MAT215: Complex Variables And Laplace Transformations

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LECTURE-02

Some important formulas

$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$$

$$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$$

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

$$\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}, \cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2}(a\sin(bt) - b\cos(bt)) + C$$

$$\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}(a\cos(bt) + b\sin(bt)) + C$$

Some important Laplace Transform formulas

- Known formula
- First Translation formula:

$$\boxed{\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}_{s\to s-a} = F(s-a)}$$

- Truncation/Second Translation formula:

$$\boxed{\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)}$$

- s-differentiation:

$$\boxed{\mathscr{L}\{(t)^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)}$$

- Partial fraction type
- Convolution type (Not very important)

Example

Example

Find the Laplace transform of $f(t) = t$.

Example

Find the Laplace transform of $f(t) = e^{at}$.

Example

a. Find the Laplace transform of $f(t) = 2e^{-2t}$.

b. Find the Laplace transform of $f(t) = 4t^3 - e^{-1}$.

c. Find the Laplace transform of $f(t) = (t^2 + 1)^2$.

d. Find the Laplace transform of $f(t) = \cosh(3t) + \sin(2t)$.

Example

a. Find the Laplace transform of $f(t) = \cos(at)$.

b. Find the Laplace transform of $f(t) = \sin(at)$.

c. Find the Laplace transform of $f(t) = \sinh(at)$.

d. Find the Laplace transform of $f(t) = \cosh(at)$.

Way-01

Find the Laplace transform of $f(t) = \sin(at)$.

$$\begin{aligned}\mathscr{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt \\ &= \text{Skip way-01}\end{aligned}$$

Way-02 (best way)

Find the Laplace transform of $f(t) = \sin(at)$.

$$
\begin{aligned}
\mathscr{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt \\
&= \int_0^\infty e^{-st}\frac{e^{i(at)} - e^{-i(at)}}{2i}dt \\
&= \frac{1}{2i}\left(\int_0^\infty e^{-st}e^{i(at)}dt - \int_0^\infty e^{-st}e^{-i(at)}dt\right) \\
&= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] \\
&= \frac{1}{2i}\frac{2ia}{s^2+a^2} \\
&= \frac{a}{s^2+a^2}
\end{aligned}
$$

Way-03

Find the Laplace transform of $f(t) = \sin(at)$.

$$\begin{aligned}
\mathscr{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt \\
&= \lim_{b\to\infty}\int_0^b e^{-st}\sin(at)dt \\
&= \lim_{b\to\infty}\left[\frac{e^{-st}(-a\cos(at)-s\sin(at))}{a^2+s^2}\right]_0^b \\
&= \lim_{b\to\infty}\left(\frac{e^{-sb}(-a\cos(ab)-s\sin(ab))}{a^2+s^2} - \frac{-a}{a^2+s^2}\right) \\
&= \frac{a}{a^2+s^2}
\end{aligned}$$

Example

- Find the Laplace transform of $f(t) = t^3$.
- Find the Laplace transform of $f(t) = t^3 e^{5t}$.
  Hint: Use the first Translation formula:

$$\boxed{\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}_{s \to s-a} = F(s-a)}$$

Example

Find the Laplace transform of $f(t) = 5\sin(4t)$.

Find the Laplace transform of $f(t) = 5\sin(4t)e^{-3t}$.
Hint: Use the first Translation formula:

$$\boxed{\mathscr{L}\{e^{at} f(t)\} = \mathscr{L}\{f(t)\}_{s \to s-a} = F(s-a)}$$

Example

Find the Laplace transform of $f(t) = (t+2)^2$.

Find the Laplace transform of $f(t) = (t+2)^2 e^t$.

Example

Find the Laplace transform of $f(t) = t\sin(at)$.

Example

Find the Laplace transform of $f(t) = t\sin(at)$.

$$
\begin{aligned}
\mathscr{L}\{t\sin(at)\} &= \int_0^\infty e^{-st}\, t\sin(at)\, dt \\
&= \int_0^\infty t \underbrace{e^{-st}\sin(at)}_{v}\, dt \\
&= \left[t\frac{e^{-st}(-a\cos(at) - s\sin(at))}{a^2+s^2} \right]_0^\infty \\
&\quad - \int_0^\infty \left[t\frac{e^{-st}(-a\cos(at) - s\sin(at))}{a^2+s^2} \right]_0^\infty dt \\
&= \frac{s}{(s^2+a^2)} \int_0^\infty e^{-st}\sin(at)\, dt + \frac{a}{(s^2+a^2)} \int_0^\infty e^{-st}\cos(at)\, dt \\
&= \frac{s}{(s^2+a^2)} \cdot \frac{a}{s^2+a^2} + \frac{a}{(s^2+a^2)} \cdot \frac{s}{s^2+a^2} \\
&= \frac{2as}{(s^2+a^2)^2}
\end{aligned}
$$

Way-02

Solution:

$$\begin{aligned}
\mathscr{L}\{t\sin(at)\} &= \int_0^\infty e^{-st} t\sin(at)dt \\
&= -\int_0^\infty e^{-st}\frac{\partial}{\partial a}(\cos(at))\,dt \\
&= -\frac{\partial}{\partial a}\int_0^\infty e^{-st}\cos(at)dt \\
&= -\frac{\partial}{\partial a}\left(\frac{s}{s^2+a^2}\right) \\
&= \frac{2as}{(s^2+a^2)^2}
\end{aligned}$$

Example

Find the Laplace transform of $f(t) = 2\sinh(3t) + 5\cosh(4t)$.

Find the Laplace transform of $f(t) = e^{2t}[2\sinh(3t) + 5\cosh(4t)]$.

Find the Laplace transform of $f(t) = te^{2t}[2\sinh(3t) + 5\cosh(4t)]$