Problem Sheet: Trigonometry and Complex Variables

(Graphs, Transformations, Modulus, Polar Form, Roots, Loci)

Instructions. Sketch neatly and label key features (amplitude, period, phase shift, intercepts, asymptotes, key points). For complex variable problems, justify algebraic steps and use geometric interpretation whenever relevant.

(1) Trigonometry

T1: (Transformations) Sketch the graph of

$$y = 2\sin\left(x - \frac{\pi}{4}\right) + 1$$

and clearly mark amplitude, period, phase shift, and vertical shift.

T2: (Scaling & reflection) Sketch

$$y = -3\cos(2x + \pi)$$

and indicate all x-intercepts and the maximum/minimum values.

T3: (Period comparison) On the same axes, sketch

$$y = \sin\left(\frac{x}{2}\right), \qquad y = \sin(2x)$$

for $x \in [-2\pi, 2\pi]$. Compare their periods and frequencies.

T4: (Tangent with shift) Sketch

$$y = -\tan\left(x - \frac{\pi}{3}\right)$$

and write down all vertical asymptotes in the interval $[-\pi, \pi]$.

T5: (Combined transformations) Sketch step-by-step

$$y = 1 + 2\cos\left(3x - \frac{\pi}{2}\right).$$

Mark at least one full period.

T6: (Modulus graph) Sketch

$$y = |\sin x|$$

for $x \in [-2\pi, 2\pi]$ and identify all points where the graph is not differentiable.

T7: (Modulus with frequency) Sketch

$$y = |\cos(2x)|$$

for $x \in [0, 2\pi]$ and determine its period.

T8: (Shifted modulus) Sketch

$$y = \left|\sin x - \frac{1}{2}\right|$$

for $x \in [0, 2\pi]$. Clearly mark the points where $\sin x - \frac{1}{2} = 0$.

T9: (Sum of moduli, optional) Sketch

$$y = |\sin x| + |\cos x|$$

for $x \in [0, 2\pi]$ and determine its minimum and maximum values (with where they occur).

T10: (Graphical solution: intersections) Using graphs, determine the number of solutions of

$$\sin x = x - 1$$

in $0 \le x \le 2\pi$. (A rough sketch with reasoning is sufficient.)

T11: (Equation with modulus) Solve in $0 \le x < 2\pi$:

$$|\sin x| = \cos x.$$

Explain briefly how the graph helps.

T12: (Sine identity via graph) Solve

$$\sin(2x) = \sin x$$

in $0 \le x < 2\pi$ using a graph first, then verify algebraically.

(2) Complex Variable

C1: (Rectangular → polar) Write each in polar form $r(\cos\theta + i\sin\theta)$ (principal argument):

$$z_1 = 1 + i\sqrt{3}, \qquad z_2 = -2 - 2i.$$

C2: (Polar → rectangular) Convert to $a + bi$:

$$z_1 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), \qquad z_2 = 3\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$

C3: (Argument & geometry) Find $\arg(z)$ and the principal value $\operatorname{Arg}(z)$ for

$$z = -1 + i\sqrt{3}.$$

Sketch z in the Argand plane.

C4: (Modulus & conjugate) Let $z = a + bi$. Express the following in terms of a, b:

$$|z|^2, \quad \bar{z}, \quad z + \bar{z}, \quad z - \bar{z}, \quad \Re(z), \quad \Im(z).$$

C5: (De Moivre) Compute (in simplest rectangular form):

$$(1-i)^8.$$

Hint: Use polar form and De Moivre's theorem.

C6: (Cube roots) Find all cube roots of

$$z = 8(\cos\pi + i\sin\pi).$$

Plot the roots on the complex plane and describe their symmetry.

C7: (Fourth roots) Find all fourth roots of

$$z = 16i.$$

Write them in both polar and rectangular form.

C8: (Roots of unity) Solve

$$z^5 = 1$$

and sketch the roots. What polygon do they form?

C9: (Circle locus) Describe geometrically and sketch:

$$|z-1| = 2.$$

C10: (Perpendicular bisector locus) Describe geometrically and sketch:

$$|z-i| = |z+i|.$$

C11: (Half-disk region) Sketch the region:

$$|z| \le 2, \qquad \Re(z) \ge 0.$$

C12: (Ellipse locus) Identify and sketch:

$$|z-1| + |z+1| = 4.$$

State the foci and major axis length.

C13: (Real condition) Show that if

$$z + \frac{1}{z} \in \mathbb{R} \quad (z \ne 0),$$

then z lies either on the real axis or on the unit circle.

C14: (Plot a complex equation) Sketch the set of all z satisfying

$$\Re\big((1-i)z\big) = 2.$$

(Write it as a line in the (x, y)-plane where $z = x + iy$.)