Linear Functions: Forms, Intuition, and Inverses

Emon Hossain
Department of Mathematics and Natural Sciences
BRAC University

Motivation

Linear functions are the simplest yet most fundamental class of functions.

Motivation

Linear functions are the simplest yet most fundamental class of functions.

- They describe **uniform change** and **straight lines**.
- They appear everywhere: physics, economics, and geometry.
- They serve as **building blocks** of linear transformations.

Definition

Linear Function

A function $f : \mathbb{R} \to \mathbb{R}$ is called **linear (affine)** if

$$f(x) = mx + c, \qquad m, c \in \mathbb{R}.$$

Definition

Linear Function

A function $f : \mathbb{R} \to \mathbb{R}$ is called **linear (affine)** if

$$f(x) = mx + c, \qquad m, c \in \mathbb{R}.$$

- m: slope — rate of change.
- c: intercept — value of f when $x = 0$.

Different Forms of Linear Equations

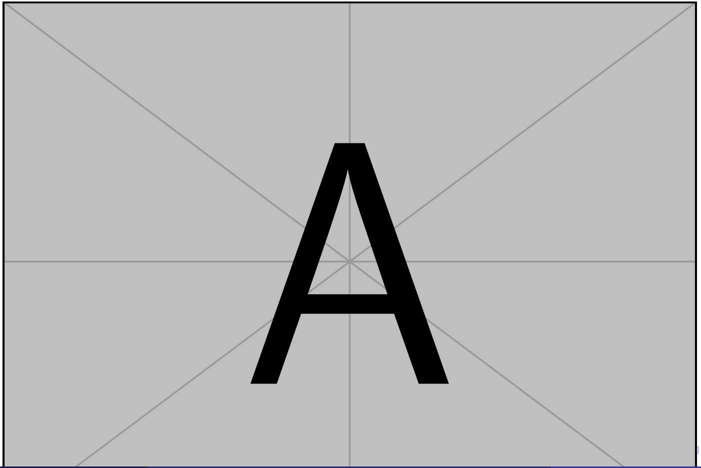
Form	**Equation**	**Use**
Slope–Intercept	$y = mx + c$	Quick graphing
Point–Slope	$y - y_1 = m(x - x_1)$	Given one point
Two–Point	$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	Two points known
General	$Ax + By + C = 0$	Algebraic relations
Vector	$\vec{r} = \vec{r_0} + t\vec{v}$	Geometry / vector form

Geometric Intuition

- The graph of $y = mx + c$ is a **straight line**.
- Equal change in $x \Rightarrow$ equal change in y.
- $m > 0$: increasing; $m < 0$: decreasing; $m = 0$: constant.

Geometric Intuition

- The graph of $y = mx + c$ is a **straight line**.
- Equal change in $x \Rightarrow$ equal change in y.
- $m > 0$: increasing; $m < 0$: decreasing; $m = 0$: constant.



Injectivity and Surjectivity

Injective

f is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$$mx_1 + c = mx_2 + c \implies m(x_1 - x_2) = 0.$$

So f is injective iff $m \neq 0$.

Injectivity and Surjectivity

Injective

f is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

$$mx_1 + c = mx_2 + c \implies m(x_1 - x_2) = 0.$$

So f is injective iff $m \neq 0$.

Surjective

For $f : \mathbb{R} \to \mathbb{R}$,

$$y = mx + c \Rightarrow x = \frac{y - c}{m}.$$

All $y \in \mathbb{R}$ have preimages iff $m \neq 0$.

Inverse Function

If $f(x) = mx + c$ with $m \neq 0$,

$$f^{-1}(x) = \frac{x - c}{m}.$$

Inverse Function

If $f(x) = mx + c$ with $m \neq 0$,

$$f^{-1}(x) = \frac{x - c}{m}.$$

Verification

$$f^{-1}(f(x)) = \frac{mx + c - c}{m} = x.$$

Thus f^{-1} and f are mirror images over $y = x$.

Composition Property

Composition

If $f(x) = m_1 x + c_1, \quad g(x) = m_2 x + c_2$, then

$$(f \circ g)(x) = m_1 m_2 x + (m_1 c_2 + c_1).$$

Hence linear functions form a group under composition when $m \neq 0$.

Matrix Viewpoint

For $T(\mathbf{x}) = A\mathbf{x}$:

- Injective $\Leftrightarrow \ker(A) = \{0\} \Leftrightarrow \det(A) \neq 0$.
- Surjective $\Leftrightarrow \mathrm{Im}(A) = \mathbb{R}^m$.
- Bijective $\Leftrightarrow$ square A with $\det(A) \neq 0$.

Matrix Viewpoint

For $T(\mathbf{x}) = A\mathbf{x}$:

- Injective $\Leftrightarrow \ker(A) = \{0\} \Leftrightarrow \det(A) \neq 0$.
- Surjective $\Leftrightarrow \mathrm{Im}(A) = \mathbb{R}^m$.
- Bijective $\Leftrightarrow$ square A with $\det(A) \neq 0$.

Example:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad A^{-1} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}.$$

Common Misconception

Beware

$f(x) = mx + c$ is not a **linear transformation** unless $c = 0$.

Common Misconception

Beware

$f(x) = mx + c$ is not a **linear transformation** unless $c = 0$.

In linear algebra, linear maps preserve:

$$T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y}).$$

An affine term $+c$ breaks this property.

Examples

Example 1

$f(x) = 3x - 5$: injective, surjective, and

$$f^{-1}(x) = \frac{x+5}{3}.$$

Examples

Example 1

$f(x) = 3x - 5$: injective, surjective, and

$$f^{-1}(x) = \frac{x+5}{3}.$$

Example 2

$f(x) = 0x + 2 \Rightarrow f(x) = 2$. Not injective, not surjective.

Advanced Problems (I)

Algebraic / Functional

1. If $f(x) = 2x + 3$, $g(x) = ax + b$ and $f \circ g = g \circ f$, find a, b.
2. Find all m, c so that $f^{-1} = f$.
3. If $f(g(x)) = x$, prove f, g are inverses and determine coefficients.

Advanced Problems (II)

Matrix and Proof Problems

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (3x + 2y, 4x + y)$. Find if T is invertible and compute T^{-1}.
2. Prove: a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is injective $\Leftrightarrow$ surjective.
3. Let $T(x, y, z) = (x + y, y + z, z + x)$. Find $\ker(T)$ and $\mathrm{Im}(T)$.

Summary

Property	$m = 0$	$m \neq 0$	**Inverse**
Injective			$f^{-1}(x) = \frac{x-c}{m}$
Surjective			–
Invertible			Exists
Graph Type	Constant	Straight Line	Reflection about $y = x$

Suggested Reading

- Seymour Lipschutz, *Schaum's Outline of Linear Algebra*.
- Stephen Abbott, *Understanding Analysis*.
- Erwin Kreyszig, *Advanced Engineering Mathematics*.
- Zill, *A First Course in Differential Equations with Linear Algebra*.

Thank You!

Questions or discussions?