MAT092: Remedial Course in Mathematics

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Lecture-06

Notation and reminders

A function $f\colon A \to B$ means $\forall x \in A$ there exists a unique $y \in B$ with $f(x) = y$.

Inj (injective): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Surj (surjective): $\forall y \in B\ \exists x \in A$ with $f(x) = y$.

Bij (bijective): both injective and surjective (equivalently invertible).

Always specify the domain and codomain.

(i) Function — Basic example (and rigorous justification)

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Justification: For each $x \in \mathbb{R}$ the formula $x \mapsto x^2$ produces exactly one real number. Thus $\forall x \in \mathbb{R}$ there exists a unique $f(x) \in \mathbb{R}$. Hence f is a function.

(i) Function — Intermediate and hard examples

Example (Intermediate). $f : \{1, 2, 3\} \to \{a, b\}$ with
$f(1) = a,\ f(2) = a,\ f(3) = b.$

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Example (Hard). Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & x \in \mathbb{Q}, \\ -x & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

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Justification: For every real x the right-hand rule returns a single real number (either x or $-x$). Hence f is a well-defined function (uniqueness and totality hold).

(i) Function — Non-examples (rigorous reasons)

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Non-example 2 (Intermediate). A "map" $g\colon \{1, 2, 3\} \to \{a, b\}$ with $g(1) = a$, $g(2) = b$ but $g(3)$ undefined.
Reason: Totality fails (some domain element has no image), so g is not a function.

Definition: Injective

$f : A \to B$ is injective if

$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Equivalently, distinct domain elements have distinct images.

(ii) Injective — Basic example with proof

Example (Basic). $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.

(ii) Injective — Basic example with proof

Example (Basic). $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$.

Proof: Suppose $f(x_1) = f(x_2)$. Then $2x_1 + 1 = 2x_2 + 1$. Subtract 1: $2x_1 = 2x_2$. Divide by 2: $x_1 = x_2$. Thus f is injective.

(ii) Injective — Intermediate example with proof

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Proof (order argument). For $x_1 < x_2$,

$$f(x_2) - f(x_1) = x_2^3 - x_1^3 = (x_2 - x_1)(x_2^2 + x_1 x_2 + x_1^2).$$

Since $x_2 - x_1 > 0$ and $x_2^2 + x_1 x_2 + x_1^2 > 0$ (the quadratic in x_2 has discriminant $-3x_1^2 \leq 0$, so it never vanishes for unequal inputs), we get $f(x_2) - f(x_1) > 0$. Hence $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, so f is strictly increasing and therefore injective.
(Alternatively: $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$.)

(ii) Injective — Hard example with proof

Example (Hard). $f : \mathbb{R} \to (0, \infty)$ given by $f(x) = e^x$.

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Example (Hard). $f : \mathbb{R} \to (0, \infty)$ given by $f(x) = e^x$.

Proof: If $e^{x_1} = e^{x_2}$, then $e^{x_1 - x_2} = 1$. Exponential function satisfies $e^t = 1 \iff t = 0$. Thus $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$. Hence f is injective.

(ii) Injective — Non-examples (with counterexamples)

Non-example (Basic). $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2.$

(ii) Injective — Non-examples (with counterexamples)

Non-example (Basic). $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$.
Counterexample: $f(2) = 4 = f(-2)$ but $2 \neq -2$. So f is not injective on $\mathbb{R}$.

Remark: If we restrict the domain to $[0, \infty)$, then x^2 becomes injective on that domain.

Definition: Surjective

$f : A \to B$ is surjective (onto) if

$$\forall y \in B \ \exists x \in A \quad f(x) = y.$$

Equivalently $\mathrm{Im}(f) = B$.

(iii) Surjective — Basic example with proof

Example (Basic). Identity $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$.

(iii) Surjective — Basic example with proof

Example (Basic). Identity $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$.

Proof: For any $y \in \mathbb{R}$ choose $x = y$. Then $f(x) = y$. So f is surjective.

(iii) Surjective — Intermediate example with proof

Example (Intermediate). $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$.

(iii) Surjective — Intermediate example with proof

Example (Intermediate). $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$.

Proof: Let $y \in [0, \infty)$ be arbitrary. Choose $x = \sqrt{y}$ (if one prefers, either $+\sqrt{y}$ or $-\sqrt{y}$ works). Then $f(x) = x^2 = y$. Hence every $y \geq 0$ has a preimage, so f is surjective onto $[0, \infty)$.

(iii) Surjective — Hard example with proof

Example (Hard). $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$.

(iii) Surjective — Hard example with proof

Example (Hard). $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$.

Proof: Let $y \in (0, \infty)$. Take $x = \ln(y)$. Then $e^x = e^{\ln y} = y$. Thus every positive real has a preimage, so f is surjective onto $(0, \infty)$.

(iii) Surjective — Non-examples (with reason)

Non-example. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

(iii) Surjective — Non-examples (with reason)

Non-example. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

Reason: There exist codomain elements (e.g. -1) for which no $x \in \mathbb{R}$ satisfies $x^2 = -1$. Hence f is not surjective onto $\mathbb{R}$.

Definition: Bijective

$f : A \to B$ is <u>bijective</u> iff it is both injective and surjective.

Equivalently: f has an inverse function $f^{-1} : B \to A$ with $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

(iv) Bijective — Basic and intermediate examples with proofs

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Proof: Injective: $x_1 + 3 = x_2 + 3 \Rightarrow x_1 = x_2$. Surjective: for any $y \in \mathbb{R}$ choose $x = y - 3$ so $f(x) = y$. Therefore bijective; inverse $f^{-1}(y) = y - 3$.

Example (Intermediate). $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \ln(x)$.

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Example (Intermediate). $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \ln(x)$.
Proof: Injective: ln is strictly increasing on $(0, \infty)$. Surjective: for any $y \in \mathbb{R}$ take $x = e^y$ (which lies in $(0, \infty)$); then $\ln(x) = y$. Thus ln is bijective with inverse e^y.

(iv) Bijective — Hard example with proof

Example (Hard). $\tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

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Example (Hard). $\tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

Proof sketch: tan is continuous and strictly increasing on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover,

$$\lim_{x \to -\pi/2^+} \tan x = -\infty, \qquad \lim_{x \to \pi/2^-} \tan x = +\infty.$$

By the intermediate value property and strict monotonicity, tan attains every real value exactly once on the interval. Hence tan is bijective onto $\mathbb{R}$; inverse is arctan.

(iv) Bijective — Non-examples and fixes

Non-example 1. $f(x) = x^2$ as a map $\mathbb{R} \to \mathbb{R}$. Not injective (collisions) and not surjective (negatives missing).

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Non-example 1. $f(x) = x^2$ as a map $\mathbb{R} \to \mathbb{R}$. Not injective (collisions) and not surjective (negatives missing).

Fix: Restrict domain and codomain to $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$. This restricted map is bijective; inverse $f^{-1}(y) = \sqrt{y}$.

Non-example 2. $f(x) = e^x$ as $f : \mathbb{R} \to \mathbb{R}$. Injective but not surjective. Fix by changing codomain to $(0, \infty)$: $e^x : \mathbb{R} \to (0, \infty)$ is bijective.

Quick in-class exercises

Prove that $\sin\colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is injective and surjective.

Determine whether $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n+1$ is bijective. Find the inverse if it exists.

Decide injectivity/surjectivity of

$$f(x) = \begin{cases} x, & x \geq 0, \\ x+1, & x < 0, \end{cases} \quad f : \mathbb{R} \to \mathbb{R}.$$

Give proofs or explicit counterexamples.

Takeaways (short)

Always state domain and codomain — injectivity and surjectivity depend on them.

Injective $\Leftrightarrow$ no two domain elements map to the same codomain element.

Surjective $\Leftrightarrow$ every codomain element is attained.

Bijective $\Leftrightarrow$ invertible (has an inverse function).