Problem Sheet 03

(Final topics: Determinants, Inverses, Systems of Equations, Permutations, Combinations, Limits, Continuity)

Instructions. Show all necessary steps clearly. Justify each method used. For system-solving problems, explicitly mention whether you are using Cramer's Rule or the inverse matrix method.

(1) Determinant of a Matrix

D1: Compute the determinant:
$$\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix}$$

D2: Find the determinant of the matrix
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$
using cofactor expansion along a suitable row or column.

D3: Evaluate
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
and state the condition under which the matrix is invertible.

D4: If
$$\det(A) = 5,$$
find
$$\det(2A), \quad \det(A^{-1}), \quad \det(A^T).$$

(2) Inverse of a Matrix

I1: Find the inverse of
$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$
using the formula for a 2×2 matrix.

I2: Determine whether the matrix
$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$
has an inverse. Justify your answer.

I3: Find A^{-1} using the adjoint method, where
$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}.$$

I4: Verify that
$$AA^{-1} = I$$
for the matrix in the previous problem.

I5: If
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
find A^{-1} (if it exists) and state clearly the condition required.

(3) Systems of Linear Equations

S1: Solve the system using **Cramer's Rule**:
$$2x + y = 5$$
$$x - y = 1$$

S2: Use Cramer's Rule to solve:
$$x + 2y = 4$$
$$3x - y = 5$$

S3: Write the following system in matrix form and solve using the **inverse matrix method**:
$$x + y + z = 6$$
$$2x - y + z = 3$$
$$x + 2y - z = 3$$

S4: Solve the system using $X = A^{-1}B$:
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}.$$

S5: Determine whether the following system has a unique solution:
$$2x + 4y = 6$$
$$x + 2y = 3$$
Give reasons based on determinants.

(4) Permutations (Distinct Objects & Restrictions)

P1: Out of a class of 30 students, how many ways are there to choose a class president, a secretary, and a treasurer? A student may hold at most one post.

P2: Suppose Ellie is choosing a secret passcode consisting of the digits $0, 1, 2, \ldots, k$ for some $k \leq 9$. She would like her passcode to use each digit at most once. What is the smallest value of k such that the number of possible permutations is at least 250,000?

P3: How many 5-digit numbers without repetition of digits can be formed using the digits $0, 2, 4, 6, 8$?

P4: Six friends go out for dinner. How many ways are there to sit them around a round table? Rotations of a sitting arrangement are considered the same, but a reflection is considered different.

(5) Combinations (Selection & Complement Counting)

C1: There are 4 balls of colour red, green, yellow and blue. In how many ways can 2 balls be selected such that at least one of them is red or blue?

C2: A team of four has to be selected from 6 boys and 4 girls. How many different ways can a team be selected if at least one boy must be in the team?

C3: There are 10 people at a party and each pair of people shakes hands exactly once. How many handshakes happen at the party?

C4: How many different diagonals does a 12-sided polygon have?

(6) Counting Geometry (Lines, Grids, Points)

G1: How many parallelograms are formed when a set of 5 parallel lines intersects a set of 4 parallel lines?

G2: How many different ways are there to color a 3×3 grid using green, red, and blue paints, using each color exactly 3 times?

G3: How many triangles can be formed using 10 points in a plane, out of which 4 are collinear?

G4: If Anna has 12 different ornaments and would like to place k of them on a necklace, and Lisa has 13 different ornaments and would like to place k of them on a necklace, for what values of k does Anna have more choices for the number of ways to place k ornaments?

(7) Basic Limits by Substitution

B1: Evaluate: $\lim_{x \to 2}(3x^2 - 5x + 1)$.

B2: Evaluate: $\lim_{x \to -1} \frac{x^2+3x+2}{x^2+1}$.

B3: Evaluate: $\lim_{x \to -2}(x^3 + 4x)$.

(8) Indeterminate Forms and Algebra Tricks

A1: Evaluate: $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

A2: Evaluate: $\lim_{x \to 2} \frac{x^3-8}{x-2}$.

A3: Evaluate: $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$.

A4: Evaluate: $\lim_{x \to 1} \frac{x^2-1}{x-1}$.

(9) One-sided Limits and Non-existence

O1: Evaluate: $\lim_{x \to 0} \frac{|x|}{x}$. State clearly whether the two-sided limit exists.

O2: Let
$$f(x) = \begin{cases} 2x + 1, & x < 1, \\ 5, & x \geq 1. \end{cases}$$
Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$. Does $\lim_{x \to 1} f(x)$ exist?

O3: Consider $f(x) = \sin\left(\frac{1}{x}\right)$. Evaluate $\lim_{x \to 0} f(x)$ (or state DNE with reason).

(10) Infinite Limits and Limits at Infinity

I1: Compute:
$$\lim_{x \to 0^+} \frac{1}{x} \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x}.$$
Does $\lim_{x \to 0} \frac{1}{x}$ exist?

I2: Evaluate: $\lim_{x \to \infty} \frac{3x^2-1}{x^2+5x+7}$.

I3: Evaluate: $\lim_{x \to \infty} \frac{5x^2+1}{2x^2-7}$.

(11) Trig Limits (Use Special Limits)

T1: Evaluate: $\lim_{x \to 0} \frac{\sin(5x)}{x}$.

T2: Evaluate: $\lim_{x \to 0} \frac{\sin(2x)}{x}$.

T3: Evaluate: $\lim_{x \to 0} \frac{1-\cos x}{x^2}$.

T4: Prove using $\epsilon - \delta$ definition that $\lim_{x \to 2}(3x - 1) = 5$.

(12) Continuity at a Point (Limit-based Definition)

C1: Let $f(x) = 3x^2 - 5x + 1$. Use the definition $\lim_{x \to a} f(x) = f(a)$ to justify that f is continuous for all $a \in \mathbb{R}$.

C2: Let $f(x) = |x|$. Check continuity at $x = 0$ by computing $\lim_{x \to 0^-} |x|$ and $\lim_{x \to 0^+} |x|$, and comparing with $f(0)$.

C3: Let
$$f(x) = \begin{cases} 2x, & x < 1, \\ 5, & x \geq 1. \end{cases}$$

Compute $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$. Does $\lim_{x \to 1} f(x)$ exist? Is f continuous at $x = 1$?

(13) Removable Discontinuity (Holes)

R1: Consider $f(x) = \dfrac{x^2 - 1}{x - 1}$.

    (a) Simplify $f(x)$ for $x \neq 1$.

    (b) Determine whether $f(1)$ is defined.

    (c) Decide whether the discontinuity at $x = 1$ is removable, and explain briefly.

(14) Jump, Infinite, and Oscillatory Discontinuities

D1: Let $f(x) = \dfrac{1}{x}$. Compute $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$. State the type of discontinuity at $x = 0$.

D2: Let $f(x) = \sin\left(\dfrac{1}{x}\right)$. Discuss whether $\lim_{x \to 0} f(x)$ exists. Hence decide continuity at $x = 0$ and name the type of discontinuity.

(15) Continuity on Intervals and Endpoints

I1: Consider $f(x) = \sqrt{x}$ on the domain $[0, 4]$.

    (a) Verify right-continuity at $x = 0$ by computing $\lim_{x \to 0^+} \sqrt{x}$ and comparing with $f(0)$.

    (b) Verify left-continuity at $x = 4$ by computing $\lim_{x \to 4^-} \sqrt{x}$ and comparing with $f(4)$.

    (c) Conclude that f is continuous on $[0, 4]$.