Applied Numerical Mathematics 137 (2019) 1–18

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

# Avoiding strange attractors in efficient parametric families of iterative methods for solving nonlinear problems ☆

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ARTICLE INFO

*Article history:*
Received 21 September 2018
Received in revised form 10 December 2018
Accepted 11 December 2018
Available online 13 December 2018

*Keywords:*
Nonlinear problems
Iterative methods with and without memory
Computational efficiency
Qualitative analysis
Feigenbaum diagrams

ABSTRACT

Searching zeros of nonlinear functions often employs iterative procedures. In this paper, we construct several families of iterative methods with memory from one without memory, that is, we have increased the order of convergence without adding new functional evaluations. The main aim of this manuscript yields in the advantage that the use of real multidimensional dynamics gives us to decide among the different classes designed and, afterwards, to select its most stable members. Moreover, we have found some elements of the family whose behavior includes strange attractors of different kinds that must be avoided in practice. In this sense, Feigenbaum diagrams have resulted an extremely useful tool. Finally, some of the designed classes with memory have been directly extended for solving nonlinear systems, getting an improvement in the efficiency in relation to other schemes with the same computational cost. These numerical tests confirm the theoretical results and show the good performance of the methods.



## 1. Introduction

When we want to model physical, chemical, etc. problems, normally nonlinearity appears, and we need numerical methods to solve them. Some clear examples can be found in Astrodynamics [4,3], modelization of chemical reactors [6], neutron transport problem (see [24]) or radioactive transfer [15]. In particular, these nonlinear problems require iterative methods to be solved.

Our goal is to obtain the solution $\alpha$ of a nonlinear equation $f(x)$, being $f : D \subset \mathbb{R} \to \mathbb{R}$. Usually these problems need iterative methods to get an estimation of their solution, as they cannot be found analytically. These iterative methods are usually fixed-point schemes that, starting from one or more initial estimations, obtain a new value that approaches our solution as

$$x_{k+1} = g(x_{k-p}, x_{k-p+1}, \ldots, x_{k-1}, x_k), k > 0, \quad p \geq 0.$$

Here, we distinguish between last iteration or last iterations. If we use only the last one ($p = 0$) we have an iterative method without memory, and if we use more tan one previous iteration ($p > 0$), the iterative method is with memory. Our interest

☆ This research was partially supported by both Ministerio de Economía y Competitividad and Generalitat Valenciana, under grants MTM2014-52016-C2-2-P and PROMETEO/2016/089, respectively.

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https://doi.org/10.1016/j.apnum.2018.12.006
0168-9274/

is in the latter one, since they allow us to obtain a faster convergence with the same functional evaluations per iteration as a method without memory, although the expression of the method is too complicated if the increase of order is very high. Some good texts about both kind of procedures can be found in [22,18–20,1].

In the last decades, literature in this field has increased, and new methods have appeared improving the existing ones, in terms of order, stability, etc. Precisely the study of stability has developed in recent years, with the emergence of dynamic techniques. These new techniques not only enrich the research environment, but also allows us to select the schemes whose behavior is better qualitatively. Specifically, most of these analyses have been carried out with the complex dynamics tools developed in the last century. See, for example the texts by Blanchard [5] or Devaney [14] and, more recently, the work of different authors [17,16,13,9,10], that have analyzed the stability of different schemes or families finding in them aspects as global convergence, convergence to undesirable attracting elements (as orbits or fixed points different from the searched roots), or even chaotic behavior. Also real dynamics has shown to be useful for this kind of analysis, as can be seen in diverse works as [2,12].

Nevertheless, these techniques can not be applied to analyze the behavior of iterative methods with memory, beyond the mere plot of the basins of attraction on specific functions. It has been very recently that the authors in [7,8] have developed a technique to study their stability on low-degree polynomials based on the real multidimensional dynamics.

Our objective in this paper is, after developing different families of iterative methods with memory by introducing accelerating parameters in a without-memory new class, to analyze its dynamical behavior and to detect how minor differences in the creation of several classes of iterative schemes with the same order of convergence generate very different qualitative behavior, varying from the chaos in strange attractors to a “boring” but desirable stable behavior. In the following, we build the discrete dynamical system associated to an iterative method with memory in order to carry out its dynamical study.

The expression of an iterative method with memory, which uses two previous iterations to calculate the following estimation, is

$$x_{k+1} = g(x_{k-1}, x_k), \quad k \geq 1,$$

where $x_0$ and $x_1$ are the initial estimations. In order to obtain the fixed points of this method, we define the auxiliary *fixed point function* $G : \mathbb{R}^2 \to \mathbb{R}^2$ by means of:

$$\begin{aligned} G(x_{k-1}, x_k) &= (x_k, x_{k+1}) \\ &= (x_k, g(x_{k-1}, x_k)), \ k = 1, 2, \ldots \end{aligned}$$

This definition can be extended in a natural way to adapt it to iterative schemes with memory using more than two previous iterations per step. Let us remark that, as $(x_{k-1}, x_k)$ is a fixed point of $G$ if

$$G(x_{k-1}, x_k) = (x_{k-1}, x_k),$$

then $x_{k+1} = x_k$ and $x_{k-1} = x_k$.

We have defined a discrete dynamical system in the real plane from function $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$G(z, x) = (x, g(z, x))$$

where $g$ is the operator of the iterative method with memory. Fixed points $(z, x)$ of $G$ satisfy $z = x$ and $x = g(z, x)$.

In the following, we recall some basic real dynamics concepts. If a fixed point $(z, x)$ of operator $G$ is different from $(r, r)$, where $r$ is a zero of $f(x)$, it is called *strange fixed point*. On the other hand, the orbit of a point $\bar{x} \in \mathbb{R}^2$ is defined as the set of successive images of $\bar{x}$ by the vector function, $\{\bar{x}, G(\bar{x}), \ldots, G^m(\bar{x}), \ldots\}$.

The dynamical performance of a point of $\mathbb{R}^2$ is classified depending on its asymptotical behavior. So, a point $x^* \in \mathbb{R}^2$ is a *k-periodic point* if $G^k(x^*) = x^*$ and $G^p(x^*) \neq x^*$, for $p = 1, 2, \ldots, k-1$. The stability of fixed points for multivariable nonlinear operators, see for example [21], satisfies the following statements.

**Theorem 1.** *Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{C}^2$. Let us assume $x^*$ is a k-periodic point. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $G'(x^*)$.*

1. *If all the eigenvalues $\lambda_j$ have $|\lambda_j| < 1$, then $x^*$ is attracting.*
2. *If one eigenvalue $\lambda_{j_0}$ has $|\lambda_{j_0}| > 1$, then $x^*$ is unstable, that is, repelling or saddle.*
3. *If all eigenvalues $\lambda_j$ have $|\lambda_j| > 1$, then $x^*$ is repelling.*

Furthermore, a fixed point is called *hyperbolic* if all the eigenvalues $\lambda_j$ of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue $\lambda_i$ such that $|\lambda_i| < 1$ and an eigenvalue $\lambda_j$ such that $|\lambda_j| > 1$, the hyperbolic point is called *saddle* point.

Moreover, a point $x$ is a *critical* point of $G$ if the associate Jacobian matrix $G'(x)$ satisfies $\det(G'(x)) = 0$. One particular case of critical points, for iterative methods of convergence order higher than two, are those fixed points whose associated eigenvalues $\lambda_j = 0$, $\forall j$ are null. These points are called *superattracting*.

Then, if $x^*$ is an attracting fixed point of function $G$, its *basin of attraction* $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \left\{ x^{(0)} \in \mathbb{R}^n : G^m\left(x^{(0)}\right) \to x^*,\ m \to \infty \right\}.$$

The set of the different basins of attraction define the *Fatou set* of the system, being its complementary in the plane the *Julia set*. One usual way to visualize these sets is meanwhile the *dynamical plane* of the method. It is built by iterating a mesh of points as starting points and painting them in different colors (orange, green, blue,...) depending on the attractor they converge to, or in black if they reach the maximum number of iterations stated without converging to any of them. The algorithms used in this manuscript for getting the dynamical planes of the different methods appear in [10].

To study the local convergence of the methods with memory, we use the following result, that can be found in [18], based on the positive root of a characteristic polynomial deduced from the error equation of the method.

**Theorem 2.** *Let $\psi$ be an iterative method with memory that generates a sequence $\{x_k\}$ of approximations to the root $\alpha$, and let this sequence converges to $\alpha$. If there exist a nonzero constant $\eta$ and nonnegative numbers $t_i$, $i = 0, 1, \ldots, m$, such that the inequality*

$$|e_{k+1}| \leq \eta \prod_{i=0}^{m} |e_{k-i}|^{t_i}$$

*holds, then the $R$-order of convergence of the iterative method $\psi$ satisfies the inequality*

$$O_R(\psi, \alpha) \geq s^*,$$

*where $s^*$ is the unique positive root of the equation*

$$s^{m+1} - \sum_{i=0}^{m} t_i s^{m-i} = 0.$$

In the following, we design three different families of methods with memory and study their order of convergence (Section 2). This order might be higher if the accelerating parameters would be approximated with higher-degree interpolating polynomials, but the main aim of this paper is to compare the stability of the resulting schemes, and it is more feasible when their iterative expressions are simpler. Then we make a multidimensional real dynamical analysis of the classes (Section 3) applied on second-degree polynomial, showing some bifurcation diagrams and dynamical planes because of the dependence on a parameter in those families. In this way, different intervals have been identified where the involved parameter defines methods with stable (convergence only to the roots) or unstable performance, with convergence to strange fixed points, strange attractors or chaotic behavior.

## 2. Modified parametric family with memory

Our starting point is the parametric family of iterative methods

$$\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)},\\
t_k &= y_k - \beta \frac{f(y_k)}{f'(x_k)},\\
x_{k+1} &= t_k - \frac{1}{\beta}\frac{f(t_k) - (\beta-1)^2 f(y_k)}{f'(x_k)},\ k = 1, 2, \ldots
\end{aligned}$$

introduced in [4], whose stability was analyzed in [13] by means of complex dynamics, finding interesting unstable behavior that limited the set of complex values of $\beta$ assuring the stability. Its local order of convergence is four, being fifth-order if $\beta = 5$; this makes it non-optimal for scalar equations but extremely efficient in case of nonlinear systems, as all the linear systems to be solved per iteration have the same matrix, that is, the Jacobian matrix of the system evaluated at the actual iteration.

### *2.1. Design and local convergence*

Now we design the new families of methods with memory. First of all, we add some accelerating parameters $\gamma_i$, $i = 1, 2, 3$, and preserve the derivatives in the iterative expression. The resulting scheme is

$$\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k) + \gamma_1 f(x_k)},\\
t_k &= y_k - \beta \frac{f(y_k)}{f'(x_k) + \gamma_2 f(x_k)},\\
x_{k+1} &= t_k - \frac{1}{\beta}\frac{f(t_k) - (\beta-1)^2 f(y_k)}{f'(x_k) + \gamma_3 f(x_k)},\ k = 1, 2, \ldots
\end{aligned} \tag{1}$$

In the following result, we analyze the convergence of this family and get its error equation depending on $\gamma_i$, $i = 1, 2, 3$. From it, we state the conditions that hold or increase its order of convergence.

**Proposition 1.** *Let an open interval $D$ be the domain of a sufficiently differentiable nonlinear function $f : D \subset \mathbb{R} \to \mathbb{R}$ and be $\alpha$ a simple zero of $f$. If $x_0$ is close enough to $\alpha$, and fixing $\gamma_3 = \gamma_2$, $\gamma_1, \gamma_2 \in \mathbb{R}$, the order 4 of convergence of (1) is got $\forall \beta \in \mathbb{R} \setminus \{0\}$ and the error equation is*

$$e_{k+1} = -(c_2 + \gamma_1)\left(c_2\left((\beta-1)\gamma_1 - 4\gamma_2\right) + (\beta-5)c_2^2 - \gamma_2^2\right)e_k^4 + \mathcal{O}(e_k^5),$$

*where $c_j = \frac{1}{j!}\frac{f''(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$.*

**Proof.** We use Taylor series expansions in first step, obtaining

$$\begin{aligned} y_k - \alpha &= (c_2 + \gamma_1)e_k^2 + \left(-2\gamma_1 c_2 - 2c_2^2 - \gamma_1^2 + 2c_3\right)e_k^3 \\ &\quad + \left(5\gamma_1 c_2^2 + c_2\left(3\gamma_1^2 - 7c_3\right) - 4c_3\gamma_1 + 4c_2^3 + \gamma_1^3 + 3c_4\right)e_k^4 + \mathcal{O}(e_k^5), \end{aligned}$$

and in the second step

$$\begin{aligned} t_k - \alpha &= \left((4\beta-2)c_2^2 - 2(\beta-1)c_3\right) - (\beta-1)(c_2+\gamma_1)e_k^2 \\ &\quad + \left(\gamma_1\left((\beta-1)\gamma_1 + \beta\gamma_2\right) + c_2\left((4\beta-2)\gamma_1 + \beta\gamma_2\right)\right)e_k^3 + \mathcal{O}(e_k^4). \end{aligned}$$

Finally, the error equation yields

$$\begin{aligned} e_{k+1} &= (\beta-1)(\gamma_2-\gamma_3)(c_2+\gamma_1)e_k^3 + \Big(\gamma_1\left(-(\beta-1)\gamma_2^2 + (\beta-1)\gamma_3^2 - (\beta-1)\gamma_1(\gamma_2-\gamma_3) + \gamma_3\gamma_2\right) \\ &\quad + c_2^2\left(-2(\beta-3)\gamma_1 + (7-5\beta)\gamma_2 + (5\beta-3)\gamma_3\right) + c_2\Big(-(\beta-1)\gamma_1^2 + \gamma_1\left((7-5\beta)\gamma_2 + (5\beta-3)\gamma_3\right) \\ &\quad - (\beta-1)\gamma_2^2 + (\beta-1)\gamma_3^2 + \gamma_2\gamma_3\Big) + 2(\beta-1)c_3(\gamma_2-\gamma_3) - (\beta-5)c_2^3\Big)e_k^4 + \mathcal{O}(e_k^5) \end{aligned}$$

So, when we fix $\gamma_2 = \gamma_3$, we obtain

$$e_{k+1} = -\left((c_2+\gamma_1)\left(c_2\left((\beta-1)\gamma_1 - 4\gamma_2\right) + (\beta-5)c_2^2 - \gamma_2^2\right)\right)e_k^4 + \mathcal{O}(e_k^5). \quad \square$$

At this point, we want to introduce memory, it is, include one more iteration to obtain the new one, to increase this fourth-order of convergence of our original method, without adding new functional evaluations. It is clear that if we fix $\gamma_1 = -c_2$, we have at least order five, but it is useless because we ignore $\alpha$ that is precisely the root that we are looking for. Moreover, if we show now the error equation with Taylor series expansions with higher order and fixing $\gamma_2 = 2\gamma_1$, we have

$$\begin{aligned} e_{k+1} &= -(c_2+\gamma_1)^2\left((\beta-5)c_2 - 4\gamma_1\right)e_k^4 + 2(c_2+\gamma_1)\Big(2(4\beta-19)\gamma_1 c_2^2 + c_2\left((4\beta-31)\gamma_1^2 - 2(\beta-6)c_3\right) \\ &\quad + (5\beta-18)c_2^3 + 10\gamma_1\left(c_3 - \gamma_1^2\right)\Big)e_k^5 + \mathcal{O}(e_k^6). \end{aligned}$$

As we can see, the term $c_2 + \gamma_1$ appears also in the term of order 5. We cannot fix $\gamma_1$ as $-c_2$ as we noted before, but we can approximate $\gamma_1$ by using divided differences or derivatives in the quotient, to obtain $\gamma_1 \approx -c_2$, and to introduce memory in the scheme. In the first case, replacing $f''(\alpha)$ by $f'[x_k, x_{k-1}]$ and $f'(\alpha)$ by $f[x_k, x_{k-1}]$, we have

$$\gamma_{1k} = -\frac{1}{2}\frac{f'(x_k) - f'(x_{k-1})}{f(x_k) - f(x_{k-1})}. \tag{2}$$

With this consideration, we get the class of iterative methods

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k) + \gamma_{1k} f(x_k)}, \\ t_k &= y_k - \beta\frac{f(y_k)}{f'(x_k) + 2\gamma_{1k} f(x_k)}, \\ x_{k+1} &= t_k - \frac{1}{\beta}\frac{f(t_k) - (\beta-1)^2 f(y_k)}{f'(x_k) + 2\gamma_{1k} f(x_k)}, \ k = 1, 2, \ldots \end{aligned} \tag{3}$$

denoted by M41. Now, we set the order of convergence of this family in this result, whose proof uses Theorem 1 from [18].

**Theorem 3.** *Let an open interval $D$ be the domain of a sufficiently differentiable nonlinear function $f: D \subset \mathbb{R} \to \mathbb{R}$ and be $\alpha$ a simple zero of $f$. If $x_0$ and $x_1$ are close enough to $\alpha$, then the order of convergence of (3) is at least $2+\sqrt{6}$. The expression of error equation is in this case*

$$e_{k+1} = -\frac{1}{4}(\beta-1)c_2(2c_2^2-3c_3)^2 e_{k-1}^2 e_k^4 + \mathcal{O}_6(e_{k-1}e_k),$$

*where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$ and $\mathcal{O}_6(e_{k-1}e_k)$ denotes the terms of the development that the addition of exponents of $e_{k-1}$ and $e_k$ is at least 6. Also, in particular, if $\beta = 1$, the error equation is*

$$e_{k+1} = \frac{1}{2}(2c_2^2-3c_3)^3 e_{k-1}^3 e_k^4 + \mathcal{O}_7(e_{k-1}e_k),$$

*being the local order $2+\sqrt{7}$.*

**Proof.** By using Taylor development around $\alpha$, we obtain

$$\begin{aligned} y_k - \alpha &= e_k - \frac{f(x_k)}{f'(x_k)+\gamma_{1k} f(x_k)} \\ &= \left(\left(c_2^2 - \frac{3c_3}{2}\right) e_{k-1} \left(-c_2^3 + \frac{5c_3c_2}{2} - 2c_4\right) e_{k-1}^2 + \mathcal{O}(e_{k-1}^3)\right) e_k^2 + \left(\frac{c_3}{2} - 2\left(c_2^3 - 2c_3c_2 + c_4\right) e_{k-1}\right. \\ &\quad \left. + \frac{1}{4}\left(8c_2^4 - 22c_3c_2^2 + 20c_4c_2 + 3c_3^2 - 10c_5\right) e_{k-1}^2 + \mathcal{O}(e_{k-1}^3)\right) e_k^3 + \mathcal{O}(e_k^4) \end{aligned}$$

and

$$\begin{aligned} 2\gamma_{1k} f(x_k) = f'(\alpha) &\left(-2c_2 + \left(2c_2^2 - 3c_3\right) e_{k-1} - \left(2c_2^3 - 5c_3c_2 + 4c_4\right) e_{k-1}^2\right. \\ &+ \left(2c_2^4 - 7c_3c_2^2 + 6c_4c_2 + 3c_3^2 - 5c_5\right) e_{k-1}^3 \\ &\left.+ \left(-2c_2^5 + 9c_3c_2^3 - 8c_4c_2^2 + \left(7c_5 - 8c_3^2\right) c_2 + 7c_3c_4 - 6c_6\right) e_{k-1}^4 + \mathcal{O}\left(e_{k-1}^5\right)\right) e_k. \end{aligned}$$

Replacing these terms on

$$t_k - \alpha = y_k - \alpha + \beta \frac{f(y_k)}{f'(x_k) + 2\gamma_1 f(x_k)},$$

we obtain the third step development, and finally the error equation remains

$$e_{k+1} = -\frac{1}{4}(\beta-1)c_2(2c_2^2-3c_3)^2 e_{k-1}^2 e_k^4 + \mathcal{O}_6(e_{k-1}e_k).$$

Then, the only positive root of $p^2 - 4p - 2$ gives us the $R$-order of the method, $p = 2+\sqrt{6}$. In the particular case of $\beta = 1$, the local order is $p = 2+\sqrt{7}$. □

On the other hand, if we use

$$\gamma_{2k} = -\frac{1}{2}\frac{f'[x_k, x_{k-1}]}{f'(x_k)} \tag{4}$$

on (3) instead of (2), we obtain another family of methods with memory, denoted by M42, whose order of convergence is set in what follows, whose demonstration is omitted. The aim of this case is to hold the derivative in the accelerating factor, as it appears in the iterative expression, instead of replacing it by an estimation. With this, we hope to obtain more stable methods than the previous ones.

**Theorem 4.** *Let an open interval $D$ be the domain of a sufficiently differentiable nonlinear function $f: D \subset \mathbb{R} \to \mathbb{R}$ and be $\alpha$ a simple zero of $f$. If $x_0$ and $x_1$ are close enough to $\alpha$, then the order of convergence of class (3) with (4) is at least $2+\sqrt{6}$. The error equation is given by*

$$e_{k+1} = -\frac{9}{4}(\beta-1)c_2c_3^2 {e_{k-1}}^2 e_k^4 + \mathcal{O}_6\left(e_{k-1}e_k\right),$$

*where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$. Again, if $\beta = 1$, the error equation is*

$$e_{k+1} = -\frac{27}{2}c_3^3 {e_{k-1}}^3 e_k^4 + \mathcal{O}_5(e_{k-1}e_k),$$

*being the local order $2+\sqrt{7}$.*

However, the most used technique to design iterative methods with memory is to replace all the derivatives by divided differences including the accelerating parameter in it,

$$f'(x_k) \approx f[x_k, x_k + \gamma f(x_k)],$$

obtaining a class of derivative-free iterative schemes.

$$\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f[x_k, x_k + \gamma_1 f(x_k)]},\\
t_k &= y_k - \beta \frac{f(y_k)}{f[x_k, x_k + \gamma_2 f(x_k)]},\\
x_{k+1} &= t_k - \frac{1}{\beta} \frac{f(t_k) - (\beta-1)^2 f(y_k)}{f[x_k, x_k + \gamma_3 f(x_k)]}, \quad k = 1, 2, \ldots
\end{aligned} \tag{5}$$

**Proposition 2.** *Let an open interval $D$ be the domain of a sufficiently differentiable nonlinear function $f : D \subset \mathbb{R} \to \mathbb{R}$ and be $\alpha$ a simple zero of $f$. If $x_0$ is close enough to $\alpha$, then in the new family without memory (5), with $\gamma_3 = \gamma_2 = 2\gamma_1$, the order of convergence reached is 4 for any $\beta \neq 0$, being the error equation*

$$e_{k+1} = (1 + f'(\alpha)\gamma_1)\left(5 - \beta(1 + f'(\alpha)\gamma_1) + 4f'(\alpha)^2{\gamma_1}^2 + 9f'(\alpha)\gamma_1\right) c_2^3 e_k^4 + \mathcal{O}(e_k^5),$$

*where $c_j = \frac{1}{j!}\frac{f''(\alpha)}{f'(\alpha)}$.*

By estimating $\gamma \equiv \gamma_1 \approx -\frac{1}{f'(\alpha)}$, we design,

$$\begin{aligned}
\gamma_k &= -\frac{1}{f[x_k, x_{k-1}]}, \quad k = 1, 2, \ldots\\
y_k &= x_k - \frac{f(x_k)}{f[x_k, x_k + \gamma_k f(x_k)]},\\
t_k &= y_k - \beta \frac{f(y_k)}{f[x_k, x_k + 2\gamma_k f(x_k)]},\\
x_{k+1} &= t_k - \frac{1}{\beta} \frac{f(t_k) - (\beta-1)^2 f(y_k)}{f[x_k, x_k + 2\gamma_k f(x_k)]}, \quad k = 1, 2, \ldots
\end{aligned} \tag{6}$$

that defines the new class, denoted by M43.

**Theorem 5.** *Let an open interval $D$ be the domain of a sufficiently differentiable nonlinear function $f : D \subset \mathbb{R} \to \mathbb{R}$ and be $\alpha$ a simple zero of $f$. If $x_0$ and $x_1$ are close enough to $\alpha$, then the order of convergence of methods with memory (6) is at least $2 + \sqrt{6}$. So,*

$$e_{k+1} = (1-\beta) c_2^5 e_{k-1}^2 e_k^4 + \mathcal{O}_6(e_{k-1} e_k),$$

*where $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$. Moreover, if $\beta = 1$, the order of convergence reaches $2 + \sqrt{7}$, being the error equation in this case*

$$e_{k+1} = 4 c_2^4 c_3 e_{k-1}^3 e_k^4 + \mathcal{O}_6(e_{k-1} e_k).$$

As in the two previous families, we obtain the same order in all cases, but this one has an interesting property that the other ones have not: it can be extended for solving nonlinear systems, not only scalar equations, preserving the order of convergence.

In the following sections, we study the stability of the three proposed families of iterative methods with memory, to know which is the best family in these terms and, moreover, to choose the $\beta$s of this class with good stability properties. To achieve this, we analyze the rational functions obtained when M41, M42 and M43 are employed on a set of second-degree polynomials $p_1(x) = x^2 - 1$, $p_2(x) = x^2 + 1$ and $p_3(x) = x^2$ (see [12]). The resulting rational functions will be denoted by $M_1^j(z, x, \beta)$, $M_2^j(z, x, \beta)$ and $M_3^j(z, x, \beta)$ (where the index $j = 1$ corresponds to the class M41 under analysis, $j = 2$ to the family M42 and $j = 3$ to the study of M43) respectively, and they are dependent on last and previous iterates $x_k$ and $x_{k-1}$ (denoted by $x$ and $z$).

## 3. Qualitative study of M41

Firstly, we construct the fixed point operator corresponding to the action of M41 on $p_1(x) = x^2 - 1$,

$$M_1^1(z, x, \beta) = \left(x, C - \frac{(x+z)(C^2 - D - 1)}{2\beta(1+xz)}\right),$$

where

$$C = -\frac{\beta(x^2-1)^2(x+z)(z^2-1)}{2(1+xz)(1+x^2+2xz)^2} + \frac{z + x(2+xz)}{1+x^2+2xz}$$

and

$$D = \frac{(\beta-1)^2(x^2-1)^2(z^2-1)}{(1+x^2+2xz)^2}.$$

Let us remark that all the fixed points of $M_1^1(z, x, \beta)$ for a specific value of $\beta$ have two equal components.

**Proposition 3.** *The set of fixed points of $M_1^1(z, x, \beta)$ and their respective character is:*

1. $\{(1, 1), (-1, -1)\}$, *both being superattracting.*
2. $(z, x) = (0, 0)$, *a saddle fixed point.*
3. *Those points $(r, r)$ being $r$ one of the real roots of $m(t) = 4 + (30-\beta)t^2 + (114+5\beta)t^4 + (268-10\beta)t^6 + (328+10\beta)t^8 + (214-5\beta)t^{10} + (66+\beta)t^{12}$. So, there exist*
   (i) *two saddle points if $\beta < -66$,*
   (ii) *none if $-66 \le \beta < 138.032649$,*
   (iii) *two non-hyperbolic points if $\beta = 138.032649$,*
   (iv) *four real roots if $\beta > 138.032649$: two are repelling and the other two are attracting for $138.032649 < \beta < 141.2412$, non-hyperbolic for $\beta = 141.2412$ and repulsive if $\beta > 141.2412$.*

**Proof.** As our objective is to get the fixed points of $M_1^1(z, x, \beta)$, we need to obtain the solutions of the equation

$$M_1^1(z, x, \beta) = (z, x),$$

so $z = x$ and

$$\frac{x(x^2-1)m(x)}{(x^2+1)^3(3x^2+1)^4} = 0.$$

Points $(1, 1)$ and $(-1, -1)$ are solutions and the eigenvalues of Jacobian matrix $M_1^{1'}(\pm 1, \pm 1, \beta)$ satisfy $\lambda_i = 0$, $i = 1, 2$, so it is proven that they are superattracting. Point $(0, 0)$ is also a strange fixed point and, in this case, $\lambda_1 = \frac{1}{2}\left(3 - \sqrt{17}\right)$ and $\lambda_2 = \frac{1}{2}\left(3 + \sqrt{17}\right)$, so it is a saddle point.

The remaining fixed points are the roots of $m(t)$, denoted by $m_i(\beta)$, $i = 1, 2, \ldots, 12$. If $\beta < -66$, the eigenvalues of the Jacobian matrix evaluated at $m_1(\beta)$ and $m_2(\beta)$ are equal in both points and one of them has absolute value lower than one, meanwhile the another is greater than one; so they are saddle points (see the absolute value of their eigenvalues depending on the parameter $\beta$ in Fig. 1, that we call stability function of the strange fixed point).

When $-66 \le \beta < 138.032649$, there are not real roots of $m(t)$. For $\beta \approx 138.032649$, only $m_1(\beta)$ and $m_3(\beta)$ are real and they have the same eigenvalues, being $\lambda_1 = 1$, $\lambda_2 = 0.555248$, so they are non-hyperbolic.

If $\beta > 138.032649$, $m_i(\beta)$, $i = 1, 2, 3, 4$ are real, being the eigenvalues of $m_1(\beta)$ and $m_4(\beta)$ and $m_2(\beta)$ and $m_3(\beta)$ equal, respectively. Both eigenvalues of $m_1(\beta)$ and $m_4(\beta)$ are, respectively, greater and lower than one, so they are saddle strange fixed points (see Fig. 2a). The eigenvalues of $m_2(\beta)$ and $m_3(\beta)$ are both lower than one in the interval $138.032649 < \beta < 141.2412$, so both are simultaneously attracting, but for $\beta > 141.2412$ both eigenvalues of $m_2(\beta)$ and $m_3(\beta)$ are higher than one, so they are repulsive strange fixed points (see Fig. 2b). □

Hereunder we study the behavior of M41 on polynomials $p_2(x)$ and $p_3(x)$. As the following results are similar to Proposition 3, the proofs are omitted. Its related rational function is

$$M_2^1(z, x, \beta) = \left(x, C - \frac{(x+z)(C^2 - D + 1)}{2(\beta xz - \beta)}\right),$$

being

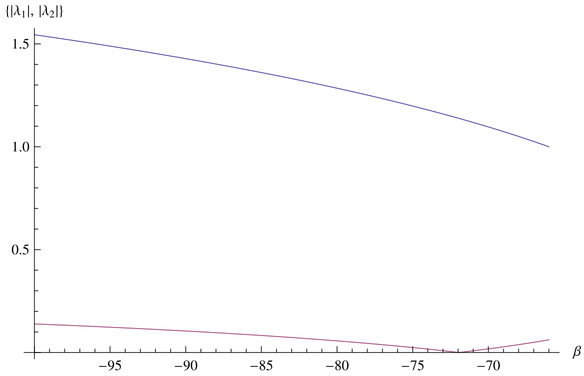



**Fig. 1.** Stability functions of $M_1^{1\prime}(m_i(\beta), m_i(\beta), \beta)$, $i = 1, 2$ for $\beta < -66$.

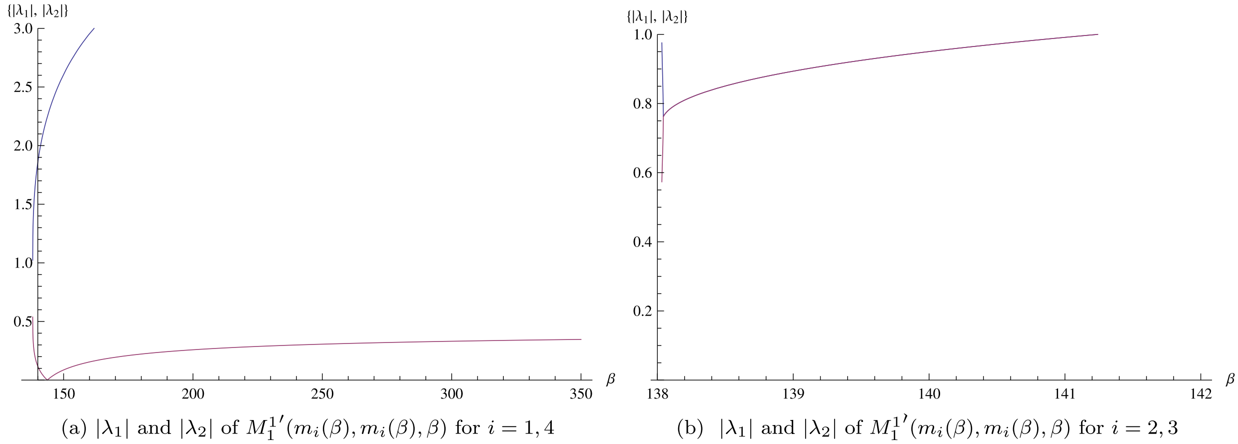



**Fig. 2.** Stability functions of $M_1^{1\prime}(m_i(\beta), m_i(\beta), \beta)$, $i = 1, 2, 3, 4$ for $\beta > 138.032649$.

$$C = \frac{-2x + (-1 + x^2)z}{-1 + x^2 + 2xz} - \frac{\beta(1 + x^2)^2(x + z)(1 + z^2)}{2(-1 + xz)(-1 + x^2 + 2xz)^2}$$

and

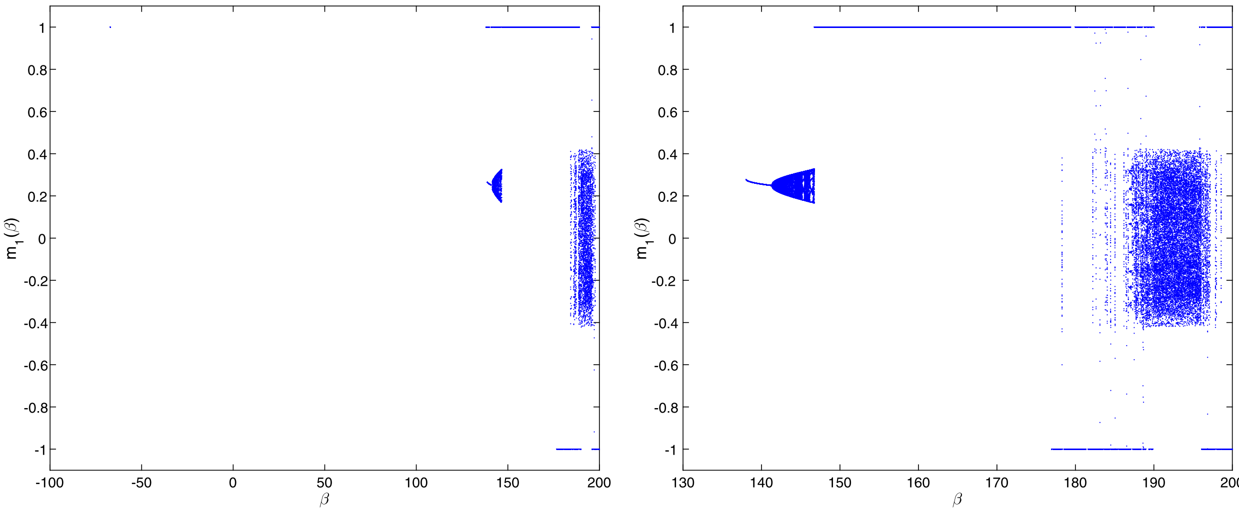
$$D = \frac{(\beta - 1)^2(1 + x^2)^2(1 + z^2)}{(-1 + x^2 + 2xz)^2}.$$

**Proposition 4.** *The set of fixed points of $M_2^1(z, x, \beta)$ and their respective character are:*

1. $(z, x) = (0, 0)$, *which is a saddle fixed point.*
2. *The real roots of* $r(t) = 4 + (-30 + \beta)t^2 + (114 + 5\beta)t^4 + (-268 + 10\beta)t^6 + (328 + 10\beta)t^8 + (-214 + 5\beta)t^{10} + (66 + \beta)t^{12}$, *so:*
   (i) *There exist two real repelling points if* $\beta \le -66$.
   (ii) *Four strange fixed points, if* $-66 < \beta < 0$*: two real repelling points and other two real roots which are: attracting points for* $-66 < \beta < -24.5740$, *non-hyperbolic points if* $\beta \approx -24.5740$, *repulsive points if* $-24.5740 < \beta \le 0$;
   (iii) *None if* $\beta = 0$, *two saddle and two repelling points for* $0 < \beta < 0.311664$.
   (iv) *One saddle and one repelling point if* $\beta = 0.311664$ *and none for* $\beta > 0.311664$.

Finally, the rational operator $M_3^1(z, x, \beta)$ associated to M41 on $p_3(x) = x^2$ is expressed as

$$M_3^1(z, x, \beta) = \left(x, -\frac{xz((4 - \beta)x^3 + 3(\beta - 8)x^2 z + (3\beta - 52)xz^2 + (\beta - 40)z^3)}{8(x + 2z)^4}\right).$$




(a) $m_1(\beta)$ (b) $m_1(\beta)$, a detail

**Fig. 3.** Feigenbaum diagrams of M41 on $p_1(x)$.

**Proposition 5.** *The set of fixed points of $M_3^1(z,x,\beta)$ and their respective character are:*

1. $(z,x)=(0,0)$, *whose stability depends on parameter $\beta$: saddle point for $\beta<-66$, non-hyperbolic if $\beta=-66$, attracting for $-66<\beta<63$, non-hyperbolic if $\beta=63$ and saddle point for $\beta>63$.*
2. *All $(x,z)\in\mathbb{R}^2$ when $\beta=-66$, which are non-hyperbolic fixed points.*

Let us remark that some pathological aspects have been already found, as $M_1^1(z,x,\beta)$ and $M_2^1(z,x,\beta)$ have strange fixed points that can be attracting and the root in case $M_3^1(z,x,\beta)$ can be saddle point. We deep in this analysis by using other tools, as bifurcation diagrams.

### *3.1. Bifurcation diagrams and phase space*

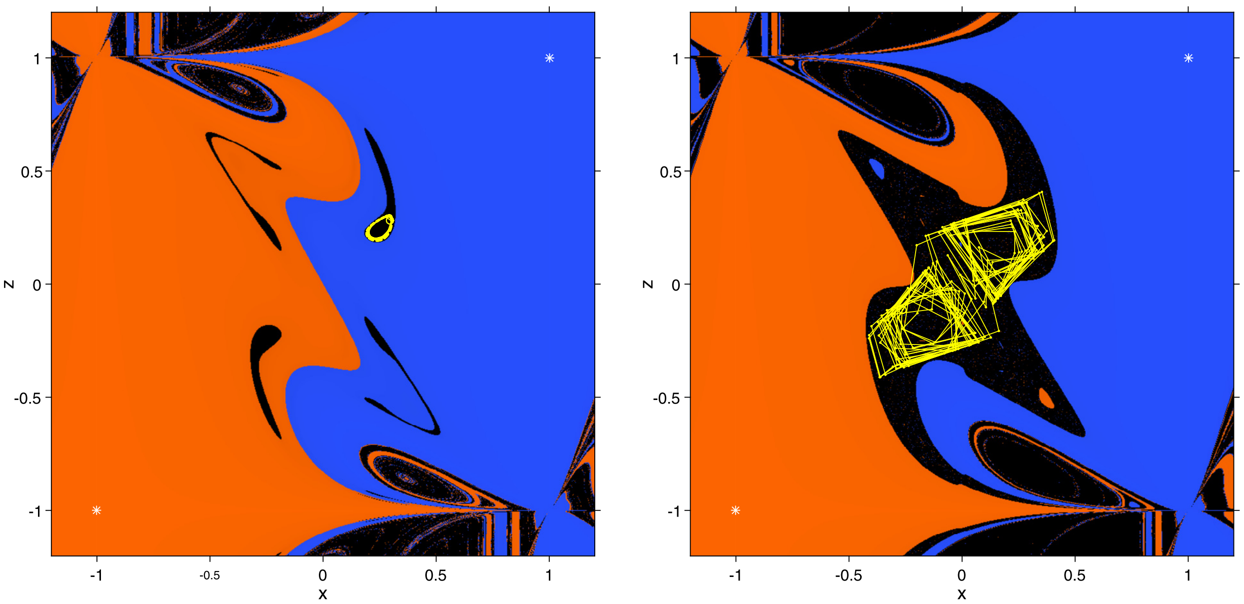
In this section we calculate the well-known Feigenbaum diagrams of the operators related to M41 acting on each polynomial $p_i(x)$, $i=1,2,3$. This is, a diagram in which we observe the bifurcation phenomena of each map, starting with an strange fixed point of it slightly perturbed, in the intervals where $\beta$ changes of stability or other behavior happen.

When we use $p_1(x)$, we have at most five real strange fixed points, that is, $(0,0)$ and the real roots of polynomial $m(t)$. All the bifurcation diagrams have a similar form, so we analyze only one of them, the corresponding to the first root of $m(t)$. In Fig. 3, we have the bifurcation diagram on an interval where all the stability changes are observed. It shows convergence to the roots of $p_1(x)$ and, in some ranges as the interval $[140,200]$, we can see period-doubling bifurcations and regions of chaotic behavior.

In Figs. 4a and 4b we show the dynamical planes associated to some values of $\beta$ in two different regions of Fig. 3 showing pathological behavior (with no convergence to the roots), observing in both cases strange attractors, which have been isolated and the convergence to these periodic orbits and strange attractors are seen in Figs. 5a and 5b. All the dynamical planes of this work have been painted by means of a mesh of $1000\times1000$ points, 400 iterations as a maximum and $10^{-4}$ as the absolute value of the difference between the iteration and the solution. The roots are represented as white stars and their respective basins of attraction colored in orange and blue colors. If any of the strange fixed points is attracting, then they are also plotted with white stars but their respective basins of attraction appear in cyan and red colors.

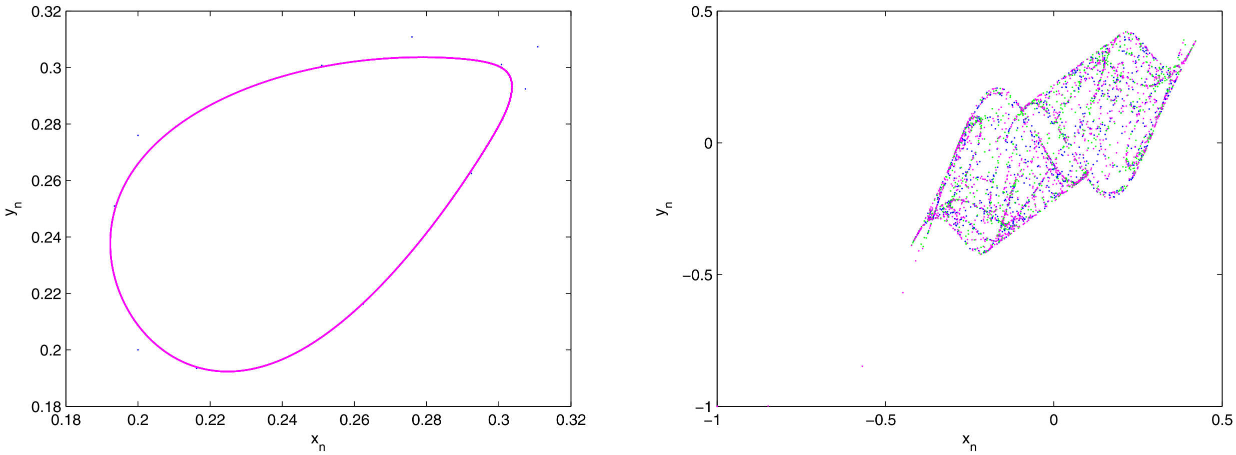
In Fig. 6, let us observe the dynamical plane corresponding to a value of $\beta$ where two strange fixed points are attracting (see Fig. 14a), although their basin of attraction is so small that it must be zoomed to be observed (see Fig. 14b). But, in spite of this chaotic behavior, for the values of $\beta$ out of the interval $[138,200]$, is mainly stable. To visualize this stable behavior, we observe in Fig. 7 some dynamical planes have been obtained by using the rational function $M_1^1(z,x,\beta)$, for different values of parameter $\beta$. In them, there is no other basin of attraction than those of the roots, being simpler when $\beta<0$.

Now, when we use $p_2(x)$, we have at most four real strange fixed points, depending on $\beta$. All the bifurcation diagrams are similar to the ones in Fig. 8, where two of them are shown; the other strange fixed points are the opposite of these $(r_{1,2}(\beta)=\pm\sqrt{s_1}(\beta)$, $r_{3,4}(\beta)=\pm\sqrt{s_2}(\beta))$, being $s_i(\beta)$, $i=1,2$ the only real roots of polynomial $r(\sqrt{t})(\beta)$ and their behavior is symmetric, so they are omitted.




(a) $\beta = 144.1132$ (b) $\beta = 190$

**Fig. 4.** Dynamical planes of M41 on $p_1(x)$ for different $\beta$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)




(a) $\beta = 144.1132$ (b) $\beta = 190$

**Fig. 5.** Basins of attraction of M41 on $p_1(x)$ for different $\beta$.

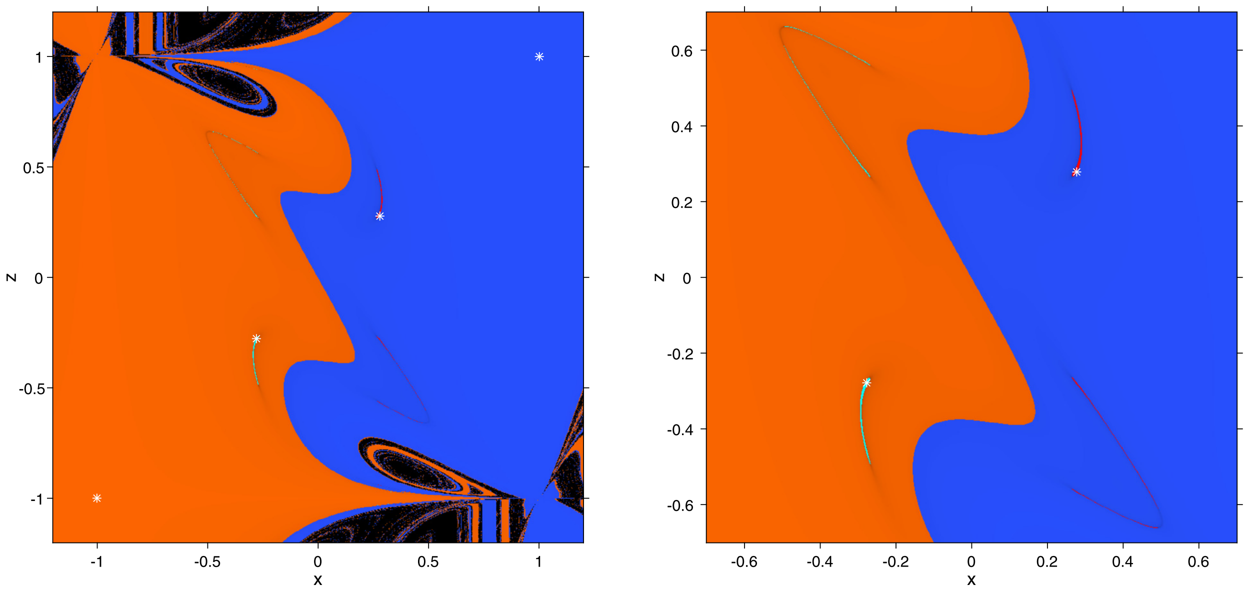
In Fig. 8, we can observe chaotic behavior with period-doubling bifurcations in $\beta \in [-20, 0]$. If we vary $\beta$ in a small interval inside this area, we can detect changes in the strange attractor which generates this chaotic behavior (see Fig. 9). We can observe how when we increase the value of $\beta$ in this blue area, the attracting area is wider in the real plane.

Finally, for $p_3(x)$, the dynamical planes show convergence to $(0, 0)$ in the interval where it is attractor (see Fig. 10), as it has been previously stated in the analysis of stability.

Summarizing, although rational operators $M_i^1$, $i = 1, 2, 3$, have many strange fixed points, the most of them are repulsive and others are attractors in small areas of parameter $\beta$. In fact, there are big regions of the real line where the methods corresponding to these values of $\beta$ show a good stability.

## 4. Qualitative study of M42

Here below, we analyze the rational operator found when M42 acts on second-degree polynomials, starting with $p_1(x) = x^2 - 1$. In this case,



(a) $\beta = 138.04$ (b) $\beta = 138.04$, a detail

**Fig. 6.** Dynamical planes of family M41 on $p_1(x)$ for $\beta = 138.04$.

$$M_1^2(z,x,\beta) = \left( x, \frac{5x - (-41+\beta)x^3 + 3(59+2\beta)x^5 + (461-15\beta)x^7 + (639+20\beta)x^9}{\left(1+x^2\right)^3 \left(1+3x^2\right)^4} + \frac{(507-15\beta)x^{11} + (203+6\beta)x^{13} - (-15+\beta)x^{15}}{\left(1+x^2\right)^3 \left(1+3x^2\right)^4} \right),$$

that only depends on one of the previous iterates. Moreover, this rational function is simpler than that of M41. However, there exist many similarities between them, as it can be deduced from the following results.

**Proposition 6.** *The fixed points of $M_1^2(z,x,\beta)$ and their stability are:*

1. $(1,1)$ *and* $(-1,-1)$ *are supperattracting.*
2. $(0,0)$ *is a saddle point.*
3. *Those points* $(r,r)$ *being r one of the real roots of the polynomial* $s_1(t) = 4 + (30-\beta)t^2 + (114+5\beta)t^4 + (268-10\beta)t^6 + (328+10\beta)t^8 + (214-5\beta)t^{1}0 + (66+\beta)t^{1}2$, *whose number varies depending on the range of parameter* $\beta$*:*
   (i) *If* $\beta < -66$, *there are only two real roots of* $s_1(t)$, *that are saddle points.*
   (ii) *No real roots of* $s_1(t)$ *for* $-66 \le \beta < 138.0326$ *exist.*
   (iii) *Two saddle points for* $\beta \approx 138.0326$.
   (iv) *For* $\beta > 138.0326$, *there exist two saddle fixed points and two more whose character varies with the range of* $\beta$*: they are attracting if* $138.0326 < \beta < 150.7075$ *and saddle if* $b \ge 150.7075$.

When M42 is applied on polynomial $p_2(x) = x^2 + 1$, the resulting rational operator is

$$M_2^2(z,x,\beta) = -\frac{x\left(5 + (-41+\beta)x^2 + 3(59+2\beta)x^4 + (-461+15\beta)x^6 + (639+20\beta)x^8\right)}{\left(1-3x^2\right)^4\left(-1+x^2\right)^3} + \frac{x\left(3(-169+5\beta)x^{10} + (203+6\beta)x^{12} + (-15+b)x^{14}\right)}{\left(1-3x^2\right)^4\left(-1+x^2\right)^3},$$

that also depend only on one of the previous iterates.

**Proposition 7.** *The fixed points of $M_2^2(z,x,\beta)$ and their stability are:*

1. $(0,0)$, *which is repelling.*
2. *The real roots of* $s_2(t) = 4 + (-30+\beta)t^2 + (114+5\beta)t^4 + (-268+10\beta)t^6 + (328+10\beta)t^8 + (-214+5\beta)t^{10} + (66+\beta)t^{12}$, *so depending on* $\beta$*:*

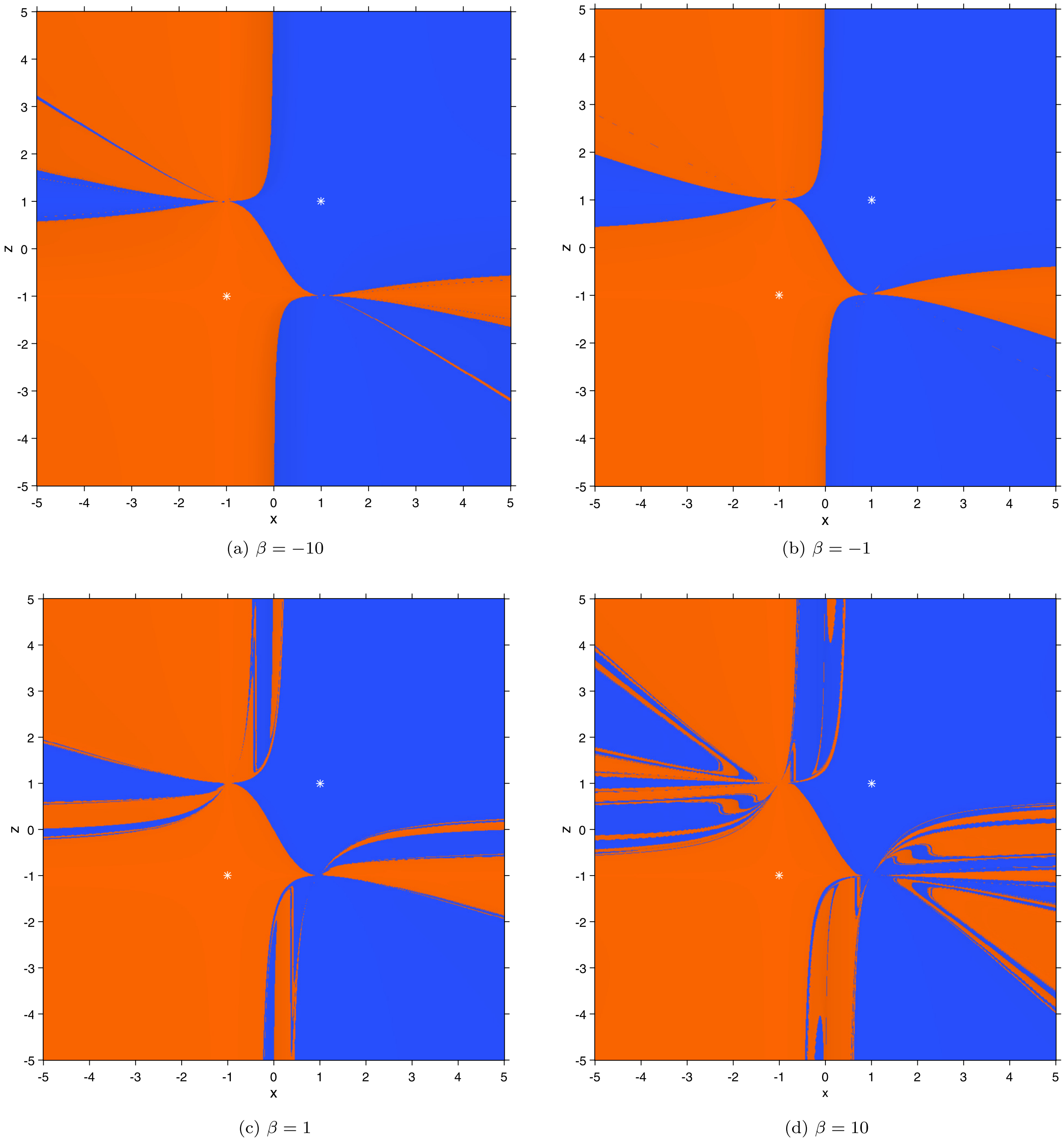
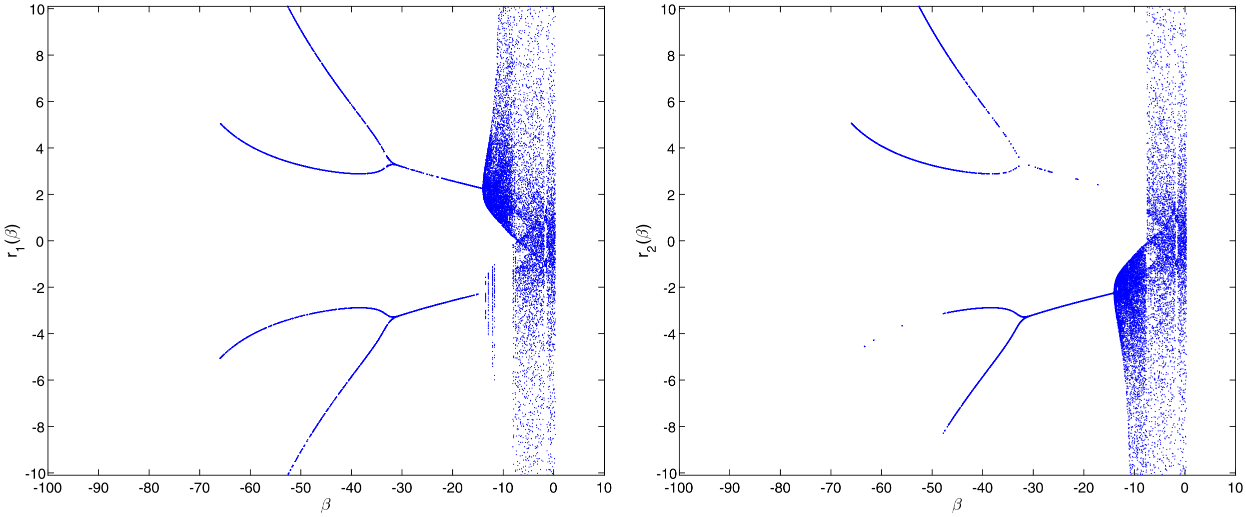


**Fig. 7.** Dynamical planes of family M41 (stable) on $p_1(x)$.

(i) *There are two repelling points for* $\beta \leq -66$.
(ii) *Four real roots appear if* $-66 < \beta < 0$, *being two of them repelling points, and the other two change their character: they are attracting points for* $-66 < \beta < -25.1096$ *(being superattractive for* $\beta = -38.4406$*), repulsive if* $-25.1096 \leq \beta < 0$.
(iii) *Four real roots exist for* $0 < \beta < 0.311664$, *two of them repelling, and the other two are attracting if* $\beta \in [0.311663, 0.311664]$, *being repulsive for the rest of values of* $\beta$.

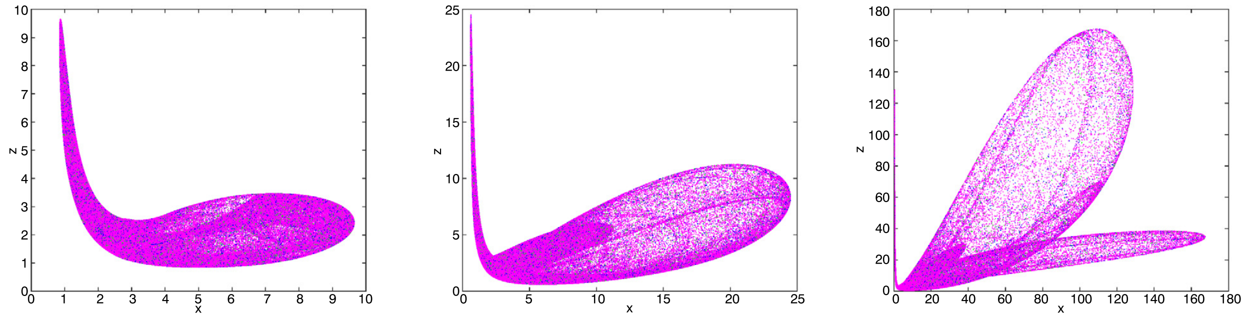
Finally, applying M42 on $p_3(x) = x^2$, the rational function is

$$M_3^2(z, x, \beta) = -\frac{1}{81}(-15 + \beta)x.$$




(a) $r_1(\beta)$ (b) $r_2(\beta)$

**Fig. 8.** Bifurcation diagrams of family M41 on $p_2(x)$ for an strange fixed point as initial estimation.




(a) $\beta = -11$ (b) $\beta = -10$ (c) $\beta = -9$

**Fig. 9.** Strange attractor in M41 on $p_2(x)$.

**Proposition 8.** *The fixed points of $M_3^2(z, x, \beta)$ and their stability are:*

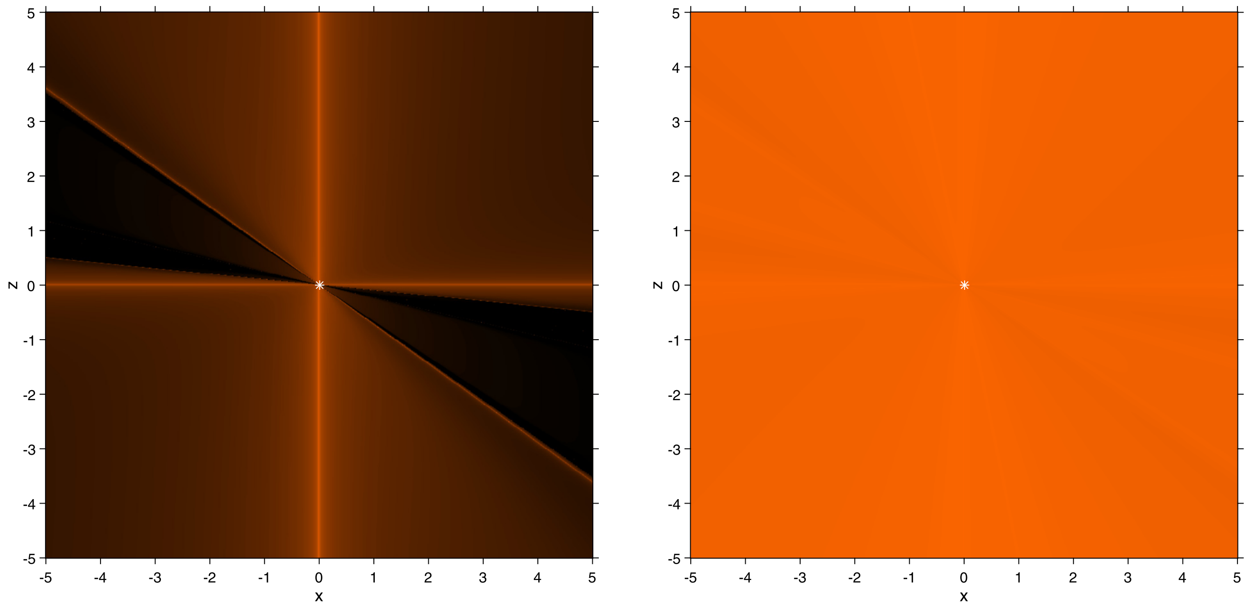
1. $(z, x) = (0, 0)$*, whose character depends on parameter $\beta$: it is superattracting if $\beta = 15$, attracting for $-66 < \beta < 96$ and repulsive for any other value of $\beta$.*
2. *All $\mathbb{R}^2$ when $\beta = -66$, being parabolic fixed points.*

### *4.1. Bifurcation diagrams and dynamical planes*

As in M41 case, we show now the Feigenbaum diagrams for M42. In the $p_1(x)$ case, we can see unstable behavior in the bifurcation diagrams, see Fig. 11. All the bifurcation diagrams of $M_1^2(z, x, \beta)$ have a similar form, so we show only two of them. In they we observe both stable and unstable behavior, the latest one appearing when the strange fixed point changes its stability: it is attracting till period-doubling bifurcations appear and also chaotic performance. This fact agrees with the qualitative behavior set in Proposition 6.

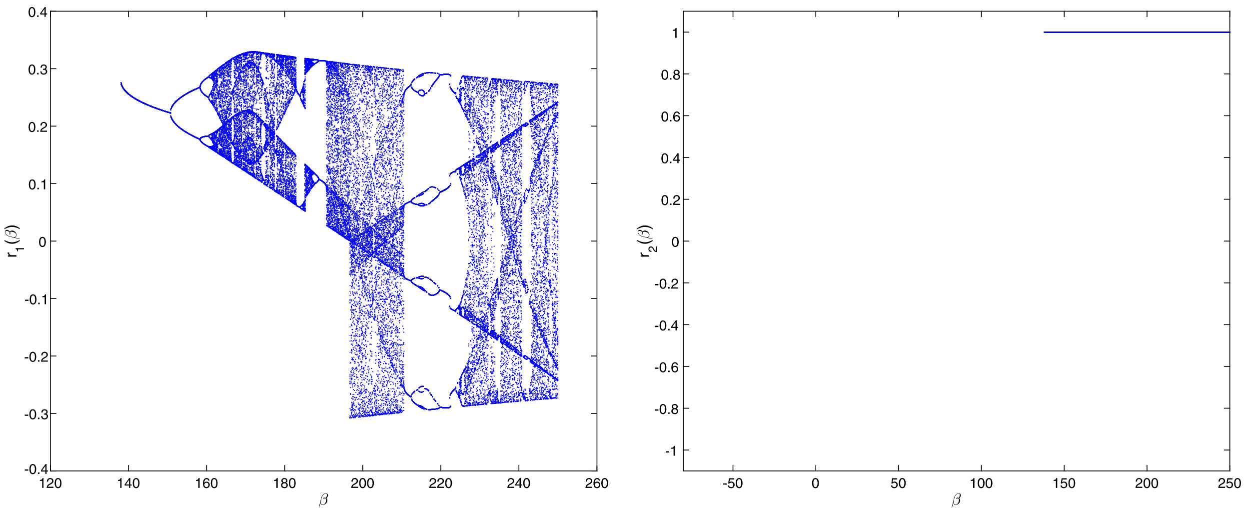
Some dynamical planes are shown in Fig. 12, where the stability is the usual behavior. In Figs. 12b and 12c, only global convergence to the roots is observed, but in Fig. 12a is can be seen as, when $\beta$ is negative and far from zero, the basins of the roots are smaller (the black regions correspond to no convergence to the roots). If $\beta = 140$, Fig. 12d, the sets of converging initial estimations of the attracting strange fixed points appear in cyan and red colors.

Secondly, for $p_2(x)$, we also observe unstable behavior in the bifurcation diagrams of $M_2^2(z, x, \beta)$, as in Fig. 13 we show some dynamical planes where two of the strange fixed points are attracting or even superattracting, being their respective basins plotted in orange and blue colors.




(a) $\beta = -60$ (b) $\beta = 60$

**Fig. 10.** Dynamical planes of family M41 on $p_3(x)$.




(a) $r_1(\beta)$, a detail (b) $r_2(\beta)$

**Fig. 11.** Feigenbaum diagrams of M42 on $p_1(x)$.

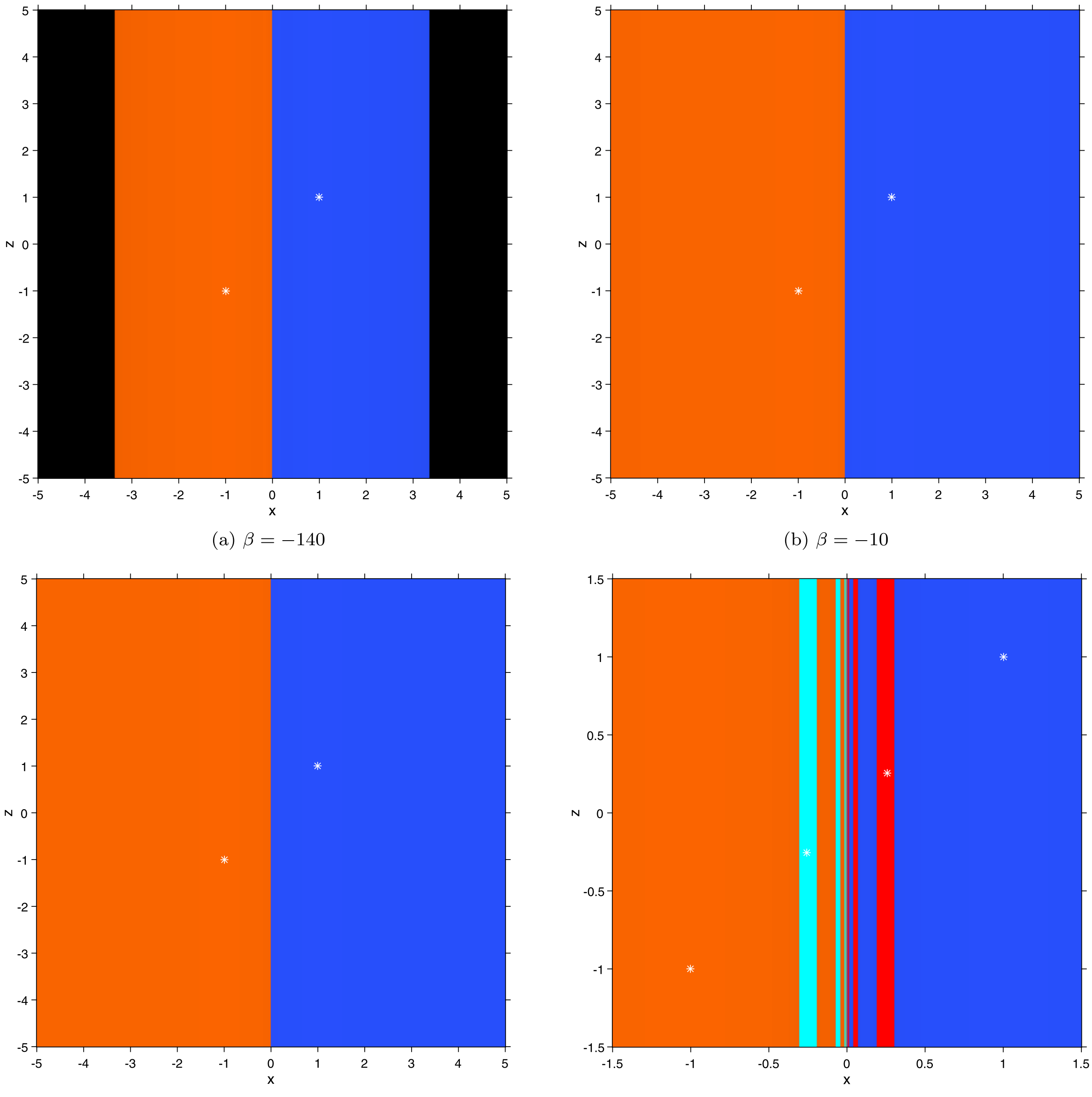
Finally, for $p_3(x)$, we only observe for $M_3^2(z, x, \beta)$ convergence to $(0, 0)$ depending on the character of the fixed point, and no unstable behavior is found. Two examples of dynamical planes are presented in Fig. 14 for values of the parameter where the double root $(0, 0)$ is attracting.

We have observed, as in the previous section, that operators associated to class M42 on quadratic polynomials have big regions of available parameter $\beta$ where the strange fixed points are complex or repelling.

To finish our analysis, we should describe the stability analysis of class M43. However, when it acts on second-degree polynomials, the rational functions obtained are exactly equal to those of M41. So, we refer to this study made in Section 3.

## 5. Numerical performance

To finish this paper, here follows we are going to run our proposed families of iterative methods with memory to approach the solution of some particular nonlinear equations, scalar or vectorial. And, to compare the results, we also




(a) $\beta = -140$

(b) $\beta = -10$

(c) $\beta = 10$

(d) $\beta = 140$, a detail

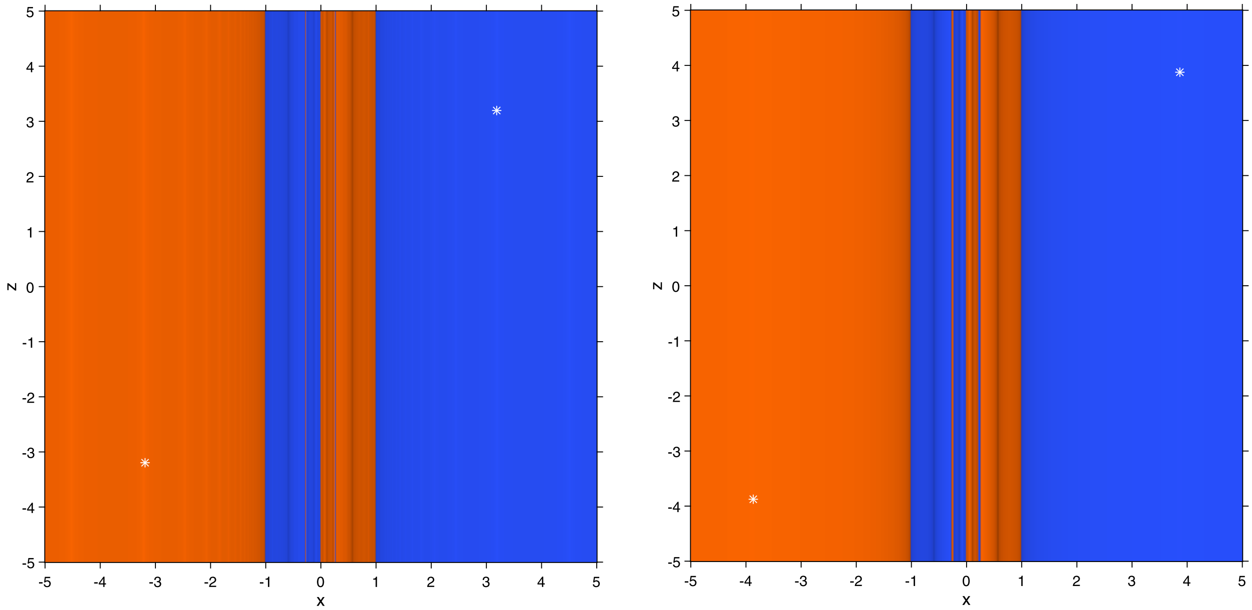
**Fig. 12.** Dynamical planes of M42 on $p_1(x)$.

obtain approximations to these solutions by using some known methods, as Newton's scheme and the original fourth-order family without memory M4.

We start with a scalar applied problem of chemical engineering. In it, we estimate the solution with low-precision calculations.

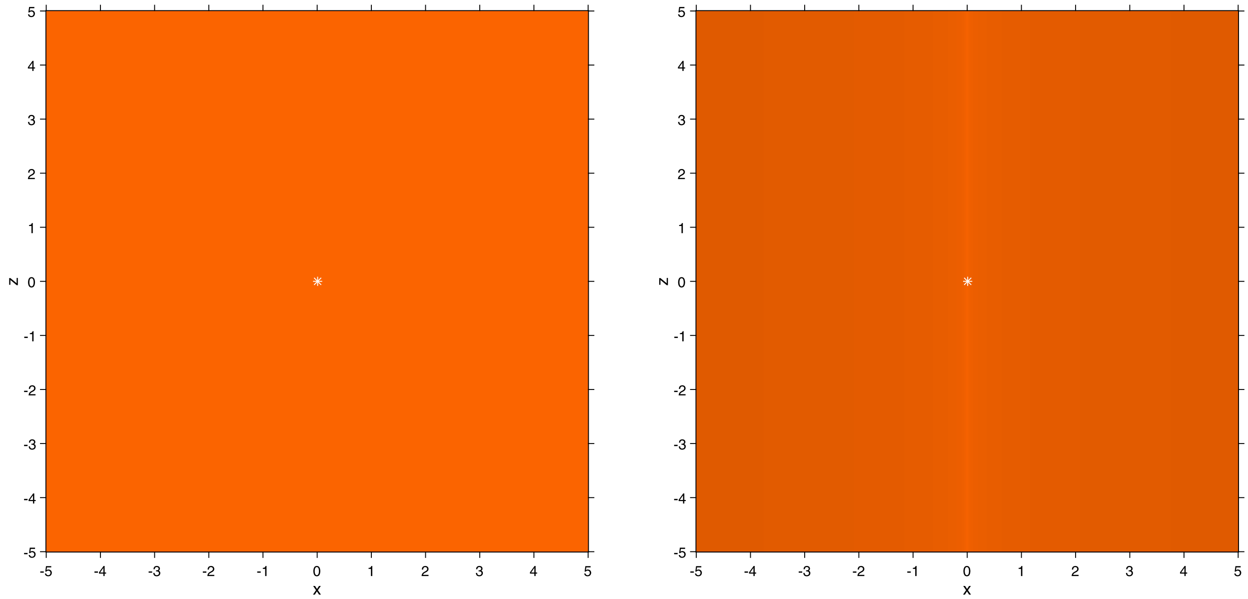
**Example 1.** In the study of friction in round-section pipes, several models are used, that show the experimental relation among the variables taking part in the flow: Reynolds number $Re = 4 \cdot 10^3$ with the longitude, the inner diameter and the rugosity of the pipe $\epsilon_r = 10^{-4}$ and its friction factor $f_f$.

$$\frac{1}{\sqrt{f_f}} = -2.0 \log_{10}\left(\frac{\epsilon_r}{3.7065} + \frac{2.5226}{Re\sqrt{f_f}}\right).$$



(a) $\beta = -30$ (b) $\beta = -38.4406$

**Fig. 13.** Dynamical planes of family M42 on $p_2(x)$.



(a) $\beta = 15$ (b) $\beta = 60$

**Fig. 14.** Dynamical planes of family M42 on $p_3(x)$.

In these conditions, $f_f \approx 0.0401$. The Colebrook–White equation (see, for example [23]) is one of the most precise and used techniques, but it is an implicit equation whose solution must be approximated by iterative techniques.

The results from Tables 1 and 2, correspond to the calculations made with Matlab $R2014b$ of the residuals $|x_{k+1} - x_k|$ and $|f(x_{k+1})|$ in the three first iterations. Moreover, a computational estimation of the order of convergence is obtained by means of ACOC introduced in [11] as

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|)}.$$

**Table 1**
Example 1, $x_0 = 0.07$ and $x_1 = 0.05$.

| Method | $\lvert x_1 - x_0\rvert$ | $\lvert x_2 - x_1\rvert$ | $\lvert x_3 - x_2\rvert$ | $\lvert f(x_1)\rvert$ | $\lvert f(x_2)\rvert$ | $\lvert f(x_3)\rvert$ | $p$ |
|---|---|---|---|---|---|---|---|
| Newton | 4.395e−2 | 1.023e−2 | 3.528e−3 | 1.385 | 2.97e−1 | 1.915e−2 | 0.7298 |
| M4 $\beta = 1$ | 4.39e−2 | 1.327e−2 | 7.002e−4 | 1.379 | 5.182e−2 | 3.93e−7 | 2.4585 |
| M4 $\beta = 5$ | 9.028e−3 | 9.864e−2 | 1.775 | 1.731 | 3.262 | 6.695 | 1.2087 |
| M41 $\beta = 1$ | 9.934e−3 | 1.397e−6 | 7.185e−8 | 9.714e−5 | 4.95e−6 | 2.98e−7 | 0.3346 |
| M41 $\beta = 5$ | 9.929e−3 | 4.277e−6 | 2.189e−7 | 2.975e−4 | 1.509e−5 | 9.08e−7 | 0.3835 |
| M42 $\beta = 1$ | 9.941e−3 | 9.077e−6 | 4.7e−7 | 6.313e−4 | 3.229e−5 | 2.068e−6 | 0.4230 |
| M42 $\beta = 5$ | 9.914e−3 | 1.967e−5 | 9.954e−7 | 1.369e−3 | 6.839e−5 | 4.36e−6 | 0.4795 |
| M43 $\beta = 1$ | 0.973e−3 | 5.953e−5 | 1.48e−13 | 4.347e−3 | 1.082e−13 | 9.43e−63 | 4.7775 |
| M43 $\beta = 5$ | 9.852e−3 | 8.057e−5 | 3.337e−14 | 5.881e−3 | 2.439e−12 | 4.223e−54 | 4.4951 |

**Table 2**
Example 1, $x_0 = 0.1$ and $x_1 = 0.05$.

| Method | $\lvert x_1 - x_0\rvert$ | $\lvert x_2 - x_1\rvert$ | $\lvert x_3 - x_2\rvert$ | $\lvert f(x_1)\rvert$ | $\lvert f(x_2)\rvert$ | $\lvert f(x_3)\rvert$ | $p$ |
|---|---|---|---|---|---|---|---|
| Newton | 1.108e−1 | 2.644e−2 | 1.008e−1 | 11.85 | 7.572 | 5.343 | 0.9343 |
| M4 $\beta = 1$ | 7.526e−1 | 15.08 | 712.4 | 6.171 | 7.175 | 8.879 | 1.286 |
| M4 $\beta = 5$ | 1.128 | 26.72 | 1411.0 | 6.133 | 7.457 | 8.91 | 1.2531 |
| M41 $\beta = 1$ | 9.938e−3 | 5.401e−6 | 2.802e−7 | 3.756e−4 | 1.919e−5 | 1.286e−6 | 0.3936 |
| M41 $\beta = 5$ | 9.92e−3 | 1.313e−5 | 6.686e−7 | 9.134e−4 | 4.581e−5 | 3.055e−6 | 0.4492 |
| M42 $\beta = 1$ | 9.97e−3 | 3.935e−5 | 2.111e−6 | 2.734e−3 | 1.436e−4 | 1.072e−5 | 0.5285 |
| M42 $\beta = 5$ | 9.877e−3 | 5.863e−5 | 2.833e−6 | 4.088e−3 | 1.931e−4 | 1.392e−5 | 0.5910 |
| M43 $\beta = 1$ | 9.521e−3 | 4.118e−4 | 3.58e−12 | 2.988e−2 | 2.616e−10 | 1.066e−46 | 5.9098 |
| M43 $\beta = 5$ | 9.496e−3 | 4.368e−4 | 2.881e−11 | 3.168e−2 | 2.105e−9 | 6.017e−41 | 5.3696 |

**Table 3**
Example 2, $x^{(0)} = (2, 2, \dots, 2)^T$.

| | Newton | M4 $\beta = 1$ | M4 $\beta = 5$ | M4 $\beta = -1$ | M43 $\beta = 1$ | M43 $\beta = 5$ | M43 $\beta = -1$ |
|---|---|---|---|---|---|---|---|
| Iterations | – | 12 | – | – | 4 | 4 | 5 |
| $\lVert x^{(k+1)} - x^{(k)}\rVert$ | – | 2.65e−69 | – | – | 9.56e−39 | 1.10e−39 | 3.04e−35 |
| $\lVert F(x^{(k+1)})\rVert$ | – | 8.71e−208 | – | – | 1.05e−181 | 2.19e−177 | 9.61e−158 |
| ACOC | – | 4.0 | – | – | – | – | – |

When the components of vector ACOC do not tend to any real value, it is marked as '-'. These data are supplied by the performance of Newton's method and cases $\beta = 1$ and $\beta = 5$ of the without-memory class M4, and the proposed families with memory M41, M42 and M43. This values of the parameters correspond to stable values in all cases, so the performance of the different methods can be compared.

In Table 1, good estimations have been used to start the processes and the precision obtained by all the high-order schemes is good, but it can be observed that those given by M43 methods decrease the residuals in ten times respect their partners. Regarding the ACOC, again M43 is the only class whose values correspond with the theoretical ones.

Bad initial estimations have been used in Table 2, with good results in cases with memory, but bad ones in their without-memory partners. Moreover, in the former case, the precision of the obtained results is slightly lower in all methods.

In what follows, we compare the multidimensional version of new class of iterative methods M43 (in fact, we test the particular cases corresponding to $\beta \in \{-1, 1, 5\}$), with Newton's scheme and fourth-order methods M4 with the same values of $\beta$. The numerical results are presented in Tables 3 to 5.

**Example 2.** Now, we define the nonlinear system of variables size defined by:

$$x_i - \cos\left(2x_i - \sum_{j=1}^{4} x_j\right) = 0, \quad i = 1, 2, \dots, n.$$

All numerical tests have been made by using the software Matlab $R2014b$ with variable precision arithmetics and 200 digits of mantissa. The stopping criterium used is that at least one of the residuals $\lVert x^{(k+1)} - x^{(k)}\rVert$ and $\lVert F(x^{(k+1)})\rVert$ reaches the tolerance $10^{-100}$ and the maximum number of iterations considered is 100.

In this example, the size of the system is $n = 20$ and the initial estimations used in for our numerical tests are $x^{(0)} = (2, 2, \dots, 2)^T$ (in case of M43, joint with $x^{(1)} = (1.2, 1.2, \dots, 1.2)^T$), $x^{(0)} = (-0.1, -0.1, \dots, -0.1)^T$ (in case of M43, joint with $x^{(1)} = (-0.2, -0.2, \dots, -0.2)^T$) and $x^{(0)} = (1, 1, \dots, 1)^T$ (in case of M43, joint with $x^{(1)} = (0.9, 0.9, \dots, 0.9)^T$). The solution of this problem is $\bar{x} \approx (0.5149, 0.5149, \dots, 0.5149)^T$.

It is observed in Tables 3 to 5 that, for this problem, the proposed class with memory M43 shows to be highly stable compared with the original class M4 and also respect Newton's method; it converges even for initial estimations far from

**Table 4**
Example 2, $x^{(0)} = (-0.1, -0.1, \ldots, -0.1)^T$.

| | Newton | M4 $\beta = 1$ | M4 $\beta = 5$ | M4 $\beta = -1$ | M43 $\beta = 1$ | M43 $\beta = 5$ | M43 $\beta = -1$ |
|---|---|---|---|---|---|---|---|
| Iterations | 54 | 10 | 5 | 6 | 5 | 5 | 5 |
| $\|x^{(k+1)} - x^{(k)}\|$ | 7.76e−82 | 8.86e−51 | 4.32e−30 | 2.63e−55 | 1.16e−84 | 3.97e−30 | 2.05e−71 |
| $\|F(x^{(k+1)})\|$ | 1.39e−163 | 4.09e−202 | 4.84e−151 | 2.61e−207 | 8.71e−208 | 2.48e−134 | 1.74e−207 |
| ACOC | – | – | – | 3.9998 | 4.5461 | 4.564 | 4.3867 |

**Table 5**
Example 2, $x^{(0)} = (1, 1, \ldots, 1)^T$.

| | Newton | M4 $\beta = 1$ | M4 $\beta = 5$ | M4 $\beta = -1$ | M43 $\beta = 1$ | M43 $\beta = 5$ | M43 $\beta = -1$ |
|---|---|---|---|---|---|---|---|
| Iterations | 7 | 4 | 4 | 4 | 3 | 3 | 3 |
| $\|x^{(k+1)} - x^{(k)}\|$ | 4.22e−69 | 3.28e−76 | 1.13e−92 | 3.18e−62 | 4.19e−27 | 9.03e−26 | 2.47e−26 |
| $\|F(x^{(k+1)})\|$ | 4.10e−138 | 3.89e−208 | 3.95e−207 | 8.71e−208 | 6.83e−127 | 2.76e−115 | 5.12e−118 |
| ACOC | 2.0000 | 4.0000 | 5.0004 | 3.9999 | 3.9992 | 3.8001 | 3.8259 |

the solution, in a very reduced number of iterations with good residuals and when the initial estimations are good, the results are also very competitive.

## 6. Conclusions

From an efficient parametric class of iterative methods with order of convergence 4 for solving systems of nonlinear equations, we design several classes with memory improving the order of convergence. By using tools of real multidimensional dynamics, we have studied the dynamical behavior of the different proposed classes on quadratic polynomials, obtaining interesting information about the qualitative performance and reliability of the members of the families. This information is used in the numerical experiments which confirm the theoretical results.

## Acknowledgements

The authors would like to thank the anonymous reviewers for their comments and suggestions that have improved the final version of this manuscript.

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