MAT216: Linear Algebra and Fourier Transformation

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LECTURE-07

Linear Combination: sum of scalar multiples of vectors

Let $(V, +, \cdot)$ be a vector space, and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ a collection of n vectors in V. Then a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a sum of scalar multiples of these vectors; in other words, a sum of the form

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$$

for some choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. A vector $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if it can be written in this form.

In general, given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, it can be quite difficult to determine simply by inspection whether or not some other vector $\mathbf{v}$ is or is not a linear combination of the given collection. One of our goals, discussed in detail below, will be to establish some systematic way of answering this question.

Example

Describe the set of all linear combinations of e_{11}, e_{12}, and e_{22}.

In the vector space $M_{2\times 2}(\mathbb{R})$, express the vector $A = \begin{pmatrix} 5 & 1 \\ -2 & 3 \end{pmatrix}$ as a linear combination of the vectors e_{11}, e_{12}, e_{21}, and e_{22}. Now do the same for the following ones:

$$e = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, f = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hint: $A = 2e + 3f + 4g$

Example

In the vector space $\mathbb{R}^3$ express the vector $v = (6, -7, 3)$ as a linear combination of the vectors $v_1 = (1, -2, 1)$, $v_2 = (2, 1, -1)$ and $v_3 = (7, -4, 1)$.
Hint: $(4 - 3t, 1 - 2t, t)$

In the vector space $\mathbb{R}^3$ express the vector $v = (1, 2, 6)$ as a linear combination of the vectors $v_1 = (2, 1, 0)$, $v_2 = (1, -1, 2)$ and $v_3 = (0, 3, -4)$.

Linear Independence

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent if

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \vec{0} \quad \iff \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

In other words, the only linear combination of the vectors that produces the zero vector is the trivial combination, where each coefficient $\alpha_i = 0$. A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly dependent iff it is not linearly independent.

Example

Show that the set of vectors $(2, 1, 2), (0, 1, -1), (4, 3, 4)$ in $\mathbb{R}^3$ is linearly independent.

Check that for $v_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ linearly independent.

Example

Given the set of vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

a) Determine whether the vectors are linearly independent or dependent.
b) If they are linearly dependent, then express one of the vectors as a linear combination of the others