MAT211: Calculus-II

Lecture Notes

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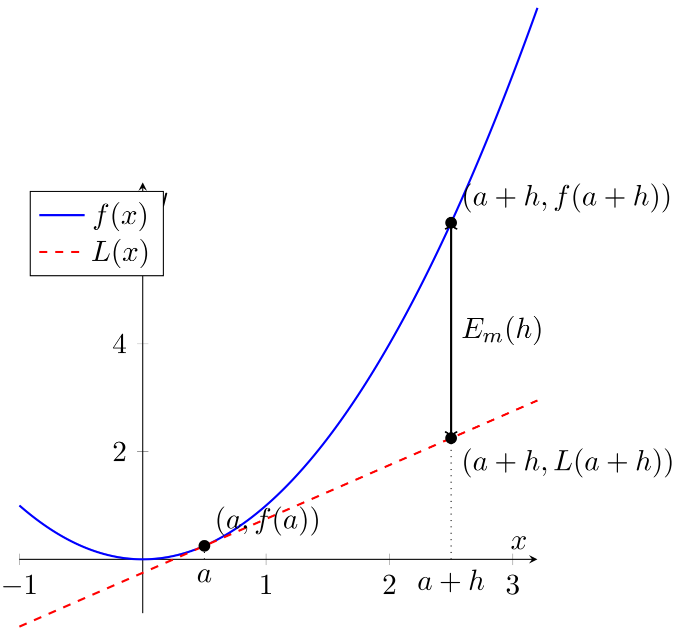
1 Lecture-01

Question: what did we mean by the derivative $f'(a)$?
Answer: Let U be an open subset of $\mathbb{R}$ and $f : U \to \mathbb{R}$ a function. Then f is differentiable at a, with derivative m, if and only if

$$\lim_{h \to 0} \frac{1}{h}\left(\underbrace{f(a+h) - f(a)}_{\Delta f} - \underbrace{mh}_{(*)}\right) = 0 \tag{1}$$

$(*)$: linear function of $\Delta x = h$.
Question: Why impose the domain U that needs to be an open set?
Answer: Write the answer.
The function mh that multiplies h by the derivative m is thus a linear function of the change in x. This definition emphasizes the idea that a function f is differentiable at a point a if the increment Δf to the function is well approximated by a linear function of the increment h to the variable, this linear function is $f'(a)h$. Let's see how we can confirm that:
Suppose $f(x)$ is our desired function and we want to approximate the function in the neighborhood at $x = a$. For this purpose, we will draw several straight lines through the point $(a, f(a))$ and see which line approximates the function best. Here, the best means, the linear line should approximate the function well at $x = a + h$. Suppose the slope of that linear function is m.



Then,

$$\frac{L(a+h) - L(a)}{a + h - a} = m$$
$$\frac{L(a+h) - f(a)}{h} = m$$
$$L(a+h) = f(a) + mh$$

Now, we want to minimize the error, $E_m(h) = f(a+h) - L(a+h)$. Hence, we need to find the optimal value for m, for which:

$$\frac{E_m(h)}{h} \to 0$$

$$\boxed{\text{optimal } m \iff \lim_{h\to 0} \frac{E_m(h)}{h} = 0}$$

A simple computation show that,

$$\lim_{h\to 0} \frac{E_m(h)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a) - mh}{h} = 0$$

which is the desired condition 1, we showed in the definition.

2 Lecture-02

Definition 2.1 (Euclidean Vector Space). A Euclidean vector space is a finite-dimensional inner product space over $\mathbb{R}$.

Definition 2.2 (Inner product space). An inner product space is a vector space V over the field F together with an inner product, that is, a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies the following three properties for all vectors $x, y, z \in V$ and all scalars $a, b \in F$.

1. (Symmetry) $\langle x, y \rangle = \langle y, x \rangle$.
2. (Linearity in the first argument) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
3. (Positive-definiteness) If $x \neq 0$ then, $\langle x, x \rangle > 0$.

Example 2.3. Verify that the dot product function $\langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an inner product.

To verify this we must prove the three conditions shown above. Let $u = (a, b)$ and $v = (c, d)$. Then,

1. $\langle u, v \rangle = ac + bd = ca + db = \langle v, u \rangle$.
2. Let $a, b \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^2$ with $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2)$. Then

$$\begin{aligned}\langle au + bv, w \rangle &= (au_1 + bv_1)w_1 + (au_2 + bv_2)w_2 \\ &= au_1w_1 + bv_1w_1 + au_2w_2 + bv_2w_2 \\ &= a(u_1w_1 + u_2w_2) + b(v_1v_2 + v_2w_2) \\ &= a\langle u, w \rangle + b\langle v, w \rangle.\end{aligned}$$

3. For any $u = (u_1, u_2)$,

$$\langle u, u \rangle = {u_1}^2 + {u_2}^2.$$

   Since squares of non-zero numbers are positive, we have ${u_1}^2 > 0, {u_2}^2 > 0$ and therefore, $\langle u, u \rangle > 0$.

Definition 2.4 (Informal). Define **norm** of a vector $x \in V$ as

$$||x|| = \sqrt{\langle x, x \rangle}.$$

Definition 2.5 (Metric Space). A metric space is an ordered pair (M, d) where M is a set and d is a **metric** (distance) on M, i.e., a function

$$d : M \times M \to \mathbb{R}$$

satisfying the following axioms for all points $x, y, z \in M$:

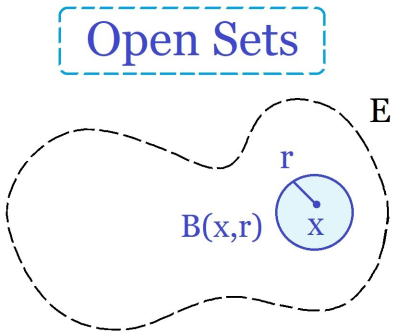
1. $d(x, y) \geq 0$;
2. $d(x, y) = 0 \iff x = y$;

3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

Example 2.6 (Tax-cab Metric)**.** In $\mathbb{R}^2$, the taxi-cab distance between $p = (p_1, p_2)$ and $q = (q_1, q_2)$ is $|q_1 - p_2| + |q_2 - p_2|$. It is easy to verify that this is a metric.

Definition 2.7 (Open ball)**.** For any $x \in \mathbb{R}^n$ and any $r > 0$, the **open ball** of radius r around x is the subset

$$B_r(x) = \{y \in \mathbb{R}^n \mid |x - y| < r\}.$$



Definition 2.8 (Open sets of $\mathbb{R}^n$)**.** A subset $U \in \mathbb{R}^n$ is open if for every point $x \in U$, there exists $r > 0$ such that the open ball $B_r(x)$ is contained in U.

Example 2.9. We prove that the Cartesian product $(a_1, b_1) \times (a_2, b_2)$ of two open intervals in $\mathbb{R}$ is an open set in $\mathbb{R}^2$ with the standard Euclidean topology.

Proof. Let d denote the Euclidean metric on $\mathbb{R}^2$:

$$d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Take an arbitrary point $p = (x, y) \in (a_1, b_1) \times (a_2, b_2)$. Then

$$a_1 < x < b_1, \qquad a_2 < y < b_2.$$

Define the four positive distances to the endpoints:

$$d_1 = x - a_1, \quad d_2 = b_1 - x, \quad d_3 = y - a_2, \quad d_4 = b_2 - y.$$

Let

$$r = \min\{d_1, d_2, d_3, d_4\} > 0.$$

We claim that the open ball $B(p, r) = \{q \in \mathbb{R}^2 : d(p, q) < r\}$ is contained in $(a_1, b_1) \times (a_2, b_2)$.

Indeed, take any point $q = (u, v) \in B(p, r)$. Then

$$|u - x| \leq d(p, q) < r, \qquad |v - y| \leq d(p, q) < r.$$

Hence

$$x - r < u < x + r, \qquad y - r < v < y + r.$$

Because $r \leq x - a_1$ and $r \leq b_1 - x$, we have

$$a_1 \leq x - r < u < x + r \leq b_1,$$

so $u \in (a_1, b_1)$. Similarly, $r \leq y - a_2$ and $r \leq b_2 - y$ imply

$$a_2 \leq y - r < v < y + r \leq b_2,$$

so $v \in (a_2, b_2)$. Consequently $q = (u, v) \in (a_1, b_1) \times (a_2, b_2)$.

Thus every point of the product has an open neighbourhood (an open ball) contained in the product. By definition, $(a_1, b_1) \times (a_2, b_2)$ is open in $\mathbb{R}^2$. □

Definition 2.10 (Closed set)**.** A closed set A is defined by its ability to contain all of its limit points, meaning if a sequence of points $\{x_n\}$ within A converges to a limit x, then that limit x must also be in A.

Definition 2.11 (Sequence)**.** A sequence is a function whose domain is $\mathbb{N}$.