

Final practice problem

1. Find the general solution of the following system of linear questions,

$$\begin{cases} x + 3y + z + t &= 2 \\ 2x + 6y + 3z + 4t &= 5 \\ 7x + 21y + 8z + 9t &= 15 \end{cases}$$

Answer:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

2. Calculate the inverse of the following matrix using Gauss-Jordan elimination (i.e. using reduced row echelon form),

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 2 & 1 \end{pmatrix}$$

3. Is the set of vectors $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \mid a + b + c = 0 \right\}$ a subspace of the vector space $V = \mathbb{R}^3$.

Answer: Yes. Check $u + v \in W$ and $\alpha u \in W$.

4. Let V be the set of all ordered pairs of real numbers and consider the following addition and scalar multiplication operations defined on the ordered pairs, $x = (x_1, x_2)$ and $y = (y_1, y_2)$:

$$x + y = (x_1 + x_2, y_1 + y_2 + 1), \quad \alpha \cdot x = (\alpha x_1, \alpha x_2)$$

5. Consider the vector space V of 2 by 2 matrix, $M_{2 \times 2}$. Now, Consider the following subsets:

- $W_1 = \{M_{2 \times 2} : \det M_{2 \times 2} = 0\}$
- $W_2 = \{M_{2 \times 2} : \det M_{2 \times 2} \neq 0\}$
- $W_3 = \{M_{2 \times 2} : \det M_{2 \times 2} = 1\}$

Verify which of the subsets are subspace of V . Explain your answer.

6. Consider three vectors,

$$u = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, w = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Is the vector $w \in \text{span}\{u, v\}$.

Answer: Yes. Check the system, $\alpha u + \beta v = w$.

7. Find the row echelon form of the following augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & a \\ 1 & 1 & 3 & 1 & b \\ 1 & 0 & 2 & -1 & c \end{array} \right)$$

Now, consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, v_4 = \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, w = \begin{pmatrix} 13 \\ 3 \\ 8 \end{pmatrix}$$

Using the previous result can you determine whether w be written as a linear combination of v_1, v_2, v_3, v_4 ?

8. Find the basis and dimension of W , where

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 7 \\ 9 \end{bmatrix} \right\}$$

9. Consider the previous question. Now, show that W can't span \mathbb{R}^3 .

10. Show that whether $S = \{u, v, w, t\}$ span \mathbb{R}^4 or not?

$$u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

11. Show that the below vectors u, v, w in \mathbb{R}^3 are linearly independent.

$$u = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

12. Consider the following matrix,

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

1. Calculate the basis of the $\text{Null}(A)$ or $\text{NullSpace}(A)$
2. Find the rank
3. Calculate the basis of the $\text{Row}(A)$ or $\text{RowSpace}(A)$
4. Calculate the basis of the $\text{Col}(A)$ or $\text{ColumnSpace}(A)$

13. Consider the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, where,

$$T \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ x + y + z \end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

14. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where,

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - y \\ y - x \\ x - z \end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

15. Consider the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where,

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

16. Consider the linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, where,

$$T \left(\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \right) = \begin{bmatrix} x - y + z + t \\ 2x - 2y + 3z + 4t \\ 3x - 3y + 4z + 5t \end{bmatrix}$$

Find the standard matrix representation and then find $\text{Ker}(T)$ and $\text{Im}(T)$. And show that $\dim \text{Ker}(T) + \dim \text{Im}(T) = \dim \mathbb{R}^3$.

Answer:

$$\text{Ker}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{Im}(T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Best of Luck!