



BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Calculus II (MAT211)

Limit and Continuity

Limit, Continuity, Algebra of Limits

FEBRUARY 22, 2026

CONDUCTED BY

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Definition of Limit

Limit

Let $U \subset \mathbb{R}^n$ be an open subset containing a neighborhood of $a \in \mathbb{R}^n$, except perhaps for the point a itself. Suppose $f : U \rightarrow \mathbb{R}^m$. We say that

$$\lim_{x \rightarrow a} f(x) = \ell$$

($f(x)$ approaches $\ell \in \mathbb{R}^m$ as x approaches a) if for every $\varepsilon > 0$ there is $\delta > 0$ so that

$$\|f(x) - \ell\| < \varepsilon \quad \text{whenever} \quad 0 < \|x - a\| < \delta.$$

Component wise Limit

💡 Proposition

$$\lim_{x \rightarrow a} f(x) = \ell$$

if and only if

$$\lim_{x \rightarrow a} f_j(x) = \ell_j$$

for all $j = 1, \dots, m$.



Examples of Limits

❓ Problem

Fix a non-zero vector $b \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $f(x) = b \cdot x$. Show that

$$\lim_{x \rightarrow a} f(x) = b \cdot a$$

Examples of Limits

② Problem

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined by $f(x) = \|x\|^2$. Show that

$$\lim_{x \rightarrow a} f(x) = \|a\|^2$$

Examples of Limits

② Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{x^2 y}{x^2 + y^2}$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

Examples of Limits

② Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{x^2}{x^4 + y^2}$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

Examples of Limits

② Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{xy}{x^4 + y^2}$. Does $\lim_{x \rightarrow 0} f(x)$ exist?

Algebra of Limits

Theorem

Suppose f and g map a neighborhood of $a \in \mathbb{R}^n$ to \mathbb{R}^m and k maps the same neighborhood to \mathbb{R} . Suppose $\lim_{x \rightarrow a} f(x) = \ell$, $\lim_{x \rightarrow a} g(x) = m$ and $\lim_{x \rightarrow a} k(x) = c$, then

- $\lim_{x \rightarrow a} (f(x) + g(x)) = \ell + m$,
- $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \ell \cdot m$,
- $\lim_{x \rightarrow a} (k(x)f(x)) = c\ell$.

Continuity

Definition

Let $U \subset \mathbb{R}^n$ be an open set containing a neighborhood of $a \in \mathbb{R}^n$, and let $f : U \rightarrow \mathbb{R}^m$. We say that f is continuous at a if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

That is, f is continuous at a if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(x) - f(a)\| < \varepsilon \quad \text{whenever} \quad \|x - a\| < \delta.$$

We say that f is continuous if it is continuous at every point in its domain.

Continuity and Open Sets

Proposition

Let $U \subset \mathbb{R}^n$ be an open set. The function $f : U \rightarrow \mathbb{R}^m$ is continuous at $a \in U$ if and only if for every open set $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is open (the preimage of an open set is open).

Composition of Functions through open sets


Proposition

Suppose $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^p$ are open, functions $f \circ g$ is defined (i.e., $g(x) \in U$ for all $x \in W$). $f : U \rightarrow \mathbb{R}^m$, $g : W \rightarrow \mathbb{R}^n$, and the composition of f and g . Then if f and g are continuous, so is $f \circ g$.

Now, Consider the first example.


Continuity and Sequences

Proposition

Suppose $U \subset \mathbb{R}^n$ is open and $f : U \rightarrow \mathbb{R}^m$. Then f is continuous at a if and only if for every sequence $\{x_k\}$ of points in U converging to a the sequence $\{f(x_k)\}$ converges to $f(a)$. 

Continuity and Level Sets

Proposition

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous. Then for any $c \in \mathbb{R}^m$, the level set 

$$f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$$

is a closed set.

Thank You!

We'd love your questions and feedback.

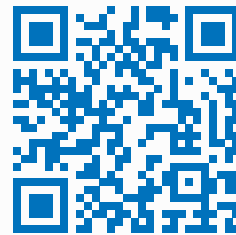
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References

- [1] THEODORE SHIFRIN, *MULTIVARIABLE MATHEMATICS: Linear Algebra, Multivariable Calculus, and Manifolds.*