MAT215: Complex Variables And Laplace Transformations

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LECTURE-01

Simple System

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If we take a mass on a spring and pull it and release, one of four things is going to happen:

- It oscillates forever: sinusoidal motion (ideal case)
- It comes to rest without oscillating (overdamped case inside a viscous fluid)
- Combines oscillation and damping, comes to rest (underdamped case inside a less viscous fluid)
- Anything else

All of these cases can be modeled with exponentials and sinusoids.

Motivation

The Fourier Transform tells us which frequencies or sinusoids are present in a function. On the other hand, the Laplace Transformation tells us which sinusoids and exponentials are present in a function. That means that the Fourier transform is just a slice of the Laplace transform.

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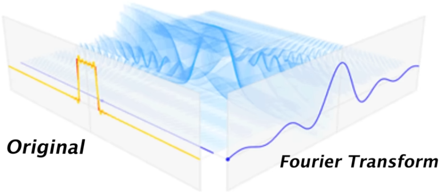
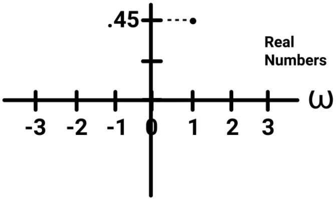


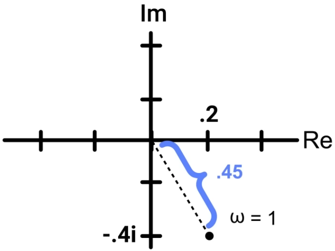
Another interesting thing is, the y intercept is the area under the curve. Why?

Because insert $\omega = 0$ give what?

Okay, now start with $f(t) = e^{-t}\sin t, t \geq 0$ then,

$$\begin{aligned}\mathscr{F}(\omega) &= \int_0^\infty e^{-t}\sin t\, e^{-i\omega t}dt \\ &= \frac{1}{1+(1+i\omega)^2} \\ &= \frac{1}{\underbrace{2-\omega^2}_{\text{real}} + \underbrace{2\omega i}_{\text{imaginary}}}\end{aligned}$$





$$\underset{\substack{\uparrow\\ \text{out}}}{X(\omega)}=\int_{-\infty}^{\infty} \underset{\substack{\uparrow\\ \text{in}}}{x(t)}e^{-i\omega t}\,dt$$

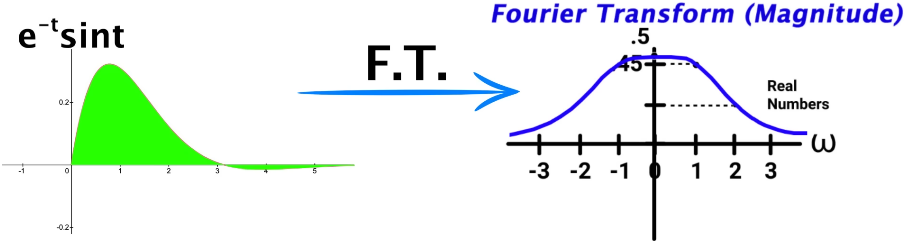
$$X(\omega)=\int_{0}^{\infty} e^{-t}\sin t\, e^{-i\omega t}\,dt$$

$$\omega = 1 \quad X(1) = \frac{1}{2-(1)^2 + 2(1)i}$$

$$\frac{1}{1+2i} = .2-.4i$$

Figure: The right image show the value of $X(1)$, and the left image plot the magnitude on top of the input

Visual Fourier



Laplace Transform

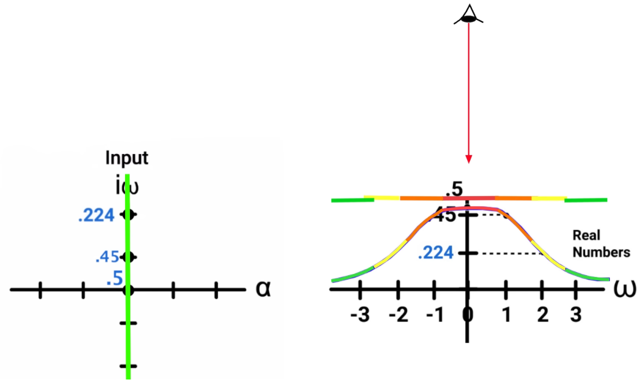
Where the Laplace Transform is,

$$
\begin{aligned}
\mathscr{L}(s) &= \int_0^\infty f(t)e^{-st}dt, s \in \mathbb{C} \\
&\overset{s=\alpha+i\omega}{=} \int_0^\infty f(t)e^{-(\alpha+i\omega)t}dt \\
&= \int_0^\infty f(t)e^{-\alpha t}e^{-i\omega t}dt
\end{aligned}
$$

That means the Laplace Transform of $f(t)$ is the Fourier transform of $f(t)e^{-\alpha t}$. Do the same computation for $f(t) = e^{-t}\sin t, t \geq 0$. Then,

$$
\begin{aligned}
\mathscr{L}(s) &= \frac{1}{1+(1+s)^2} \\
&= \frac{1}{1+(1+\alpha+i\omega)^2}
\end{aligned}
$$

If $\alpha = 0$, we get the exact Fourier Transform.

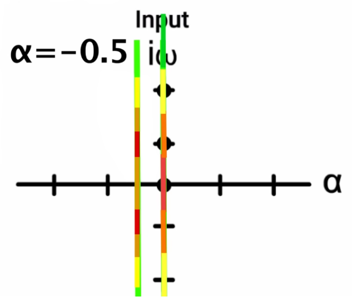


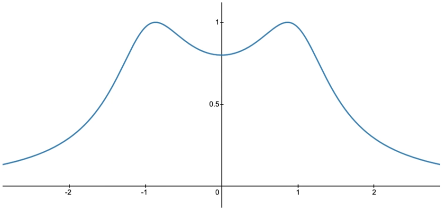
Here the green line represents the slice where $\alpha = 0$, which is the Fourier Transform. But we can't see the magnitude plot this time. So, we use color to represent magnitude. Brighter color means higher magnitude.

Laplace Transform

$$X(s)=\int_0^{\infty} x(t)e^{0.5t}e^{-iwt}dt$$

$$\mathcal{F}\{(e^{-t}\sin t)e^{0.5t}\}$$





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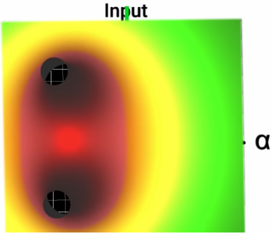

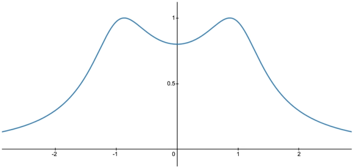

See the animation for the complete visualization: `https://youtube.com/clip/UgkxEzdqZqm4uYFIgYPmoLHf6dGBKRtj7TXh?si=VI5mkUaOHnu5nGqc`. Where the 3D plot show the magnitude of the output.

All we care about the pole which happen at $\alpha = -1$ with the imaginary coordinate $\omega = i$ and $\omega = -i$, which matches the coefficient or angular frequency of our sinusoid, $\sin t$ (because at that point the magnitude shoots to infinity). The real component -1 matches the coefficient in the exponential term, e^{-t}.
Here is the full animation: `https://youtube.com/clip/UgkxCi_tdc5Wwr0rwwwkra50Si3TYPwJgIxn?si=sUwSRDF242P1nY-K`

By the way, we ignore the convergence issue for simplicity

What if our pole was at on the real or imaginary axis?

Explore more, untill Khatam, Tata, Bye Bye, Gaya!