Cauchy-Goursat Theorem

Statement: Let $f(z)$ be analytic in a region R and on its boundary C. Then

$$\oint_C f(z)dz = 0$$

Example

Evaluate

$$\oint_C \frac{z^2}{z-1} dz$$

where C is the circle $|z-2| = 3$.
Hint: Use path deformation,

$$|z-2| = 3 \to |z-1| = 1$$

Example

Interesting question: Evaluate

$$\oint_C \frac{1}{z-a}dz$$

Where C is any simple closed curve and
i) $z = a$ is outside C
ii) $z = a$ is inside C

Example

Interesting question: Evaluate

$$\oint_C \frac{1}{(z-a)^{n+1}}dz$$

Where C is any simple closed curve and
i) $z = a$ is outside C

$$\oint_C \frac{1}{(z-a)^{n+1}}dz = 0$$

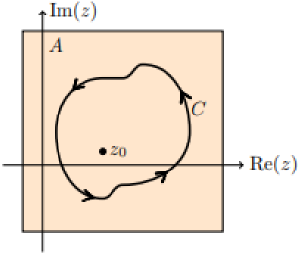
ii) $z = a$ is inside C

$$\oint_C \frac{1}{(z-a)^{n+1}}dz = \begin{cases} 2\pi i, & n = 0 \\ 0, & n > \text{otherwise} \end{cases}$$

Cauchy Integral Formula

(Cauchy's integral formula) Suppose C is a simple closed curve and the function $f(z)$ is analytic on a region containing C and its interior. We assume C is oriented counterclockwise. Then for any z_0 inside C:

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\, dz} \tag{1}$$

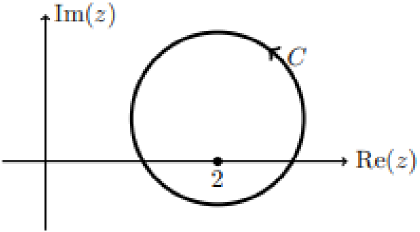


Cauchy's integral formula: simple closed curve C, $f(z)$ analytic on and inside C.

This is remarkable: it says that knowing the values of f on the boundary curve C means we know everything about f inside C!! This is probably unlike anything you've encountered with functions of real variables.

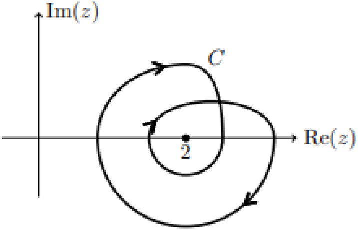
Example

Compute $\displaystyle\int_c \frac{e^{z^2}}{z-2}\,dz$, where C is the curve shown.



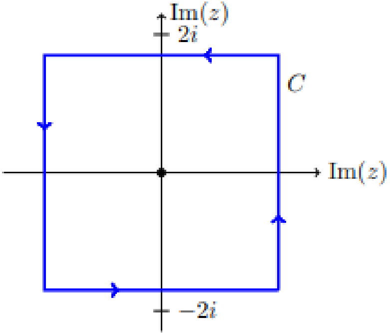
Example

Do the same integral as the previous examples with C the curve shown.



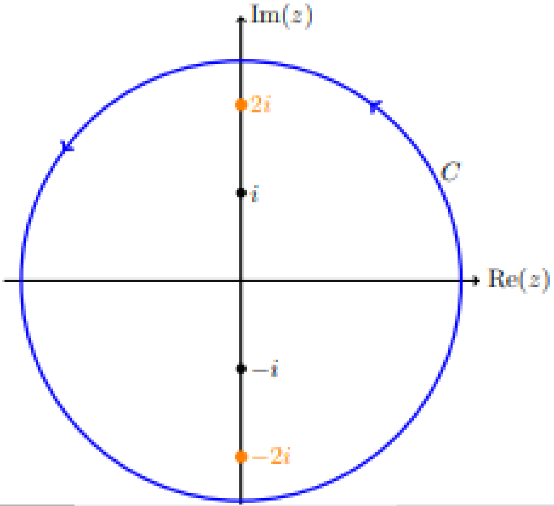
Example

Compute $\displaystyle\int_C \frac{\cos(z)}{z(z^2+8)}\,dz$ over the contour shown.



Example

Compute $\displaystyle\int_C \frac{z}{z^2+4}\,dz$ over the curve C shown below.



General version

General version:

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Example:

Evaluate $\displaystyle \oint_C \frac{e^z}{\left(z^2+\pi^2\right)^2} dz$ where C is the circle $|z-i| = 4$

Did you notice that for multiple poles we can use partial fractions?

We have a better method to handle that!

Behavior of functions near zeros and poles

The basic idea is that near a zero of order n, a function behaves like $(z-z_0)^n$

$$f(z) \approx a_n(z-z_0)^n$$

because

$$f(z) = a_n(z-z_0)^n \left(1 + \frac{a_{n+1}}{a_n}(z-z_0) + \cdots\right)$$

and near a pole of order n, a function behaves like $\frac{1}{(z-z_0)^n}$

$$f(z) \approx \frac{b_n}{(z-z_0)^n}$$

because

$$f(z) = \frac{b_n}{(z-z_0)^n} \left(1 + \frac{b_{n-1}}{b_n}(z-z_0) + \cdots\right)$$

Motivation

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

The residue of f at z_0 is b_1. We denoted it by $\mathrm{Res}(f, z_0) = b_1$. Why is it so important?

$$\oint_C f(z)dz = \oint_C \left(\cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots \right) dz$$

which is nothing but $2\pi i b_1 = 2\pi i \,\mathrm{Res}(f, z_0)$.

Find the Residue

Statement: The residue of a function $f(z)$ at $z = z_0$, is the constant $\boldsymbol{a_{-1}}$. However, in the case where $\boldsymbol{z = z_0}$ is a pole of order **n**, there is a simple formula for a_{-1} given by

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n f(z) \right) \right\}$$

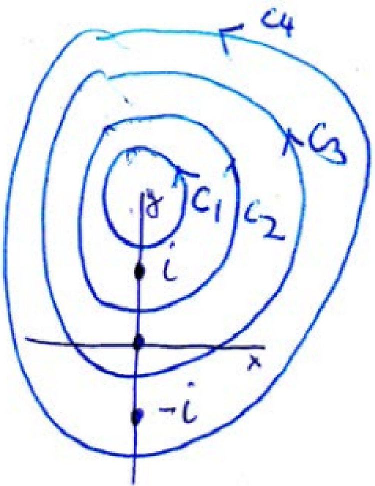
Cauchy Residue Theorem

Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also, suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then,

$$\oint_C f(z)dz = 2\pi i \sum_i \text{Res}(f, z_i)$$

Example

Example: Compute $\oint \frac{1}{z(z^2+1)} dz$ over each contours.



Example

Evaluate

$$\oint_C \frac{2+3\sin(\pi z)}{z(z-1)^2}\,dz$$

where C is a square having vertices at $3+3i$, $3-3i$, $-3+3i$ and $-3-3i$.