Exponential, Logarithmic, and Trigonometric Functions

Emon Hossain
Department of Mathematics & Natural Sciences
BRAC University

Fall 2025

Motivation

- Exponential, logarithmic, and trigonometric functions describe many natural phenomena:
  - Growth and decay
  - Oscillation and periodic motion
  - Sound, light, and signal behavior
- These functions are fundamental for calculus, physics, and engineering.

Definition

Definition

For any $a > 0, a \neq 1,$

$$f(x) = a^x$$

is called the **exponential function**.

Definition

Definition

For any $a > 0, a \neq 1$,

$$f(x) = a^x$$

is called the **exponential function**.

Natural Base

When $a = e \approx 2.71828$,

$$f(x) = e^x$$

is the **natural exponential function**.

Properties of Exponential Functions

- Domain: $\mathbb{R}$
- Range: $(0, \infty)$
- $a^{x+y} = a^x a^y$
- $a^{-x} = \dfrac{1}{a^x}$
- $(a^x)^y = a^{xy}$

Definition and Inverse Relationship

Definition

$$\log_a y = x \iff a^x = y, \quad a > 0, a \neq 1, y > 0$$

Definition and Inverse Relationship

Definition

$$\log_a y = x \iff a^x = y, \quad a > 0, a \neq 1, y > 0$$

Natural Logarithm

$$\ln x = \log_e x$$

Definition and Inverse Relationship

Definition

$$\log_a y = x \iff a^x = y, \quad a > 0, a \neq 1, y > 0$$

Natural Logarithm

$$\ln x = \log_e x$$

$$a^{\log_a x} = x, \quad \log_a(a^x) = x$$

Properties of Logarithms

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
- $\log_a(x^r) = r \log_a x$
- $\log_a x = \dfrac{\log_b x}{\log_b a}$

Examples

Evaluate

$$\log_2 8 = 3$$

$$\ln e^2 = 2$$

$$\log_{10} 0.01 = -2$$

Examples

Evaluate

$$\log_2 8 = 3$$
$$\ln e^2 = 2$$
$$\log_{10} 0.01 = -2$$

Observe

Exponential and logarithmic functions are inverses — their graphs are mirror images across $y = x$.

Example

- Convert $2^x = 128$ to logarithmic form and solve it.
- Convert $\log_{10} x = 2.9$ to exponential form and solve it.
- Solve $\log_2(\log_5 x) = 2$
- Solve $4\log_x 2 - \log_x 4 = 2$
- Solve $(\log_5 x)^2 - \log_5 x^3 = 18$
- Solve $3(2^{2x}) - 2^{x+1} - 8 = 0$

How to change base of the Logarithm? Take $x = \log_b a$ then $b^x = a$. Now, change the base,

$$\log_c b^x = \log_c a \implies x = \frac{\log_c a}{\log_c b}$$

- Solve $\log_3 x = \log_9(x+6)$

Application

- The volume of water in a container, $V\ \mathrm{cm}^3$, at time t minutes, is given by the formula

$$V = 2000\mathrm{e}^{-kt}$$

  When $V = 1000, t = 15$. Find the value of k. Find the value of V when $t = 22$.
- Sketch the graph of $y = 3e^{-2x} - 5$.
- Sketch the graph of $y = \ln(2x + 5)$.
- Find the inverse of the function $f(x) = 2e^{-4x} + 3, x \in \mathbb{R}$ and find the domain of the inverse function.

Definition

Right Triangle Definition

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Definition

Right Triangle Definition

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Reciprocal functions:

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Domain, Range, and Periodicity

Function	Domain	Range	Period
$\sin x$	$\mathbb{R}$	$[-1, 1]$	2π
$\cos x$	$\mathbb{R}$	$[-1, 1]$	2π
$\tan x$	$x \neq \frac{\pi}{2} + k\pi$	$\mathbb{R}$	π

Fundamental Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Fundamental Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x$$

Example

Find $\sin\theta$ if $\cos\theta = \frac{3}{5}$ and θ is in the first quadrant.

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \frac{4}{5}$$

Interconnections

- Exponential Logarithmic: $\log_a(a^x) = x$
- Exponential Trigonometric: Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

- Logarithmic Trigonometric: Appears in calculus

$$\int \tan x \, dx = -\ln|\cos x| + C$$

Exercises

1. Simplify $2^{3x} \cdot 2^{2x}$
2. Solve for x: $\log_3(x-1) = 2$
3. Find the range of $y = \sin(2x)$
4. Express $\log_2 8$ in exponential form
5. Show that $e^{ix}e^{-ix} = 1$
6. Prove $\tan x = \frac{\sin x}{\cos x}$

Thank You!

Questions or Discussion?