

# Cauchy-Goursat Theorem

**Statement:** Let  $f(z)$  be analytic in a region  $R$  and on its boundary  $C$ . Then

$$\oint_C f(z) dz = 0$$

## Example

Evaluate

$$\oint_C \frac{z^2}{z-1} dz$$

where  $C$  is the circle  $|z-2|=3$ .

**Hint:** Use path deformation,

$$|z-2|=3 \rightarrow |z-1|=1$$

## Example

**Interesting question:** Evaluate

$$\oint_C \frac{1}{z-a} dz$$

Where  $C$  is any simple closed curve and

- i)  $z = a$  is outside  $C$
- ii)  $z = a$  is inside  $C$

## Example

**Interesting question:** Evaluate

$$\oint_C \frac{1}{(z-a)^{n+1}} dz$$

Where  $C$  is any simple closed curve and

i)  $z = a$  is outside  $C$

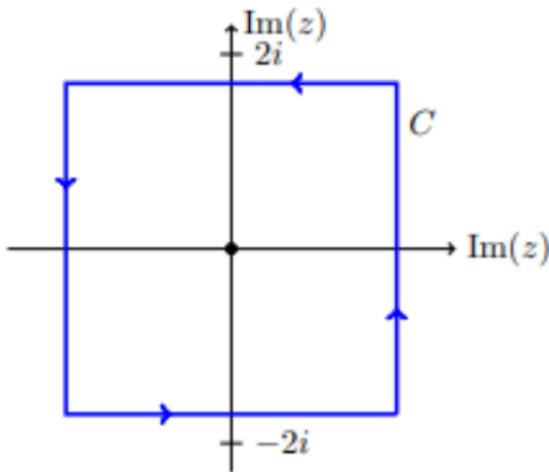
$$\oint_C \frac{1}{(z-a)^{n+1}} dz = 0$$

ii)  $z = a$  is inside  $C$

$$\oint_C \frac{1}{(z-a)^{n+1}} dz = \begin{cases} 2\pi i, & n=0 \\ 0, & n > \text{otherwise} \end{cases}$$

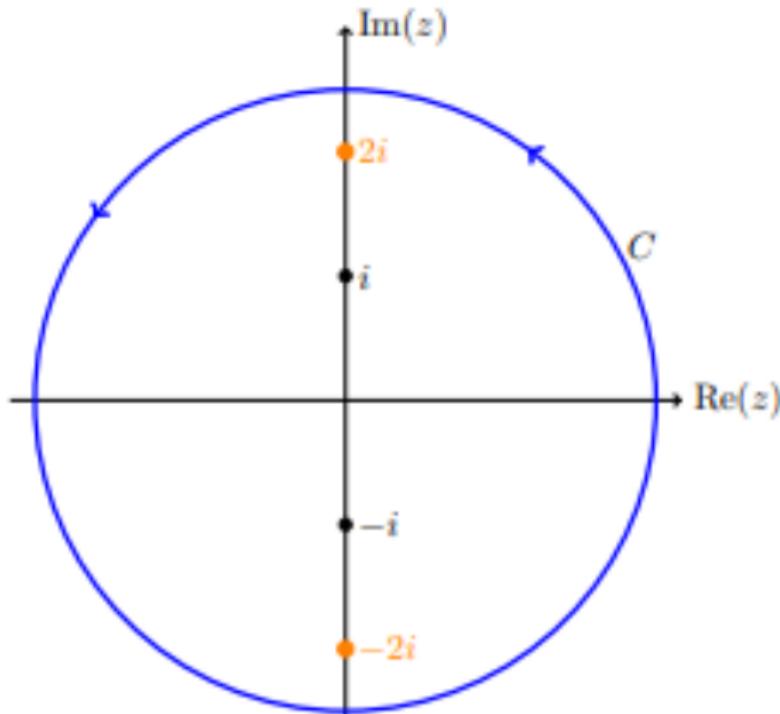
## Example

Compute  $\int_C \frac{\cos(z)}{z(z^2 + 8)} dz$  over the contour shown.



## Example

Compute  $\int_C \frac{z}{z^2 + 4} dz$  over the curve  $C$  shown below.



## General version

**General version:**

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

**Example:**

Evaluate  $\oint_C \frac{e^z}{(z^2 + \pi^2)^2} dz$  where  $C$  is the circle  $|z - i| = 4$

Did you notice that for multiple poles we can use partial fractions?

We have a better method to handle that!

## Behavior of functions near zeros and poles

The basic idea is that near a zero of order  $n$ , a function behaves like  $(z - z_0)^n$

$$f(z) \approx a_n(z - z_0)^n$$

because

$$f(z) = a_n(z - z_0)^n \left( 1 + \frac{a_{n+1}}{a_n}(z - z_0) + \dots \right)$$

and near a pole of order  $n$ , a function behaves like  $\frac{1}{(z - z_0)^n}$

$$f(z) \approx \frac{b_n}{(z - z_0)^n}$$

because

$$f(z) = \frac{b_n}{(z - z_0)^n} \left( 1 + \frac{b_{n-1}}{b_n}(z - z_0) + \dots \right)$$

# Motivation

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The residue of  $f$  at  $z_0$  is  $b_1$ . We denoted it by  $\text{Res}(f, z_0) = b_1$ . Why is it so important?

$$\oint_C f(z) dz = \oint_C \left( \cdots + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{(z - z_0)} + a_0 + a_1(z - z_0) + \cdots \right) dz$$

which is nothing but  $2\pi i b_1 = 2\pi i \text{Res}(f, z_0)$ .

## Find the Residue

**Statement:** The residue of a function  $f(z)$  at  $z = z_0$ , is the constant  $a_{-1}$ . However, in the case where  $z = z_0$  is a pole of order  $n$ , there is a simple formula for  $a_{-1}$  given by

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)) \right\}$$

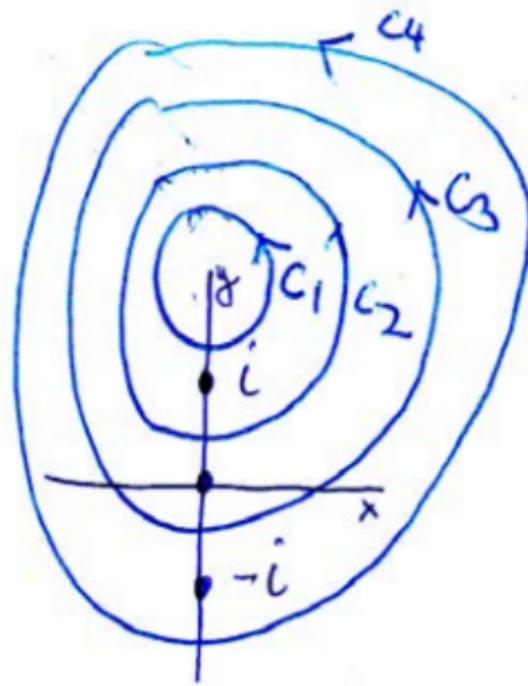
## Cauchy Residue Theorem

Suppose  $f(z)$  is analytic in the region  $A$  except for a set of isolated singularities. Also, suppose  $C$  is a simple closed curve in  $A$  that doesn't go through any of the singularities of  $f$  and is oriented counterclockwise. Then,

$$\oint_C f(z) dz = 2\pi i \sum_i \text{Res}(f, z_i)$$

## Example

Example: Compute  $\oint \frac{1}{z(z^2+1)} dz$  over each contours.



## Example

Evaluate

$$\oint_C \frac{2 + 3 \sin(\pi z)}{z(z - 1)^2} dz$$

where  $C$  is a square having vertices at  $3 + 3i$ ,  $3 - 3i$ ,  $-3 + 3i$  and  $-3 - 3i$ .