MAT215: Complex Variables And Laplace Transformations

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Lecture-03

Example

- Find the Laplace transform of the function $f(t) = te^{-2t}\sin(3t)$.
- Find the Laplace transform of the function $f(t) = te^{-3t}\sin(2t)\sin(5t)$.
  Use the formula:

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Reverse Relation!

What will be the laplace transform,

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$$

Example

- Find the Laplace transform of the function $f(t) = \frac{\sin(2t)}{t}$
- Find the Laplace transform of the function $f(t) = \frac{\sin(2t)}{t}e^{2t}$

Use the formula: :

$$\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Unit Step Function

- The unit step function is defined as:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

- The Laplace transform of the unit step function is given by:

$$\mathscr{L}\{f(t-a)u(t-a)\} = F(s)e^{-as}$$

$$\mathscr{L}\{f(t)u(t-a)\} = \mathscr{L}\{f(t+a)u(t)\} = e^{-as}\mathscr{L}\{f(t+a)\}$$

Derivation

$$
\begin{aligned}
\mathscr{L}\{f(t-a)u(t-a)\}(s) &= \int_0^\infty e^{-st} f(t-a)u(t-a)dt \\
&= \int_a^\infty e^{-st} f(t-a)dt \\
&\overset{t=\tau+a}{=} \int_0^\infty e^{-s(\tau+a)} f(\tau)d\tau \\
&= e^{-as}\int_0^\infty e^{-s\tau} f(\tau)d\tau \\
&= e^{-as}F(s)
\end{aligned}
$$

Example

Example

- $$\mathscr{L}\{(2t-3)u(t-1)\}$$
- $$\mathscr{L}\{e^{-2t}u(t-1)\}$$
- $$\mathscr{L}\{\cos(2t)u(t-\pi)\}$$

Piecewise Functions

Example

- Find the Laplace transform of the function:

$$f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ \cos(t), & t \geq \pi \end{cases}$$

- Find the Laplace transform of the function:

$$f(t) = \begin{cases} 5\sin(t), & 0 \leq t < \pi \\ -4\cos(t), & t \geq \pi \end{cases}$$

- Find the Laplace transform of the function:

$$f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ 4, & 2 \leq t < 4 \\ e^t, & t \geq 4 \end{cases}$$

Convolution

What is Convolution?

$$[1,2,3] \star [4,5,6] = [4,13,28,27,18]$$

Convolution

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$$[1,2,3] \star [4,5,6] = [4,13,28,27,18]$$

Slide 6,5,4 then multiply and add

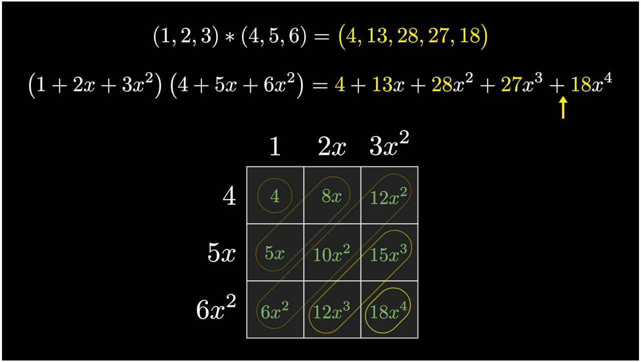

Figure: Convolution Example

Convolution

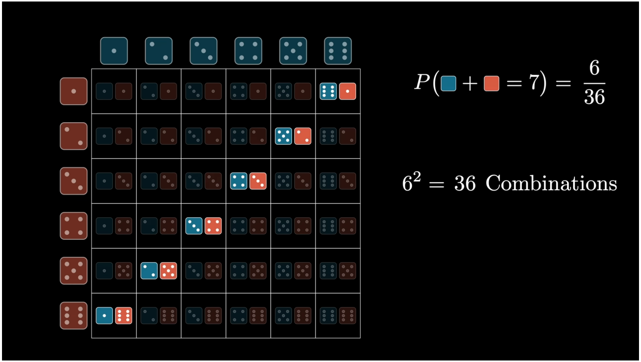

Figure: Convolution Example

Convolution

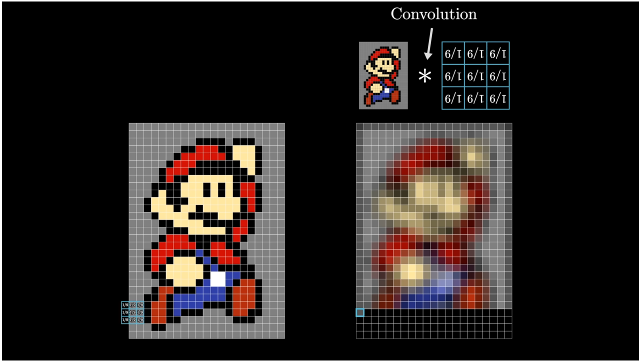

Figure: Convolution Example

https://youtube.com/clip/UgkxyW-Gg-Albj__rrUiPBRgonzPh_Z2_KIm?si=bx6zQsKoZxjE6PFB

Convolution Integral

The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

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Convolution Theorem:

$$\mathscr{L}\{f * g\}(s) = F(s)G(s)$$

where $F(s) = \mathscr{L}\{f(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$.

Example

Question: Find the Laplace Transformation of $\frac{1}{s^2}\frac{1}{s^+1}$.
Answer: We already knew that,

$$\mathscr{L}\left\{\frac{1}{s^2}\right\} = t\,u(t), \qquad \mathscr{L}\left\{\frac{1}{s^2+1}\right\} = \sin t\,u(t)$$

Now, use the convolution formula,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s^2+1}\right\} &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \\ &= \int_{-\infty}^{\infty}(t-\tau)\,u(t-\tau)\sin(\tau)\,u(\tau)d\tau \\ &\stackrel{A}{=} \int_0^t (t-\tau)\sin\tau d\tau\end{aligned}$$

See the footnote for A[1].

[1] $u(t-\tau)$ force $t-\tau \geq 0 \implies t \geq \tau$ and $u(\tau)$ force $\tau \geq 0$. Hence, $0 \leq \tau \leq t$