MAT216: Linear Algebra and Fourier Transformation

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LECTURE-11

Eigen system

As we know that the is no difference between a matrix and a Linear Transformation. Consider a linear transformation,
$T : (\mathbb{R}^2, +, \cdot) \to (\mathbb{R}^2, +, \cdot)$ defined by,

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 2y \end{bmatrix}$$

Then the corresponding matrix is,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

To visualize the transformation, use this:
https://www.geogebra.org/m/YCZa8TAH

Eigen system

We are interested in those vectors (from the domain) which doesn't change direction after transformation.

$$A\vec{v} = \lambda \vec{v}$$
$$A\vec{v} - \lambda \vec{v} = 0$$
$$(A - \lambda I)\vec{v} = 0$$

This is a homogeneous system. For nontrivial solution ($\mathbf{v} \neq \mathbf{0}$) to exist, the matrix $(A - \lambda I)$ must be singular. (Why?)

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation.

Calculate eigen values and vectors

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Step 1: Compute the characteristic polynomial

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$$

Set this to zero:

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0$$

So the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$.

continued...

Step 2: Find eigenvectors For $\lambda = 1$:

$$(A - I)\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \Rightarrow v_1 + v_2 = 0 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 3$:

$$(A - 3I)\mathbf{v} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow -v_1 + v_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = \left| \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{vmatrix},$$

$$= (2-\lambda)\big[(3-\lambda)(9-\lambda) - 16\big] = -\lambda^3 + 14\lambda^2 - 35\lambda + 22.$$

The roots of the characteristic polynomial are 2, 1, and 11, which are the only three eigenvalues of A. These eigenvalues correspond to the

eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, or any nonzero multiple thereof.

Example

Example

Do the same for,

$$A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$$

Similar Matrix

Two square matrices A and B of the same size $n \times n$ are called similar if there exists an **invertible matrix** P such that:

$$B = P^{-1}AP$$

This is denoted as:

$$A \sim B$$

In words: Matrix B is similar to A if B is obtained from A by a change of basis via P.

Why It Matters: Similarity captures the idea that two matrices **represent the same linear transformation**, but **in different bases**.

Properties of similar matrices

Properties of Similar Matrices:
1. Same Characteristic Polynomial:

$$\det(A - \lambda I) = \det(B - \lambda I)$$

So they have the **same eigenvalues** (including algebraic multiplicity).
2. Same Minimal Polynomial
3. Same Determinant and Trace:

$\det(A) = \det(B)$

$\text{tr}(A) = \text{tr}(B)$

4. Same Rank
5. Same Eigenvalues: But **not necessarily** the same eigenvectors.

Properties of similar matrices

6. Same Jordan Canonical Form
7. Similarity is an Equivalence Relation:

   Reflexive: $A \sim A$

   Symmetric: If $A \sim B$, then $B \sim A$

   Transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$

8. Diagonalization: A matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$D = P^{-1}AP$$

This happens when A has n linearly independent eigenvectors.

Example

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then compute:

$$B = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

So $A \sim B$, and B is diagonal.