Quadratic Function Analysis (Without Calculus)

Emon Hossain

Department of Mathematics and Natural Sciences
BRAC University

General Form:

$$f(x) = ax^2 + bx + c, \quad (a \neq 0)$$

Shape:

- Opens upward if $a > 0$
- Opens downward if $a < 0$

General Form:

$$f(x) = ax^2 + bx + c, \quad (a \neq 0)$$

Shape:

- Opens upward if $a > 0$
- Opens downward if $a < 0$

Vertex Form (by completing the square):

$$\begin{aligned} f(x) &= a(x^2 + \tfrac{b}{a}x) + c \\ &= a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2 - 4ac}{4a} \end{aligned}$$

$$\boxed{f(x) = a(x-h)^2 + k}, \quad h = -\tfrac{b}{2a}, \; k = c - \tfrac{b^2}{4a}$$

Vertex: $(h, k) = \left(-\frac{b}{2a},\ c - \frac{b^2}{4a}\right)$
Axis of Symmetry: $x = h$
Extreme Value:

$$\begin{cases} \text{Minimum at } (h, k), & a > 0, \\ \text{Maximum at } (h, k), & a < 0. \end{cases}$$

Vertex: $(h, k) = \left(-\frac{b}{2a},\ c - \frac{b^2}{4a}\right)$
Axis of Symmetry: $x = h$
Extreme Value:

$$\begin{cases} \text{Minimum at } (h, k), & a > 0, \\ \text{Maximum at } (h, k), & a < 0. \end{cases}$$

Example:

$$f(x) = 2x^2 - 8x + 3 = 2(x-2)^2 - 5$$

$$\Rightarrow \text{Vertex } (2, -5), \quad \text{Minimum value } f_{\min} = -5$$

$$f(x) = -x^2 + 6x - 8 = -(x-3)^2 + 1$$

$$\Rightarrow \text{Vertex } (3,1), \quad f_{\max} = 1$$

$$\text{Axis: } x = 3, \quad a = -1 < 0 \Rightarrow \text{opens downward.}$$

$$\Delta = b^2 - 4ac$$

Case	Roots	Graph
$\Delta > 0$	2 real distinct	Cuts x-axis twice
$\Delta = 0$	1 real double root	Touches x-axis
$\Delta < 0$	No real roots	Lies above/below x-axis

From $f(x) = a(x-h)^2 + k$:

$$\text{Range: } \begin{cases} [k, \infty), & a > 0 \\ (-\infty, k], & a < 0 \end{cases}$$

From $f(x) = a(x-h)^2 + k$:

$$\text{Range: } \begin{cases} [k, \infty), & a > 0 \\ (-\infty, k], & a < 0 \end{cases}$$

Example:

$$f(x) = x^2 + 4x + 7 = (x+2)^2 + 3$$

$$\Rightarrow \text{Vertex } (-2, 3), \text{ Range } [3, \infty)$$

Height of a projectile:

$$h(t) = -5t^2 + 20t + 1 = -5(t-2)^2 + 21$$

$$\Rightarrow \text{Maximum height } = 21 \text{ m at } t = 2 \text{ s.}$$

Concept	**Formula / Meaning**
Vertex	$(h, k) = (-\frac{b}{2a}, c - \frac{b^2}{4a})$
Axis of symmetry	$x = -\frac{b}{2a}$
Extreme value	$f(h) = k$
Range	$[k, \infty)$ or $(-\infty, k]$
Discriminant	$\Delta = b^2 - 4ac$

For what value of k will the quadratic

$$f(x) = x^2 - 6x + k$$

have its minimum value equal to 4?

For what value of k will the quadratic

$$f(x) = x^2 - 6x + k$$

have its minimum value equal to 4?

$$f(x) = (x-3)^2 + (k-9)$$

Minimum value $= k - 9 = 4 \Rightarrow k = 13$.
Hence $f(x) = x^2 - 6x + 13$ has min $f(x) = 4$ at $x = 3$.

The sum of a number and its reciprocal is given by:

$$S = x + \frac{1}{x}, \quad x > 0.$$

Find the minimum value of S.

The sum of a number and its reciprocal is given by:

$$S = x + \frac{1}{x}, \quad x > 0.$$

Find the minimum value of S.
Multiply by x:

$$Sx = x^2 + 1.$$

$$S = x + \frac{1}{x} \Rightarrow Sx - x^2 = 1 \Rightarrow x^2 - Sx + 1 = 0$$

For real x, discriminant ≥ 0:

$$S^2 - 4 \geq 0 \Rightarrow S \geq 2.$$

Hence $\min S = 2$ at $x = 1$.

The product of two positive numbers is 36. Find the minimum possible value of their sum.

The product of two positive numbers is 36. Find the minimum possible value of their sum.
Let the numbers be x and $\frac{36}{x}$.

$$S = x + \frac{36}{x}, \quad x > 0$$

$$Sx = x^2 + 36 \Rightarrow S = x + \frac{36}{x}$$

By discriminant method:

$$S^2 - 4 \cdot 36 \geq 0 \Rightarrow S \geq 12.$$

Hence $\min S = 12$ when $x = 6$.

Find the minimum value of

$$f(x) = 4x^2 - 12x + 11.$$

Find the minimum value of

$$f(x) = 4x^2 - 12x + 11.$$

$$f(x) = 4(x^2 - 3x) + 11 = 4\left[(x - \tfrac{3}{2})^2 - \tfrac{9}{4}\right] + 11$$
$$f(x) = 4(x - \tfrac{3}{2})^2 + 2.$$

Minimum value $= 2$ at $x = \frac{3}{2}$.

The function $f(x) = 2x^2 - 8x + m$ has a minimum value of 10. Find m.

The function $f(x) = 2x^2 - 8x + m$ has a minimum value of 10. Find m.

$$f(x) = 2(x-2)^2 + (m-8)$$

Minimum value $= m - 8 = 10 \Rightarrow m = 18$.
Hence $f(x) = 2x^2 - 8x + 18$ has $\min f(x) = 10$ at $x = 2$.

If $f(x) = a(x-1)^2 + b(x-2)^2 + c(x-3)^2$ has a minimum at $x = 2$, find a relation among a, b, c.

If $f(x) = a(x-1)^2 + b(x-2)^2 + c(x-3)^2$ has a minimum at $x = 2$, find a relation among a, b, c.
At $x = 2$ (vertex), slope of symmetry coefficients of $(x-2)$ vanish.
Without calculus, compare symmetry:
Let $x = 2 + t$ and $x = 2 - t$, then $f(2+t) = f(2-t)$ for all t.
After simplification:

$$(a + c) = b.$$

Hence the condition: $b = a + c$.

Find all p for which the minimum value of

$$f(x) = x^2 + (p-4)x + p$$

is equal to 1.

Find all p for which the minimum value of

$$f(x) = x^2 + (p-4)x + p$$

is equal to 1.

$$f(x) = (x + \tfrac{p-4}{2})^2 - \tfrac{(p-4)^2}{4} + p$$

Minimum value $= -\frac{(p-4)^2}{4} + p = 1$

$$\Rightarrow (p-4)^2 = 4(p-1)$$

$$\Rightarrow p^2 - 8p + 16 = 4p - 4$$

$$\Rightarrow p^2 - 12p + 20 = 0 \Rightarrow p = 10 \text{ or } 2.$$

Hence $p = 2$ or 10.

Concept	Formula / Meaning
Vertex	$(h, k) = (-\frac{b}{2a}, c - \frac{b^2}{4a})$
Axis of symmetry	$x = -\frac{b}{2a}$
Extreme value	$f(h) = k$
Range	$[k, \infty)$ or $(-\infty, k]$
Discriminant	$\Delta = b^2 - 4ac$

1. Find the vertex, axis, and range of $f(x) = 3x^2 + 6x - 9$.
2. For $f(x) = -2x^2 + 8x - 5$, find its maximum and range.
3. If $f(x) = x^2 - 6x + k$ has a minimum value of 2, find k.
4. Find the least value of $x + \frac{9}{x}$ for $x > 0$.
5. If $f(x) = a(x-1)^2 + b(x-2)^2 + c(x-3)^2$ has vertex at $x = 2$, find b in terms of a, c.

1. Seymour Lipschutz & Murray Spiegel, *Schaum's Outline of College Algebra*, McGraw-Hill.
2. Erwin Kreyszig, *Advanced Engineering Mathematics*, Wiley.
3. Dennis Zill, *A First Course in Differential Equations*, Cengage.
4. Lial, Hornsby & Schneider, *Precalculus: Graphs and Models*, Pearson.