MAT216: Linear Algebra and Fourier Transformation

Emon Hossain[1]

[1]Lecturer
MNS department
Brac University

LECTURE-13

Periodic functions

Square wave:

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

Sawtooth wave:

$$f(x) = x \quad \text{on } [-\pi, \pi]$$

Trigonometric Identity

Prove the following identities:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

Orthogonal Basis

Consider the orthogonal basis,

$$\mathscr{B} = \{1, \cos(kx), \sin(kx)\}_{k=1}^{\infty}$$

And for any 2π-periodic function $f(x)$ we get,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

Then we can derive the coefficient in the same fashion and will get

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Example

Find the Fourier series of the following function,

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

Hint: First, get the periodic extension. Carefully observe the domain $[-\pi, \pi]$ and then find the Fourier series.

Example

Find the Fourier series of the following function,

$$f(x) = \begin{cases} 1, & \pi < x < 2\pi \\ -1, & 0 < x < \pi \end{cases}$$

Hint: First, get the periodic extension. Carefully observe the domain $[0, 2\pi]$ and then find the Fourier series.

General Formula

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{x\pi}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(k\frac{x\pi}{L}\right)$$

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x)\cos(kx)dx$$

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x)\sin(kx)dx$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx$$

Example

Find the Fourier series of the following function,

$$f(x) = x^2, -1 < x < 1$$

Hint: First, get the periodic extension. Carefully observe the domain $[-1, 1]$ and then find the Fourier series.