

MAT216: Linear Algebra and Fourier Transformation

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LECTURE-13

Periodic functions

A function $f(x)$ is said to be periodic with period P if

$$f(x + P) = f(x)$$

for all values of x in the domain of f . The smallest positive value of P for which this equation holds is called the fundamental period of f .

Question: Find the period for the following Trigonometric function:

$$f(x) = a \sin^b(cx + d) + e$$

Answer: If b is an even integer, then the period is $\frac{\pi}{c}$. Otherwise, the period is $\frac{2\pi}{c}$.

Periodic functions

Square wave:

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

Sawtooth wave:

$$f(x) = x \quad \text{on } [-\pi, \pi]$$

Examples of odd and even functions

$f(x) = x^2$ is an even function since $f(-x) = (-x)^2 = x^2 = f(x)$

$f(x) = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(x) = \sin x$ is an odd function since $\sin(-x) = -\sin x$

$f(x) = \cos x$ is an even function since $\cos(-x) = \cos x$

$f(x) = 3|\sin(x)|$ is an even function since $|\sin(-x)| = |\sin x|$

$f(x) = 1 + \sin(x)$ is neither even nor odd since

$f(-x) = 1 + \sin(-x) = 1 - \sin x \neq f(x)$ and $f(-x) \neq -f(x)$

Example of complicated odd and even functions

Example

Determine whether the function $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}$ is even, odd, or neither.

Example

Determine whether the function $f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 \leq x < \pi \end{cases}$ is even, odd, or neither.

Continued...

Example

Determine whether the function $f(x) = \begin{cases} 2, & -\pi < x < 0 \\ -2, & 0 \leq x < \pi \end{cases}$ is even, odd, or neither.

Example

Determine whether the function $f(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$ is even, odd, or neither.

Example

Determine whether the function $f(x) = \begin{cases} 2\sin x, & 0 < x < \pi \\ 0, & \text{otherwise} \end{cases}$ is even, odd, or neither.

Trigonometric Identity

Prove the following identities:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

Orthogonal Basis

Consider the orthogonal basis,

$$\mathcal{B} = \{1, \cos(kx), \sin(kx)\}_{k=1}^{\infty}$$

And for any 2π -periodic function $f(x)$ we get,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

Then we can derive the coefficient in the same fashion and will get

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

Example

Find the Fourier series of the following function,

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

Hint: First, get the periodic extension. Carefully observe the domain $[-\pi, \pi]$ and then find the Fourier series.

Example

Find the Fourier series of the following function,

$$f(x) = \begin{cases} 1, & \pi < x < 2\pi \\ -1, & 0 < x < \pi \end{cases}$$

Hint: First, get the periodic extension. Carefully observe the domain $[0, 2\pi]$ and then find the Fourier series.

General Formula

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k \frac{x\pi}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(k \frac{x\pi}{L}\right)$$

$$a_k = \frac{1}{L} \int_{-L}^L f(x) \cos(kx) dx$$

$$b_k = \frac{1}{L} \int_{-L}^L f(x) \sin(kx) dx$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

Example

Find the Fourier series of the following function,

$$f(x) = x^2, -1 < x < 1$$

Hint: First, get the periodic extension. Carefully observe the domain $[-1, 1]$ and then find the Fourier series.