MAT216: Linear Algebra and Fourier Transformation

Emon Hossain[1]

[1]Lecturer
MNS department
Brac University

LECTURE-12

Orthogonal Set

Definition

A set of vectors $\{\mathbf{v}_i \mid 1 \leq i \leq n\}$ is orthogonal if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$ and orthonormal if $\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

For ease of notation, we define the the Kronecker delta function δ_{ij} to be the discrete function $\delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right.$. The matrix associated with the Kronecker delta, $[\delta_{ij}] = I$. Note that an orthonormal set is an orthogonal set of unit vectors.

Orthogonal basis

Definition

An Orthonormal Basis is an orthonormal set of vectors, which is also a basis.

Example

Suppose, we are given five points, $x_1 = -2, x_2 = -1, x_3 = 0, x_4 = 1$, and $x_5 = 2$. Show that $x \perp x^2$ where,

$$\langle P(x), Q(x) \rangle = \sum_{i=1}^{5} P(x_i) Q(x_i)$$

Example

Example

Show that, $S = \{1, \sin(x), \cos(x)\}$ is an orthogonal set where the inner product defined by,

$$\langle p(x), q(x) \rangle = \int_{-\pi}^{\pi} p(x) q(x) dx$$

Projection

Suppose we wish to project the vector $(3,2,1)$ onto the vector $(1,2,3)$. Compute,

$$\begin{aligned}\text{proj}_{(1,2,3)}((3,2,1)) &= \frac{\langle (3,2,1),(1,2,3)\rangle}{\langle (1,2,3),(1,2,3)\rangle}(1,2,3) = \frac{3\cdot 1 + 2\cdot 2 + 1\cdot 3}{1\cdot 1 + 2\cdot 2 + 3\cdot 3}(1,2,3) \\ &= \frac{10}{14}(1,2,3) = \left(\frac{5}{7},\frac{10}{7},\frac{15}{7}\right)\end{aligned}$$

Let us double-check that the projection is orthogonal. That is $\vec{w} - \text{proj}_{\vec{v}}(\vec{w})$ ought to be orthogonal to $\vec{v}$. That is,

$$(3,2,1) - \text{proj}_{(1,2,3)}((3,2,1)) = \left(3-\frac{5}{7}, 2-\frac{10}{7}, 1-\frac{15}{7}\right) = \left(\frac{16}{7},\frac{4}{7},\frac{-8}{7}\right)$$

ought to be orthogonal to $(1,2,3)$.

Example

We compute the inner product, and we had better get zero:

$$\left\langle \left(\frac{16}{7}, \frac{4}{7}, \frac{-8}{7}\right), (1,2,3) \right\rangle = \frac{16}{7}\cdot 1 + \frac{4}{7}\cdot 2 - \frac{8}{7}\cdot 3 = 0.$$

Example

The vectors $(1,1)$ and $(1,-1)$ form an orthogonal basis of $\mathbb{R}^2$. Suppose we wish to represent $(3,4)$ in terms of this basis, that is, we wish to find a_1 and a_2 such that

$$(3,4) = a_1(1,1) + a_2(1,-1)$$

We compute:

$$a_1 = \frac{\langle (3,4),(1,1)\rangle}{\langle (1,1),(1,1)\rangle} = \frac{7}{2}, \quad a_2 = \frac{\langle (3,4),(1,-1)\rangle}{\langle (1,-1),(1,-1)\rangle} = \frac{-1}{2}.$$

$$(3,4) = \frac{7}{2}(1,1) + \frac{-1}{2}(1,-1)$$

Example

Example

The vectors $(1,2,3)$ and $(3,0,-1)$ are orthogonal, and so they are an orthogonal basis of a subspace S :

$$S = \text{span}\{(1,2,3),(3,0,-1)\}$$

Let us find the vector in S that is closest to $(2,1,0)$. That is, let us find $\text{proj}_S((2,1,0))$.

$$\begin{aligned}
\text{proj}_S((2,1,0)) &= \frac{\langle(2,1,0),(1,2,3)\rangle}{\langle(1,2,3),(1,2,3)\rangle}(1,2,3) + \frac{\langle(2,1,0),(3,0,-1)\rangle}{\langle(3,0,-1),(3,0,-1)\rangle}(3,0,-1) \\
&= \frac{2}{7}(1,2,3) + \frac{3}{5}(3,0,-1) \\
&= \left(\frac{73}{35}, \frac{4}{7}, \frac{9}{35}\right)
\end{aligned}$$

Gram-Schmidt Process

(Gram-Schmidt). If $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ is a linearly independent list of vectors in W, then there exists an orthogonal list $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of vectors in W such that

$$\operatorname{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\} = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$$

for $j = 1, \ldots, p$. More specifically,

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\langle \mathbf{x}_p, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{x}_p, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 - \cdots - \frac{\langle \mathbf{x}_p, \mathbf{v}_{p-1} \rangle}{\langle \mathbf{v}_{p-1}, \mathbf{v}_{p-1} \rangle}\mathbf{v}_{p-1}
\end{aligned}$$

Gram-Schmidt Process

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for V. Normalizing each $\mathbf{v}_j$ results in an orthonormal basis.

We can write this in compact notation by,

$$\begin{aligned} v_1 &= x_1 \\ v_k &= x_k - \sum_{i=2}^{k-1} \mathrm{Proj}_{v_i} x_k \\ &= x_k - \sum_{i=2}^{k-1} \frac{\langle x_k, v_i \rangle}{\langle v_i, v_i \rangle} v_i \end{aligned}$$

Example

Example

Assume that

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

Apply the Gram Schmidt process to $\{\mathrm{v1}, \mathrm{v2}, \mathrm{v3}\}$.

For visualization: `https://www.geogebra.org/m/zjcz6grm`. And for Fourier idea: `https://www.desmos.com/calculator/4bmaga18bg`. Reason of inner product: `https://math.stackexchange.com/questions/3491286`.