UNIFORM CONVERGENCE

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1. Pointwise Convergence of a Sequence

Let E be a set and Y be a metric space. Consider functions $f_n : E \to Y$ for $n = 1, 2, \ldots$. We say that the sequence (f_n) **converges pointwise** on E if there is a function $f : E \to Y$ such that $f_n(p) \to f(p)$ for every $p \in E$. Clearly, such a function f is unique and it is called the **pointwise limit** of (f_n) on E. We then write $f_n \to f$ on E. For simplicity, we shall assume $Y = \mathbb{R}$ with the usual metric.

Let $f_n \to f$ on E. We ask the following questions:

(i) If each f_n is bounded on E, must f be bounded on E? If so, must $\sup_{p\in E} |f_n(p)| \to \sup_{p\in E} |f(p)|$?

(ii) If E is a metric space and each f_n is continuous on E, must f be continuous on E?

(iii) If E is an interval in $\mathbb{R}$ and each f_n is differentiable on E, must f be differentiable on E? If so, must $f_n' \to f'$ on E?

(iv) If $E = [a, b]$ and each f_n is Riemann integrable on E, must f be Riemann integrable on E? If so, must $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$?

These questions involve interchange of two processes (one of which is 'taking the limit as $n \to \infty$') as shown below.

(i) $\displaystyle \lim_{n\to\infty} \sup_{p\in E} |f_n(p)| = \sup_{p\in E} \Big| \lim_{n\to\infty} f_n(p) \Big|$.

(ii) For $p \in E$ and $p_k \to p$ in E, $\displaystyle \lim_{k\to\infty} \lim_{n\to\infty} f_n(p_k) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(p_k)$.

(iii) $\displaystyle \lim_{n\to\infty} \frac{d}{dx}(f_n) = \frac{d}{dx}\Big(\lim_{n\to\infty} f_n \Big)$.

(iv) $\displaystyle \lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \Big(\lim_{n\to\infty} f_n(x) \Big) dx$.

Answers to these questions are all negative.

Examples 1.1. (i) Let $E := (0, 1]$ and define $f_n : E \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 0 & \text{if } 0 < x \le 1/n, \\ 1/x & \text{if } 1/n \le x \le 1. \end{cases}$$

Then $|f_n(x)| \le n$ for all $x \in E$, $f_n \to f$ on E, where $f(x) := 1/x$. Thus each f_n is bounded on E, but f is not bounded on E.

(ii) Let $E := [0, 1]$ and define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/(nx+1)$. Then each f_n is continuous on E, $f_n \to f$ on E, where $f(0) := 1$ and $f(x) := 0$ if $0 < x \le 1$. Clearly, f is not continuous on E.

(iii) (a) Let $E := (-1, 1)$ and define $f_n : E \to \mathbb{R}$ by $f_n(x) := 1/(nx^2 + 1)$. Then each f_n is differentiable on E and $f_n \to f$ on $(-1, 1)$, where $f(0) := 1$ and $f(x) := 0$ if $0 < |x| < 1$. Clearly, f is not differentiable on E.

(b) Let $E := \mathbb{R}$ and define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) := (\sin nx)/n$. Then each f_n is differentiable, $f_n \to f$ on $\mathbb{R}$, where $f \equiv 0$. But $f_n'(x) = \cos nx$ for $x \in \mathbb{R}$, and (f_n') does not converge pointwise on $\mathbb{R}$. For example, $(f_n'(\pi))$ is not a convergent sequence.

(c) Let $E := (-1, 1)$ and define

$$f_n(x) := \begin{cases} [2 - (1+x)^n]/n & \text{if} \quad -1 < x < 0, \\ (1-x)^n/n & \text{if} \quad 0 \le x < 1. \end{cases}$$

Then each f_n is differentiable on E. (In particular, we have $f_n'(0) = -1$ by L'Hôpital's Rule.) Also, $f_n \to f$ on $(-1, 1)$, where $f \equiv 0$. Also,

$$f_n'(x) = \begin{cases} -(1+x)^{n-1} & \text{if} \quad -1 < x < 0, \\ -(1-x)^{n-1} & \text{if} \quad 0 \le x < 1. \end{cases}$$

Further, $f_n' \to g$ on $(-1, 1)$, where $g(0) := -1$ and $g(x) := 0$ for $0 < |x| < 1$. Clearly, $f' \ne g$.

(iv) (a) Let $E := [0, 1]$ and define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 1 & \text{if } x = 0, 1/n!, 2/n!, \ldots, n!/n! = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then each f_n is Riemann integrable on $[0, 1]$ since it is discontinuous only at a finite number of points.

Also, $f_n \to f$ on $[0, 1]$, where $f(x) := \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$

For if $x = p/q$ with $p \in \{0, 1, 2, \ldots, q\} \subset \mathbb{N}$, then for all $n \ge q$, we have $n!x \in \{0, 1, 2, \ldots\}$ and so $f_n(x) = 1$, while if x is an irrational number, then $f_n(x) = 0$ for all $n \in \mathbb{N}$. We have seen that the Dirichlet function f is not Riemann integrable.

(b) Let $E := [0, 1]$, and define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) := n^3 x e^{-nx}$. Then each f_n is Riemann integrable and $f_n \to f$ on $[0, 1]$, where $f \equiv 0$. (Use L'Hôpital's Rule repeatedly to show that $\lim_{t\to\infty} t^3/e^{st} = 0$ for any $s \in \mathbb{R}$ with $s > 0$.) However, using Integration by Parts, we have

$$\int_0^1 x e^{-nx} dx = \frac{1}{n^2} - \frac{1}{n^2 e^n} - \frac{1}{n e^n} \quad \text{for each } n \in \mathbb{N},$$

so that $\int_0^1 f_n(x) dx = n - (n/e^n) - (n^2/e^n) \to \infty$.

(c) Let $E := [0, 1]$ and define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) := n^2 x e^{-nx}$. As above, each f_n is Riemann integrable, and $f_n \to f$ on $[0, 1]$, where $f \equiv 0$, and $\int_0^1 f_n(x) dx = 1 - (1/e^n) - (n/e^n) \to 1$, which is not equal to $\int_0^1 f(x) dx = 0$.

2. Uniform Convergence of a Sequence

In an attempt to obtain affirmative answers to the questions posed at the beginning of the previous section, we introduce a stronger concept of convergence.

Let E be a set and consider functions $f_n : E \to \mathbb{R}$ for $n = 1, 2, \ldots$. We say that the sequence (f_n) of functions **converges uniformly** on E if there is a function $f : E \to \mathbb{R}$ such that for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ satisfying

$$n \geq n_0,\ p \in E \Longrightarrow |f_n(p) - f(p)| < \epsilon.$$

Note that the natural number n_0 mentioned in the above definition may depend upon the given sequence (f_n) of functions and on the given positive number ϵ, but it is independent of $p \in E$. Clearly, such a function f is unique and it is called the **uniform limit** of (f_n) on E. We then write $f_n \rightrightarrows f$ on E. Obviously, $f_n \rightrightarrows f$ on $E \Longrightarrow f_n \to f$ on E, but the converse is not true : Let $E := (0, 1]$ and define $f_n(x) := 1/(nx+1)$ for $0 < x \leq 1$. If $f(x) := 0$ for $x \in (0, 1]$, then $f_n \to f$ on $(0, 1]$, but $f_n \not\rightrightarrows f$ on $(0, 1]$. To see this, let $\epsilon := 1/2$, note that there is no $n_0 \in \mathbb{N}$ satisfying

$$|f_n(x) - f(x)| = \frac{1}{nx+1} < \frac{1}{2} \text{ for all } n \geq n_0 \text{ and for all } x \in (0, 1],$$

since $1/(nx + 1) = 1/2$ when $x = 1/n, n \in \mathbb{N}$.

A sequence (f_n) of real-valued functions defined on a set E is said to be **uniformly Cauchy** on E if for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ satisfying

$$m, n \geq n_0,\ p \in E \Longrightarrow |f_m(p) - f_n(p)| < \epsilon.$$

Proposition 2.1. (Cauchy Criterion for Uniform Convergence of a Sequence) *Let (f_n) be a sequence of real-valued functions defined on a set E. Then (f_n) is uniformly convergent on E if and only if (f_n) is uniformly Cauchy on E.*

Proof. $\Longrightarrow$) Let $f_n \rightrightarrows f$. For all $m, n \in \mathbb{N}$ and $p \in E$, we have

$$|f_m(p) - f_n(p)| \leq |f_m(p) - f(p)| + |f(p) - f_n(p)|.$$

$\Longleftarrow$) For each $p \in E$, $(f_n(p))$ is a Cauchy sequence in $\mathbb{R}$, and so it converges to a real number which we denote by $f(p)$. Let $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ satisfying

$$m, n \geq n_0,\ p \in E \Longrightarrow |f_m(p) - f_n(p)| < \epsilon.$$

For any $m \geq n_0$ and $p \in E$, letting $n \to \infty$, we have $|f_m(p) - f(p)| \leq \epsilon$. $\square$

We have the following useful test for checking the uniform convergence of (f_n) when its pointwise limit is known.

Proposition 2.2. (Test for Uniform Convergence of a Sequence) *Let f_n and f be real-valued functions defined on a set E. If $f_n \to f$ on E, and if there is a sequence (a_n) of real numbers such that $a_n \to 0$ and $|f_n(p) - f(p)| \leq a_n$ for all $p \in E$, then $f_n \rightrightarrows f$ on E.*

Proof. Let $\epsilon > 0$. Since $a_n \to 0$, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Longrightarrow a_n < \epsilon$, and so $|f_n(p) - f(p)| < \epsilon$ for all $p \in E$. $\square$

Example 2.3. Let $0 < r < 1$ and $f_n(x) := x^n$ for $x \in [-r, r]$. Then$f_n(x) \to 0$ for each $x \in [-r, r]$. Since $r^n \to 0$ and $|f_n(x) - 0| \leq r^n$ for all $x \in [-r, r]$, (f_n) is uniformly convergent on $[-r, r]$.

Let us now pose the four questions stated in the last section with 'convergence' replaced by 'uniform convergence'. We shall answer them one by one, but not necessarily in the same order.

Uniform Convergence and Boundedness.

Proposition 2.4. *Let f_n and f be real-valued functions defined on a set E. If $f_n \rightrightarrows f$ on E and each f_n is bounded on E, then f bounded on E.*

Proof. There is $n_0 \in \mathbb{N}$ such that $n \geq n_0$, $p \in E \Longrightarrow |f_n(p) - f(p)| < 1$. Also, since f_{n_0} is bounded on E, there is α_0 such that $p \in E \Longrightarrow |f_{n_0}(p)| \leq \alpha_0$. Hence $p \in E \Longrightarrow |f(p)| \leq |f(p) - f_{n_0}(p)| + |f_{n_0}(p)| < 1 + \alpha_0$. $\square$

The converse of the above result is not true, that is, each f_n as well as f bounded on E and $f_n \to f \not\Longrightarrow f_n \rightrightarrows f$ on E. For example, let $E := (0, 1], f_n(x) := 1/(nx + 1)$ and $f \equiv 0$.

Given a set E, let $B(E)$ denote the set of all real-valued bounded functions defined on E. For f, g in $B(E)$, define $d(f, g) := \sup\{|f(p) - g(p)| : p \in E\}$. Then it is easy to see that d is a metric on $B(E)$, known as the **sup-metric** on $B(E)$. Also, by Proposition 2.2, for f_n and f in $B(E)$, we have $f_n \rightrightarrows f$ on E if and only if $d(f_n, f) \to 0$, that is, (f_n) converges to f in the sup-metric on $B(E)$. Similarly, (f_n) is uniformly Cauchy on E if and only if $d(f_n, f_m) \to 0$ as $n, m \to \infty$, that is, (f_n) is a Cauchy sequence in the sup-metric on $B(E)$. Thus Propositions 2.1 and 2.4 show that $B(E)$ is a complete metric space. Also, under the hypotheses of Proposition 2.4, we have

$$\left| \sup_{p \in E} |f_n(p)| - \sup_{p \in E} |f(p)| \right| \leq d(f_n, f) \to 0,$$

and so, $\sup_{p \in E} |f_n(p)| \to \sup_{p \in E} |f(p)|$.

Uniform Convergence and Integration.

Proposition 2.5. *Let (f_n) be a sequence of real-valued functions defined on $[a, b]$. If $f_n \rightrightarrows f$ on $[a, b]$ and each f_n is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$ and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.*

Proof. Since $f_n \rightrightarrows f$ and each f_n is bounded, we see that f is bounded on $[a, b]$ by Proposition 2.4. For $n \in \mathbb{N}$, let $\alpha_n := d(f_n, f)$, where d denotes the sup-metric on $B([a, b])$. For each $n \in \mathbb{N}$ and $x \in [a, b]$, we have $|f_n(x) - f(x)| \leq \alpha_n$, that is, $f_n(x) - \alpha_n \leq f(x) \leq f_n(x) + \alpha_n$, and so

$$L(f_n) - \alpha_n(b - a) \leq L(f) \leq U(f) \leq U(f_n) + \alpha_n(b - a).$$

But since f_n is Riemann integrable, we have $L(f_n) = U(f_n)$, and hence $0 \leq U(f) - L(f) \leq 2\alpha_n(b - a) \to 0$ as $n \to \infty$. Thus $L(f) = U(f)$, that is, f is Riemann integrable on $[a, b]$. Also,

$$\int_a^b f_n(x)dx - \alpha_n(b - a) \leq \int_a^b f(x)dx \leq \int_a^b f_n(x)dx + \alpha_n(b - a),$$

that is, $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| \leq \alpha_n(b - a) \to 0$ as $n \to \infty$. $\square$

The converse of the above result is not true, that is, each f_n as well as f Riemann integrable on $[a, b]$, $f_n \to f$ on $[a, b]$ and $\int_a^b f_n(x)dx \to \int_a^b f(x)dx \not\Longrightarrow f_n \rightrightarrows f$. For example, if $f_n(x) := 1/(nx + 1)$ for $x \in [0, 1]$,

$f(0) := 1$, $f(x) := 0$ for $x \in (0,1]$, then $f_n \not\rightrightarrows f$ on $[0,1]$, but f is integrable and

$$\int_0^1 f_n(x)dx = \frac{\ln(nx+1)}{n}\Big|_0^1 = \frac{\ln(1+n)}{n} \to 0 = \int_0^1 f(x)dx.$$

Uniform Convergence and Continuity.

Proposition 2.6. *Let (f_n) be a sequence of real-valued functions defined on a metric space E. If $f_n \rightrightarrows f$ on E and each f_n is continuous on E, then f is continuous on E.*

Proof. Let $\epsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $p \in E \Longrightarrow |f_{n_0}(p) - f(p)| < \epsilon/3$. Consider $p_0 \in E$. Since f_{n_0} is continuous at p_0, there is $\delta > 0$ such that $p \in E$, $d(p, p_0) < \delta \Longrightarrow |f_{n_0}(p) - f_{n_0}(p_0)| < \epsilon/3$. and hence

$$|f(p) - f(p_0)| \le |f(p) - f_{n_0}(p)| + |f_{n_0}(p) - f_{n_0}(p_0)| + |f_{n_0}(p_0) - f(p_0)| < \epsilon,$$

establishing the continuity of f at $p_0 \in E$. □

The converse of the above result is not true, that is, each f_n as well as f continuous on a metric space E, $f_n \to f$ on $E \not\Longrightarrow f_n \rightrightarrows f$. For example, let $f_n(x) := nxe^{-nx}$ and $f(x) := 0$ for $x \in [0,1]$. Since $f_n(0) = 0$ and for $x \in (0,1]$, $f_n(x) \to 0$ as $n \to \infty$ by L'Hôpital's Rule, we see that $f_n \to f$. But there is no $n_0 \in \mathbb{N}$ such that $n \ge n_0$, $x \in [0,1] \Longrightarrow |nxe^{-nx} - 0| < 1$, since $|nxe^{-nx}| = e^{-1}$ for $x = 1/n$, $n \in \mathbb{N}$. However, the following partial converse holds.

Proposition 2.7. (Dini's Theorem) *Let (f_n) be a sequence of real-valued functions defined on a compact metric space E. If $f_n \to f$ on E, each f_n and f are continuous on E, and (f_n) is a monotonic sequence (that is, $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$, or $f_n \ge f_{n+1}$ for all $n \in \mathbb{N}$), then $f_n \rightrightarrows f$ on E.*

For a proof, see Theorem 7.13 of [3].

The following examples show that neither the compactness of the metric space E nor the continuity of the function f can be dropped from Dini's Theorem: (i) $E := (0,1]$ and $f_n(x) := 1/(nx+1)$, $x \in E$, (ii) $E := [0,1]$ and $f_n(x) := x^n$, $x \in E$.

Uniform Convergence and Differentiation. Answers to the questions regarding differentiation posed in the last section are not affirmative even when $f_n \rightrightarrows f$ on an interval of $\mathbb{R}$.

(a) Let $f_n(x) := \sqrt{x^2 + (1/n^2)}$ and $f(x) := |x|$ for $x \in [-1,1]$. Since

$$|f_n(x) - f(x)| = \left|\sqrt{x^2 + (1/n^2)} - \sqrt{x^2}\right| \le \left|\sqrt{x^2 + (1/n^2) - x^2}\right| = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in [-1,1]$, Proposition 2.2 shows that $f_n \rightrightarrows f$ on $[-1,1]$. Although each f_n is differentiable on $[-1,1]$, the limit function f is not.

(b) In Example 1.1 (iii) (b), $f_n \rightrightarrows f$ on $\mathbb{R}$, each f_n differentiable, but (f_n') does not converge pointwise.

(c) In Example 1.1 (iii) (c), $f_n \rightrightarrows f$ on $(-1,1)$ and each f_n as well as f is differentiable on $(-1,1)$, and $f_n' \to g$ on $(-1,1)$, where $g \ne f'$.

However, if we assume the uniform convergence of the 'derived sequence' (f_n') along with the convergence of the sequence (f_n) at only one point of the interval, we have a satisfactory answer.

Proposition 2.8. *Let (f_n) be a sequence of real-valued functions defined on $[a,b]$. If (f_n) converges at one point of $[a,b]$, each f_n is continuously differentiable on $[a,b]$ and (f_n') converges uniformly on $[a,b]$, then there is $f:[a,b]\to\mathbb{R}$ such that f is continuously differentiable on $[a,b]$, $f_n'\to f'$ on $[a,b]$ and in fact, $f_n \rightrightarrows f$ on $[a,b]$.*

Proof. Let $x_0\in[a,b]$ and $c_0\in\mathbb{R}$ be such that $f_n(x_0)\to c_0$. Also, let each f_n be continuously differentiable and $g:[a,b]\to\mathbb{R}$ be such that $f_n'\rightrightarrows g$ on $[a,b]$. By Proposition 2.6, the function g is continuous on $[a,b]$. Define $f:[a,b]\to\mathbb{R}$ by

$$f(x):=c_0+\int_{x_0}^{x} g(t)dt \quad \text{for } x\in[a,b].$$

By part (ii) of the Fundamental Theorem of Calculus (FTC), f' exists on $[a,b]$ and $f'(x)=g(x)$ for $x\in[a,b]$. Thus f is continuously differentiable on $[a,b]$ and $f_n'\to g=f'$. Also, by part (i) of the FTC, we have

$$f_n(x)=f_n(x_0)+\int_{x_0}^{x} f_n'(t)dt \qquad \text{for } x\in[a,b].$$

Hence for $n\in\mathbb{N}$ and $x\in[a,b]$, we obtain

$$\begin{aligned}
|f_n(x)-f(x)| &\le |f_n(x_0)-c_0|+\Big|\int_{x_0}^{x}\big(f_n'(t)-g(t)\big)dt\Big| \\
&\le |f_n(x_0)-c_0|+|x-x_0|\sup_{t\in[a,b]}|f_n'(t)-g(t)| \\
&\le |f_n(x_0)-c_0|+(b-a)d(f_n',g).
\end{aligned}$$

Thus $f_n\rightrightarrows f$ on $[a,b]$ by Proposition 2.2. □

The converse of the above result is not true, that is, each f_n as well as f continuously differentiable on $[a,b]$, $f_n\rightrightarrows f$ on $[a,b]$, $f_n'\to f'$ on $[a,b]$ $\not\Longrightarrow$ $f_n'\rightrightarrows f'$. For example, let $f_n(x):=(nx+1)e^{-nx}/n$ and $f(x):=0$ for $x\in[0,1]$. Since $f_n'(x)=-nxe^{-nx}$ for each $n\in\mathbb{N}$ and all $x\in[0,1]$, each f_n is monotonically decreasing on $[0,1]$. As $f_n(0)=1/n$, we obtain $|f_n(x)-f(x)|\le 1/n$ for all $x\in[0,1]$, and so $f_n\rightrightarrows f$ on $[0,1]$. Also, we have seen after the proof of Proposition 2.6 that $f_n'\to f'$, but $f_n'\not\rightrightarrows f'$ on $[0,1]$.

Remark 2.9. Proposition 2.8 holds if we drop the word 'continuously' appearing (two times) in its statement, but then the proof is much more involved. See Theorem 7.17 of [3].

The results in Propositions 2.4, 2.5, 2.6 and 2.8 are summarized in the following theorem.

Theorem 2.10. (i) *The uniform limit of a sequence of real-valued bounded functions defined on a set is bounded.*

(ii) *The uniform limit of a sequence of Riemann integrable functions defined on $[a,b]$ is Riemann integrable, and its Riemann integral is the limit of the sequence of termwise Riemann integrals, that is, if (f_n) is uniformly convergent to f on $[a,b]$ and each f_n is Riemann integrable on $[a,b]$, then the function f is Riemann integrable on $[a,b]$ and $\int_a^b f(x)dx=\lim_{n\to\infty}\int_a^b f_n(x)dx$.*

(iii) *The uniform limit of a sequence of continuous functions defined on a metric space is continuous.*

(vi) *If a sequence of continuously differentiable functions defined on* $[a, b]$ *is convergent at one point of* $[a, b]$ *and if the 'derived' sequence is uniformly convergent on* $[a, b]$*, then the given sequence converges uniformly on* $[a, b]$*, the uniform limit is continuously differentiable on* $[a, b]$ *and its derivative is the limit of the sequence of termwise derivatives, that is, if* (f_n) *converges at one point of* $[a, b]$*, each* f_n *is continuously differentiable on* $[a, b]$ *and* (f_n') *is uniformly convergent on* $[a, b]$*, then* (f_n) *converges uniformly to a continuously differentiable function* f *on* $[a, b]$*, and* $f'(x) = \lim_{n\to\infty} f_n'(x)$ *for all* $x \in [a, b]$.

3. Uniform Convergence of a series

The reader is assumed to be familiar with the elementary theory of series of real numbers. (See, for example, Chapter 9 of [1], or Chapter 3 of [3].)

Let (f_k) be a sequence of real-valued functions defined on a set E. Consider the sequence (s_n) of real-valued functions on E defined by

$$s_n := f_1 + \cdots + f_n = \sum_{k=1}^{n} f_k.$$

Note: Just as the sequence (f_k) determines the sequence (s_n), so does (s_n) determine (f_k): If we let $s_0 = 0$, then we have $f_k = s_k - s_{k-1}$ for all $k \in \mathbb{N}$.

We say that the series $\sum_{k=1}^{\infty} f_k$ **converges pointwise** on E if the sequence (s_n) converges pointwise on E, and we say that the series $\sum_{k=1}^{\infty} f_k$ **converges uniformly** on E if the sequence (s_n) converges uniformly on E. For $n \in \mathbb{N}$, the function s_n is called the nth **partial sum** of the series $\sum_{k=1}^{\infty} f_k$ and if $s_n \to s$, then the function s is called its **sum**.

Results about convergence / uniform convergence of sequences of functions carry over to corresponding results about convergence / uniform convergence of series of functions.

Proposition 3.1. (Cauchy Criterion for Uniform Convergence of a Series) *Let* (f_k) *be a sequence of real-valued functions defined on a set* E*. Then the series* $\sum_{k=1}^{\infty} f_k$ *converges uniformly on* E *if and only if for every* $\epsilon > 0$*, there is* $n_0 \in \mathbb{N}$ *such that*

$$m \geq n \geq n_0,\, p \in E \Longrightarrow \Big| \sum_{k=n}^{m} f_k(p) \Big| < \epsilon.$$

Proof. Use Proposition 2.1 for the sequence (s_n) of partial sums. □

Proposition 3.2. (Weierstrass M-Test for Uniform Absolute Convergence of a Series) *Let* (f_k) *be a sequence of real-valued functions defined on a set* E*. Suppose there is a sequence* (M_k) *in* $\mathbb{R}$ *such that* $|f_k(p)| \leq M_k$ *for all* $k \in \mathbb{N}$ *and all* $p \in E$*. If* $\sum_{k=1}^{\infty} M_k$ *is convergent, then* $\sum_{k=1}^{\infty} f_k$ *converges uniformly and absolutely on* E*.*

Proof. Note that

$$\Big|\sum_{k=n}^{m} f_k(p)\Big| \le \sum_{k=n}^{m} |f_k(p)| \le \sum_{k=n}^{m} M_k \quad \text{for all } m \ge n,$$

and use Proposition 3.1. □

Examples 3.3. (i) Consider the series $\sum_{k=0}^{\infty} x^k$, where $x \in (-1, 1)$. If $r < 1$, then the series converges uniformly on $\{x \in \mathbb{R} : |x| \le r\}$ since the series $\sum_{k=0}^{\infty} M_k$ is convergent, where $M_k := r^k$ for $k = 0, 1, \ldots$

(ii) For $k \in \mathbb{N}$, let $f_k(x) := (-1)^k(x+k)/k^2$, where $x \in [0, 1]$. We show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[0, 1]$. For $k \in \mathbb{N}$, let $g_k(x) := (-1)^k x/k^2$, where $x \in [0, 1]$. Letting $M_k := 1/k^2$ for $k \in \mathbb{N}$, we observe that the series $\sum_{k=0}^{\infty} M_k$ is convergent. Hence the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on $[0, 1]$. Also, the series $\sum_{k=1}^{\infty}(-1)^k/k$ converges uniformly on $[0, 1]$, being a convergent series of constants. Since $f_k(x) = g_k(x) + (-1)^k/k$ for $k \in \mathbb{N}$ and $x \in [0, 1]$, the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[0, 1]$.

This example also shows that the converse of Weierstrass' M-test does not hold: If $M_k := \sup_{x\in[0,1]} |f_k(x)| = (1+k)/k^2$ for $k \in \mathbb{N}$, then $\sum_{k=1}^{\infty} M_k$ does not converge, since $\sum_{k=1}^{\infty} 1/k^2$ converges, but $\sum_{k=1}^{\infty} 1/k$ diverges.

Proposition 3.4. (Dirichlet's Test for Uniform Conditional Convergence of a Series) *Let (f_k) be a monotonic sequence of real-valued functions defined on a set E such that $f_k \rightrightarrows 0$ on E. If (g_k) is a sequence of real-valued functions defined on E such that the partial sums of the series $\sum_{k=1}^{\infty} g_k$ are uniformly bounded on E, then the series $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E. In particular, the series $\sum_{k=1}^{\infty}(-1)^k f_k$ converges uniformly on E.*

Proof. For each $p \in E$, the series $\sum_{k=1}^{\infty} f_k(p)g_k(p)$ converges in $\mathbb{R}$ by Dirichlet's Test for conditional convergence of a series of real numbers. (See, for example, Proposition 9.20 of [1], or Theorem 3.42 of [3].) For $p \in E$, let $H(p) := \sum_{k=1}^{\infty} f_k(p)g_k(p)$. Also, for $n \in \mathbb{N}$, let $G_n := \sum_{k=1}^{n} g_k$ and $H_n := \sum_{k=1}^{n} f_k g_k$. Further, let $\beta \in \mathbb{R}$ be such that $|G_n(p)| \le \beta$ for all $n \in \mathbb{N}$ and all $p \in E$. Then by using the partial summation formula

$$\sum_{k=1}^{n} f_k g_k = \sum_{k=1}^{n-1}(f_k - f_{k+1})G_k + f_n G_n \quad \text{for all } n \ge 2,$$

we have $|H(p) - H_n(p)| \le 2\beta|f_{n+1}(p)|$ for all $p \in E$. Since $f_{n+1} \rightrightarrows 0$ on E, it follows that $H_n \rightrightarrows H$ on E, that is, the series $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

In particular, letting $g_k(p) := (-1)^k$ for all $k \in \mathbb{N}$ and $p \in E$, and noting that $|G_n(p)| \le 1$ for all $n \in \mathbb{N}$ and all $p \in E$, we obtain the uniform convergence of the series $\sum_{k=1}^{\infty}(-1)^k f_k$ on E. □

Example 3.5. Let $E := [0, 1]$ and $f_k(x) := x^k/k$ for $k \in \mathbb{N}$ and $x \in [0, 1]$. Then (f_k) is a momotonically decreasing sequence and since $|f_k(x)| \le 1/k$ for $k \in \mathbb{N}$ and $x \in [0, 1]$, we see that $f_k \rightrightarrows 0$ on $[0, 1]$ by Proposition 2.2. Hence the series $\sum_{k=1}^{\infty}(-1)^k x^k/k$ converges uniformly on $[0, 1]$.

Results regarding the boundedness, Riemann integrability, continuity and differentiability of the sum function of a convergent series of functions can be easily deduced from the corresponding results for the sequence of its partial sums.

Theorem 3.6. (i) *The sum function of a uniformly convergent series of real-valued bounded functions defined on a set is bounded.*

(ii) *The sum function of a uniformly convergent series of Riemann integrable functions defined on $[a, b]$ is Riemann integrable, and the series can be integrated term by term, that is, if $\sum_{k=1}^{\infty} f_k$ is uniformly convergent on $[a, b]$ and each f_k is Riemann integrable on $[a, b]$, then the function $\sum_{k=1}^{\infty} f_k$ is Riemann integrable on $[a, b]$ and*

$$\int_a^b \Big(\sum_{k=1}^{\infty} f_k(x)\Big)dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)dx.$$

(iii) *The sum function of a uniformly convergent series of real-valued continuous functions defined on a metric space is continuous.*

(vi) *If a series of continuously differentiable functions defined on $[a, b]$ is convergent at one point of $[a, b]$ and if the 'derived' series is uniformly convergent on $[a, b]$, then the given series converges uniformly on $[a, b]$, the sum function is continuously differentiable on $[a, b]$ and the series can be differentiated term by term, that is, if $\sum_{k=1}^{\infty} f_k$ converges at one point of $[a, b]$, each f_k is continuously differentiable on $[a, b]$ and $\sum_{k=1}^{\infty} f_k'$ is uniformly convergent on $[a, b]$, then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuously differentiable function, and*

$$\Big(\sum_{k=1}^{\infty} f_k(x)\Big)' = \sum_{k=1}^{\infty} f_k'(x) \text{ for all } x \in [a, b].$$

Proof. The results follow by applying Theorem 2.10 to the sequence of partial sums of the given series. □

4. Two Celebrated Theorems on Uniform Approximation

We have seen in Proposition 2.6 that a uniform limit of a sequence of continuous functions on a metric space is continuous. In this section, we reverse the procedure and ask whether every continuous function on a closed and bounded interval of $\mathbb{R}$ is the uniform limit of a sequence of some 'special' continuous functions.

For a function $f : [0, 1] \to \mathbb{R}$ and $n \in \mathbb{N}$, we define the nth **Bernstein polynomial** of f by

$$B_n(f) := \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big)x^k(1-x)^{n-k}.$$

Theorem 4.1. (Polynomial Approximation Theorem of Weierstrass) *If $f : [0, 1] \to \mathbb{R}$ is continuous, then $B_n(f) \rightrightarrows f$ on $[0, 1]$. Consequently, every real-valued continuous function on $[0, 1]$ is the uniform limit of a sequence of real-valued polynomial functions.*

For a proof, see Theorem 7.26 of [3], or Corollary 3.12 of [2].

Remark 4.2. Theorem 4.1 can be used to prove that every real-valued continuous function on any closed and bounded interval $[a, b]$ is the uniform limit of a sequence of real-valued polynomial functions. Let $\phi : [0, 1] \to [a, b]$ be defined by $\phi(x) := (1 - x)a + xb$ for $x \in [0, 1]$. Then ϕ is a bijective continuous function and its continuous inverse $\phi^{-1} : [a, b] \to [0, 1]$ is given by $\phi^{-1}(t) = (t - a)/(b - a)$ for $t \in [a, b]$. Given a continuous real-valued function g on $[a, b]$, consider the continuous function $f := g \circ \phi$ defined on $[0, 1]$. If (P_n) is a sequence of polynomial functions such that $P_n \rightrightarrows f$ on $[0, 1]$, and if we let $Q_n := P_n \circ \phi^{-1}$, then since $Q_n(t) = P_n\big((t - a)/(b - a)\big)$ for $t \in [a, b]$, each Q_n is a polynomial function, and $Q_n \rightrightarrows f \circ \phi^{-1} = g$ on $[a, b]$.

Instead of polynomials, let us now consider trigonometric polynomials for approximating a function. They are given by

$$a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \quad \text{for } n \in \mathbb{N},$$

where $a_0, a_1, a_2, \dots, b_1, b_2, \dots$ are real numbers. For a Riemann integrable function f on $[-\pi, \pi]$, we define the **Fourier coefficients** of f by

$$a_0(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt, \quad \text{and for } k \in \mathbb{N},$$

$$a_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\, dt, \quad b_k(f) := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\, dt.$$

The series $a_0(f) + \sum_{k=1}^{\infty}\big(a_k(f)\cos kx + b_k(f)\sin kx\big)$ of functions defined on $[-\pi, \pi]$ is called the **Fourier series** of the function f. For $n = 0, 1, 2, \dots$, let $s_n(f)$ denote the nth partial sum of this series, and consider the **arithmetic means** of these partial sums given by

$$\sigma_n(f) := \frac{s_0(f) + s_1(f) \cdots + s_n(f)}{n + 1} \quad \text{for } n = 0, 1, 2 \dots$$

Theorem 4.3. (Trigonometric Polynomial Approximation Theorem of Fejér) *If $f : [-\pi, \pi] \to \mathbb{R}$ is continuous and $f(-\pi) = f(\pi)$, then $\sigma_n(f) \rightrightarrows f$ on $[-\pi, \pi]$. Consequently, every real-valued continuous function on $[-\pi, \pi]$ having the same value at $-\pi$ and π is the uniform limit of a sequence of real-valued trigonometric polynomial functions.*

For a proof, see Theorem 8.15 and Exercise 8.15 of [3], or Theorem 3.13 of [2].

References

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[2] B. V. Limaye, *Functional Analysis*, New Age International, 2nd Ed., New Delhi, 1996.

[3] W. Rudin, *Principles of Mathematical Analysis*, 3rd Ed., McGraw Hill, New Delhi, 1976.