1. Set theory and Logic
   1. Fundamental concepts
      1. Basic Notation
         1. Sets <- Capital letters
         2. Objects (elements) <- lowercase letters
         3. Logical Identity <- = (equality symbol)
         4. A ⊂ B <- A is a subset of B , inclusion
            1. Every element of A is also an element of B
         5. A \subsetneq B <- A is a proper subset of B, proper inclusion
            1. A \subset B and A is different from B
      2. The Union of Sets and the meaning of ‘or’
         1. Union of A and B ( A \cup B) = { x | x \in A or x \in B}
            1. P or Q : P or Q or both
      3. The Intersection of Sets, the Empty Set and the Meaning of “if .. Then”
         1. Intersection of A and B ( A \cap B) = { x | x \in A and x \in B}
         2. Empty set \phi : the set having no elements
            1. A \cup \phi = A
            2. A \cap \phi = \phi
         3. Disjoint (a set A, a set B) : A \cap B = \phi
         4. If (Hypothesis), then (conclusion)
            1. Vacuously true : no case for which the hypothesis holds
      4. Contrapositive and Converse
         1. Contrapositive ( ‘If P, then Q ‘ statement) : If Q is not true, then P is not true
         2. Converse ( ‘If P, then Q ‘ statement) : If Q, then P
         3. P <-> Q : P holds iff Q holds
      5. Negation ( a statement P ) : not P
      6. The difference of two sets
         1. Difference of two sets A, B ( Complement of B relative to A)
            1. A – B = { x | x \in A and x \notin B}
      7. Rules of set theory
         1. Distributive law (sets A, B, C)
            1. A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
            2. A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
         2. Demorgan’s laws
            1. A – (B \cup C) = (A – B) \cap (A – C)
            2. A – (B \cap C) = (A – B) \cup (A – C)
      8. Collection of the sets
         1. Power set of A ( \mathcal{P} (A) ): set of all subsets of A
         2. Collection of sets : a set whose elements are sets
      9. Arbitrary unions and intersections
         1. Given a collection \mathcal{A} of sets,
            1. Union of the elements of \mathcal{A}

\bigcup\_{A \in \mathcal{A}} A = { x | x \in A for at least one A \in \mathcal{A}}

* + - * 1. Intersection of the elements of \mathcal{A}

\bigcap\_{A \in \mathcal{A}} A = { x | x \in A for every A \in \mathcal{A}}

* + - * 1. \bigcup\_{A \in \mathcal{A}} A = \phi
        2. \bigcap\_{A \in \mathcal{A}} A = (Not defined)
    1. Cartesian Product
       1. Ordered pair of objects a , b : a \times b
          1. All ordered pairs of real numbers is a plane
       2. Cartesian product ( sets A , B) : A \times B = {a \times b | a \in A and b \in B}
  1. Function
     1. A rule of assignment : a subset r of the cartesian product C x D of two sets, having the property that each element of C appears as the first coordinate of at most one ordered pair belonging to r.
        1. [c \times d \in r and c \times d’ \in r] -> [d = d’]
        2. Domain (a rule of assignment r) : { c | there exists d \in D s.t. c \times d \in r}
        3. Image ( a rule of assignment r) : {d | there exists c \in C s.t. c \times d \in r}
     2. Function f : a rule of assignment r, together with a set B that contains the image set of r.
        1. Domain (a function f) : Domain of the rule r
        2. Image set (a function f) : the image set of r
        3. Range ( a function f) : the set B
           1. f is a function having domain A and range B : f : A -> B
        4. value ( function f, element a of Domain (f) ) : unique element of B the rule determining f assigns to a
     3. Restriction of function f | A\_{0} : If f : A -> B and if A\_{0} is a subset of A, we define the restriction of f to A\_{0} to be the function mapping A\_{0} into B whose rule is {(a,f(a)) | a \in A\_{0}}
     4. Composite g \bullet f : given functions f : A->B and g : B->C, the composite of f and g is the function g \bullet f : A -> C defined by equation (g \bullet f)(a) = g(f(a))
     5. Injective : A function f : A -> B is said to be injective (or one -to- one) if for each pair of distinct points of A, their images under f are distinct.
     6. Surjective : f is said to map A onto B if every element of B is the image of some element of A under the function f.
     7. Bijective : f is said to be one-to-one correspondence if f is both injective and surjective
        1. If f is bijective, there exists a function from B to A, called the inverse of f. f^{-1}
        2. (Criterion of bijection) : Let f : A->B. if there are functions g : B->A and h:B->A s.t. g(f(a)) = a for every a in A and f(h(b)) = b for every b in B, then f is bijective and g = h = f^{-1}
     8. Image of A\_{0} under f : Let f:A->B. if A\_{0} is a subset of A, we denote by f(A\_{0}) the set of all images of points of A\_{0} under the function.
        1. Preimage of B\_{0} under f : if B\_{0} is a subset of B, f^{-1}(B\_{0}) is the set of all elements of A whose images under f lie in B\_{0}.
        2. A\_{0} \subseteq f^{-1}(f(A\_{0})) (equality if f is injective)
        3. F(f^{-1}(B\_{0})) \subseteq B\_{0} (equality if f is surjective)
  2. Relations
     1. A relation on a set A : a subset C of the cartesian product A \times A
     2. Equivalence relation on a set A : a relation C on A with
        1. (Reflexivity) : xCx for every x in A
        2. (Symmetry) if xCy, then yCx
        3. (Transitivity) if xCy and yCz, then xCz
     3. Equivalence class determined by x : given an equivalence relation ~ on a set A and an element x of A, we define a certain subset E of A, by the equation E = { y | y ~ x }
        1. Two equivalence classes E and E’ are either disjoint or equal.
     4. Partition of a set A : a collection of disjoint nonempty subsets of A whose union is all of A.
        1. Given any partition \mathcal{D} of A, there is exactly one equivalence relation on A from which it is derived.
     5. Order relations ( a relation C on a set A)
        1. (Comparability) For every x and y in A for which x \neq y, either xCy or yCx.
        2. (nonreflexivity) : For no x in A does the relation xCx hold.
        3. (Transitivity) If xCy and yCz, then xCz
     6. Open interval in X : if X is a set and < is an order relation on X, and if a < b, we use the notation (a, b) to denote the set {x | a < x < b }
        1. a is the immediate predecessor of b, and b is the immediate predecessor of a if (a, b) is empty
     7. Suppose that A and B are two sets with order relations <\_{A} and <\_{B} respectively, We say that A and B have the same order type if there is a bijective correspondence between them that preserves order
        1. Preserve order : a\_{1} <\_{A} a\_{2} -> f(a\_{1}) <\_{B} f(a\_{2})
     8. Dictionary order relation : Suppose A and B are two sets with order relations <\_{A} and <\_{B} respectively, define an order relation on A \times B by defining a\_{1} \times b\_{1} < a\_{2} times b\_{2} if a\_{1}<\_{A}a\_{2} or if a\_{1} = a\_{2} and b\_{1} <\_{B} b\_{2}.
     9. Largest element of A\_{0} (an element b) : Suppose that A is a set ordered by relation <. Let A\_{0} be a subset of A. b \in A\_{0} and if x \le b for every x \in A\_{0}
     10. Smallest element of A\_{0} (an element a) : a \in A\_{0} and if a \le x for every x \in A\_{0}
     11. Bounded above (a subset A\_{0} of A) : there is an element b of A s.t. x \le b for every x \in A\_{0}.
         1. Upper bound for A\_{0} : b
         2. The least upper bound of A\_{0} [supremum of A\_{0}, sup A\_{0}] : a smallest element of the set of all upper bounds for A\_{0}
         3. Least upper bound property ( an ordered set A) : every nonempty subset A\_{0} of A that is bounded above has a least upper bound
     12. Bounded below (a subset A\_{0} of A) : there is an element a of A s.t. a \le x for every x \in A\_{0}.
         1. Lower bound for A\_{0} : a
         2. The greatest lower bound of A\_{0} [infimum of A\_{0}, inf A\_{0}] : a largest element of the set of all lower bounds for A\_{0}
         3. Greatest lower bound property (an ordered set A) : every nonempty subset A\_{0} of A that is bounded below has the greatest lower bound property.
  3. The Integers and the real numbers
     1. Binary operation on a set A : a function f mapping A \times A into A
     2. Assume the existence of the set of real numbers \mathbb{R} with +, \cdot, <
        1. Algebraic properties
           1. (x+y) + z = x + (y+z) , (x \cdot y) \cdot z = x \cdot (y \cdot z) for all x, y, z in \mathbb{R}
           2. x + y = y + x, x \cdot y = y \cdot x for all x,y in \mathbb{R}
           3. There exists a unique element of \mathbb{R} called zero, denoted by 0, s.t. x + 0 = x for all x \in \mathbb{R}, There exists a unique element of \mathbb{R} called one, denoted by 1, s.t. x \cdot 1 = x for all x \in \mathbb{R}
           4. (Negative of x) For each x in \mathbb{R}, there exists a unique y in \mathbb{R} s.t. x + y = 0,
           5. (Reciprocal of x) For each x in \mathbb{R} different from 0, there exists a unique y in \mathbb{R} s.t. x\cdot y = 1.
           6. x \cdot (y+z) = (x \cdot y) + (x \cdot z) for all x, y, z \in \mathbb{R}.
        2. A Mixed Algebraic and Order Property
           1. if x > y, then x + z > y + z. If x > y and z > 0, then x \cdot z > y \cdot z
        3. Order properties
           1. The order relation < has the least upper bound property.
           2. If x < y, there exists an element z s.t. x < z and z < y
        4. Subtraction operation : z-x = z + (-x)
        5. Quotient z/x = z \cdot (1/x)
        6. Laws of inequalities : If x > y and z < 0, then x \cdot z < y \cdot z
        7. Field : only algebraic properties of above
        8. Ordered field : algebraic property and a mixed algebraic and order property of above
        9. Linear continuum : only order properties of above
     3. Inductive ( a subset of \mathbb{R}) : it contains the number 1 and if for every x in A, the number x + 1 is also in A.
     4. Positive Integers Z\_{+} = \bigcap\_{A \in \mathcal{A}} A
        1. Z\_{+} is inductive
        2. (Principle of Induction) : if A is an inductive set of positive integers, then A = Z\_{+}
     5. Integers \mathbb{Z} : a set consisting of Z\_{+}, the number 0, the negatives of the elements of Z\_{+}
     6. Rational numbers \mathbb{Q} : a set of quotients of integers
     7. Section S\_{n} of the positive integers : S\_{n+1} = {1,…,n}, S\_{1} = \phi
     8. (Well-ordering property) : Every nonempty subset of \mathbb{Z}\_{+} has a smallest element.
     9. (Strong induction principle) : Let A be a set of positive integers. Suppose that for each positive integer n, the statement S\_{n} \subset A implies the statement n \in A, then A = \mathbb{Z}\_{+}.
     10. Proofs using least upper bound axiom
         1. Archimedian ordering property of the real line : \mathbb{Z}\_{+} of positive integers has no upper bound in \mathbb{R}
         2. Greatest lower bound property
         3. Existence of a unique positive square root for every positive real number
  4. Cartesian Products
     1. Let \mathcal{A} be a nonempty collection of sets.
        1. Indexing function (\mathcal{A}) : a surjective function f from some set J to \mathcal{A}.
           1. Index set : J
           2. Indexed family of sets {A\_{\alpha}}\_{\alpha \in J} : the collection \mathcal{A} with indexing function f

A\_{\alpha} = f(\alpha)

* + 1. \bigcup\_{\alpha \in J} A\_{\alpha} = {x | for at least one \alpha \in J, x \in A\_{\alpha}}
    2. \bigcap\_{\alpha \in J} A\_{\alpha} = {x | for every \alpha \in J, x \in A\_{\alpha}}
    3. M-tuple of elements of X : Let m be a positive integer. Given a set, a function \mathbf{x} : {1,…,m} -> X
       1. (x\_{1}, …, x\_{m})
       2. Ith coordinate of \mathbf{x} : value of \mathbf{x} at I
       3. Cartesian product ({A\_{1}, …, A\_{m}} being a family of sets indexed with the set {1,…,m}) :
          1. Let X = A\_{1} \cup … \cup A\_{m}. then \prod\_{I = 1}^{m} A\_{i} or A\_{1} \times … \times A\_{m} = set of all m-tuples (x\_{1} , …, x\_{m}) of elements of X s.t. x\_{i} \in A\_{i} for each i.
       4. X^{m} <- M-tuple of elements of X
    4. \omega -tuple of elements of a set X : \mathbf{x} : \mathbb{Z}\_{+} -> X
       1. Also called sequence , infinite sequence, (x\_{1}, x\_{2}, …) , (x\_{n})\_{n \in \mathbb{Z}\_{+}\_
       2. Ith coordinate of \mathbf{x} : value of \mathbf{x} at I
       3. Cartesian product ({A\_{1}, A\_{2}, … } being a family of sets indexed with positive integers) :
       4. Let X = Union of the sets in this family. then \prod\_{I = \mathbb{Z}\_{+}} A\_{i} or A\_{1} \times A\_{2} \times… = set of all \omega -tuples (x\_{1} , x\_{2}… ) of elements of X s.t. x\_{i} \in A\_{i} for each i.
       5. X^{\omega} <- 1.5.5 \omega -tuple of elements of a set X
  1. Finite sets
     1. Finite (a set) : there is a bijective correspondence of A with some section of the positive integers.
        1. Cardinality 0 : A is empty
        2. Cardinality n : there is a bijection f:A->{1,..,n} for some positive integer n
     2. (Thm 6.1) Let A be a set. Suppose that there exists a bijection f:A-> {1,..,n} for some n \in \mathbb{Z}\_{+}. Let B be a proper subset of A. then there exists no bijection g : B->{1,…,n}. But provided B \neq \phi, there does exist a bijection h:B->{1,…,m} for some m<n.
        1. Let n be a positive integer. Let A be a set. Let a\_{0} be an element of A. There exists a bijective correspondence f of the set A with the set {1,…,n+1} iff there exists a bijective correspondence g of the set A-{a\_{0}} with the set {1,…,n}.
        2. (Cor) If A is finite, there is no bijection of A with a proper subset of itself.
        3. (Cor) Z\_{+} is not finite.
        4. (Cor) The cardinality of a finite set A is uniquely determined by A.
        5. (Cor) If B is a subset of the finite set A, then B is finite. If B is a proper subset of A, then the cardinality of B is less than the cardinality of A.
        6. (Cor) Let B be a nonempty set. Then the following are equivalent.
           1. B is finite.
           2. There is a surjective function from a section of the positive integers onto B.
           3. There is an injective function from B into a section of the positive integers.
        7. (Cor) Finite unions and finite cartesian products of finite sets are finite.
           1. If A and B are finite, so is A \cup B.
  2. Countable and Uncountable Sets
     1. Infinite : not finite
     2. Countably infinite : there is a bijective correspondence f : A-> \mathbb{Z}\_{+}
     3. Countable (a set) : it is finite or countably finite
     4. Uncountable (a set) : not countable
     5. (Thm 7.1) : Let B be a nonempty set. Then the following are equivalent.
        1. B is countable.
        2. There is a surjective function f : \mathbb{Z}\_{+} -> B
        3. There is an injective function g : \B -> \mathbb{Z}\_{+}
        4. (Lem) If C is an infinite subset of \mathbb{Z}\_{+}, then C is countably infinite.
           1. H(n) = smallest element of [C- H({1,…,n-1})], H(1) = smallest element of C
     6. Principle of recursive definition : Let A be a set. A recursive formula determines a unique function h: \mathbb{Z}\_{+} -> A.
        1. Recursive formula : a formula that defines h(1) as a unique element of A, and for i>1 defines h(i) uniquely as an element of A in terms of the values of h for positive integers less than I.
     7. (Cor) A subset of a countable set is countable.
     8. (Cor) the set \mathbb{Z}\_{+} \times \mathbb{Z}\_{+} is countably infinite.
     9. (Thm 7.5) A countable union of countable sets is countable.
     10. (Thm 7.6) A finite product of countable sets is countable.
     11. (Thm 7.7) Let X be the two element set {0,1}. Then the set X^{\omega} is uncountable.
     12. (Thm 7.8) Let A be a set. There is no injective map f: \mathcal{P} (A) -> A, and there is no surjective map g: A->\mathcal{P}
  3. The principle of Recursive Definition
     1. (Thm 8.3) There exists a unique function h: \mathbb{Z}\_{+} -> C satisfying recursive formula for all I \in \mathbb{Z}\_{+}.
        1. (Lem)Given n \in \mathbb{Z}\_{+}, there exists a function f:{1,…,n} ->C that satisfies the recursive formula for all I in its domain.
        2. (Lem)Suppose that f:{1,…,n} -> C and g : {1,…,m} ->C both satisfy recursive formula for all I in their respective domains. Then f(i) = g(i) for all I in both domains.
     2. (Thm 8.4) (Principle of recursive definition) : let A be a set; let a\_{0} be an element of A. Suppose \rho is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function h: \mathbb{Z}\_{+} -> A s.t. recursion formula for h satisfied.
        1. Recursion formula for h :
           1. H(1) = a\_{0}
           2. H(i) = \rho(h | {1,…,i-1}) for I >1
  4. Infinite sets and the axiom of choice
     1. (Thm 9.1) Let A be a set. The following statements about A are equivalent.
        1. There exists an injective function f : \mathbb{Z}\_{+} -> A
        2. There exists a bijection of A with a proper subset of itself.
        3. A is infinite.
        4. Pf) Axiom of choice is needed., recursion formula, c: \mathcal{B} -> \bigcup\_{B \in \mathcal{B}} B = A
     2. (Axiom of choice) : Given a collection \mathcal{A} of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of \mathcal{A}
        1. A set C s.t. C is contained in the union of the elements of \mathcal{A}, and for each A \in \mathcal{A}, the set C\cap A contains a single element.
     3. (Existence of a choice function)(Lem)
        1. Given a collection \mathcal{B} of nonempty sets not necessarily disjoint, there exists a function c : \mathcal{B} -> \bigcup\_{B \in \mathcal{B}} B s.t. c(B) is an element of B, for each B \in \mathcal{B}.
           1. A function c is called a choice function for the collection \mathcal{B} .
     4. Finite axiom of choice : given a finite collection \mathcal{A} of disjoint nonempty sets, there exists a set \mathcal{C} consisting of exactly one element from each element of \mathcal{A}.
        1. Weaker form of axiom of choice.
  5. Well-Ordered sets
     1. Well-ordered ( a set A with an order relation < ) : every nonempty subset of A has a smallest element.
        1. Constructing well-ordered sets
           1. If A is a well-ordered set, then any subset of A is well-ordered in the restricted order relation.
           2. If A and B are well-ordered sets, then A \times B is well-ordered in the dictionary order.
     2. (Thm 10.1) Every nonempty finite ordered set has the order type of a section {1,…,n} of \mathbb{Z}\_{+}, so it is well-ordered.
     3. (Well-ordering theorem) If A is a set, there exists an order relation on A that is well-ordering.
        1. Pf) choice axiom
     4. (Cor) There exists an uncountable well-ordered set.