BRAC University

Department of Mathematics and Natural Sciences

LECTURE ON

Calculus II (MAT211)

Limit and Continuity

Limit, Continuity, Algebra of Limits

FEBRUARY 22, 2026

CONDUCTED BY

Emon Hossain

Lecturer, BRAC University, Dhaka-1212

emon.hossain@bracu.ac.bd | mdemon7475@gmail.com

For updates subscribe on @emonhossainraihan

Definition of Limit

Limit

Let $U \subset \mathbb{R}^n$ be an open subset containing a neighborhood of $a \in \mathbb{R}^n$, except perhaps for the point a itself. Suppose $f : U \to \mathbb{R}^m$. We say that

$$\lim_{x \to a} f(x) = \ell$$

($f(x)$ approaches $\ell \in \mathbb{R}^m$ as x approaches a) if for every $\varepsilon > 0$ there is $\delta > 0$ so that

$$\|f(x) - \ell\| < \varepsilon \qquad \text{whenever} \qquad 0 < \|x - a\| < \delta.$$

Component wise Limit

Proposition

$$\lim_{x \to a} f(x) = \ell$$

if and only if

$$\lim_{x \to a} f_j(x) = \ell_j$$

for all $j = 1, \ldots, m$.

Examples of Limits

Problem

Fix a non-zero vector $b \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = b \cdot x$. Show that

$$\lim_{x \to a} f(x) = b \cdot a$$

Examples of Limits

Problem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|^2$. Show that

$$\lim_{x \to a} f(x) = \|a\|^2$$

Examples of Limits

Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x^2 y}{x^2+y^2}$. Does $\lim_{x \to 0} f(x)$ exist?

Examples of Limits

Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{x^2}{x^4+y^2}$. Does $\lim_{x\to 0} f(x)$ exist?

Examples of Limits

Problem

Let $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{xy}{x^4+y^2}$. Does $\lim_{x\to 0} f(x)$ exist?

Algebra of Limits

Theorem

Suppose f and g map a neighborhood of $a \in \mathbb{R}^n$ to $\mathbb{R}^m$ and k maps the same neighborhood to $\mathbb{R}$. Suppose $\lim_{x \to a} f(x) = \ell$, $\lim_{x \to a} g(x) = m$ and $\lim_{x \to a} k(x) = c$, then

- $\lim_{x \to a}(f(x) + g(x)) = \ell + m$,
- $\lim_{x \to a}(f(x) \cdot g(x)) = \ell \cdot m$,
- $\lim_{x \to a}(k(x) f(x)) = c\ell$.

Continuity

Definition

Let $U \subset \mathbb{R}^n$ be an open set containing a neighborhood of $a \in \mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$. We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

That is, f is continuous at a if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(x) - f(a)\| < \varepsilon \qquad \text{whenever} \qquad \|x - a\| < \delta.$$

We say that f is continuous if it is continuous at every point in its domain.

Continuity and Open Sets

Proposition

Let $U \subset \mathbb{R}^n$ be an open set. The function $f : U \to \mathbb{R}^m$ is continuous at $a \in U$ if and only if for every open set $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is open (the preimage of an open set is open).

Composition of Functions through open sets

Proposition

Suppose $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^p$ are open, functions $f \circ g$ is defined (i.e., $g(x) \in U$ for all $x \in W$). $f : U \to \mathbb{R}^m$, $g : W \to \mathbb{R}^n$, and the composition of Then if f and g are continuous, so is $f \circ g$.

Now, Consider the first example.

Continuity and Sequences

Proposition

Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$. Then f is continuous at a if and only if for every sequence $\{x_k\}$ of points in U converging to a the sequence $\{f(x_k)\}$ converges to $f(a)$.

Continuity and Level Sets

Proposition

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous. Then for any $c \in \mathbb{R}^m$, the level set

$$f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$$

is a closed set.

Thank You!

We'd love your questions and feedback.

Emon Hossain

Lecturer, BRAC University, Dhaka-1212

emon.hossain@bracu.ac.bd | mdemon7475@gmail.com

@emonhossainraihan

(Lectures, walkthroughs, and course updates)

Scan for the channel

References

[1] THEODORE SHIFRIN, *MULTIVARIABLE MATHEMATICS: Linear Algebra, Multivariable Calculus, and Manifolds*.