha$. The graph of $y(t)$ is given below. For each problem, estimate graphically y_1 and y_2 obtained by Euler's Method with $h = 0.25$ and $h = 0.5$, respectively. Explain graphically how y_1 and y_2 are derived.



(a) $y(t)$



(b) $y(t)$

4. Consider using the 2nd order Taylor Method. For each of the following initial-value problems,
(1) find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ (use the true solution $y(t)$ to find M) and $|\tau_{i+1}(h)|$ with given h; and
(2) find h such that $|\tau_{i+1}(h)| \le 0.01$.

a. $y' = \dfrac{e^t}{y},\ 0 \le t \le 1,\ y(0) = 1,\ y(t) = \sqrt{2e^t - 1},\ h = 0.1$
b. $y' + ty = ty^2,\ 0 \le t \le 2,\ y(0) = 0.5,\ y(t) = \left(1 + e^{t^2/2}\right)^{-1},\ h = 0.2$
c. $y' = te^{t+y} - 1,\ 0 \le t \le 2,\ y(0) = -1,\ y(t) = -t - \ln(e - t^2/2),\ h = 0.2$
d. $y' = \sin(t) - y,\ \pi \le t \le 2\pi,\ y(\pi) = 1,\ y(t) = \frac{1}{2}e^{\pi - t} + \frac{1}{2}\sin(t) - \frac{1}{2}\cos(t),\ h = \frac{\pi}{10}$

e. $y' + \dfrac{4y}{t} = t^4,\ 1 \le t \le 2,\ y(1) = 1,\ y(t) = \frac{1}{9}t^5 + \frac{8}{9}t^{-4},\ h = 0.1$

5. Consider the initial-value problem: $y' = \dfrac{t}{y},\ 0 \le t \le 5,\ y(0) = 1.$ The true solution is $y(t) = \sqrt{t^2+1}$.

For each of the methods: Euler's Method, the 2nd order Taylor Method and the 4th order Taylor Method,

(1) find an upper bound (as small as possible) of the truncation error $|\tau_{i+1}(h)|$ (use the true solution $y(t)$ to find M) and $|\tau_{i+1}(h)|$ with $h = 0.1$; and

(2) find h such that $|\tau_{i+1}(h)| \le 0.01$.