1. Set theory and Logic
   1. Fundamental concepts
      1. Basic Notation
         1. Sets <- Capital letters
         2. Objects (elements) <- lowercase letters
         3. Logical Identity <- = (equality symbol)
         4. A ⊂ B <- A is a subset of B , inclusion
            1. Every element of A is also an element of B
         5. A \subsetneq B <- A is a proper subset of B, proper inclusion
            1. A \subset B and A is different from B
      2. The Union of Sets and the meaning of ‘or’
         1. Union of A and B ( A \cup B) = { x | x \in A or x \in B}
            1. P or Q : P or Q or both
      3. The Intersection of Sets, the Empty Set and the Meaning of “if .. Then”
         1. Intersection of A and B ( A \cap B) = { x | x \in A and x \in B}
         2. Empty set \empty : the set having no elements
            1. A \cup \empty = A
            2. A \cap \empty = \empty
         3. Disjoint (a set A, a set B) : A \cap B = \empty
         4. If (Hypothesis), then (conclusion)
            1. Vacuously true : no case for which the hypothesis holds
      4. Contrapositive and Converse
         1. Contrapositive ( ‘If P, then Q ‘ statement) : If Q is not true, then P is not true
         2. Converse ( ‘If P, then Q ‘ statement) : If Q, then P
         3. P <-> Q : P holds iff Q holds
      5. Negation ( a statement P ) : not P
      6. The difference of two sets
         1. Difference of two sets A, B ( Complement of B relative to A)
            1. A – B = { x | x \in A and x \notin B}
      7. Rules of set theory
         1. Distributive law (sets A, B, C)
            1. A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
            2. A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
         2. Demorgan’s laws
            1. A – (B \cup C) = (A – B) \cap (A – C)
            2. A – (B \cap C) = (A – B) \cup (A – C)
      8. Collection of the sets
         1. Power set of A ( \mathcal{P} (A) ): set of all subsets of A
         2. Collection of sets : a set whose elements are sets
      9. Arbitrary unions and intersections
         1. Given a collection \mathcal{A} of sets,
            1. Union of the elements of \mathcal{A}

\bigcup\_{A \in \mathcal{A}} A = { x | x \in A for at least one A \in \mathcal{A}}

* + - * 1. Intersection of the elements of \mathcal{A}

\bigcap\_{A \in \mathcal{A}} A = { x | x \in A for every A \in \mathcal{A}}

* + - * 1. \bigcup\_{A \in \mathcal{A}} A = \empty
        2. \bigcap\_{A \in \mathcal{A}} A = (Not defined)
    1. Cartesian Product
       1. Ordered pair of objects a , b : a \times b
          1. All ordered pairs of real numbers is a plane
       2. Cartesian product ( sets A , B) : A \times B = {a \times b | a \in A and b \in B}
  1. Function
     1. A rule of assignment : a subset r of the cartesian product C x D of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r.
        1. [c \times d \in r and c \times d’ \in r] -> [d = d’]
        2. Domain (a rule of assignment r) : { c | there exists d \in D s.t. c \times d \in r}
        3. Image ( a rule of assignment r) : {d | there exists c \in C s.t. c \times d \in r}
     2. Function f : a rule of assignment r, together with a set B that contains the image set of r.
        1. Domain (a function f) : Domain of the rule r
        2. Image set (a function f) : the image set of r
        3. Range ( a function f) : the set B
           1. f is a function having domain A and range B : f : A -> B
        4. value ( function f, element a of Domain (f) ) : unique element of B the rule determining f assigns to a
     3. Restriction of function f | A\_{0} : If f : A -> B and if A\_{0} is a subset of A, we define the restriction of f to A\_{0} to be the function mapping A\_{0} into B whose rule is {(a,f(a)) | a \in A\_{0}}
     4. Composite g \bullet f : given functions f : A->B and g : B->C, the composite of f and g is the function g \bullet f : A -> C defined by equation (g \bullet f)(a) = g(f(a))
     5. Injective : A function f : A -> B is said to be injective (or one -to- one) if for each pair of distinct points of A, their images under f are distinct.
     6. Surjective : f is said to map A onto B if every element of B is the image of some element of A under the function f.
     7. Bijective : f is said to be one-to-one correspondence if f is both injective and surjective
        1. If f is bijective, there exists a function from B to A, called the inverse of f. f^{-1}
        2. (Criterion of bijection) : Let f : A->B. if there are functions g : B->A and h:B->A s.t. g(f(a)) = a for every a in A and f(h(b)) = b for every b in B, then f is bijective and g = h = f^{-1}
     8. Image of A\_{0} under f : Let f:A->B. if A\_{0} is a subset of A, we denote by f(A\_{0}) the set of all images of points of A\_{0} under the function.
        1. Preimage of B\_{0} under f : if B\_{0} is a subset of B, f^{-1}(B\_{0}) is the set of all elements of A whose images under f lie in B\_{0}.
        2. A\_{0} \subseteq f^{-1}(f(A\_{0})) (equality if f is injective)
        3. F(f^{-1}(B\_{0})) \subseteq B\_{0} (equality if f is surjective)
  2. Relations
     1. A relation on a set A : a subset C of the cartesian product A \times A
     2. Equivalence relation on a set A : a relation C on A with
        1. (Reflexivity) : xCx for every x in A
        2. (Symmetry) if xCy, then yCx
        3. (Transitivity) if xCy and yCz, then xCz
     3. Equivalence class determined by x : given an equivalence relation ~ on a set A and an element x of A, we define a certain subset E of A, by the equation E = { y | y ~ x }
        1. Two equivalence classes E and E’ are either disjoint or equal.
     4. Partition of a set A : a collection of disjoint nonempty subsets of A whose union is all of A.
        1. Given any partition \mathcal{D} of A, there is exactly one equivalence relation on A from which it is derived.
     5. Order relations ( a relation C on a set A)
        1. (Comparability) For every x and y in A for which x \neq y, either xCy or yCx.
        2. (nonreflexivity) : For no x in A does the relation xCx hold.
        3. (Transitivity) If xCy and yCz, then xCz
     6. Open interval in X : if X is a set and < is an order relation on X, and if a < b, we use the notation (a, b) to denote the set {x | a < x < b }
        1. a is the immediate predecessor of b, and b is the immediate predecessor of a if (a, b) is empty
     7. Suppose that A and B are two sets with order relations <\_{A} and <\_{B} respectively, We say that A and B have the same order type if there is a bijective correspondence between them that preserves order
        1. Preserve order : a\_{1} <\_{A} a\_{2} -> f(a\_{1}) <\_{B} f(a\_{2})
     8. Dictionary order relation : Suppose A and B are two sets with order relations <\_{A} and <\_{B} respectively, define an order relation on A \times B by defining a\_{1} \times b\_{1} < a\_{2} times b\_{2} if a\_{1}<\_{A}a\_{2} or if a\_{1} = a\_{2} and b\_{1} <\_{B} b\_{2}.
     9. Largest element of A\_{0} (an element b) : Suppose that A is a set ordered by relation <. Let A\_{0} be a subset of A. b \in A\_{0} and if x \le b for every x \in A\_{0}
     10. Smallest element of A\_{0} (an element a) : a \in A\_{0} and if a \le x for every x \in A\_{0}
     11. Bounded above (a subset A\_{0} of A) : there is an element b of A s.t. x \le b for every x \in A\_{0}.
         1. Upper bound for A\_{0} : b
         2. The least upper bound of A\_{0} [supremum of A\_{0}, sup A\_{0}] : a smallest element of the set of all upper bounds for A\_{0}
         3. Least upper bound property ( an ordered set A) : every nonempty subset A\_{0} of A that is bounded above has a least upper bound
     12. Bounded below (a subset A\_{0} of A) : there is an element a of A s.t. a \le x for every x \in A\_{0}.
         1. Lower bound for A\_{0} : a
         2. The greatest lower bound of A\_{0} [infimum of A\_{0}, inf A\_{0}] : a largest element of the set of all lower bounds for A\_{0}
         3. Greatest lower bound property (an ordered set A) : every nonempty subset A\_{0} of A that is bounded below has the greatest lower bound property.
  3. The Integers and the real numbers
     1. Binary operation on a set A : a function f mapping A \times A into A
     2. Assume the existence of the set of real numbers \mathbb{R} with +, \cdot, <
        1. Algebraic properties
           1. (x+y) + z = x + (y+z) , (x \cdot y) \cdot z = x \cdot (y \cdot z) for all x, y, z in \mathbb{R}
           2. x + y = y + x, x \cdot y = y \cdot x for all x,y in \mathbb{R}
           3. There exists a unique element of \mathbb{R} called zero, denoted by 0, s.t. x + 0 = x for all x \in \mathbb{R}, There exists a unique element of \mathbb{R} called one, denoted by 1, s.t. x \cdot 1 = x for all x \in \mathbb{R}
           4. (Negative of x) For each x in \mathbb{R}, there exists a unique y in \mathbb{R} s.t. x + y = 0,
           5. (Reciprocal of x) For each x in \mathbb{R} different from 0, there exists a unique y in \mathbb{R} s.t. x\cdot y = 1.
           6. x \cdot (y+z) = (x \cdot y) + (x \cdot z) for all x, y, z \in \mathbb{R}.
        2. A Mixed Algebraic and Order Property
           1. if x > y, then x + z > y + z. If x > y and z > 0, then x \cdot z > y \cdot z
        3. Order properties
           1. The order relation < has the least upper bound property.
           2. If x < y, there exists an element z s.t. x < z and z < y
        4. Subtraction operation : z-x = z + (-x)
        5. Quotient z/x = z \cdot (1/x)
        6. Laws of inequalities : If x > y and z < 0, then x \cdot z < y \cdot z
        7. Field : only algebraic properties of above
        8. Ordered field : algebraic property and a mixed algebraic and order property of above
        9. Linear continuum : only order properties of above
     3. Inductive ( a subset of \mathbb{R}) : it contains the number 1 and if for every x in A, the number x + 1 is also in A.
     4. Positive Integers Z\_{+} = \bigcap\_{A \in \mathcal{A}} A
        1. Z\_{+} is inductive
        2. (Principle of Induction) : if A is an inductive set of positive integers, then A = Z\_{+}
     5. Integers \mathbb{Z} : a set consisting of Z\_{+}, the number 0, the negatives of the elements of Z\_{+}
     6. Rational numbers \mathbb{Q} : a set of quotients of integers
     7. Section S\_{n} of the positive integers : S\_{n+1} = {1,…,n}, S\_{1} = \empty
     8. (Well-ordering property) : Every nonempty subset of \mathbb{Z}\_{+} has a smallest element.
     9. (Strong induction principle) : Let A be a set of positive integers. Suppose that for each positive integer n, the statement S\_{n} \subset A implies the statement n \in A, then A = \mathbb{Z}\_{+}.
     10. Proofs using least upper bound axiom
         1. Archimedian ordering property of the real line : \mathbb{Z}\_{+} of positive integers has no upper bound in \mathbb{R}
         2. Greatest lower bound property
         3. Existence of a unique positive square root for every positive real number
  4. Cartesian Products
     1. Let \mathcal{A} be a nonempty collection of sets.
        1. Indexing function (\mathcal{A}) : a surjective function f from some set J to \mathcal{A}.
           1. Index set : J
           2. Indexed family of sets {A\_{\alpha}}\_{\alpha \in J} : the collection \mathcal{A} with indexing function f

A\_{\alpha} = f(\alpha)

* + 1. \bigcup\_{\alpha \in J} A\_{\alpha} = {x | for at least one \alpha \in J, x \in A\_{\alpha}}
    2. \bigcap\_{\alpha \in J} A\_{\alpha} = {x | for every \alpha \in J, x \in A\_{\alpha}}
    3. M-tuple of elements of X : Let m be a positive integer. Given a set, a function \mathbf{x} : {1,…,m} -> X
       1. (x\_{1}, …, x\_{m})
       2. Ith coordinate of \mathbf{x} : value of \mathbf{x} at I
       3. Cartesian product ({A\_{1}, …, A\_{m}} being a family of sets indexed with the set {1,…,m}) :
          1. Let X = A\_{1} \cup … \cup A\_{m}. then \prod\_{I = 1}^{m} A\_{i} or A\_{1} \times … \times A\_{m} = set of all m-tuples (x\_{1} , …, x\_{m}) of elements of X s.t. x\_{i} \in A\_{i} for each i.
       4. X^{m} <- M-tuple of elements of X
    4. \omega -tuple of elements of a set X : \mathbf{x} : \mathbb{Z}\_{+} -> X
       1. Also called sequence , infinite sequence, (x\_{1}, x\_{2}, …) , (x\_{n})\_{n \in \mathbb{Z}\_{+}\_
       2. Ith coordinate of \mathbf{x} : value of \mathbf{x} at I
       3. Cartesian product ({A\_{1}, A\_{2}, … } being a family of sets indexed with positive integers) :
       4. Let X = Union of the sets in this family. then \prod\_{I = \mathbb{Z}\_{+}} A\_{i} or A\_{1} \times A\_{2} \times… = set of all \omega -tuples (x\_{1} , x\_{2}… ) of elements of X s.t. x\_{i} \in A\_{i} for each i.
       5. X^{\omega} <- 1.5.5 \omega -tuple of elements of a set X
  1. Finite sets
     1. Finite (a set) : there is a bijective correspondence of A with some section of the positive integers.
        1. Cardinality 0 : A is empty
        2. Cardinality n : there is a bijection f:A->{1,..,n} for some positive integer n
     2. (Thm 6.1) Let A be a set. Suppose that there exists a bijection f:A-> {1,..,n} for some n \in \mathbb{Z}\_{+}. Let B be a proper subset of A. then there exists no bijection g : B->{1,…,n}. But provided B \neq \empty, there does exist a bijection h:B->{1,…,m} for some m<n.
        1. Let n be a positive integer. Let A be a set. Let a\_{0} be an element of A. There exists a bijective correspondence f of the set A with the set {1,…,n+1} iff there exists a bijective correspondence g of the set A-{a\_{0}} with the set {1,…,n}.
        2. (Cor) If A is finite, there is no bijection of A with a proper subset of itself.
        3. (Cor) Z\_{+} is not finite.
        4. (Cor) The cardinality of a finite set A is uniquely determined by A.
        5. (Cor) If B is a subset of the finite set A, then B is finite. If B is a proper subset of A, then the cardinality of B is less than the cardinality of A.
        6. (Cor) Let B be a nonempty set. Then the following are equivalent.
           1. B is finite.
           2. There is a surjective function from a section of the positive integers onto B.
           3. There is an injective function from B into a section of the positive integers.
        7. (Cor) Finite unions and finite cartesian products of finite sets are finite.
           1. If A and B are finite, so is A \cup B.
  2. Countable and Uncountable Sets
     1. Infinite : not finite
     2. Countably infinite : there is a bijective correspondence f : A-> \mathbb{Z}\_{+}
     3. Countable (a set) : it is finite or countably finite
     4. Uncountable (a set) : not countable
     5. (Thm 7.1) : Let B be a nonempty set. Then the following are equivalent.
        1. B is countable.
        2. There is a surjective function f : \mathbb{Z}\_{+} -> B
        3. There is an injective function g : \B -> \mathbb{Z}\_{+}
        4. (Lem) If C is an infinite subset of \mathbb{Z}\_{+}, then C is countably infinite.
           1. H(n) = smallest element of [C- H({1,…,n-1})], H(1) = smallest element of C
     6. Principle of recursive definition : Let A be a set. A recursive formula determines a unique function h: \mathbb{Z}\_{+} -> A.
        1. Recursive formula : a formula that defines h(1) as a unique element of A, and for i>1 defines h(i) uniquely as an element of A in terms of the values of h for positive integers less than I.
     7. (Cor) A subset of a countable set is countable.
     8. (Cor) the set \mathbb{Z}\_{+} \times \mathbb{Z}\_{+} is countably infinite.
     9. (Thm 7.5) A countable union of countable sets is countable.
     10. (Thm 7.6) A finite product of countable sets is countable.
     11. (Thm 7.7) Let X be the two element set {0,1}. Then the set X^{\omega} is uncountable.
     12. (Thm 7.8) Let A be a set. There is no injective map f: \mathcal{P} (A) -> A, and there is no surjective map g: A->\mathcal{P}
  3. The principle of Recursive Definition
     1. (Thm 8.3) There exists a unique function h: \mathbb{Z}\_{+} -> C satisfying recursive formula for all I \in \mathbb{Z}\_{+}.
        1. (Lem)Given n \in \mathbb{Z}\_{+}, there exists a function f:{1,…,n} ->C that satisfies the recursive formula for all I in its domain.
        2. (Lem)Suppose that f:{1,…,n} -> C and g : {1,…,m} ->C both satisfy recursive formula for all I in their respective domains. Then f(i) = g(i) for all I in both domains.
     2. (Thm 8.4) (Principle of recursive definition) : let A be a set; let a\_{0} be an element of A. Suppose \rho is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function h: \mathbb{Z}\_{+} -> A s.t. recursion formula for h satisfied.
        1. Recursion formula for h :
           1. H(1) = a\_{0}
           2. H(i) = \rho(h | {1,…,i-1}) for I >1
  4. Infinite sets and the axiom of choice
     1. (Thm 9.1) Let A be a set. The following statements about A are equivalent.
        1. There exists an injective function f : \mathbb{Z}\_{+} -> A
        2. There exists a bijection of A with a proper subset of itself.
        3. A is infinite.
        4. Pf) Axiom of choice is needed., recursion formula, c: \mathcal{B} -> \bigcup\_{B \in \mathcal{B}} B = A
     2. (Axiom of choice) : Given a collection \mathcal{A} of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of \mathcal{A}
        1. A set C s.t. C is contained in the union of the elements of \mathcal{A}, and for each A \in \mathcal{A}, the set C\cap A contains a single element.
     3. (Existence of a choice function)(Lem)
        1. Given a collection \mathcal{B} of nonempty sets not necessarily disjoint, there exists a function c : \mathcal{B} -> \bigcup\_{B \in \mathcal{B}} B s.t. c(B) is an element of B, for each B \in \mathcal{B}.
           1. A function c is called a choice function for the collection \mathcal{B} .
     4. Finite axiom of choice : given a finite collection \mathcal{A} of disjoint nonempty sets, there exists a set \mathcal{C} consisting of exactly one element from each element of \mathcal{A}.
        1. Weaker form of axiom of choice.
  5. Well-Ordered sets
     1. Well-ordered ( a set A with an order relation < ) : every nonempty subset of A has a smallest element.
        1. Constructing well-ordered sets
           1. If A is a well-ordered set, then any subset of A is well-ordered in the restricted order relation.
           2. If A and B are well-ordered sets, then A \times B is well-ordered in the dictionary order.
     2. (Thm 10.1) Every nonempty finite ordered set has the order type of a section {1,…,n} of \mathbb{Z}\_{+}, so it is well-ordered.
     3. (Well-ordering theorem) If A is a set, there exists an order relation on A that is well-ordering.
        1. Pf) choice axiom
     4. (Cor) There exists an uncountable well-ordered set.
     5. Section S\_{\alpha} of X by \alpha : Let X be a well-ordered set. Given \alpha \in X, S\_{\alpha} = {x | x \in X and x < \alpha}
     6. (Thm 10.3) If A is a countable subset of S\_{\Omega}, then A has an upper bound in S\_{\Omega}.
        1. (Lem) There exists a well-ordered set A having a largest element \Omega, s.t. the section S\_{\Omega} of A by \Omega is uncountable but every other section of A is countable.
  6. The Maximum principle
     1. Strict partial order on A (a relation < on A)
        1. (Nonreflexivity) The relation a<a never holds
        2. (Transitivity) If a<b and b<c, then a<c
     2. (The maximum principle) Let A be a set; let < be a strict partial order on A. Then there exists a maximal simply ordered subset of B.
        1. Pf) well-ordering theorem
     3. Let A be a set and let < be a strict partial order on A.
        1. Upper bound on B ( B a subset of A) : an element c of A s.t. for every b in B, either b = c or b < c.
        2. Maximal element of A : an element m of A s.t. for no element a of A does the relation m < a hold.
     4. (Zorn’s Lemma) : Let A be a set that is strictly partially ordered. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.
     5. Partial order on A : let < be a strict partial order on a set A. Then if we define a \le b either a < b or a = b, then the relation \le is a partial order.

1. Topological spaces and continuous functions
   1. Topological spaces
      1. Topology on a set X : a collection \Tau of subsets of X having following properties.
         1. \empty and X are in \Tau.
         2. Union of the elements of any subcollection of \Tau is in \Tau.
         3. Intersection of the elements of any finite subcollection of \Tau is in \Tau.
      2. Topological space : ordered pair (X, \Tau)
         1. X <- ordered pair (X, \Tau)
      3. Open set of X (a subset U of X) : Let X is a topological space with topology \Tau, if U belongs to the collection \Tau.
      4. Sort
         1. Discrete topology ( a set X) : a collection of all subsets of X
         2. Indiscrete topology ( a set X) : {X, \empty} ~ trivial topology
         3. Finite complement topology \Tau\_{f} ( a set X) : Collection of all subsets U of X s.t. X-U either is finite or is all of X.
      5. Suppose \Tau and \Tau’ are two topologies on a given set X.
         1. \Tau’ is finer than \Tau : \Tau \subset \Tau’
         2. \Tau’ is strictly finer than \Tau : \Tau \subsetneq \Tau’
         3. \Tau’ is coarser than \Tau : \Tau’ \subset \Tau
         4. \Tau’ is strictly coarser than \Tau : \Tau’ \subsetneq \Tau
         5. \Tau is comparable with \Tau’ : \Tau’ \subset \Tau or \Tau \subset \Tau’
   2. Basis for a Topology
      1. Basis for a topology on X (a set X) : a collection \mathcal{B} of subsets of X (called basis elements) s.t.
         1. For each x \in X, there is at least one basis element B containing x.
         2. If x belongs to the intersection of two basis elements B\_{1} and B\_{2}, then there is a basis element B\_{3} containing x s.t. B\_{3} \subset B\_{1} \cap B\_{2}.
      2. Topology \Tau generated by \mathcal{B}
         1. Open in X ( a subset U of X) : for each x \in U, there is a basis element B \in \mathcal{B} s.t. x \in B and B \subset U.
      3. (Lem 13.1) : Let X be a set; let \mathcal{B} be a basis for a topology \Tau on X. then \Tau equals the collection of all unions of elements of \mathcal{B}.
      4. (Lem 13.2) : Let X be a topological space. Suppose that \mathcal{C} is a collection of open sets of X s.t. for each open set U of X and each x in U , there is an element C of \mathcal{C} s.t. x \in C \subset U. Then \mathcal{C} is a basis for the topology of X.
      5. (Lem 13.3) Let \mathcal{B} and \mathcal{B} be bases for the topologies \Tau and \Tau’ respectively on X. Then the following are equivalent.
         1. \Tau’ is finer than \Tau.
         2. For each x \in X and each basis element B \in \mathcal{B} containing x, there is a basis element B’ \in \mathcal{B}’ s.t. x \in B’ \subset B.
      6. Standard topology on the real line : Topology generated by \mathcal{B}
         1. \mathcal{B} is the collection of all open intervals in the real line, (a,b) = {x | a < x < b}.
      7. Lower limit topology \mathbb{R}\_{l} on \mathbb{R} : Topology generated by \mathcal{B}’.
         1. \mathcal{B}’ is the collection of all half-open intervals
      8. K-topology \mathbb{R}\_{K} on \mathbb{R} : Topology generated by \mathcal{B}’’
         1. \mathcal{B}’’ is the collection of all open intervals (a,b) , along with all sets of the form (a,b) – K
            1. K = {x | x = 1/n, for n \in \mathbb{Z}\_{+}}
      9. (Lem 13.4) Topologies of \mathbb{R}\_{l} and \mathbb{R}\_{K} are strictly finer than the standard topology on \mathbb{R}, but not comparable with one another.
      10. Subbasis \mathcal{S} for a topology on X : a collection of subsets of X whose union equals X
          1. Topology generated by the subbasis \mathcal{S} : the collection \Tau of all unions of finite intersections of elements of \mathcal{S}
   3. Order topology
      1. Order topology : If X is a simply ordered set, there is a standard topology for X, defined using order relation.
      2. Suppose X is a set having a simple order relation <. Given elements a and b of X s.t. a <b,
         1. Open interval (a,b) = {x| a<x<b}
         2. Closed interval [a,b] = {x | a \le x \le b}
         3. Half-open intervals]
            1. (a,b]= {x | a < x \le b }
            2. [a,b) = {x | a \le x < b}
      3. Order topology : Let X be a set with simple order relation, assume X has more than one element. Let \mathcal{B} be the collection of all sets of the following types. The coillection \mathcal{B} is a basis for a topology on X, which is order topology.
         1. All open intervals in X
         2. All intervals of the form [a\_{0},b) , where a\_{0} is the smallest element (if any) of X.
         3. All intervals of the form (a, b\_{0}], where b\_{0} is the largest element (if any) of X.
      4. If X is an ordered set and a is an element of X, there are four subsets of X that are called rays determined by a.
         1. Open rays
            1. (a, +\infty) = {x | x > a}
            2. (-\infty, a) = {x | x < a}
         2. Closed rays
            1. [a, + \infty) = {x | x \ge a}
            2. (-\infty, a] = {x | x \le a}
         3. A topology generated using open rays as a subbasis contains the order topology
   4. Product Topology on X \times Y
      1. Product topology on X \times Y ( topological spaces X and Y) : topology having as basis the collection \mathcal{B} of all sets of the form U \times V, where U is an open subset of X and V is an open subset of Y.
      2. (Thm 15.1) If \mathcal{B} is a basis for the topology of X and \mathcal{C} is a basis for the topology of Y, then the collection \mathcal{D} = {B \times C | B \in \mathcal{B} and C \in \mathcal{C}} is a basis for the topology of X \times Y.
         1. Pf) (Lem 13.2)
      3. Projections of X \times Y onto its first and second factors
         1. \pi\_{1} : X \times Y -> X , \pi\_{1} (x,y) = x
         2. \pi\_{2} : X \times Y -> Y, \pi\_{2} (x,y) = y
      4. (Thm 15.2) The collection \mathcal{S} = {\pi^{-1}\_{1}(U) | U open in X} \cup {\pi^{-1}\_{2}(V) | V open in Y} is a subbasis for the product topology on X \times Y.
   5. Subspace topology
      1. Subspace topology : Let X be a topological space with topology \Tau. If Y is a subset of X, the collection \Tau\_{Y} = {Y \cap U | U \in \Tau} is a topology on Y.
         1. Subspace of X : Y with this topology
      2. (Lem 16.1) If \mathcal{B} is a basis for the topology of X then the collection \mathcal{B}\_{Y} = {B \cap Y | B \in \mathcal{B}} is a basis for the subspace topology on Y.
         1. Pf) (Lem 13.2)
      3. Open in Y ( a set U ) : U belongs to the topology of Y.
      4. (Lem 16.2) Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.
      5. (Thm 16.3) If A is a subspace of X and B is a subspace of Y, then the product topology on A \times B is the same as the topology A \times B inherits as a subspace of X \times Y.
      6. Ordered square I^{2}\_{0} : set I \times I in the dictionary order topology (I = [0,1])
         1. Dictionary order topology of it \neq subset topology inherited from \mathbb{R}^{2}
      7. Convex in X (a subset Y of an ordered set X) : for each pair of points a<b of Y, the entire interval (a,b) of points of X lies in Y.
      8. (Thm 16.4) : Let X be an ordered set in the order topology; Let Y be a subset of X that is convex in X, Then the order topology on Y is the same as the topology Y inherits as a subspace of X.]
   6. Closed sets and limit points
      1. Closed
         1. Closed ( a subset A of a topological space X) : if the set X-A is open
         2. (Thm 17.1) Let X be a topological space, then the following conditions hold
            1. \empty and X are closed.
            2. Arbitrary intersections of closed sets are closed
            3. Finite unions of closed sets are closed.
         3. Closed in Y ( a set A) : If Y is a subspace of X, if A is a subset of Y and if A is closed in the subspace topology of Y.
         4. (Thm 17.2) Let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y.
         5. (Thm 17.3) Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.
      2. Closure and Interior of a set
         1. Interior ( a subset A of a topological space X) : union of all open sets contained in A
         2. Closure ( a subset A of a topological space) : intersection of all closed sets containing A.
         3. Int A \subset A \subset \bar{A}
         4. (Thm 17.4) Let Y be a subspace of X. Let A be a subset of Y. Let \bar{A} denote the closure of A in X. Then the closure of A in Y equals \bar{A} \cap Y.
            1. Pf) Thm 17.2
         5. Intersects (sets A, B) : the intersection A \cap B is not empty.
         6. (Thm 17.5) Let A be a subset of the topological space X.
            1. x \in \bar{A} iff every open set U containing x intersects A.
            2. Suppose the topology of X is given by a basis, then x \in \bar{A} iff every basis element B containing x intersects A.
         7. U is a neighborhood of x (a set U, an element x) : U is an open set containing x
      3. Limit points
         1. Limit point x of A ( a point x in X , a subset A of the topological space X) : every neighborhood of x intersects A in some point other than x itself.
            1. x is a limit point of A if it belongs to the closure A – {x}
         2. (Thm 17.6) Let A be a subset of the topological space X. Let A’ be the set of all limit points of A. Then \bar{A} = A \cup A’.
            1. Pf) Thm 17.5.
            2. (Cor) A subset of a topological space is closed iff it contains all its limit points.
      4. Hausdorff spaces
         1. Converges to the point x of X ( a sequence x\_{1}, x\_{2}, … of the points of the space X) : corresponding to each neighborhood U of x, there is a positive integer N s.t. x\_{n} \in U for all n \ge N.
         2. Hausdorff space ( a topological space X) : for each pair x\_{1}, x\_{2} of distinct points of X, there exists neighborhoods U\_{1}, U\_{2} of x\_{1} and x\_{2}, respectively, that are disjoint.
         3. (Thm 17.8) Every finite point set in a Hausdorff space X is closed.
         4. T\_{1} axiom : Every finite point sets in a space are closed
         5. (Thm 17.9) Let X be a space satisfying the T1 axiom. Let A be a subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points of A.
         6. (Thm 17.10) If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.
            1. x\_{n} -> x <= Limit x of the sequence x\_{n}
         7. (Thm 17.11) Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.
   7. Continuous functions
      1. Continuity of a Function
         1. Continuous ( A function f: X ->Y, Topological spaces X , Y) : For each open subset V of Y, the set f^{-1}(V) is an open subset of X.
            1. F is continuous relative to specific topologies on X and Y.
         2. (Thm 18.1) Let X and Y be topological spaces. Let f : X ->Y. then the following are equivalent.
            1. f is continuous.
            2. For every subset A of X, one has f(\bar{A}) \subset \bar{f(A)}.
            3. For every closed set B of Y, the set f^{-1}(B) is closed in X.
            4. For each x \in X and each neighborhood V of f(x), there is a neighborhood U of x s.t. f(U) \subset V.

f is continuous at the point x.

* + 1. Homeomorphism
       1. Homeomorphism ( f : X -> Y, topological spaces X, Y) : Let f be bijection. Both the function and the inverse function f^{-1} : Y ->X are continuous,
       2. Topological property of X : any property of X expressed in terms of the topology of X yields, via the correspondence f, the corresponding property for the space Y.
       3. Topological imbedding f of X in Y ( topological spaces X and Y, f : X->Y injective) : the function f’ : X -> f(X) , which is bijective, happens to be homeomorphism
       4. Unit circle s^{1} = { x \times y | x ^{2} + y^{2} = 1} , [0,1)
          1. F : [0,1) -> s^{1} , (cos 2\pi t, sin 2\pi t)
    2. Constructing Continuous functions
       1. (Rules for constructing continuous functions)(Thm 18.2) : Let X, Y, Z be topological spaces.
          1. (Constant function) If f : X->Y maps all of X into the single point y\_{0} of Y, then f is continuous.
          2. (Inclusion) If A is a subspace of X, the inclusion function j : A ->X is continuous.
          3. (Composites) If f : X->Y and g : Y ->Z are continuous, then the map g \bullet f : X -> Z is continuous.
          4. (Restricting the domain) If f : X->Y is continuous, and if A is a subspace of X, then the restricted function f | A : A->Y is continuous.
          5. (Restricting or expanding the range) Let f : X -> Y be continuous. If Z is a subspace of Y containing the image set f(X), then the function g : X ->Z obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function h: X ->Z obtained by expanding the range of f is continuous.
          6. (Local formulation of continuity) The map f: X->Y is continuous if X can be written as the union of open sets U\_{\alpha} s.t. f | U\_{\alpha} is continuous for each \alpha.
       2. (The pasting lemma)(Thm 18.3) : Let X = A \cup B, where A and B are closed in X. Let f : A -> Y and g : B -> Y be continuous. If f (x) = g (x) for every x \in A \cap B, then f and g combine to give a continuous function h : X -.Y, defined by setting h(x) = f(x) if x \in A, and h(x) = g(x) if x \in B.
       3. (Map into products) (Thm 18.4) Let f : A -> X \times Y be given by the equation f(a) = (f\_{1}(a), f\_{2}(a)). Then f is continuous iff the function f\_{1} : A -> X and f\_{2} : A -> Y are continuous.
          1. Maps f\_{1} and f\_{2} are called the coordinate functions of f.
       4. (Uniform Limit Theorem on a Real space) : If a sequence of continuous real-valued functions of a real variable converges uniformly to a limit function, then the limit function is necessarily continuous.
  1. Product Topology
     1. J-tuple of elements of X ( an Index set J, a set X) : a function \mathbf{x} : J -> X
        1. \alpha th coordinate x\_{\alpha} of \mathbf{x} : the value of \mathbf{x} at \alpha
        2. (x\_{\alpha})\_{\alpha \in J} 🡸 \mathbf{x}
     2. Cartesian product \prod\_{\alpha \in J} A\_{\alpha} ( an indexed family {A\_{\alpha}}\_{\alpha \in J} ) : a set of all J-tuples (x\_{\alpha})\_{\alpha \in J} of elements of X s.t. x\_{\alpha} \in A\_{\alpha} for each \alpha \in J.
        1. Set of all functions \mathbf{x} : J -> \bigcup\_{\alpha \in J} A\_{\alpha} s.t \mathbf{x} (\alpha) \in A\_{\alpha} for each \alpha \in J.
     3. Box topology( {X\_{\alpha}}\_{\alpha \in J}) : the topology generated by this basis
        1. Basis for a topology on the product space \prod\_{\alpha \in J} X\_{\alpha} the collection of all sets of the form \prod\_{\alpha \in J} U\_{\alpha} where U\_{\alpha} is open in X\_{\alpha}.
        2. Projection mapping associated with the index \beta : \pi\_{\beta}((x\_{\alpha})\_{\alpha \in J}) = x\_{\beta}
     4. Product topology : The topology generated by the subbasis \mathcal{S}
        1. \mathcal{S} = \bigcup\_{\beta \in J} \mathcal{S}\_{\beta}
           1. \mathcal{S}\_{\beta} = {\pi^{-1}\_{\beta}(U\_{\beta}) | U\_{\beta} open in X\_{beta}}
        2. Product space : \prod\_{\alpha \in J} X\_{\alpha}
     5. (Comparison of the box and product topologies) (Thm 19.1) : The box topology on \prod X\_{\alpha} has as basis all sets of the form \prod U\_{\alpha}, where U\_{\alpha} is open in X\_{\alpha} for each \alpha . The product topology on \prod X\_{\alpha} has as basis all sets of the form \prod U\_{\alpha} , where U\_{\alpha} is open in X\_{\alpha} for each \alpha and U\_{\alpha} equals X\_{\alpha} except for finitely many values of \alpha.
        1. Box topology is finer than the product topology, but pretty much the same.
        2. When considering the product \prod X\_{\alpha}, the product topology is usually assumed.
     6. (Thm 19.2) Suppose the topology on each space X\_{\alpha} is given by a basis \mathcal{B}\_{\alpha}. The collection of all sets of the form \prod\_{\alpha \in J} B\_{\alpha} where B\_{\alpha} \in \mathcal{B}\_{\alpha} for each \alpha, will serve as a basis for the box topology on \prod\_{\alpha \in J} X\_{\alpha}.
        1. The collection of all sets of the same form, where B\_{\alpha} \in \mathcal{B}\_{\alpha} for finitely many indices \alpha and B\_{\alpha} = X\_{\alpha} for all the remaining indices, will serve as a basis for the product topology \prod\_{\alpha \in J} X\_{\alpha}.
     7. (Thm 19.3) Let A\_{\alpha} be a subspace of X\_{\alpha}. For each \alpha \in J. Then \prod A\_{\alpha} is a subspace of \prod X\_{\alpha} if both products are given the box topology, or if both products are given the product topology.
     8. (Thm 19.4) If each space is a Hausdorff space, then \prod X\_{\alpha} is a Hausdorff space in both the box and product topologies.
     9. (Thm 19.5) Let {X\_{\alpha}} be an indexed family of spaces ; let A\_{\alpha} \subset X\_{\alpha} for each \alpha. If \prod X\_{\alpha} is given either the product or the box topology, then \prod \bar{A\_{\alpha}} = \bar{\prod A\_{\alpha}}.
     10. (Thm 19.6) Let f: A => \prod\_{\alpha \in J} X\_{\alpha} be given by e equation f(a) = (f\_{\alpha} (a) ) \_{\alpha \in J} where f\_{\alpha} : A -> X\_{\alpha} for each \alpha . Let \prod X\_{\alpha} have the product topology, then the function f is continuous iff each function f\_{\alpha} is continuous.
  2. Metric topology
     1. Metiric on a set X : a function d : X \times X -> R having the following properties.
        1. d(x,y) \ge 0 for all x, y \in X; equality iff x = y.
        2. d(x,y) = d(y,x) for all x,y \in X.
        3. (Triangle inequality) d(x,y) + d(y,z) \ge d(x,z) for all x,y,z \in X.
     2. Distance between x and y : d(x,y)
        1. B\_{d} (x,\epsilon) : \epsilon -ball centered at x
     3. Metric topology induced by d ( a metric d on the set X) : Collection of all \epsilon -balls B\_{d} (x,\epsilon) for x \in X and \epsilon > 0 being a basis for a topology on X.
     4. Metrizable (a topological space X) : there exists a metric d on the set X that induces the topology of X.
     5. Metric space : a metrizable space X together with a specific metric d that gives the topology of X.
     6. Bounded ( a subset A of a metric space X with metric d) : there is some number M s.t. d(a\_{1}, a\_{2}) \le M for every pair a\_{1}, a\_{2} of points of A.
        1. Diameter of A : if A is bounded and nonempty, diam A = sup{d(a\_{1}, a\_{2}) | a\_{1}, a\_{2} \subset A}
     7. Standard bounded metric corresponding to d ( a metric space X with metric d) : \bar{d} : X \times X -> \mathbb{R} by the equation \bar{d} (x,y) = min{d(x,y) , 1}. Then \bar{d} is a metric that induces the same topology as d.
     8. Norm of \mathbf{x} (\mathbf{x} = (x\_{1}, …, x\_{n} ) in \mathbb{R}^{n} ) : \Vert x \Vert = ( x^{2}\_{1} + … + x^{2}\_{n} )^{frac{1}{2}}
        1. Euclidean metric d on \mathbb{R}^{n} : d(x,y) = \Vert \mathbf{x} - \mathbf{y} \Vert = [(x\_{1} – y\_{1})^{2} + … + (x\_{n} – y\_{n})^{2}]^{frac{1}/{2}}
        2. Squate metric \rho : \rho (\mathbf{x}, \mathbf{y}) = max{ \vert x\_{1} – y\_{1} \vert , …, \vert x\_{n} – y\_{n} \vert }
     9. (Lem 20.2) Let d and d’ be two metrics on the set X; Let \Tau and \Tau’ be the topologies they induce, respectively. Then \Tau’ is finer than \Tau iff for each x in X and each \epsilon >0, there exists a \delta >0 s.t. B\_{d’} (x,\delta) \subset B\_{d} (x,\epsilon).
        1. Pf) (Lem 13.3)
     10. (Thm 20.3) The topologies on \mathbb{R}^{n} induced by the Euclidean metric d and the squate metric \rho are the same as the product topology on \mathbb{R}^{n}.
     11. Uniform metric on \mathbb{R}^{J} ( an index set J) : Given points \mathbf{x} = (x\_{\alpha})\_{\alpha \in J} and \mathbf{y} = (y\_{\alpha})\_{\alpha \in J} of \mathbb{R}^{J}. Define a metric \bar{\rho} on \mathbb{R}^{J} by the equation \bar{\rho} (\mathbf{x} , \mathbf{y}) = sup{\bar{d} (x\_{\alpha} , y\_{\alpha}) | \alpha \in J}, where \bar{d} is the standard bounded metric on \mathbb{R}.
         1. Uniform topology : Topology \bar{\rho} induces
     12. (Thm 20.4) Uniforn topology on \mathbb{R}^{J} is finer than the product topology and coarser than the box topology; these three topologies are all different if J is infinite.
     13. (Thm 20.5) Let \bar{d} (a,b) = min{\vert a-b \vert , 1} be the standard bounded metric on \mathbb{R}. if \mathbf{x} and \mathbf{y} are two points of \mathbb{R}^{\omega}, define D(x,y) = sup{frac{\bar{d} (x\_{i}, y\_{i})} {i}}. Then D is a metric that induces the product topology on \mathbb{R}^{\omega}.
  3. Metric topology continued
     1. (Thm 21.1) Let f : X->Y; let X and Y be metrizable with metrics d\_{X} and d\_{Y}, respectively. Then continuity of f is equivalent to the requirement that given x \in X and given \epsilon > 0, there exists \delta > 0 s.t. d\_{X} (x,y) < \delta -> d\_{Y} (f\_{x}, f\_{y}) < \epsilon.
     2. (Sequenece Lemma) (Lem 21.2) : Let X be a topological space; let A \subset X. if there is a sequence of points of A converging to x, then x \in \bar{A}; the converse holds if X is metrizable.
        1. Pf) Thm 17.5
     3. (Thm 21.3) Let f : X -> Y. If the function f is continuous, then for every convergent sequence x\_{n} -> x in X, the sequence f(x\_{n}) converges to f(x). the converse holds if X is metrizable.
        1. Pf) Sequence lemma
     4. a countable basis at the point x ( a space X) : there is a countable collection {U\_{n}}\_{n \in \mathbb{Z}\_{+}} of neighborhoods of x s.t. any neighborhood U of x contains at least one of the sets U\_{n}.
     5. First countability axiom : A space X that has a countable basis at each of its points
     6. (Lem 21.4) Addition, subtraction, and multiplication operations are continuous functions from \mathbb{R} \times \mathbb{R} into \mathbb{R}, and the quotient operation is a continuous function from \mathbb{R} \times (\mathbb{R} – {0}) into \mathbb{R}.
     7. (Thm 21.5) If X is a topological space, and if f,g : X -> \mathbb{R} are continuous functions, then f + g, f – g, and f \cdot g are continuous. If g(x) \neq 0 for all x, then f/g is continuous.
        1. Pf) (Thm 18.4)
     8. Converges uniformly to the function f: X -> Y ( f\_{n} : X -> Y a sequence of functions from the set X to the metric space Y) : Let d be the metric for Y. Given \epsilon >0, there exists an integer N s.t. d(f\_{n} (x) , f (x)) < \epsilon for all n > N and all x in X;
     9. (Uniform limit theorem) (Thm 21.6) : Let f\_{n} : X -> Y be a sequence of continuous functions from the topological space X to the metric space Y. if (f\_{n}) converges uniformly to f, then f is continuous.
     10. Examples of spaces not metrizable
         1. \mathbb{R}^{\omega} in the box topology is not metrizable.
         2. \An uncountable product of \mathbb{R} with itself is not metrizable.
  4. Quotient topology
     1. Quotient map ( The map p) : Let X and Y be topological spaces; let p : X -> Y be a surjective map. If, A subset U of Y is open in Y iff p^{-1}(U) is open in Y, p is a quotient map.
     2. Saturated with respect to the surjective map p : X -> Y ( a subset C of X) : C contains every set p^{-1}({y}) that it intersects.
     3. If p : X->Y is a surjective continuous map that is either open or closed, then p is a quotient map.
        1. Open map (a map f : X -> Y) : for each open set U of X the set f(U) is open in Y
        2. Closed map (a map f : X -> Y) : for each closed set A of X the set f(A) is closed in Y
     4. Quotient topology induced by p : If X is a space and A is a set and if p : X -> A is a surjective map, then there exists exactly one topology \Tau on A relative to which p is a quotient map.
     5. Quotient space of X (a topological space X) : Let X\* be a partition of X into disjoint subsets whose union is X. Let p : X -> X\* be the surjective map that carries each point of X to the element of X\* containing it. In the quotient topology induced by p, the space X\* is called a quotient space of X.
     6. (Thm 22.1) Let p : X->Y be a quotient map; Let A be a subspace of X that is saturated with respect to p; let q : A -> p(A) be the map obtained by restricting p.
        1. If A is either open or closed in X, then q is a quotient map.
        2. If p is either an open map or a closed map, then q is a quotient map.
     7. (Thm 22.2) Let p : X -> Y be a quotient map. Let Z be a space and let g : X -> Z be a map that is constant on each set \rho^{-1}({y}), for y \in Y. Then g induces a map f : Y -> Z s.t. f \bullet p = g. The induced map f is continuous iff g is continuous; f is a quotient map iff g is a quotient map.
        1. (Cor) Let g : X ->Z be a surjective continuous map. Let X\* be the following collection of subsets of X : X\* = {g^{-1}({z}) | z \in Z}. Give X\* the quotient topology.
           1. The map g induces a bijective continuous map f: X\* -> Z, which is a homeomorphism iff g is a quotient map.
           2. If Z is Hausdorff, so is X\*.

1. Connectedness and Compactness
   1. Connected spaces :
      1. Separation of X ( a topological space X) : a pair U, V of disjoint nonempty open subsets of X whose union is X.
      2. Connected ( a topological space X) if there does not exist a separation of X.
         1. The only subsets of X that are both open and closed in X are the empty set and X itself.
      3. (Lem 23.1) If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.
      4. (Lem 23.2) If the sets C and D form a separation of X, and if Y is a connected subspace of X then Y lies entirely within either C or D.
      5. (Thm 23.3) The union of a collection of connected subspaces of X that have a point in common is connected.
      6. (Thm 23.4) Let A be a connected subspace of X. If A \subset B \subset \bar{A}, then B is also connected.
         1. Pf) Lem 23.2
      7. (Thm 23.5) The image of a connected space under a continuous map is connected.
      8. (Thm 23.6) A finite cartesian product of connected spaces is connected.
      9. An arbitrary product of connected spaces is connected in the product topology.
   2. Connected subspaces of the real line.
      1. Linear continnum ( a simply ordered set L) : if following holds.
         1. L has the least upper bound property.
         2. If x < y, there exists z s.t. x < z < y.
      2. (Thm 24.1) If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.
         1. Pf) convex
         2. (Cor) The real line \mathbb{R} is connected and so are intervals and rays in \mathbb{R}.
      3. (Intermediate value theorem ) (Thm 24.3): Let f : X -> Y be a continuous map, where X is a connected space and Y is an ordered set in the oerder topology. If a and b are two points of X and if r is a point of Y lying between f(a) and f(b), then there exists a point c of X s.t. f(c) = r
      4. Path in X from x to Y ( points x, y of the space X) : continuous map f : [a,b] -> X of some closed interval in the real line into X, s.t. f(a) = x and f(b) = y.
      5. Path connected (a space X) : every pair of points of X can be joined by a path in X.
         1. Path connected space X is connected.
      6. Examples
         1. Unit ball B^{n} in \mathbb{R}^{n} : B^{n} = {\mathbf{x} | \Vert x \Vert \le 1}
            1. \Vert \mathbf{x} \Vert = \Vert (x\_{1}, …, x\_{n}) \Vert = (x^{2}\_{1} + .. + x^{2}\_{n} )^{fact{1}{2}}
            2. Unit ball is path connected.
         2. Punctured Euclidean space : \mathbb{R}^{n} – {\mathbf{0}}
            1. \mathbf{0} : origin of \mathbb{R}^{n}.
            2. If n > 1, it is path connected.
         3. Unit sphere S^{n-1} in \mathbb{R}^{n} : S^{n-1} = {\mathbf{x} | \Vert \mathbf{x} \Vert = 1}
            1. If n > 1, it is path connected
         4. Ordered square I^{2}\_{0} is connected but not path connected.
         5. Topologist’s sine curve : S = {x \times sin(1/x) | 0<x \le 1}
            1. Connected but not path connected
   3. Components and Local connectedness
      1. components of X (a space X) : Equivalent classes where an equivalence relation on X by setting x ~ y if there is a connected subspace of X containing both x and y.
         1. Also called connected components
      2. (Thm 25.1) The components of X are connected disjoint subspaces of X whose union is X, s.t. each nonempty connected subspace of X intersects only one of them.
      3. Path components of X (a space X) : Equivalent classes where an equivalence relation on X by defining x ~ y if there is a path in X from x to y.
         1. Pf ) (pasting lemma)
      4. (Thm 25.2) The path components of X are path-connected disjoint subspaces of X whose union is X, s.t. each nonempty path connected subspace of X intersects only one of them.
      5. Locally connected at x (a point x in a space X) : For every neighborhood U of x, there is a connected neighborhood V of x contained by U.
      6. Locally connected (a space X): X is locally connected at each of its points.
      7. Locally path connected at x (a point x in a space X) : For every neighborhood U of x, there is a path connected neighborhood V of x contained by U.
      8. Locally path connected (a space X): X is locally path connected at each of its points.
      9. (Thm 25.3) A space X is locally connected iff for every open set U of X, each component of U is open in X.
      10. (Thm 25.4) A space X is locally path connected iff for every open set U of X, each path component of U is open in X.
      11. (Thm 25.5) If X is a topological space, each path component of X lies in a component of X. If X is a locally path connected, then the components and the path components of X are the same.
   4. Compact spaces
      1. Covering of X (a collection \mathcal{A} of subsets of a space X) : the union of the elements of \mathcal{A} is equal to X.
         1. Also called ‘to cover X’.
         2. Open covering of X if its elements are open subsets of X.
      2. Compact (a space X) : every open covering \mathcal{A} of X contains a finite subcollection that also covers X
      3. Cover Y (a subspace Y of X) : If, for a collection \mathcal{A} of subsets of X, the union of its elements contains Y.
      4. (Lem 26.1) Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.
      5. (Thm 26.2) Every closed subspace of a compact space is compact.
      6. (Thm 26.3) Every compact subspace of a Hausdorff space is closed.
      7. (Lem 26.4) If Y is a compact subspace of the Hausdorff space X and x\_{0} is not in Y, then there exist disjoint open sets U and V of X containing x\_{0} and Y, tespectively.
      8. (Thm 26.5) The image of a compact space under a continuous map is compact.
      9. (Thm 26.6) Let f : X -> Y be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.
         1. (Thm 26.2) (Thm 26.5) (Thm 26.3)
      10. (Thm 26.7) The product of finitely many compact spaces is compact)
          1. (Tube Lemma) (Lem 26.8) : Consider the product space X \times Y, where Y is compact. If N is an open set of X \times Y containing the slice x\_{0} \times Y of X \times Y, then N contains some tube W \times Y about x\_{0} \times Y, where W is a neighborhood of x\_{0} in X.
      11. Finite intersection property (a collection \mathcal{C} of subsets of X) : For every finite subcollection {C\_{1}, …, C\_{n}} of \mathcal{C}, the intersection C\_{1} \cap … \cap C\_{n} is nonempty.
      12. (Thm 26.9) Let X be a topological space. Then X is compact iff for every collection \mathcal{C} of closed sets in X having the finite intersection property, the intersection \bigcup\_{C \in \mathcal{C}} C of all the elements of \mathcal{C} is nonempty.
          1. Nested sequence C\_{1} \supset C\_{2} \supset … \supset C\_{n} \supset .. of closed sets in a compact space X
   5. Compact subspaces of the Real line
      1. (Thm 27.1) Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.
         1. Pf) Subspace topology equals order topology
         2. (Cor) Every closed interval in \mathbb{R} is compact.
      2. (Thm 27.3) A subspace A of \mathbb{R}^{n} is compact iff it is closed and is bounded in the Euclidean metric d or the square metric \rho.
      3. (Extreme value theorem) (Thm 27.4) Let f: X -> Y be continuous, where Y is an ordered set in the order topology. If X is compact, then there exists points c and d in X s.t. f(c) \le f(x) \le f(d) for every x \in X.
      4. Distance from x to A ( a point x \in X, nonempty subset A of X) : For a metric space (X,d), d(x,A) = inf {d(x,a) | a \in A}
         1. D(x,A) is continuous
      5. (Lebesgue number lemma)(Lem 27.5) Let \mathcal{A} be an open covering of the metric space (X,d). If X is compact, there is a \delta > 0 s.t. for each subset of X having diameter less than \delta, there exists an element of \mathcal{A} containing it.
         1. Lebesgue number for the covering \mathcal{A} : \delta
      6. Uniformly continuous (a function f : X -> Y) : Let metric spaces (X,d\_{X}), (Y,d\_{y}), if given \epsilon >0, there is a \delta >0 s.t. for every pair of points x\_{0}, x\_{1} of X, d\_{X} (x\_{0}, x\_{1}) < \delta -> d\_{Y} (f (x\_{0}), f (x\_{1}) ) < \epsilon
      7. (Uniform continuity theorem) (Thm 27.6) : Let f : X -> Y be a continuous map of the compact metric space (X,d\_{X}) to the metric space (Y,d\_{Y}). Then f is uniformly continuous.
         1. Pf) (Lebesgue number lemma)
      8. Isolated point x of X (a point x of a space X) : one-point set {x} is open in X.
      9. (Thm 27.7) Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.
         1. Pf) (Thm 26.9)
         2. (Cor 27.8) Every closed interval in \mathbb{R} is uncountable.
   6. Limit point compactness
      1. Limit point compact (a space X) : every infinite subset of X has a limit point.
      2. (Thm 28.1) Compactness implies limit point compactness, but not conversely.
      3. Subsequence ( a sequence (x\_{n}) of points of a topological space X) : n\_{1} < n\_{2} < … < n\_{i} < … is an increasing sequence of positive integer, then the sequence (yi) defined by setting y\_{i} = x\_{n}
      4. Sequentially compact ( a space X) : every sequence of points of X has a convergent subsequence
      5. (Thm 28.2) Let X be a metrizable space. Then the following are equivalent.
         1. X is compact.
         2. X is limit point compact.
         3. X is sequentially compact.
         4. Pf) If X is sequentially compact, the (Lebesgue number lemma) holds for X.
   7. Local compactness
      1. Locally compact at x ( a point x in a space X) : there is some compact subspace C of X that contains a neighborhood of x.
      2. Locally compact (a space X) : X is locally compact at each of its points
      3. (Thm 29.1) Let X be a space. Then X is locally compact Hausdorff iff there exists a space Y satisfying the following conditions. If Y and Y’ are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y’ that equals the identity map on X.
         1. X is a subspace of Y.
         2. The set Y-X consists of a single point.
         3. Y is a compact Hausdorff space.
      4. Compactification Y of X ( a compact Hausdorff space Y and a proper subspace X of Y) : closure of X equals Y
      5. One-point compactification Y of X( a compact Hausdorff space Y and a proper subspace X of Y ): Y-X equals a single point
         1. X has a one-point compactification Y iff X is a locally compact Hausdorff space that is not itself compact.
         2. Riemann sphere : \mathbb{C} \cup {\infty} , one-point compactification of \mathbb{R}^{2}
      6. (Thm 29.2) Let X be a Hausdorff space. Then X is locally compact iff given x in X, and given a neighborhood U of x, there is a neighborhood V of x s.t. \bar{V} is compact and \bar{V} \subset U.
         1. (Cor 29.3) Let X be locally compact Hausdorff; let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.
         2. (Cor 29.4) A space X is homeomorphic to an open subspace of a compact Hausdorff space iff X is locally compact Hausdorff.
            1. Pf) (Thm 29.1) (Cor 29.3)
2. Countability and Separation Axioms
   1. Countability axioms
      1. a countable basis at the point x ( a space X) : there is a countable collection \mathcal{B} of neighborhoods of x s.t. any neighborhood U of x contains at least one of the elements of \mathcal{B}.
      2. First countability axiom : A space X that has a countable basis at each of its points
         1. Also called ‘first-countable’
      3. (Thm 30.1) Let X be a topological space.
         1. Let A be a subset of X. If there is a sequence of points of A converging to x, then x \in \bar{A} ; the converse holds if X is first-countable.
         2. Let f : X -> Y. If f is continuous, then for every convergent sequence x\_{n} -> x in X, the sequence f (x\_{n}) converges to f(x). The converse holds if X is first-countable.
      4. Second countability axiom : A space X has countable basis for its topology
         1. Also called ‘second-countable’
      5. (Thm 30.2) A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.
      6. Dense in X ( a subset A of a space X) : \bar{A} – X
      7. (Thm 30.3) Suppose that X has a countable basis. Then
         1. Every open covering of X contains a countable subcollection covering X.
         2. There exists a countable subset of X that is dense in X.
      8. Lindelöf space : A space for which every open coveing contains a countable subcovering
      9. Separable (a space) : a space has a countable dense subset
         1. Not a good term
   2. Separation Axioms
      1. Regular ( a space X) : Suppose that one-point sets are closed in X. For each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B.
         1. Regular space is Hausdorff space.
      2. Normal ( a space x) : Suppose that one-point sets are closed in X. For each pair A, B of disjoint closed sets of X, there exists disjoint open sets containing A and B, respectively.
         1. Normal space is Regular space.
      3. (Lem 31.1) Let X be a topological space. Let one-point sets in X be closed.
         1. X is regular iff given a point given a point x of X and a neighborhood U of x, there is a neighborhood V of x s.t. \bar{V} \subset U.
         2. X is normal iff given al closed set A and an open set U containing A, there is an open set V containing A s.t. \bar{V} \subset U.
      4. (Thm 31.2)
         1. A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.
         2. A subspace of a regular space is regular, a product of regular spaces is regular.
         3. Pf) (Lem 31.1), (Thm 19.5)
      5. Example
         1. Sorgenfrey plane \mathbb{R}^{2}\_{l} is not normal.
   3. Normal spaces
      1. (Thm 32.1) Every regular space with a countable basis is normal.
      2. (Thm 32.2) Every metrizable space is normal.
      3. (Thm 32.3) Every compact Hausdorff space is normal.
         1. Pf) (Lem 26.4)
      4. (Thm 32.4) Every well-ordered set X is normal in the order topology.
      5. Examples not normal
         1. If J is uncountable, the product space \mathbb{R}^{J} is not normal.
         2. The product space S\_{\Omega} \times \bar{S\_{\Omega}} is not normal.
   4. Urysohn lemma
      1. (Urysohn lemma) (Thm 33.1) Let X be a normal space; let A and B be disjoint closed subsets of X. Let [a,b] be a closed interval in the real line. Then there exists a continuous map f : X -> [a,b] s.t. f(x) = a for every x in A, and f(x) = b for every x in B.
         1. Pf) (Thm 10.1)
      2. A and B can be separated by a continuous function ( two subsets A,B of the topological space X) : there is a continuous function f : X -> [0,1] s.t. f(A) = {0} and f(B) = {1}.
      3. Completely regular ( a space X) : one-point sets are closed in X and if for each point x\_{0} and each closed set A not containing x\_{0}, there is a continuous function f : X->[0,1] s.t. f(x\_{0}) = 1 and f(A) = {0}
      4. (Thm 33.2) A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.
         1. Examples
            1. \mathbb{R}^{2}\_{l}, S\_{\Omega} \times \bar{S\_{\Omega}} is completely regular.
   5. Urysohn Metrization Theorem
      1. (Urysohn Metrization Theorem) (Thm 34.1) Every regular space X with a countable basis is metrizable.
         1. Pf) (Thm 20.5) , (There exists a countable collection of continuous functions f\_{n} : X -> [0,1] having the property that given any point x\_{0} of X and any neighborhood U of x\_{0}, there exists an index n s.t. f\_{n} is positive at x\_{0} and vanishes outside U) , (Urysohn lemma), imbedding
      2. (Imbedding Theorem) (Thm 34.2)
         1. Let X be a space in which one-point sets are closed. Suppose that {f\_{\alpha}}\_{\alpha \in J} is an indexed family of continuous function f\_{\alpha} : X -> \mathbb{R} satisfying the requirement that for each point x\_{0} of X and each neighborhood U of x\_{0}, there is an index \alpha s.t. f\_{\alpha} is positive at x\_{0} and vanishes outside U.
            1. Separate points from closed sets in X : a family of continuous functions behaving as {f\_{\alpha}}\_{\alpha \in J}
         2. Then the function F : X->R^{J} defined by F(x) = (f\_{\alpha} (x) )\_{\alpha \in J} is an imbedding of X in \mathbb{R}^{J}.
         3. If f\_{\alpha} maps X into [0,1] for each \alpha, then F imbeds X in [0,1]^{J}.
      3. (Thm 34.3) A space X is completely regular iff it is homeomorphic to a subspace of [0,1]^{J} for some J.
   6. Tietze Extension Theorem
      1. (Tietze Extension Theorem) (Thm 35.1) : Let X be a normal space; Let A be a closed subspace of X.
         1. Any continuous map of A into the closed interval [a,b] of \mathbb{R} may be extended to a continuous map of all of X into [a,b]
         2. Any continuous map of A into \mathbb{R} may be extended to a continuous map of all of X into \mathbb{R}.
         3. Pf) (Urysohn lemma), (Weierstrass M-test), uniform convergence
   7. Imbeddings of Manifolds
      1. M-manifold : a Hausdorff space X with a countable basis s.t. each point x of X has a neighborhood that is homeomorphic with an open subset of \mathbb{R}^{m}.
         1. Curve : 1-manifold
         2. Surface : 2-manifold
      2. Support of \phi (\phi : X -> \mathbb{R}) : closure of the set \phi^{-1} ( \mathbb{R} – {0} )
      3. Partition of unity dominated by {U\_{i}} ( a finite index open covering {U\_{1}, …, U\_{n}} of the space X) : Indexed family of continuous functions \phi\_{i} : X -> [0,1] for i = 1, …, n if :
         1. (support \phi\_{i}) \subset U\_{i} for each i.
         2. \sum\_{i = 1}^{n} \phi\_{i} (x) = 1 for each x.
      4. (Existence of finite partitions of unity) (Thm 36.1) : Let {U\_{1}, …, U\_{n}} be a finite open covering of the normal space X. Then there exists a partition of unity dominated by {U\_{i}}.
         1. Pf) (Urysohn lemma)
      5. (Thm 36.2) If X is a compact m-manifold, then X can be imbedded in \mathbb{R}^{N} for some positive integer N.
         1. Pf) (compact & Hausdorff -> normal)
3. Tychonoff Theorem
   1. Tychonoff Theorem
      1. (Lem 37.1) Let X be a set; let \mathcal{A} be a collection of subsets of X having the finite intersection property. Then there is a collection \mathcal{D} of subsets of X s.t. \mathcal{D} contains \mathcal{A}, and \mathcal{D} has the finite intersection property, and no collection of subsets of X that properly contains \mathcal{D} has this property.
         1. A collection \mathcal{D} satisfying the conclusion of this theorem is maximal with respect to the finite intersection property.
         2. Pf) (Zorn’s lemma)
      2. (Lem 37.2) Let X be a set; Let \mathcal{D} be a collection of subsets of X that is maximal with respect to the finite intersection property. Then
         1. Any finite intersection of elements of \mathcal{D} is an element of \mathcal{D}.
         2. If A is a subset of X that intersects every element of \mathcal{D}, then A is an element of \mathcal{D}.
      3. (Tychonoff Theorem) (Thm 37.3) An arbitrary product of compact spaces is compact in the product topology.
         1. Pf) (Lem 37.1) (Lem 37.2)
   2. Stone- \check{C} ech Compactification
      1. Compactification Y of space X : a compact Hausdorff space Y containing X as a subspace s.t. \bar{X} = Y.
      2. Equivalent (Two compactifications Y\_{1} and Y\_{2} of a space X) : there is a homeomorphism h : Y\_{1} -> Y\_{2} s.t. h(x) = x for every x \in X.
      3. (Lem 38.1) Let X be a space; suppose that h: X->Z is an imbedding of X in the compact Hausdorff space Z. Then there exists a corresponding compactification Y of X; it has the property that there is an imbedding H : Y -> Z that equals h on X. The compactification Y is uniquely determined up to equivalence.
         1. Compactification Induced by imbedding h : Y
      4. (Thm 38.2) Let X be a completely regular space. There exists a compactification Y of X having the property that every bounded continuous map f : X -> \mathbb{R} extends uniquely to a continuous map of Y into \mathbb{R}.
         1. Pf) (Tychonoff Theorem), (Thm 34.2) (Lem 38.3)
      5. (Lem 38.3) Let A \subset X; let f : A -> Z be a continuous map of A into the Hausdorff space Z. There is at most one extension of f to a continuous function g:\bar{A} ->Z.
      6. (Thm 38.4) Let X be a completely regular space; let Y be a compactification of X satisfying the extension property of (Thm 38.2). Given any continuous map f: X->C of X into a compact Hausdorff space C, the map f extends uniquely to a continuous map g : Y -> C.
         1. Pf) (Completely regular -> imbedded in [0,1]^{J} for some J)
      7. (Thm 38.5) Let X be a completely regular space. If Y\_{1} and Y\_{2} are two compactifications of X satisfying the extension property of (Thm 38.2), then Y\_{1} and Y\_{2} are equivalent.
         1. Pf) (Preceding theorem)
      8. Stone- \check{C} ech compactification of X ( a completely regular space X) : \beta (X), a compactification of X satisfying the extension condition of (Thm 38.2)
         1. Any continuous map f : X->C of X into a compact Hausdorff space C extends uniquely to a continuous map g : \beta (X) -> C.
4. Metrization Theorems and Paracompactness
   1. Local finiteness
      1. Locally finite in X (a collection \mathcal{A} of subsets of a topological space X) : Every point of X has a neighborhood that intersects only finitely many elements of \mathcal{A}.
      2. (Lem 39.1) Let \mathcal{A} be a locally finite collection of subsets of X. Then;
         1. Any subcollection of \mathcal{A} is locally finite.
         2. The collection \mathcal{B} = {\bar{A} }\_{A \in \mathcal{A}} of the closures of the elements of \mathcal{A} is locally finite.
         3. \bar{\bigcup\_{A \in \mathcal{A}}} = \bigcup\_{A \in \mathcal{A}} \bar{A}
      3. Locally finite indexed family in X ( an indexed family {A\_{\alpha}}\_{\alpha \in J} ) : every x \in X has a neighborhood that intersects A\_{\alpha} for only finitely many values of \alpha.
      4. Countably locally finite ( a collection \mathcal{B} of subsets of X) : \mathcal{B} can be written as the countable union of collections \mathcal{B}\_{n} , each of which is locally finite.
      5. Refinement \mathcal{B} of \mathcal{A} ( a collection \mathcal{B} of subsets of X) : for each element B of \mathcal{B}, there is an element A of \mathcal{A} containing B.
         1. Also called ‘ to refine \mathcal{A}’
         2. Open refinement of \mathcal{A} ( a collection \mathcal{B} of subsets of X) : elements of \mathcal{B} are open sets
         3. Closed refinement of \mathcal{A} ( a collection \mathcal{B} of subsets of X) : elements of \mathcal{B} are closed sets
      6. (Lem 39.2) Let X be a metrizable space. If \mathcal{A} is an open covering of X, then there is an open covering \Epsilon of X that is countably locally finite.
         1. Pf) (well-ordering theorem)
   2. Nagata-Smirnov Metrization Theorem
      1. G\_{\delta} set in X ( a subset A of a space X) : it equals the intersection of a countable collection of open subsets of X.
      2. (Lem 40.1) Let X be a regular space with a basis \mathcal{B} that is countably locally finite. Then X is normal, and every closed set in X is a G\_{\delta} set in X.
         1. Pf) (Lem 39.1) (Thm 32.1)
      3. (Lem 40.2) Let X be normal; let A be a closed G\_{\delta} set in X. Then there is a continuous function f : X-> [0,1] s.t. f(x) = 0 for x \in A and f(x) > 0 for x \notin A.\
      4. (Nagata-Smirnov Metrization Theorem) (Thm 40.3) : A space X is metrizable iff X is regular and has a basis that is countably locally finite.
         1. Pf) (Lem 39.2)
   3. Paracompactness
      1. Paracompact ( a space X) : every open covering \mathcal{A} of X has a locally finite open refinement \mathcal{B} that covers X.
         1. \mathbb{R}^{n}
      2. (Thm 41.1) Every paracompact Hausdorff space X is normal.
      3. (Thm 41.2) Every closed subspace of a paracompact space is paracompact.
      4. (Lem 41.3) Let X be regular. Then the following conditions on X are equivalent; Every open covering of X has a refinement that is;
         1. An open covering of X and countably locally finite.
         2. A covering of X and locally finite.
         3. A closed covering of X and locally finite
         4. An open covering of X and locally finite
         5. Pf) (Lem 39.1)
      5. (Thm 41.4) Every metrizable space is paracompact
         1. Pf) (Lem 39.2)
      6. (Thm 41.5) Every regular Lindelöf space is paracompact.
         1. Pf) (Preceding lemma)
      7. Partition of unity on X, dominated by {U\_{\alpha}} ( a indexed open covering {U\_{\alpha}}\_{\alpha \in J} of the space X) : Indexed family of continuous functions \phi\_{\alpha} : X -> [0,1] for i = 1, …, n if :
         1. (support \phi\_{\alpha}) \subset U\_{\alpha} for each \alpha
         2. The indexed family {Support \phi\_{\alpha} } is locally finite
         3. \sum \phi\_{\alpha} (x) = 1 for each x.
      8. (Shrinking lemma)(Lem 41.6) Let X be a paracompact Hausdorff space; Let {U\_{\alpha}}\_{\alpha \in J} be an indexed family of open sets covering X. Then there exists a locally finite indexed family {V\_{\alpha}}\_{\alpha \in J} of open sets covering X s.t. \bar{V}\_{\alpha} \subset U\_{\alpha} for each \alpha.
         1. Family {\bar{V}\_{\alpha}} is a ‘precise refinement’ of the faminy {U\_{\alpha}} \bar{V}\_{\alpha} \subset U\_{\alpha} for each \alpha.
      9. (Thm 41.7) Let X be a paracompact Hausdorff space; let {U\_{\alpha}}\_{\alpha \in J} be an indexed open covering of X. Then there exists a partition of unity on X dominated by {U\_{\alpha}}.
         1. Pf) (Shrinking lemma)
      10. (Thm 41.8) Let X be a paracompact Hausdorff space; let \mathcal{C} be a collection of subsets of X; for each C \in \mathcal{C}, let \epsilon\_{C} be a positive number. If \mathcal{C} is locally cinite, there is a continuous function f: X->\mathbb{R} s.t. f(x) >0 for all x, and f(x) \le \epsilon\_{C} for x \in C.
   4. Smirnov Metrization Theorem
      1. Locally metrizable (a space X) : Every point x of X has a neighborhood U that is metrizable in the subspace topology.
      2. (Smirnov metrization theorem) (Theorem 42.1) Space X is metizable iff it is a paracompact Hausdorff space that is locally metrizable.
         1. Pf) similar to (Thm 40.3)
5. Complete Metric spaces and Function spaces
   1. Complete Metric Spaces
      1. Cauchy sequence (x\_{n}) in (X,d) ( a sequence (x\_{n}) of points of X, a metric space ( X,d)) : it has the property that given \epsilon > 0 , there is an integer N s.t. d(x\_{n}, x\_{m}) < \epsilon whenever n, m \ge N
      2. Complete ( a metric space (X,d) ) : Every Cauchy sequence in X converges.
         1. If X is complete under the metric d, then X is complete under the standard bounded metric corresponding to d.
      3. (Lem 43.1) : A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.
      4. (Thm 43.2) Euclidean space \mathbb{R}^{k} is complete in either of its usual metrics, the Euclidean metric d or the square metric \rho.
         1. Pf) (Thm 28.2)
      5. (Lem 43.3) Let X be the product space X = \prod X\_{\alpha} ; let x\_{n} be a sequence of points of X. Then x\_{n} -> X iff \pi\_{\alpha} (\mathbf{x}\_{n}) -> \pi\_{\alpha} (\mathbf{x}) for each \alpha.
      6. (Thm 43.4) There is a metric for the product space \mathbb{R}^{\omega} relative to which \mathbb{R}^{\omega} is complete.
         1. Pf) D(x,y) = sup{\bar{d}(x\_{i}, y\_{i}) / i}
      7. Uniform metric on Y^{J} corresponding to the metric d on Y : \bar{\rho} (\mathbf{x}, \mathbf{y}) = sup{\bar{d} (x\_{\alpha}, y\_{\alpha}) | \alpha \in J)
         1. Let (Y,d) be a metric space; let \bar{d} (a,b) = min{d(a,b), 1} be the standard bounded metric on Y derived from d. If \mathbf{x} = (x\_{\alpha})\_{a \in J} and \mathbf{y} = (y\_{\alpha})\_{a \in J} are points of the cartesian product Y^{J}.
      8. (Thm 43.5) If the space Y is complete in the metric d, then the space Y^{J} is complete in the uniform metric \bar{\rho} corresponding to d.
         1. Pf) (if (Y,d) is complete, so is (Y, \bar{d}))
      9. Bounded ( a function f : X -> Y) : its image f(X) is a bounded subset of the metric space (Y,d)
      10. (Thm 43.6) Let X be a topological space and let (Y,d) be a metric space. The set \mathcal{C} (X,Y) of continuous functions is closed in Y^{X} under the uniform metric. So is the set \mathcal{B} (X,Y) of bounded functions. Therefore, if Y is complete, these spaces are complete in the uniform metric.
          1. Pf) (Uniform limit theorem, Thm 21.6)
      11. Sup metric ( a metric \rho) : \rho ( f, g) = sup { d(f(x), g(x)) | x \in X} If (Y,d) is a metric space, one can define another metric on the set \mathcal{B} (X,Y) of bounded functions from X to Y . \rho is well-defined, for the set f(X) U g(X) is bounded if both f(x) and g(X) are.
          1. If X is a compact space, then every continuous function f: X -> Y is bounded, hence the sup metric is defined on \mathcal{C} (X, Y)
      12. (Thm 43.7) Let (X,d) be a metric space, There is an isometric imbedding of X into a complete metric space.
          1. Isometric imbedding of X in Y : an imbedding f:X->Y have the property that for every pair of points x\_{1}, x\_{2} of X, d\_{Y} (f(x\_{1}), f(x\_{2})) = d\_{X} (x\_{1}, x\_{2})
          2. Pf) ( a fixed point : an element of the function’s domain that is mapped to itself by the function)
      13. Completion \bar{h(X)} of X ( a metric space X) : a subspace \bar{h(x)} of Y, if h: X->Y is an isometric imbedding of X into a complete metric space Y
          1. Completion of X is uniquely determined up to an isometry.
   2. Space-filling Curve
      1. (Thm 44.1) Let I = [0,1]. There exists a continuous map f: I -> I^{2} whose image fills up the entire square I^{2}.
         1. Pf) (closed -> complete -> \mathcal{C}(I,I^{2}) is complete)
   3. Compactness in Metric spaces
      1. Every compact metric space is complete
         1. Pf) (Lem 43.1)
      2. Totally bounded ( a metric spacx (X,d)) : for every \epsilon >0, there is a finite covering of X by \epsilon -balls.
      3. (Thm 45.1) A metric space (X,d) is compact iff it is complete and totally bounded.
         1. Pf) (Every compact metric space is complete)
      4. Equicontinuous \mathcal{F} at x\_{0} (a subset \mathcal{F} of the function space \mathcal{C} (X,Y), a x\_{0} \in X) : Let (Y,d) be a metric space. If given \epsilon >0, there is a neighborhood U of x\_{0} s.t. for all x \in U and all f \in \mathcal{F}, d(f(x), f(x\_{0})) < \epsilon .
         1. Equicontinuous (a set \mathcal{F}) : the set \mathcal{F} is equicontinuous at x\_{0} for each x\_{0} \in X.
      5. (Lem 45.2) Let X be a space; Let (Y,d) be a metric space. If the subset \mathcal{F} of \mathcal{C} (X,Y) is totally bounded under the uniform metric corresponding to d, then \mathcal{F} is equicontinuous under d.
      6. (Lem 45.3) Let X be a pace; let (Y,d) be a metric space; assume X and Y are compact. If the subset \mathcal{F} of \mathcal{C}(X,Y0 is equicontinuous under d, then \mathcal{F} is totally bounded under the uniform and sup metrics corresponding to d.
      7. Pointwise bounded \mathcal{F} under d ( a subset \mathcal{F} of the function space \mathcal{C} (X,Y) , a metric space (Y,d)) : for each x \in X, the subset \mathcal{F}\_{a} = {f(a) | f \in \mathcal{F}} of Y is bounded under d.
      8. (Ascoli’s Theorem, classical version) (Thm 45.4) : Let X be a compact space; Let (\mathbb{R},d) denote euclidean space in either the square metric or the Euclidean metric; give \mathcal{C} (X,\mathbb{R}^{n}) the corresponding uniform topology. A subspace \mathcal{F} of \mathcal{C} (X,\mathbb{R}^{n}) has compact closure iff \mathcal{F} is equicontinuous and pointwise bounded under d.
         1. Pf) (Thm 45.1) (Lem 45.2) (Lem 45.3)
         2. (Cor) Let X be compact; let d denote either the square metric or the Euclidean metric on \mathbb{R}^{n} ; give \mathcal{C} (X,\mathbb{R}^{n}) the corresponding uniform topology. A subspace \mathcal{F} of \mathcal{C} (X,\mathbb{R}^{n}) is compact iff it is closed, bounded under the sup metric \rho, and equicontinuous under d.
   4. Pointwise and Compact convergence
      1. Topology of pointwise convergence : given a point x of the set X and an open set U of space Y, let S(x,U) = {f | f \in Y^{X} and f (x) \in U}. The sets S(x,U) are a subbasis for topology on Y^{X}.
         1. Also called ‘point-open topology’
      2. (Thm 46.1) A sequence f\_{n} of functions converges to the function f in the topology of pointwise convergence iff for each x in X, the sequence f\_{n} (x) of points of Y converges to the point f(x).
         1. Pf) (Lem 43.3)
      3. Topology of compact convergence : Let (Y,d) be a metric space; let X be a topological space. Given an element f of Y^{X}, a compact subspace C of X, and a number \epsilon >0, let B\_{C} (f, \epsilon) denote the set of all those elements g of Y^{X} for which sup{d(f(x), g(x)) | x \in C} < \epsilon. The sets B\_{C} (f, \epsilon ) form a basis for a topology on Y^{X}.
         1. Also called ‘Topology of uniform convergence on compact sets’
      4. (Thm 46.2) A sequence f\_{n} : X -> Y of functions converges to the function f in the topology of compact convergence iff for each compact subspace C of X, the sequence f\_{n} | C converges uniformly to f | C.
      5. Compactly generated (a space X) : it satisfies the following condition.
         1. A set A is open in X if A \cap C is open in C for each compact subspace C of X.
      6. (Lem 46.3) If X is locally compact, or if X satisfies the first countability axiom, then X is compactly generated.
      7. (Lem 46.4) If X is compactly generated, then a function f: X->Y is continuous if for each compact subspace C of X, the restricted function f | C is continuous.
      8. (Thm 46.5) Let X be a compactly generated space; let (Y,d) be a metric space. Then \mathcal{C} (X,Y) is closed in Y^{X} in the topology of compact convergence.
         1. Pf) ( Uniform limit theorem)
         2. (Cor 46.6) Let X be a compactly generated space; let (Y,d) be a metric space. If a sequence of continuous functions f\_{n} : X -> Y converges to f in the topology of compact convergence, then f is continuous.
      9. (Thm 46.7) Let X be a space; let (Y,d) be a metric space. For the function space Y^{X}, one has the following inclusions of topologies :
         1. (Uniform) \supseteq (Compact convergence) \supseteq (Pointwise convergence)
         2. If X is compact, the first two coincide, and if X is discrete, the second two coincide.
      10. Compact-open topology : Let X and Y be topological spaces. If C is a compact subspace of X and U is an open subset of Y, define S(C,U) = {f | f \in \mathcal{C} (X,Y) and f(C) \subset U}. The sets S(C,U) forms a subbasis for a topology on \mathcal{C} (X,Y).
      11. (Thm 46.8) Let X be a space and let (Y,d) be a metric space. On the set \mathcal{C} (X,Y), the compact-open topology and the topology of compact convergence coincide.
          1. (Cor 46.9) Let Y be a metric space. The compact convergence topology on \mathcal{C} (X,Y) does not depend on the metric of Y. Therefore if X is compact, the uniform topology on \mathcal{C} (X,Y) does not depend on the metric of Y.
      12. (Thm 46.10) Let X be a locally compact Hausdorff; let \mathcal{C} (X,Y) have the compact-open topology. Then the map e: X \times \mathcal{C} (X,Y) -> Y defined by the equation e(x,f) = f(x) is continuous.
          1. Evaluation map : the map e
      13. Map F induced by f ( a function f : X \times Z -> Y) : There is a corresponding function F : Z -> \mathcal{C} (x<Y), defined by the equation (F(z))(x) = f(x,z). Conversely, given F : Z -> \mathcal{C} (X,Y), this equation defines a corresponding function f : X \times Z -> Y.
      14. (Thm 46.11) Let X and Y be spaces; give \mathcal{C} (X,Y) the compact-open topology. If f: X \times Z -> Y is continuous, then so is the induced function F: Z -> \mathcal{C} (X,Y). The converse holds if X is locally compact Hausdorff.
          1. Pf) (Tube lemma)
      15. Homotopic (two functions f and g in \mathcal{C} (X,Y) ) : There is a continuous map h : X \times [0,1] -> Y s.t. h(x,0) = f(x) and h(x,1) = g(x) for each x \in X.
          1. Homotopy between f and g : the map h
   5. Ascoli’s Theorem
      1. (Ascoli’s Theorem) (Thm 47.1) : Let X be a space and let (Y,d) be a metric space. Give \mathcal{C} (X,Y) the topology of compact convergence; Let \mathcal{F} be a subset of \mathcal{C} (X,Y).
         1. If \mathcal{F} is equicontinuous under d and the set \mathcal{F}\_{a} = { f(a) | f\in \mathcal{F} } has compact closure for each a \in X, then \mathcal{F} is contained in a compact subspace of \mathcal{C} (X,Y).
         2. The converse holds if X is locally compact Hausdorff.
         3. Pf) (Tychonoff Theorem) (Thm 46.8) (Thm 46.10) (Thm 45.1) (Lem 45.2)
6. Baire Spaces and Dimension Theory
   1. Baire Spaces
      1. A has Empty interior ( a subset A of a space X) : A contains no open set of X other than the empty set.
         1. A has empty interior if every point of A is a limit point of the complement of A.
         2. The complement of A is dense in X.
      2. Baire space ( a space X) : if the following condition holds.
         1. Given any countable collection {A\_{n}} of closed sets of X each of which has empty interior in X, their union \bigcup A\_{n} also has empty interior in X.
      3. First category in X ( a subset A of a space X) : it was contained in the union of a countable collection of closed sets of X having empty interiors in X.
         1. Second category in X (a subset A of a space X) : A is not on the first category in X.
         2. A space X is a Baire space iff every nonempty open set in X is of the second category.
      4. (Lem 48.1) X is a Baire space iff given any countable collection {U\_{n}} of open sets in X, each of which is dense in X, their intersection \bigcap U\_{n} is also dense in X.
      5. (Baire Category theorem) (Thm 48.2) : If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.
      6. (Lem 48.3) Let C\_{1} \supset C\_{2} \supset … be a nested sequence of nonempty closed sets in the complete metric space X. If diam C\_{n} -> 0, then \bigcap C\_{n} \neq \empty
      7. (Lem 48.4) Any open subspace Y of a Baire space X is itself a Baire space.
      8. (Thm 48.5) Let X be a space; let (Y,d) be a metric space. Let f\_{n} : X -> Y be a sequence of continuous functions s.t. f\_{n} (x) -> f(x) for all x \in X, where f : X-> Y. If X is a baire space, the set of points at which f is continuous is dense in X
         1. Pf) (Lem 48.4)
   2. Nowhere-Differentiable function
      1. (Thm 49.1) Let h : [0,1] -> \mathbb{R} be a continuous function. Given \epsilon >0, there is a function g : [0,1] -> \mathbb{R} with \vert h(x) -> g(x) \vert < \epsilon for all x, s.t. g is continuous and nowhere differentiable.
         1. Pf) (Lemma 48.1) (Complete metric space -> Baire space)
            1. piecewise-linear (a function g) : a function whose graph is a broken line segment; each line segment in the graph of g has slope at least \alpha in absolute value
            2. sawtooth graph
   3. Introduction to Dimension Theory
      1. \mathcal{A} has order m+1 ( a collection \mathcal{A} of subsets of the space X) : some points of X lies in m+1 elements of \mathcal{A}, and no point of X lies in more than m+1 elements of \mathcal{A}.
      2. Finite dimensional ( a space X) : There is some integer m s.t. for every open covering \mathcal{A} of X, there is an open covering \mathcal{B} of X that refines \mathcal{A} and has order at most m+1.
      3. Topological dimension of X (a space X) : the smallest value of m for a space X to be finite dimensional.
         1. Dim X : Topological dimension of X
         2. Example
            1. Any compact subspace X of \mathcal{R} has topological dimension at most 1.
            2. Any compact subspace X of \mathcal{R}^{2} has topological dimension at most 2.
      4. (Thm 50.1) Let X be a space having finite dimension. If Y is a closed subspace of X, then Y has finite dimension and dim Y \le dim X.
      5. (Thm 50.2) Let X = Y \cup Z, where Y and Z are closed subspaces of X having finite topological dimension. Then dim X = max {dim Y, dim Z}
         1. Pf) (Thm 50.1)
         2. (Cor 50.3) Let X = Y\_{1} \cup … \cup Y\_{k}, where each Y\_{i} is a closed subspace of X and is finite dimensional. Then dim X = max{dim Y\_{1} , .., dim Y\_{k}}.
      6. Arc : a space homeomorphic to the closed unit interval
         1. Endpoints of Arc : the points p and q s.t. A – {p} and A-{q} are connected.
         2. Linear graph G : a Hausdorff space that is written as the union of finitely many arcs, each pair of which intersects in at most a common end point.
         3. Edges of G : arcs in the collection
         4. Vertices of G : endpoints of arcs
         5. Each edge of G is closed in G; G has topological dimension 1.
         6. Every finite linear graph can be imbedded in \mathbb{R}^{3}
      7. Geometrically independent (a set {\mathbf{x}\_{0},…, \mathbf{x}\_{k}} of points of \mathbb{R}^{N} ) : if the equation \sum\_{I = 0}^{k} a\_{i}mathbf{x}\_{i} = \mathbf{0} and \sum\_{I = 0}^{k} a\_{i} = 0 holds only if each a\_{i} = 0.
      8. Plane P determined by these points (a set of all points \mathbf{x} of of \mathbb{R}^{N} ) : Let {\mathbf{x}\_{0},…, \mathbf{x}\_{k}} be a set of points of \mathbb{R}^{N} that is geometrically independent.\mathbf{x} = \sum\_{I = 0}^{k} t\_{i}mathbf{x}\_{i} where \sum\_{I = 0}^{k} t\_{i} = 1.
         1. mathbf{x} = \mathbf{x}\_{0} + \sum\_{I = 0}^{k} a\_{i}( mathbf{x}\_{i} - \mathbf{x}\_{0}) for some scalars a\_{1} , …, a\_{k}
         2. Translation T of \mathbb{R}^{N} : a homeomorphism T : \mathbb{R}^{N} -> \mathbb{R}^{N} ; T(\mathbf{x}) = \mathbf{x – x\_{0}}
         3. K-plane P in \mathbb{R}^{N} : a plane P
      9. A in general position in \mathbb{R}^{N} (a set A of points of \mathbb{R}^{N}) : every subset of A containing N + 1 or fewer points is geometrically independent
      10. (Lem 50.4) Given a finite set {\mathbf{x}\_{1},…, \mathbf{x}\_{n}} of points of \mathbb{R}^{N} and given \delta >0, there exists a set {\mathbf{y}\_{1},…, \mathbf{y}\_{n}} of points of \mathbb{R}^{N} in general position in \mathbb{R}^{N}, s.t. \vert x\_{i} – y\_{i} \vert < \delta for all i.
          1. Pf) (Def of Baire space)
      11. (The imbedding Theorem) (Thm 50.5) Every compact metrizable space X of topological dimension m can be imbedded in \mathbb{R}^{2m+1}
      12. (Thm 50.6) Every compact subspace of \mathbb{R}^{N} has topological dimension at most N.
          1. Pf) M-cube : homeomorphic to the product (0,1)^{M}
          2. (Cor 50.7) Every compact m-manifold has topological dimension at most m.
          3. (Cor 50.8) Every compact m-manifold con be imbedded in \mathbb{R}^{2m+1}
          4. (Cor 50.9) Let X be a compact metrizable space. Then X can be imbedded in some Euclidean space \mathbb{R}^{N} iff X has finite topological dimension.

Algebraic Topology

1. Fundamental Group\
   1. Homotopy of paths
      1. f is homotopic to f’ (two continuous maps f, f’ of the space X into the space Y) : Let I = [0,1]. there is a continuous map F : X \times I -> Y s.t. F(x,0) = f(x) and F(x,1) = f’(x).
         1. Homotopy between f and f’ : the map F
         2. f \simeq f’ : f is homotopic to f’
         3. nulhomotopic f : If f \simeq f’ and f’ is a constant map
      2. Path in X from x\_{0} to x\_{1} ( a continuous map f : [0,1] -> X ) : f(0) = x\_{0} , f(1) = x\_{1}
         1. Initial point : x\_{0}
         2. Final point : x\_[1]
      3. Path homotopic ( two paths f, f’ : [0,1] ->X ) : They have the same initial point x\_{0} and the same final point x\_{1} and there is a continuous map F : I \times I -> X for each s \in I and each t \in I s.t.
         1. F (s, 0) = f(s) and F (s, 1) = f’(s)
         2. F (0,t) = x\_{0} and F (1, t) = x\_{1}
            1. F : a path homotopy between f and f’
            2. f \simeq\_{p} f’ : f is path homotopic to f’
      4. (Lem 51.1) The relations \simeq and \simeq\_{p} are equivalence relations.
      5. [ f ] : path-homotopy equivalence class of a path f
      6. Example
         1. Straight-line homotopy ( two maps f, g X -> \mathbb{R}^{2}) : F(x,t) = (1-t) f(x) + t g(x)
         2. Let A be any convex subspace of \mathbb{R}^{n}. Then any two paths f, g in A from x\_{0} to x\_{1} are path homotopic in A.
      7. Product f \* g of f and g ( a path f in X from x\_{0} to x\_{1}, a path g in X from x\_{1} to x\_{2} ) : a path h given by the equations
         1. h(s) = \begin{cases} f(2s) for s \in [0,frac{1}{2}] , \\ g(2s – 1) for s \in [frac{1}{2}, 1] \end{cases}
         2. The function h is well-defined and continuous.
            1. Pf) pasting lemma
         3. h is a path in X from x\_{0} to x\_{2}
      8. (Groupoid properties of \* )(Thm 51.2) The operation \* has the following properties.
         1. (Associativity) If [ f ] \* ( [ g ] \* [ h ] ) is defined, so is ( [ f ] \* [ g ] ) \* [ h ] ,and they are equal.
         2. (Right and left identities) Given x \in X, let e\_{x} denote the constant path e\_{x} : I -> X carrying all of I to the point x. If f is a path in X from x\_{0} to x\_{1}, then [ f ] \* [ e\_{x\_{1}} ] = [ f ] and [ e\_{x\_{0}} ] \* [ f ] = [ f ]
         3. (Inverse) Given the path f in X from x\_{0} to x\_{1}, let \bar{f} be the path defined by \bar{f} (s) = f (1-s) . It is called the reverse of f. Then [ f ] \* [ \bar{f} ] = [ e\_{x\_{0}} ] and [ \bar{f} ] \* [ f ] = [ e\_{x\_{1}} ]
         4. Pf) convex
      9. (Thm 51.3) Let f be a path in X, and let a\_{0}, …, a\_{n} be numbers s.t. 0 = a\_{0} < a\_{1} < … < a\_{n} = 1. Let f\_{i} : I -> X be the path that equals the positive linear map of I onto [a\_{i-1}, a\_{i}] followed by f. Then [ f ] = [ f\_{1} ] \* .. \* [ f\_{n} ]
   2. Fundamental Group
      1. Suppose G and G’ are groups, written multiplicatively.
         1. Homomorphism (a map f : G -> G’) : f (x \cdot y) = f (x) \cdot f (y).
            1. f( e ) = e’
            2. f ( x^{-1} ) = f ( x ) ^ {-1}
            3. Kernel of f : the set f^{-1} (e’) , which is subgroup of G
            4. Monomorphism : an injective homomorphism
            5. Epimorphism : a surjective homomorphism
            6. Isomorphism : a bijective homomorphism
         2. Left coset xH of H in G (a subgroup H of a group G ) : the set of all products xh , for h \in H
         3. Normal subgroup H of G (a subgroup H of a group G ) : x \cdots h \cdots x^{-1} \in H for each x \in G and each h \in H.
         4. G/H : a partition which is the collection of all coset forms.
      2. Quotient of G by H (a subgroup H of a group G ) : a group (G/H, \cdot ) which \cdot is a well-defined operation (x H) \cdot (y H) = (x \cdot y) H.
         1. If f : G -> G’ is an epimorphism, its kernel N is a normal subgroup of G and f induces an isomorphism G/N -> G’ ; x N -> f(x) for each x \in G.
      3. Loop based at x\_{0} ( a point x\_{0} of a space X ) : a path in X that begins and ends at x\_{0}
      4. Fundamental group of X relative to the base point x\_{0} ( a point x\_{0} of a space X ) : The set of path homotopy classes of loops based at x\_{0}
         1. \pi\_{1} (X, x\_{0}) <- Fundamental group of X relative to the base point x\_{0}
         2. The operation \* restricted to this set satisfies axioms for a group.
         3. Also called ‘ First homotopy group of X ‘
         4. Example
            1. For a convex subset X of \mathbb{R}^{n} \pi\_{1} (X, x\_{0}) is trivial group.
            2. Unit ball B^{n} in \mathbb{R}^{n} has trivial fundamental group.
      5. \alpha – hat : let \alpha be a path in X from x\_{0} to x\_{1}. A map \alpha – hat is \hat{\alpha} : \pi\_{1} (X, x\_{0}) -> \pi\_{1} (X, x\_{1}) by the equation \hat{\alpha}([ f ]) – [ \bar{\alpha} ] \* [ f ] \* [ \alpha ].
      6. (Thm 52.1) The map \hat{\alpha} is a group isomorphism.
         1. (Cor 52.2) If X is path connected and x\_{0} and x\_{1} are two points of X, then \pi\_{1} (X, x\_{0}) is isomorphic to \pi\_{1} (X, x\_{1})
      7. Simply connected ( a space X) : X is a path-connected space and if \pi\_{1} (X, x\_{0}) is the trivial (one-element) group for some x\_{0} \in X, and hence for every x\_{0} \in X.
         1. \pi\_{1} (X, x\_{0}) : \pi\_{1} (X, x\_{0}) is the trivial group.
      8. (Lem 52.3) In a simply connected space X, any two paths having the same initial and final points are path homotopic.
      9. h : (X, x\_{0}) -> (Y, y\_{0}) <- a continuous map that carries the point x\_{0} of X to the point y\_{0} of Y
      10. Homomorphism h\_{\*} induced by h relative to the base point x\_{0} (a continuous map h : (X, x\_{0}) -> (Y, y\_{0}) ) : h\_{\*} : \pi\_{1} (X, x\_{0}) -> \pi\_{1} (Y, y\_{0}) by the equation h\_{\*} ([ f ]) = [h \bullet f ]
      11. (Thm 52.4) If h : (X, x\_{0}) -> (Y, y\_{0}) and k : (Y, y\_{0}) -> (Z, z\_{0}) are continuous, then (k \bullet h)\_{\*} = k\_{\*} \bullet h\_{\*} . If i: (X, x\_{0}) -> (X, x\_{0}) is the identity map, then i\_{\*} is the identity homomorphism.
          1. (Cor) If h : (X, x\_{0}) -> (Y, y\_{0}) is a homomorphism of X with Y, then h\_{\*} is an isomorphism of \pi\_{1} (X, x\_{0}) with \pi\_{1} (Y, y\_{0})
   3. Covering Spaces
      1. Evenly covered by p ( an open set U of B, a map p : E -> B ) : Let p be continuous surjective map. The inverse image p^{-1} (U) can be written as the union of disjoint open sets V\_{\alpha} in E s.t. for each \alpha, the restriction p to V\_{\alpha} is a homeomorphism of V\_{\alpha} onto U.
         1. A partition of p^{-1} (U) into slices : the collection {V\_{\alpha}}
      2. Covering map ( a continuous surjective map p : E -> B) : Every point b of B has a neighborhood U that is evenly covered by p.
         1. a covering space E of B : if a covering map p : E -> B exists
         2. If p : E->B is a covering map, then for each b \in B the subspace p^{-1}(b) of E has the discrete topology.
         3. If p : E ->B is a covering map, then p is an open map.
      3. (Thm 53.1) The map p : \mathbb{R} -> S^{-1} given by the equation p(x) = (cos 2 \pi x , sin 2 \pi x ) is a covering map.
      4. A local homeomorphism p of E with B ( a map p : E-> B) : each point e of E has a neighborhood that is mapped homeomorphically by p onto an open subset of B.
         1. If p: E -> B is a covering map, then p is a local homeomorphism of E with B.
         2. Example
            1. The map p : \mathbb{R}\_{+} -> S^{1} ; p(x) = (cos 2 \pi x , sin 2 \pi x )
      5. (Thm 53.2) : Let p : E->B be a covering map. If B\_{0} is a subspace of B, and if E\_{0} = p^{-1}(B\_{0}), then the map p\_{0} : E\_{0} -> B\_{0} obtained by restricting p is a covering map.
      6. (Thm 53.3) If p : E -> B and p’ : E’ -> B’ are covering maps, then p \times p’ : E \times E’ -> B \times B’ is a covering map.
         1. Example
            1. For a Torus T = S^{-1} \times S^{-1}, p \times p : \mathbb{R} \times \mathbb{R} -> T
            2. Let b\_{0} = p(0) of S^{1}; a figure-eight space B\_{0} = a subspace (S^{1} \times b\_{0}) U (b\_{0} \times S^{1}) of S^{1} \times S^{1} ; an infinite grid E\_{0} = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} ). Then the map p\_{0} : E\_{0} -> B\_{0} = p \times p | E\_{0} is a covering map.
   4. Fundamental group of the circle
      1. Lifting of f ( a continuous mapping f of some space X into B) : Let p : E-.B be a map. A lifting of f is a map \tilde{f} : X -> E s.t. p \bullet \tilde{f} = f.
      2. (Lem 54.1) Let p : E -> B be a covering map, let p(e\_{0}) = b\_{0}. Any path f : [0,1] -> B beginning at b\_{0} has a unique lifting to a path \bar{f} in E beginning at e\_{0}.
         1. Pf) (Lebesgue number lemma) (pasting lemma)
      3. (Lem 54.2) Let p : E->B be a covering map; let p(e\_{0}) = b\_{0}. Let the map F : I \times I -> B be continuous, with F (0,0) = b\_{0}. There is a unique lifting of F to a continuous map \tilde{F} : I \times I -> E s.t. \tilde{F} (0,0) = e\_{0}. If F is a path homotopy, then \tilde{F} is a path homotopy.
         1. Pf) (Lem 54.1) (Lebesgue number lemma) (pasting lemma)
      4. (Thm 54.3) Let p : E -> B be a covering map ; let p(e\_{0}) = b\_{0}. Let f and g be two paths in B from b\_{0} to b\_{1}; let \tilde{f} and \tilde{g} be their respective liftings to paths in E beginning ar e\_{0}. If f and g are path homotopic, then \tilde{f} and \tilde{g} end at the same point of E and path homotopic.
      5. Lifting correspondence \phi derived from the covering map p ( a covering map p: E -> B) : Let b\_{0} \in B. Choose e\_{0} so that p(e\_{0}} = b\_{0}. Given an element [ f ] of \pi\_{1} (B,b\_{0}) , let \tilde{F} be the lifting of f to a path in E that begins at e\_{0}. Let \phi([ f ]) denote the end point \tilde{f} (1) of \tilde{f}. Then \phi is a well-defined set map \phi : \pi\_{1} (B,b\_{0}) -> p^{-1} (b\_{0}).
      6. (Thm 54.4) Let p: E->B be a covering map; let p(e\_{0}) = b\_{0}. If E is a path connected, then the lifting correspondence \phi : \pi\_{1} (B,b\_{0}) -> p^{-1} (b\_{0}). Is surjective. If E is simply connected, it is bijective.
      7. (Thm 54.5) The fundamental group of S^{1} is isomorphic to the additive group of integers.
         1. Pf) (Thm 53.1)
      8. Cyclic group (a group G) : Let x be an element of G. the set of all elements of the form x^{m}, for m \in \mathbb{Z}, equals G
         1. The symbol x^{n} denotes the n-fold product of x with itself, x^{-n} denotes the n-fold product of x^{-1} with itself, and x^{0} denotes the identity element of G.
         2. x : a generator of G
      9. Order of the group : Cardinality of a group
      10. (Thm 54.6) Let p : E->B be a covering map ; let p(e\_{0}) = b\_{0}.
          1. The homomorphism p\_{\*} : \pi\_{1} (E,e\_{0}) -> \pi\_{1} (B,b\_{0}) is a monomorphism.
          2. Let H = p\_{\*}( \pi\_{1} (E,e\_{0}) ). The lifting correspondence \phi induces an injective map \Phi : \pi\_{1} (B,b\_{0}) / H -> p^{-1} (b\_{0}) of the collection of right cosets of H into p^{-1} (b\_{0}), which is bijective if E is path connected.
          3. If f is a loop in B based at b\_{0}, then [ f ] \in H iff f lifts to a loop in E based at e\_{0}.
             1. Pf) \phi ( [ f ] ) = \phi ( [ g ] ) iff [ f ] \in H \* [ g ].
   5. Retractions and Fixed Points
      1. Retraction of X onto A ( a subset A of X ) : a continuous map r : X -> A s.t. r |A is the identity map of A.
         1. Retract A of X : If such a map r exists
      2. (Lem 55.1) If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion j : A ->X is injective.
      3. (No-retraction theorem) (Thm 55.2) : There is no retraction of B^{2} onto S^{1}.
         1. B^{2} is a unit ball.
      4. (Lem 55.3) Let h : S^{1} -> X be a continuous map. Then the following conditions are equivalent.
         1. h is nullhomotopic.
         2. h extends to a continuous map k : B^{2} -> X.
         3. h\_{\*} is the trivial homomorphism of fundamental groups.
      5. (Cor 55.4) The inclusion map j : S^{1} -> \mathbb{R}^{2} - \mathbf{0} is not nullhomotopic. The identity map I : S^{1} -> S^{1} is not nullhomotopic.
      6. (Thm 55.5) Given a nonvanishing vector field on B^{2}, there exists a point of S^{1} where the vector field points directly inward and a point of s^{1} where it points directly outward.
         1. Vector field on B^{2} : an ordered pair (x, v(x)), where x is in B^{2} and v is a continuous map of B^{2} into \mathbb{R}^{2}.
         2. Nonvanishing (a vector field) : v(x) \neq \mathbf{0} for every x
         3. Pf) (Cor 55.4)
      7. (Brower fixed-point theorem for the disc) (Thm 55.6) : If f : B^{2} -> B^{2} is continuous, then there exists a point x \in B^{2} s.t. f(x) = x
         1. (Cor 55.7) Let A be a 3 by 3 matrix of positive real numbers. Then A has a positive real eigenvalue.
      8. (Thm 55.8) There is an \epsilon > 0 s.t. for every open covering \mathcal{A} of T by sets of diameter less than \epsilon, some point of T belongs to at least three elements of \mathcal{A}.
         1. T = {(x,y) | x \ge 0 and y \ge 0 and x + y \le 1}
         2. Pf) T is homeomorphic to B^{2}, a partition of unity,
   6. Fundamental Theorem of Algebra
      1. (Fundamental theorem of algebra) (Thm 56.1) A polynomial equation x^{n} + a\_{n-1} x^{n-1} + … + a\_{1}x + a\_{0} = 0 of degree n > 0 with real or complex coefficients has at least one (real or complex) root.
         1. Pf) g: S^{1} -> \mathbb{R}^{2} - \mathbf{0} ; g(z) = z^{n} is not nullhomotopic.
   7. Borsuk-Ulam Theorem
      1. Antipode (a point x in S^{n}) : -x
      2. Antipode-preserving h ( a map h : S^{n} -> S^{m}) : h(-x) = -h(x) for all x \in S^{n}.
      3. (Thm 57.1) if h : S^{1} -> S^{1} is continuous and antipode-preserving, then h is not nullhomotopic.
         1. Pf) covering map,
      4. (Thm 57.2) There is no continuous antipode-preserving map g : S^{2} -> S^{1}.
         1. Pf) (Thm 57.1)
      5. (Borsuk-Ulam theorem for S^{2}) (Thm 57.3) Given a continuous map f : S^{2} -> \mathbb{R}^{2}, there is a point x of S^{2} s.t. f(x) = f(-x).
      6. (The bisection theorem) (Thm 57.4) Given two bounded polygonal regions in \mathbb{R}^{2}, there exists a line in \mathbb{R}^{2} that bisects each of them.
         1. Pf) (Borsuk-Ulam theorem)
   8. Deformation Retracts and Homotopy type
      1. (Lem 58.1) Let h, k : (X, x\_{0}) -> (Y, y\_{0}) be continuous maps, If h and k are homotopic, and if the image of the base point x\_{0} of X remains fixed at y\_{0} during the homotopy, then the homomorphisms h\_{\*} and k\_{\*} are equal.
      2. (Thm 58.2) The inclusion map j : S^{n} -> \mathbb{R}^{n+1} - \mathbf{0} induces an isomorphism of fundamental groups.
         1. Pf) (Lem 58.1)
      3. Deformation retract of X ( a subspace A of X) : the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy.
         1. Deformation retraction H of X onto A ( a homotopy H) : There is a continuous map H : X \times I -> X s.t. H(x,0) = x and H(x,1) \in A for all x \in X, and H(a,t) = a for all a \in A.
         2. The map r : X -> A defined by the equation r(x) = H(x,1) is a retraction of X onto A.
         3. H is a homotopy between the identity map of X and the map j \bullet r, where j : A -> X is inclusion.
      4. (Thm 58.3) Let A be a deformation retract of X ; let x\_{0} \in A. Then the inclusion map j : (A, x\_{0}) -> (X, x\_{0}) induces an isomorphism of fundamental groups.
      5. Homotopy equivalences ( continuous maps f : X -> Y, g : Y -> X) : the map g \bullet f : X -> X is homotopic to the identity map of X, and the map f \bullet g : Y -> Y is homotopic to the identity map of Y.
         1. Each is said to be a ‘homotopy inverse’ of the other.
         2. Relation of homotopy equivalence is an equivalence relation.
         3. Same Homotopy type : Spaces that are homotopy equivalent
      6. (Lem 58.4) Let h, k : X -> Y be continuous maps; let h(x\_{0}) = y\_{0} and k(x\_{0}) = y\_{1}. If h and k are homotopic, there is a path \alpha in Y from y\_{0} to y\_{1} s.t. k\_{\*} = \hat{\alpha} \bullet h\_{\*}. Indeed, if H : X \times I -> Y is the homotopy between h and k, then \alpha is the path \alpha(t) = H(x\_{0}, t)
         1. (Cor 58.5) Let h, k : X ->Y be homotopic continuous maps; let h(x\_{0}) = y\_{0} and k(x\_{0}) = y\_{1}. If h\_{\*} is injective, or surjective, or trivial, so is k\_{\*}.
         2. (Cor 58.6) Let h : X -> Y. If h is nullhomotopic, then h\_{\*} is the trivial homomorphism.
      7. (Thm 58.7) Let f : X -> Y be continuous; let f(x\_{0}) = y\_{0}. If f is a homotopy equivalence, then f\_{\*} : \pi\_{1} (X, x\_{0}) ->\pi\_{1} (Y, y\_{0}) is an isomorphism.
   9. Fundamental Group of S^{n}
      1. (Thm 59.1) Suppose X = U \cup V, where U and V are open sets of X. Suppose that U \cap V is path connected, and that x\_{0} \in U \cap V. Let I and j be the inclusion mappings of U and V, respectively, into X. Then the images of the induced homomorphisms i\_{\*} : \pi\_{1} (U, x\_{0}) ->\pi\_{1} (X, x\_{0}) and j\_{\*} : \pi\_{1} (V, x\_{0}) ->\pi\_{1} (X, x\_{0}) generate \pi\_{1} (X, x\_{0})
         1. Pf) there is a subdivision a\_{0} < a\_{1} < … < a\_{n} of the unit interval s.t. f(a\_{i}) \in U \cap V and f([a\_{i-1}, a\_{i}]) is contained either in U or in V, for each i., (Lebesgue number lemma)
         2. (Cor 59.2 ) Suppose X = U \cup V, where U and V are open sets of X; suppose X \cap V is nonempty and path connected, If U and V are simply connected, then X is simply connected.
      2. (Thm 59.3) If n \ge 2, then n-sphere S^{n} is simply connected.
         1. Stereographic projection f : (S^{n} – p) -> \mathbb{R}^{n} ; f(x) = f(x\_{1}, …, x\_{n+1}) = frac{1}{1-x\_{n+1}} (x\_{1}, …, x\_{n}) where p = (0,…,0,1) \in \mathbb{R}^{n+1}
         2. Pf) (Cor 59.2)
   10. Fundamental Groups of Some Surfaces
       1. (Thm 60.1) \pi\_{1} (X \times Y, x\_{0} \times y\_{0}) is isomorphic with \pi\_{1}(X,x\_{0}) \times \pi\_{1} (Y, y\_{0}).
          1. (Cor 60.2) The fundamental group of the torus T = S^{1} \times S^{1} is isomorphic to the group \mathbb{Z} \times \mathbb{Z}.
       2. Projective plane P^{2} : quotient space obtained from S^{2} by identifying each point x of S^{2} with its antipodal point -x.
       3. (Thm 60.3) The projective plane P^{2} is a compact surface, and the quotient map p: S^{2} -> P^{2} is a covering map.
          1. Pf) S^{2} is normal and p is a closed map -> P^{2} is Hausdorff
          2. (Cor 60.4) \pi\_{1} (P^{2}, y) is a group of order 2.
             1. Pf) (Thm 54.4)
       4. Projective n-space : quotient space obtained from S^{n} by identifying each point x of S^{2} with its antipodal point -x
          1. The projective plane P^{n} is a compact surface, and the quotient map p: S^{n} -> P^{n} is a covering map.
       5. (Lem 60.5) The fundamental group of the figure eight is not abelian.
       6. (Thm 60.6) The fundamental group of the double torus is not abelian.
          1. Double torus T#T : the surface obtained by taking two copies of the torus, deleting a small open disc from each of them, and pasting the remaining pieces together along their edges.
          2. Pf) Figure eight X is a retract of T#T.
          3. (Cor 60.7) The 2-sphere, torus, projective plane and double torus are topologically distinct.
2. Separation Theorems in the Plane
   1. Jordan Separation Theorem
      1. (Lem 61.1)