MAT092

Emon Hossain[1]

[1]Lecturer
MNS department
Brac University

PRE-CALCULUS

Today

How **derivatives** create a **slope field** and guide the original function.

How **Riemann sums** approximate **area** and lead to the definite integral.

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Core idea: derivative gives slope

If a curve is given by $y = y(x)$, then

$$\left.\frac{dy}{dx}\right|_{(x,y)} = \text{(slope of the tangent line at that point).}$$

A **slope field** is a picture of these slopes:

At many points (x, y), draw a small line segment with slope $\dfrac{dy}{dx}$.

Curves that follow these tiny line segments are **solutions**.

Easy example: $\frac{dy}{dx} = x$

Consider the differential equation

$$\frac{dy}{dx} = x.$$

Meaning:

At every point (x, y), the slope depends only on x.

So along the vertical line $x = c$, every little segment has slope c:

$$x = -2 \Rightarrow \text{slope } = -2, \quad x = -1 \Rightarrow \text{slope } = -1, \quad x = 0 \Rightarrow \text{slope } = 0, \quad x = 1$$

Slope field picture (concept)

For $x < 0$ slopes are negative $\Rightarrow$ solution curves go **down**.

At $x = 0$ slope is 0 $\Rightarrow$ curves are **flat** there.

For $x > 0$ slopes are positive $\Rightarrow$ curves go **up**.

Larger $|x|$ gives steeper segments $\Rightarrow$ curves become **steeper** as $|x|$ increases.

Takeaway: Even without solving, the slope field predicts the qualitative shape.

Recovering the original function

Now integrate:

$$\frac{dy}{dx} = x \quad \Longrightarrow \quad y = \int x\,dx = \frac{x^2}{2} + C.$$

So the solution family is

$$y = \frac{x^2}{2} + C \qquad \text{(parabolas).}$$

Geometric match with slope field:

Horizontal tangent at $x = 0$ (because slope $= 0$ there).

Curves go down for $x < 0$ and up for $x > 0$.

Steeper for large $|x|$.

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Core idea: area by rectangles

The definite integral

$$\int_a^b f(x)\,dx$$

represents (for $f \geq 0$) the **area under the curve** from $x = a$ to $x = b$.
A **Riemann sum** approximates this area using rectangles:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n}.$$

As $n \to \infty$, rectangles become very thin and the approximation becomes exact.

Easy example: $f(x) = x$ on $[0, 1]$

We want to approximate

$$\int_0^1 x\,dx.$$

Take $n = 4$ equal subintervals:

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

Using **left endpoints**:

$$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}.$$

Heights are

$$f(0) = 0, \quad f\left(\frac{1}{4}\right) = \frac{1}{4}, \quad f\left(\frac{1}{2}\right) = \frac{1}{2}, \quad f\left(\frac{3}{4}\right) = \frac{3}{4}.$$

Compute the left Riemann sum ($n = 4$)

Left Riemann sum:

$$\sum_{i=0}^{3} f(x_i)\,\Delta x = \left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right)\frac{1}{4}.$$

Compute:

$$0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = \frac{0}{4} + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} = \frac{6}{4} = \frac{3}{2}.$$

So

$$\left(\frac{3}{2}\right)\left(\frac{1}{4}\right) = \frac{3}{8}.$$

Since $f(x) = x$ is increasing on $[0,1]$, left rectangles **underestimate** the true area:

$$\frac{3}{8} < \int_0^1 x\,dx.$$

Exact value (and the limit idea)

Exact integral:

$$\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Meaning:

(1) Riemann sums give approximations (rectangles).

(2) Increasing n makes Δx smaller.

(3) The limit becomes the exact area:

$$\int_0^1 x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Key intuition: integration is <u>adding up many tiny areas</u>.

Big picture connection

Derivative (local)

$$\frac{dy}{dx} = \text{instantaneous slope at a point.}$$

A slope field visualizes these local slopes everywhere.

Integral (global)

$$\int_a^b f(x)\,dx = \text{total accumulated area/change on an interval.}$$

Riemann sums show how "adding tiny pieces" becomes the integral.

Quick in-class exercises

(1) For $\dfrac{dy}{dx} = 2x$, what slopes do you get at $x = -1, 0, 1$?

(2) Approximate $\displaystyle\int_0^1 x\,dx$ using $n = 2$ left rectangles. Compare with $\frac{1}{2}$.