MAT092

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LECTURE-01

Lecture Goals

By the end of this lecture, you should be able to:

- Read and interpret limit notation (two-sided and one-sided).
- Compute limits using substitution, factoring, rationalizing, and special trig limits.
- Recognize when a limit does **not** exist (jump, oscillation, infinity).
- See the **rigorous** meaning of "approaches" via ε–δ.

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What does a limit mean? (Intuition)

$$\lim_{x \to a} f(x) = L$$

means:

As x gets closer to a (not necessarily equal), $f(x)$ gets closer to L.

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Key idea: The limit depends on the behavior <u>near</u> a, not at a.

What does a limit mean? (Intuition)

$$\lim_{x \to a} f(x) = L$$

means:

> *As x gets closer to a (not necessarily equal), $f(x)$ gets closer to L.*

Key idea: The limit depends on the behavior near a, not at a.
Important: It is possible that $f(a)$ is undefined or different from L.

One-sided limits

$$\lim_{x\to a^-} f(x) \quad \text{(from the left)}, \qquad \lim_{x\to a^+} f(x) \quad \text{(from the right)}.$$

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$$\lim_{x \to a^-} f(x) \quad \text{(from the left)}, \qquad \lim_{x \to a^+} f(x) \quad \text{(from the right)}.$$

Fact: The two-sided limit exists iff the one-sided limits exist and are equal:

$$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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Example 1: Direct substitution (polynomials)

Evaluate:

$$\lim_{x \to 2} (3x^2 - 5x + 1).$$

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Evaluate:

$$\lim_{x\to 2}(3x^2 - 5x + 1).$$

Solution: Polynomials are continuous everywhere, so substitute $x = 2$:

$$\lim_{x\to 2}(3x^2 - 5x + 1) = 3(2)^2 - 5(2) + 1 = 12 - 10 + 1 = 3.$$

Example 2: Rational function where denominator is nonzero

Evaluate:

$$\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 + 1}.$$

Example 2: Rational function where denominator is nonzero

Evaluate:

$$\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 + 1}.$$

Solution: Since $(-1)^2 + 1 = 2 \neq 0$, substitute:

$$\frac{(-1)^2 + 3(-1) + 2}{(-1)^2 + 1} = \frac{1 - 3 + 2}{2} = \frac{0}{2} = 0.$$

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When substitution gives $\frac{0}{0}$

If

$$\lim_{x \to a} f(x)$$

gives $\frac{0}{0}$, it is **indeterminate**. You must simplify first using:

- factoring and canceling,
- rationalizing,
- trig identities (sometimes).

Example 3: Factoring (removable discontinuity)

Evaluate:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$$

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Evaluate:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$$

Solution:

$$\frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{x-3} = x + 3 \quad (x \neq 3).$$

So,

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3}(x+3) = 6.$$

Example 4: Factoring a cubic

Evaluate:

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}.$$

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Evaluate:

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Solution: Use $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$:

$$\frac{x^3 - 8}{x - 2} = \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = x^2 + 2x + 4.$$

Thus,

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 2^2 + 2(2) + 4 = 4 + 4 + 4 = 12.$$

Example 5: Rationalizing

Evaluate:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}.$$

Example 5: Rationalizing

Evaluate:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}.$$

Solution: Multiply by the conjugate:

$$\frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} = \frac{1}{\sqrt{x} + 2} \quad (x \neq 4).$$

So,

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}.$$

Example 6: Absolute value (piecewise thinking)

Evaluate:

$$\lim_{x \to 0} \frac{|x|}{x}.$$

Example 6: Absolute value (piecewise thinking)

Evaluate:

$$\lim_{x \to 0} \frac{|x|}{x}.$$

Solution:

$$\frac{|x|}{x} = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Thus,

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since left $\neq$ right, the two-sided limit **does not exist**.

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Example 7: Jump discontinuity

Let

$$f(x) = \begin{cases} 2x+1, & x < 1, \\ 5, & x \geq 1. \end{cases}$$

Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

Example 7: Jump discontinuity

Let

$$f(x) = \begin{cases} 2x+1, & x < 1, \\ 5, & x \geq 1. \end{cases}$$

Find $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x+1) = 3, \qquad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 5 = 5.$$

So $\lim_{x \to 1} f(x)$ **does not exist**.

Example 8: Oscillation (no single value approached)

Consider $f(x) = \sin\left(\frac{1}{x}\right)$. Evaluate $\lim_{x \to 0} f(x)$.

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Solution (idea): As $x \to 0$, $\frac{1}{x}$ becomes very large and $\sin(\frac{1}{x})$ keeps oscillating between -1 and 1 infinitely often. Therefore the function does not approach a single number.

$$\boxed{\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ does not exist.}}$$

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Example 9: Infinite limit (vertical asymptote)

Evaluate:

$$\lim_{x \to 0^+} \frac{1}{x} \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x}.$$

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Evaluate:

$$\lim_{x \to 0^+} \frac{1}{x} \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x}.$$

Solution:

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

So the two-sided limit at 0 does not exist (it blows up differently).

Example 10: Limit at infinity (horizontal asymptote)

Evaluate:

$$\lim_{x\to\infty} \frac{3x^2-1}{x^2+5x+7}.$$

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Evaluate:

$$\lim_{x\to\infty} \frac{3x^2-1}{x^2+5x+7}.$$

Solution: Divide numerator and denominator by x^2:

$$\frac{3-\frac{1}{x^2}}{1+\frac{5}{x}+\frac{7}{x^2}} \xrightarrow[x\to\infty]{} \frac{3-0}{1+0+0} = 3.$$

$$\boxed{\lim_{x\to\infty} \frac{3x^2-1}{x^2+5x+7} = 3.}$$

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A key special limit (used later in calculus)

A fundamental limit is:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(Here x is measured in **radians**.)

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A fundamental limit is:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(Here x is measured in **radians**.)
Using it, we can compute many trig limits by rewriting expressions.

Example 11: Using $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Evaluate:

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Evaluate:

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Solution: Rewrite:

$$\frac{\sin(5x)}{x} = \frac{\sin(5x)}{5x} \cdot 5.$$

Now as $x \to 0$, we have $5x \to 0$, so

$$\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1.$$

Hence

$$\boxed{\lim_{x \to 0} \frac{\sin(5x)}{x} = 5.}$$

Example 12: A trig simplification

Evaluate:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}.$$

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Evaluate:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

Solution: Multiply by the conjugate:

$$\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)}.$$

Then

$$\frac{\sin^2 x}{x^2(1+\cos x)} = \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x}.$$

As $x \to 0$:

$$\left(\frac{\sin x}{x}\right)^2 \to 1, \qquad \cos x \to 1.$$

So

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.}$$

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The ε–δ definition (rigor)

We write

$$\lim_{x \to a} f(x) = L$$

if:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \implies \quad |f(x) - L| < \varepsilon.$$

The ε–δ definition (rigor)

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$$\lim_{x \to a} f(x) = L$$

if:

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

Interpretation:

- You (the challenger) choose how close you want $f(x)$ to be to L (that is ε).
- I (the solver) must produce a distance δ so that if x is within δ of a, then $f(x)$ is within ε of L.

Check :`https://betterexplained.com/articles/an-intuitive-introduction-to-limits/`

Example 13: ε–δ proof for a linear function

Prove using ε–δ that:

$$\lim_{x \to 2} (3x - 1) = 5.$$

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Proof: Let $\varepsilon>0$ be given. We want $|3x-1-5|<\varepsilon$ whenever $0<|x-2|<\delta$.

$$|3x-1-5|=|3x-6|=3|x-2|.$$

So it is enough to ensure:

$$3|x-2|<\varepsilon \quad\Longleftarrow\quad |x-2|<\frac{\varepsilon}{3}.$$

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$$|3x - 1 - 5| = |3x - 6| = 3|x - 2|.$$

So it is enough to ensure:

$$3|x - 2| < \varepsilon \quad \Longleftarrow \quad |x - 2| < \frac{\varepsilon}{3}.$$

Choose:

$$\boxed{\delta = \frac{\varepsilon}{3}.}$$

Then $0 < |x - 2| < \delta$ implies $|3x - 1 - 5| < \varepsilon$.

$\square$

Example 14: ε–δ proof for a quadratic

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Proof: Let $\varepsilon > 0$. We want $|x^2 - 1| < \varepsilon$ when $0 < |x-1| < \delta$.
Factor:

$$|x^2 - 1| = |x-1||x+1|.$$

We need to control $|x+1|$.

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We need to control $|x+1|$.
Assume additionally $|x-1| < 1$. Then $x \in (0,2)$, so $|x+1| < 3$.
Hence, under $|x-1| < 1$:

$$|x^2 - 1| = |x-1||x+1| < 3|x-1|.$$

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Assume additionally $|x-1| < 1$. Then $x \in (0,2)$, so $|x+1| < 3$.
Hence, under $|x-1| < 1$:

$$|x^2 - 1| = |x-1||x+1| < 3|x-1|.$$

So it suffices to guarantee:

$$3|x-1| < \varepsilon \quad \Longleftarrow \quad |x-1| < \frac{\varepsilon}{3}.$$

Choose:

$$\boxed{\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}.}$$

Example 15 (Challenge): Find a δ in terms of ε

Show (rigorously) that:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

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Hint: For $x \neq 3$,

$$\frac{x^2-9}{x-3} = x+3.$$

So you want: $|x+3-6| < \varepsilon$.

Example 15 (Challenge): Find a δ in terms of ε

Show (rigorously) that:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

Hint: For $x \neq 3$,

$$\frac{x^2 - 9}{x - 3} = x + 3.$$

So you want: $|x + 3 - 6| < \varepsilon$.
Solution:

$$|x + 3 - 6| = |x - 3|.$$

So choose

$$\boxed{\delta = \varepsilon.}$$

Then $0 < |x - 3| < \delta$ implies $\left| \frac{x^2 - 9}{x - 3} - 6 \right| < \varepsilon$.

□

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Practice (compute these limits)

Compute:

1. $\lim_{x \to -2} (x^3 + 4x)$
2. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$
3. $\lim_{x \to 0} \frac{\sin(2x)}{x}$
4. $\lim_{x \to 0} \frac{|x|}{x}$ (state DNE properly)
5. $\lim_{x \to \infty} \frac{5x^2 + 1}{2x^2 - 7}$

Answers

1. Substitute: $(-2)^3 + 4(-2) = -8 - 8 = -16$.
2. Factor: $\frac{(x-1)(x+1)}{x-1} = x + 1 \Rightarrow 2$.
3. $\frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2 \Rightarrow 2$.
4. Left limit $= -1$, right limit $= 1 \Rightarrow$ DNE.
5. Divide by x^2: $\frac{5+1/x^2}{2-7/x^2} \Rightarrow \frac{5}{2}$.

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Key Takeaways

Limits describe **approach** behavior near a point.

If left and right limits disagree, the limit **does not exist**.

$\frac{0}{0}$ means: **simplify first** (factor/rationalize).

ε–δ makes “approaches” fully rigorous.

Next: Continuity & how limits define it.

Questions?