MAT092

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LECTURE-02

Lecture Goals

By the end of this lecture, you should be able to:

- Understand the definition of continuity at a point.
- Identify different types of discontinuities.
- Check continuity for polynomials, rational and piecewise functions.
- Understand continuity using ε–δ rigor.

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What does continuity mean?

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Intuitively:

No holes

No jumps

No infinite breaks

Limit-based definition (core idea)

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This means three things must hold:

(i) $f(a)$ is defined

(i) $\lim_{x \to a} f(x)$ exists

(i) $\lim_{x \to a} f(x) = f(a)$

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Example 1: Polynomial functions

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Solution: Polynomials are continuous for all real numbers.

$$\lim_{x \to a} f(x) = f(a) \quad \text{for all } a \in \mathbb{R}.$$

$$\Rightarrow \text{Continuous everywhere.}$$

Example 2: Rational function (continuous domain)

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Simplify:

$$f(x) = x + 1 \quad (x \neq 1).$$

At $x = 1$:

$$f(1) \text{ is undefined.}$$

$$\Rightarrow f \text{ is continuous for } x \neq 1, \text{ but discontinuous at } x = 1.$$

(Removable discontinuity)

Example 3: Absolute value

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$$f(x) = \frac{1}{x}.$$

At $x = 0$:

$$\lim_{x \to 0^+} f(x) = +\infty, \quad \lim_{x \to 0^-} f(x) = -\infty.$$

$\Rightarrow f$ has an infinite discontinuity at $x = 0$.

Example 6: Oscillatory discontinuity

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$$\Rightarrow f \text{ is discontinuous at } x = 0.$$

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Continuity on intervals

A function is:

- Continuous on (a, b) if continuous at every point in the interval
- Continuous on $[a, b]$ if:
  - Continuous on (a, b)
  - Right-continuous at a
  - Left-continuous at b

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$$\lim_{x \to 4^-} \sqrt{x} = 2 = f(4).$$

Example 7: Endpoint continuity

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$$f(x) = \sqrt{x}, \quad x \in [0, 4].$$

$$\lim_{x \to 0^+} \sqrt{x} = 0 = f(0).$$

$$\lim_{x \to 4^-} \sqrt{x} = 2 = f(4).$$

$$\Rightarrow f \text{ is continuous on } [0, 4].$$

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Formal ε–δ definition

A function f is continuous at $x = a$ if:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that

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This is exactly the ε–δ definition of:

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To ensure $2|x - 3| < \varepsilon$, choose:

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Thus $|x - 3| < \delta$ implies $|f(x) - f(3)| < \varepsilon$.

$\square$

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Assume $|x-1| < 1 \Rightarrow x \in (0,2)$ so $|x+1| < 3$.
Thus:

$$|x^2 - 1| < 3|x-1|.$$

Choose:

$$\boxed{\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}}.$$

□

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Key Takeaways

Continuity means: limit equals function value.

Three checks: existence of $f(a)$, limit, and equality.

Polynomials and absolute values are continuous everywhere.

Rational functions are continuous where denominators $\neq 0$.

ε–δ makes continuity mathematically precise.

Questions?