Numerical Algorithms (2021) 87:1017–1060
https://doi.org/10.1007/s11075-020-00997-4

ORIGINAL PAPER

Check for updates

# New fourth- and sixth-order classes of iterative methods for solving systems of nonlinear equations and their stability analysis

**Munish Kansal[1] · Alicia Cordero[2] · Sonia Bhalla[3] · Juan R. Torregrosa[2]**

Received: 4 October 2019 / Accepted: 4 August 2020 / Published online: 3 September 2020


## Abstract

In this paper, a two-step class of fourth-order iterative methods for solving systems of nonlinear equations is presented. We further extend the two-step class to establish a new sixth-order family which requires only one additional functional evaluation. The convergence analysis of the proposed classes is provided under several mild conditions. A complete dynamical analysis is made, by using real multidimensional discrete dynamics, in order to select the most stable elements of both families of fourth- and sixth-order of convergence. To get this aim, a novel tool based on the existence of critical points has been used, the parameter line. The analytical discussion of the work is upheld by performing numerical experiments on some application-oriented problems. We provide an implementation of the proposed scheme on nonlinear optimization problem and zero-residual nonlinear least-squares problems taken from the constrained and unconstrained testing environment test set. Finally, based on numerical results, it has been concluded that our methods are comparable with the existing ones of similar nature in terms of order, efficiency, and computational time and also that the stability results provide the most efficient member of each class of iterative schemes.

**Keywords** Systems of nonlinear equations · Order of convergence · Multipoint iterative methods · Stability analysis

**Mathematics Subject Classification 2010** 65H10 · 65Y20

This research was partially supported by PGC2018-095896-B-C22 (MCIU/AEI/FEDER, UE), Generalitat Valenciana PROMETEO/2016/089.

✉ Juan R. Torregrosa
jrtorre@mat.upv.es

Extended author information available on the last page of the article.

## 1 Introduction

Systems of nonlinear equations are of immense importance for applications in many areas of science and engineering. For a given nonlinear system, $G(X) = 0$, where $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, we are interested to find a vector $X^* = (x_1^*, x_2^*, \cdots, x_n^*)^T$ such that $G(X^*) = 0$ , where $G(X) = (g_1(X), g_2(X), \ldots, g_n(X))^T$ is a Fréchet differentiable function and $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. The classical Newton's method is the most basic procedure to solve systems of nonlinear equations. It is given by:

$$X^{(k+1)} = X^{(k)} - \{G'(X^{(k)})\}^{-1}G(X^{(k)}),\ k = 0, 1, \ldots \tag{1}$$

where $\{G'(X^{(k)})\}^{-1}$ is the inverse of first-order Fréchet derivative of the function $G$ evaluated in $(X^{(k)})$. Assuming that the function $G$ is continuously differentiable and the initial approximation is close enough to the solution, then this method converges quadratically. In literature, there are a variety of higher-order methods which improve the order of convergence of Newton's scheme. For example, several authors have developed third-order methods [1–4] each requiring evaluation of one $G$, two $G'$, and two matrix inversion per iteration. Cordero and Torregrosa [5] derived two more third-order methods, one of which requires one $G$, and three $G'$ whereas the other requires one $G$, and four $G'$ evaluations, and two matrix inversions. Another third-order method by Darvishi and Barati [6] utilizes two $G$, two $G'$, and two matrix inversions per iteration. Darvishi and Barati [7] and Potra and Pták [8] have proposed third-order methods that require two $G$, one $G'$, and one matrix inversion. Babajee et al. [9] have presented fourth-order method which consumes one $G$, two $G'$, and two matrix inversions per iteration. The fourth-order method by Cordero et al. [10] is developed using two evaluations of the function and the Jacobian, and one matrix inversion, whereas the authors in [11] propose another fourth-order method utilizing three $G$, one $G'$, and one matrix inversion per iteration. Another fifth-order method proposed by Cordero et al. [12] requires three evaluations of the function and only one Jacobian evaluation, with the solution of three linear systems with the same matrix of coefficients per iteration.

In pursuit of faster algorithms, researchers have also developed fifth- and sixth-order methods for example [13–16]. In [14], Narang et al. extended the existing Babajee's fourth-order scheme [17] to solve systems of nonlinear equations and developed a sixth-order convergent family of Chebyshev-Halley type methods. Their scheme requires two $G$, two $G'$ evaluations, and the solution of two linear systems per iteration. One can notice that while the researchers are making an attempt to improve the order of convergence of an iterative method, it mostly leads to an increase in the computational cost per step. The computational cost is especially high if the method involves the use of second-order Fréchet derivative $G''(X)$. This is a major limitation of the higher-order methods. Thus, while developing new iterative methods, we should try to keep the computational cost low. With this intention, we have made an attempt to develop a family of three-step sixth-order family of methods requiring two $G$, two $G'$, and one matrix inversion per iteration. This family of methods is compared to be more efficient than existing methods. These have been found to be effective in solving particularly large-scale systems of nonlinear equations.

The outline of the paper is as follows. In Section 2, a parametric family of fourth-order methods is presented along with their convergence analysis. In Section 3, we present a class of new sixth-order schemes by adding a step to the fourth-order family and its convergence analysis. Section 4 and Section 5 are devoted to the analysis of the real dynamics of the proposed classes and the selection of their most stable elements. In Section 6, we test the consistency of convergence behavior of the methods and examine the theoretical results with the help of various numerical experiments. Finally, Section 7 contains concluding remarks.

## 2 Development of fourth-order scheme

In this section, we extended to the context of nonlinear systems $G(X) = 0$, the two-point fourth-order Chebyshev-Halley type methods proposed by Behl and Kanwar [18] for solving nonlinear equations. For this purpose, we write the generalized form as:

$$\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}), \\ X^{(k+1)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Gamma(X^{(k)}) &= \{G'(X^{(k)})\}^{-1}G(X^{(k)}), \\ \mu(X^{(k)}) &= A_1 I + 3A_2\Omega(X^{(k)}) + 9A_3\left(\Omega(X^{(k)})\right)^2 - 27A_4\left(\Omega(X^{(k)})\right)^3, \\ \eta(X^{(k)}) &= 2\left(A_5 I - 3A_6\Omega(X^{(k)}) + 9A_7\left(\Omega(X^{(k)})\right)^2 - 27A_8\left(\Omega(X^{(k)})\right)^3\right), \\ \Omega(X^{(k)}) &= \{G'(X^{(k)})\}^{-1}G'(Y^{(k)}), \end{aligned}$$

and

$$\begin{array}{ll} A_1 = 54a^3 - 135a^2 + 102a - 23, & A_2 = -54a^3 + 135a^2 - 112a + 29, \\ A_3 = 18a^3 - 45a^2 + 38a - 11, & A_4 = (a-1)^2(2a-1), \\ A_5 = 27a^3 - 54a^2 + 33a - 4, & A_6 = 27a^3 - 54a^2 + 35a - 6, \\ A_7 = 9a^3 - 18a^2 + 11a - 2, & A_8 = a(a-1)^2, \end{array}$$

being $a$ a free disposable parameter, $G'(X^{(k)})$ the Jacobian matrix of $G$ evaluated in the iterate $X^{(k)}$, and $I$ the identity matrix of size $n \times n$. This class of schemes is denoted by $PM_4$.

### 2.1 Convergence analysis

In order to analyze the convergence of the proposed class (2), we need some tools and procedures introduced in [20].

Let $e^{(k)} = X^{(k)} - X^*$ be the error in the $k$-th step of sequence of approximations generated by an iterative method. The equation:

$$e^{(k+1)} = L(e^{(k)})^{\rho} + O((e^{(k)})^{\rho+1}), \tag{3}$$

is called the error equation. The smallest exponent $\rho$ of $e^{(k)}$ represents the order of convergence and $L$ represents a $\rho$-linear function, $L \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n \times \overset{\rho}{\ldots\ldots} \times \mathbb{R}^n, \mathbb{R}^n)$. The next result (see [19]) establishes the Taylor expansion of function $G$.

**Lemma 1** *Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be p-times Fréchet differentiable in a convex set $D \subseteq \mathbb{R}^n$, then for any $X,\ h \in \mathbb{R}^n$, the following expression holds:*

$$\begin{aligned} G(X+h) &= G(X) + G'(X)h + \frac{1}{2!}G''(X)h^2 + \frac{1}{3!}G'''(X)h^3 + \ldots.. \\ &+ \frac{1}{(p-1)!}G^{(p-1)}(X)h^{(p-1)} + R_p. \end{aligned}$$

*where*

$$||R_p|| \leq \sup_{0\leq u\leq 1} \frac{1}{p!}||G^{(p)}(X+uh)||||h||^p \text{ and } h^p = \overbrace{(h, h, \ldots, h)}^{p}.$$

*For any $X,\ X^{(k)} \in D$, the Taylor series of $G(X)$ about $X^{(k)}$ is defined as follows:*

$$\begin{aligned} G(X) &= G(X^{(k)}) + G'(X^{(k)})(X - X^{(k)}) + \frac{1}{2!}G''(X^{(k)})(X - X^{(k)})^2 \\ &+ \frac{1}{3!}G'''(X^{(k)})(X - X^{(k)})^3 + \cdots + \frac{1}{(p-1)!}G^{(p-1)}(X^{(k)})(X - X^{(k)})^{(p-1)} \\ &+ O(||X - X^{(k)}||^p). \end{aligned}$$

Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood D. The $l$-$th$ derivative of $G$ at $u \in \mathbb{R}^n$, $l \geq 1$, is the $l$-linear function $G^{(l)}(u) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ such that $G^{(l)}(u)(v_1, \ldots, v_l) \in \mathbb{R}^n$. It is easy to observe that

1. $G^{(l)}(u)(v_1, \ldots, v_{l-1}, v_l) \in \mathcal{L}(\mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^n)$ denotes the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$.
2. $G^{(l)}(u)(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) = G^{(l)}(u)(v_1, \ldots, v_l)$, for all permutations $\sigma$ of $\{1, 2, \ldots, l\}$.

From the above properties, we can use the following notation:

(a) $G^{(l)}(u)(v_1, \ldots, v_l) = G^{(l)}(u)v_1, \ldots, v_l$
(b) $G^{(l)}(u)v^{l-1}G^{(p)}v^p = G^{(l)}(u)G^{(p)}(u)v^{p+l-1}$, where $v \in \mathbb{R}^n$ and $v^p = \overbrace{(v, v, \ldots, v)}^{p}$.

On the other side, for $(X^* + h) \in \mathbb{R}^n$ lying in a neighborhood of a solution $X^*$ of $G(X) = 0$, we can apply Taylor's series expansion and assuming that the Jacobian matrix $G'(X^*)$ is non-singular, we have:

$$G(X^* + h) = G'(X^*)\left[h + \sum_{l=2}^{p-1} C_l h^l\right] + O(h^p), \tag{4}$$

where $C_l = \frac{1}{l!}\{G'(X^*)\}^{-1}G^{(l)}(X^*)$ for $l \geq 2$. It is clear that $C_l h^l \in \mathbb{R}^{n\times n}$, since $G^{(l)}(X^*) \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$ and $\{G'(X^*)\}^{-1} \in \mathcal{L}(\mathbb{R}^n)$. In addition, $G'$ can be expressed as:

$$G'(X^* + h) = G'(X^*)\left[I + \sum_{l=2}^{p-2} lC_l h^{l-1}\right] + O(h^{p-1}), \tag{5}$$

where $I$ is the identity matrix. Therefore, $lC_l h^{l-1} \in \mathbb{R}^{n\times n}$. From (5), we have:

$$\{G'(X^*+h)\}^{-1} = \{G'(X^*)\}^{-1}\left(I + C_2^* h + C_3^* h^2 + C_4^* h^3 + C_5^* h^4 + \ldots\right) + O(h^p), \tag{6}$$

where

$$\begin{aligned} C_2^* &= -2C_2, \\ C_3^* &= 4C_2^2 - 3C_3, \\ C_4^* &= -4C_4 + 6C_2C_3 + 6C_3C_2 - 8C_2^3, \\ &\vdots \end{aligned}$$

Let $e^{(k)} = X^{(k)} - X^*$ be the error at $k^{th}$ iteration. The equation $e^{(k+1)} = M(e^{(k)})^\rho + O((e^{(k)})^{\rho+1})$, where $M$ is a $\rho$-linear function $M \in \mathcal{L}(\mathbb{R}^n \times \cdots \times \mathbb{R}^n, \mathbb{R}^n)$, is called error equation, being $\rho$ its convergence order.

**Theorem 1** *Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood $D$ of $X^*$ which is solution of the system $G(X) = 0$. Consider that initial guess $X^{(0)}$ is sufficiently close to the required solution $X^*$ and $G'(X)$ is continuous and nonsingular in $X^*$. Then, the proposed iterative scheme (2) has local fourth-order of convergence for all $a \in \mathbb{R}$.*

*Proof* Let $e^{(k)} = X^{(k)} - X^*$ be the error at $k$th iteration. With the help of Taylor's series expansion, one can obtain the following expansions of the function $G(X^{(k)})$ and its first-order derivative $G'(X^{(k)})$ around the point $X^*$.

$$G(X^{(k)}) = G'(X^*)\Big(e^{(k)} + C_2(e^{(k)})^2 + C_3(e^{(k)})^3 + C_4(e^{(k)})^4 + C_5(e^{(k)})^5 + C_6(e^{(k)})^6 + O((e^{(k)})^7)\Big). \tag{7}$$

$$G'(X^{(k)}) = G'(X^*)\left(I + 2C_2e^{(k)} + 3C_3(e^{(k)})^2 + 4C_4(e^{(k)})^3 + 5C_5(e^{(k)})^4 + 6C_6(e^{(k)})^5 + O((e^{(k)})^6)\right), \tag{8}$$

where $C_j = \frac{1}{j!}\{G'(X^*)\}^{-1}G^{(j)}(X^*)$ for $j = 2, 3, 4\ldots$. From (8), one can obtain the expression of the inverse:

$$\begin{aligned} \{G'(X^{(k)})\}^{-1} = \{G'(X^*)\}^{-1}&\left(I - 2C_2e^{(k)} + \left(4C_2^2 - 3C_3\right)(e^{(k)})^2\right. \\ &+ \left(-8C_2^3 + 12C_3C_2 - 4C_4\right)(e^{(k)})^3 \\ &+ \left(16C_2^4 - 36C_3C_2^2 + 16C_4C_2 + 9C_3^2 - 5C_5\right)(e^{(k)})^4 \\ &+ \left. O\left((e^{(k)})^5\right)\right). \end{aligned} \tag{9}$$

By using expressions (7) and (9), one gets:

$$
\begin{aligned}
&\{G'(X^{(k)})\}^{-1}G(X^{(k)})\\
&= e^{(k)} - C_2(e^{(k)})^2 + 2\Big(C_2^2 - C_3\Big)(e^{(k)})^3 + \Big(4C_2C_3 + 3C_3C_2 - 4C_2^3 - 3C_4\Big)(e^{(k)})^4\\
&\quad + \Big(-4C_5 + 6C_2C_4 - 8C_2^2C_3 + 6C_3^2 + 4C_4C_2 - 6C_2C_3C_2 + 8C_2^4 - 6C_3C_2^2\Big)(e^{(k)})^5\\
&\quad - \Big(C_6 - 2C_2C_5 - 5C_5C_2 + 4C_2^2C_4 + 8C_4C_2^2 - 3C_3C_4 - 4C_4C_3 + 6C_2C_3^2 + 6C_3C_2C_3 - 8C_2^3C_3\\
&\quad + 8C_2C_4C_2 - 12C_2^2C_3C_2 + 9C_3^2C_2 - 12C_2C_3C_2^2 + 16C_2^5 - 12C_3C_2^3\Big)(e^{(k)})^6 + O((e^{(k)})^7).
\end{aligned}
\tag{10}
$$

The first step of the proposed scheme (2) can be rewritten as:

$$
Y^{(k)} - X^* = X^{(k)} - X^* - \frac{2}{3}\{G'(X^{(k)})\}^{-1}G(X^{(k)}). \tag{11}
$$

In view of (10), (11) yields as:

$$
\begin{aligned}
Y^{(k)} - X^* &= \frac{1}{3}e^{(k)} + \frac{2}{3}C_2(e^{(k)})^2 - \frac{2}{3}\left(2C_2^2 - 2C_3\right)(e^{(k)})^3 - \frac{2}{3}\left(-4C_2^3 + 7C_3C_2 - 3C_4\right)(e^{(k)})^4\\
&\quad - \frac{2}{3}\left(8C_2^4 - 20C_3C_2^2 + 10C_4C_2 + 6C_3^2 - 4C_5\right)(e^{(k)})^5 - \frac{2}{3}\left(-16C_2^5 + 52C_3C_2^3 - 28C_4C_2^2\right.\\
&\quad \left. - 33C_3^2C_2 + 13C_5C_2 + 17C_3C_4 - 5C_6\right)(e^{(k)})^6 + O\left((e^{(k)})^7\right).
\end{aligned}
\tag{12}
$$

Furthermore, the Taylor series expansions of $G(Y^{(k)})$ and $G'(Y^{(k)})$ around the point $X^*$ are given as:

$$
\begin{aligned}
G(Y^{(k)}) &= G'(X^*)\left[(Y^{(k)} - X^*) + C_2(Y^{(k)} - X^*)^2 + C_3(Y^{(k)} - X^*)^3 + O((Y^{(k)} - X^*)^4)\right],\\
&= G'(X^*)\left[\frac{1}{3}e^{(k)} + \frac{7}{9}C_2(e^{(k)})^2 - \frac{1}{27}\left(24C_2^2 - 37C_3\right)(e^{(k)})^3 + \frac{1}{81}\left(180C_2^3 - 288C_3C_2\right.\right.\\
&\quad \left. + 163C_4\right)(e^{(k)})^4 - \frac{1}{243}\left(1296C_2^4 - 2916C_3C_2^2 + 1272C_4C_2 + 864C_3^2 - 649C_5\right)(e^{(k)})^5\\
&\quad + \frac{1}{729}\left(9072C_2^5 - 26352C_3C_2^3 + 12384C_4C_2^2 + 15552C_3^2C_2 - 4992C_5C_2 - 7632C_3C_4 + 2431C_6\right)(e^{(k)})^6\\
&\quad \left. + O\left((e^{(k)})^7\right)\right],
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
G'(Y^{(k)}) = G'(X^*)\Big(I &+ \frac{2}{3}C_2e^{(k)} + \frac{1}{3}\Big(4C_2^2 + C_3\Big)(e^{(k)})^2\\
&+ \Big(-\frac{8}{3}C_2^3 + 4C_3C_2 + \frac{4C_4}{27}\Big)(e^{(k)})^3\\
&+ \frac{1}{81}\Big(432C_2^4 - 864C_3C_2^2 + 396C_4C_2 + 216C_3^2 + 5C_5\Big)(e^{(k)})^4\\
&+ O\Big((e^{(k)})^5\Big)\Big).
\end{aligned}
\tag{14}
$$

Also, by using (9) and (14), one can have:

$$\begin{aligned}\Omega(X^{(k)}) &= I - \frac{4C_2}{3}e^{(k)} + \left(4C_2^2 - \frac{8C_3}{3}\right)(e^{(k)})^2 \\ &- \frac{8}{27}\left(36C_2^3 - 45C_3C_2 + 13C_4\right)(e^{(k)})^3 \\ &+ \frac{4}{81}\left(540C_2^4 - 999C_3C_2^2 + 363C_4C_2 + 216C_3^2 - 100C_5\right)(e^{(k)})^4 \\ &+ O\left((e^{(k)})^5\right),\end{aligned} \tag{15}$$

$$\begin{aligned}(\Omega(X^{(k)}))^2 &= I - \frac{8C_2}{3}e^{(k)} + \frac{8}{9}\left(11C_2^2 - 6C_3\right)(e^{(k)})^2 \\ &- \frac{16}{27}\left(54C_2^3 - 57C_3C_2 + 13C_4\right)(e^{(k)})^3 \\ &+ \frac{8}{81}\left(990C_2^4 - 1575C_3C_2^2 + 467C_4C_2 + 288C_3^2 - 100C_5\right)(e^{(k)})^4 \\ &+ O\left((e^{(k)})^5\right),\end{aligned} \tag{16}$$

and

$$\begin{aligned}(\Omega(X^{(k)}))^3 &= I - 4C_2e^{(k)} + \left(\frac{52C_2^2}{3} - 8C_3\right)(e^{(k)})^2 \\ &- \frac{8}{27}\left(224C_2^3 - 207C_3C_2 + 39C_4\right)(e^{(k)})^3 \\ &+ \frac{4}{27}\left(1584C_2^4 - 2247C_3C_2^2 + 571C_4C_2 + 360C_3^2 - 100C_5\right)(e^{(k)})^4 \\ &+ O\left((e^{(k)})^5\right).\end{aligned} \tag{17}$$

In view of (10), (15), (16), and (17), the second step of scheme (2), one gets the following error equation:

$$\begin{aligned}X^{(k+1)} - X^* &= \frac{1}{9}\left(-72a^2C_2^3 + 144aC_2^3 - 63C_2^3 - 9C_3C_2 + C_4\right)(e^{(k)})^4 \\ &+ \frac{2}{27}\Big(216a^3C_2^4 - 648a^2C_3C_2^2 - 648aC_2^4 + 1296aC_3C_2^2 + 378C_2^4 \\ &- 540C_3C_2^2 - 30C_4C_2 - 27C_3^2 + 4C_5\Big)(e^{(k)})^5 + O\left((e^{(k)})^6\right).\end{aligned} \tag{18}$$

This implies that the scheme (2) achieves fourth order of convergence. This completes the proof. □

## 3 The sixth-order scheme and its convergence analysis

In this section, we propose the following parametric family of sixth-order iterative scheme based on (2) by introducing an additional step requiring only one new function evaluation as follows:

$$\begin{aligned} Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}), \\ Z^{(k)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}), \\ X^{k+1} &= Z^{(k)} - \left(pI + \Omega(X^{(k)})\left(qI + r\Omega(X^{(k)})\right)\right)\{G'(X^{(k)})\}^{-1}G(Z^{(k)}), \end{aligned} \tag{19}$$

where $p, q, r$ are free disposable parameters, $\Gamma(X^{(k)})$, $\mu(X^{(k)})$, $\eta(X^{(k)})$, and $A_i's$ (for $i = 1, 2, \ldots, 8$) are defined as previously in scheme (2). In the rest of the manuscript, this family will be denoted by $PM_6$.

In the following result, we establish the local order of convergence of class (19).

**Theorem 2** *Let $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable in an open neighborhood $D$ of $X^*$ which is a solution of $G(X) = 0$. Consider that initial guess $X^{(0)}$ is sufficiently close to the required zero $X^*$ and $G'(X)$ is continuous and nonsingular in $X^*$. Then, for all $a \in \mathbb{R}$ and $r \in \mathbb{R}$, the local order of convergence of the sequence $\{X^{(k)}\}$ generated by the proposed iterative scheme (19) is six for $p = r + \frac{5}{2}$ and $q = -\frac{3}{2} - 2r$.*

*Proof* Using the error (18), expand the function $G(Z^{(k)})$ around the point $X^*$, one obtains:

$$\begin{aligned} G(Z^{(k)}) \;=\; & G'(X^*)\Big[(Z^{(k)} - X^*) + C_2(Z^{(k)} - X^*)^2 + C_3(Z^{(k)} - X^*)^3 \\ & +\; O((Z^{(k)} - X^*)^4)\Big]. \end{aligned} \tag{20}$$

Pre-multiply the (20) with (9) and after simplification, one can have:

$$\begin{aligned} &\{G'(X^{(k)})\}^{-1}G(Z^{(k)}) \\ &= \frac{1}{9}\Big(-72a^2C_2^3 + 144aC_2^3 - 63C_2^3 - 9C_3C_2 + C_4\Big)(e^{(k)})^4 + \frac{2}{27}\Big(216a^3C_2^4 + 216a^2C_2^4 \\ &\quad - 648a^2C_3C_2^2 - 1080aC_2^4 + 1296aC_3C_2^2 + 567C_2^4 - 513C_3C_2^2 - 33C_4C_2 - 27C_3^2 + 4C_5\Big)(e^{(k)})^5 \\ &\quad + O\Big((e^{(k)})^6\Big). \end{aligned} \tag{21}$$

In view of (15) and (21), one can have:

$$\begin{aligned}
&\left(pI + \Omega(X^{(k)})\left(qI + r\Omega(X^{(k)})\right)\right)\{G'(X^{(k)})\}^{-1}G(Z^{(k)})\\
&= (p+q+r) - \frac{4}{3}\left(C_2q + 2C_2r\right)e^{(k)} + \frac{4}{9}\left(9C_2^2q - 6C_3q\right.\\
&\left. + 22C_2^2r - 12C_3r\right)(e^{(k)})^2 - \frac{8}{27}\left(36C_2^3q - 45C_3C_2q + 13C_4q\right.\\
&\left. + 108C_2^3r - 114C_3C_2r + 26C_4r\right)(e^{(k)})^3 + \frac{4}{81}\left(540C_2^4q\right.\\
&-999C_3C_2^2q + 363C_4C_2q + 216C_3^2q - 100C_5q + 1980C_2^4r\\
&\left. - 3150C_3C_2^2r + 934C_4C_2r + 576C_3^2r - 200C_5r\right)(e^{(k)})^4\\
&+O\left((e^{(k)})^5\right).
\end{aligned} \tag{22}$$

Substituting (22) in the last step of method (19), one obtains:

$$\begin{aligned}
X^{(k+1)} - X^* = &\frac{1}{9}\left(9(8(a-2)a+7)C_2^3 + 9C_3C_2 - C_4\right)(p+q+r-1)(e^{(k)})^4\\
&+ \frac{2}{27}\left(-216a^3C_2^4 + 648(a-2)aC_3C_2^2 + 27C_3^2 - 4C_5\right)(p+q+r-1)\\
&+ \frac{2}{27}\left(-216a^2C_2^4p - 360a^2C_2^4q - 504a^2C_2^4r\right.\\
&+ 1080aC_2^4p + 1368aC_2^4q\\
&+ 1656aC_2^4r - 648aC_2^4 - 63C_2^4(9p + 11q + 13r - 6)\\
&+ C_4C_2(33p + 35q + 37r - 30)\\
&\left.+ 513C_3C_2^2p + 495C_3C_2^2q + 477C_3C_2^2r - 540C_3C_2^2\right)(e^{(k)})^5\\
&+ \frac{1}{81}\left(5184a^4C_2^5 - 10368a^3C_3C_2^3 + 5616(a-2)aC_4C_2^2\right.\\
&+ 7776(a-2)aC_3^2C_2\\
&- 42C_6)(p+q+r-1)\\
&+ \frac{1}{81}\left(-6480a^3C_2^5p - 4752a^3C_2^5q - 3024a^3C_2^5r\right.\\
&+ 9072a^3C_2^5 + 288a^2C_2^5(45p + 60q + 79r - 36) - 5832a^2C_3C_2^3p\\
&- 12744a^2C_3C_2^3q - 19656a^2C_3C_2^3r - 3888a^2C_3C_2\\
&+ 432aC_3C_2^3(99p + 131q + 163r - 54)\\
&- 27216aC_2^5p - 41040aC_2^5q - 57168aC_2^5r + 14256aC_2^5\\
&+ 2C_5C_2(159p + 175q + 191r - 135)\\
&+ 3C_3C_4(207p + 215q + 223r - 198) + 13770C_2^5p\\
&- 22275C_3C_2^3p + 4230C_4C_2^2p
\end{aligned}$$

$$
\begin{aligned}
&+ 5751C_3^2C_2p + 20574C_2^5q - 27567C_3C_2^3q + 3930C_4C_2^2q \\
&+ 5319C_3^2C_2q + 28386C_2^5r - 32715C_3C_2^3r + 3614C_4C_2^2r \\
&+ 4887C_3^2C_2r - 6966C_2^5 + 14418C_3C_2^3 - 4626C_4C_2^2 \\
&- 6318C_3^2C_2\Big)(e^{(k)})^6 + O\left((e^{(k)})^7\right).
\end{aligned}
\tag{23}
$$

To achieve the sixth order of convergence, the coefficients of $(e^{(k)})^4$ and $(e^{(k)})^5$ must be equal to zero, which reduces to the following linear system:

$$
\begin{aligned}
p + q + r &= 1, \\
3p + 5q + 7r &= 0.
\end{aligned}
\tag{24}
$$

Now, solving the above system of equations, one gets $p = r + \frac{5}{2}$ and $q = -\frac{3}{2} - 2r$. Furthermore, (3) reduces to the following final error equation:

$$
\begin{aligned}
X^{(k+1)} - X^* = \frac{1}{81}&\left(9(8(a-2)a+7)C_2^3 + 9C_3C_2 - C_4\right)\Big(2C_2^2(8r-27) \\
&+ C_3\Big)(e^{(k)})^6 + O\left((e^{(k)})^7\right).
\end{aligned}
\tag{25}
$$

This completes the proof of Theorem 2. □

Finally, using the conditions of the Theorem 2, we obtain the sixth-order scheme as follows:

$$
\begin{aligned}
Y^{(k)} &= X^{(k)} - \frac{2}{3}\Gamma(X^{(k)}), \\
Z^{(k)} &= X^{(k)} - \eta(X^{(k)})^{-1}\mu(X^{(k)})\Gamma(X^{(k)}), \\
X^{k+1} &= Z^{(k)} - \left((r + \frac{5}{2})I + \Omega(X^{(k)})\left((-\frac{3}{2} - 2r)I \right.\right. \\
&\left.\left. + \; r\Omega(X^{(k)})\right)\right)\{G'(X^{(k)})\}^{-1}G(Z^{(k)}),
\end{aligned}
\tag{26}
$$

where $a, r \in \mathbb{R}$.

## 4 Stability analysis of class $PM_4$

From the early works of Fatou and Julia [21, 22] at the end of the nineteenth century about the qualitative performance of Newton's iterative method on quadratic polynomials (by means of the analysis of its associate rational function), discrete dynamics has shown to be a powerful tool for the study of the stability of iterative schemes for solving nonlinear problems. Nowadays, when a class of iterative methods depending on one or more parameters is designed, the analysis of its performance on quadratic or cubic polynomials allows us to select the most stable members of the class (see, for example, [23–30]).

These elements are also numerically checked in order to test their performance on other functions. Although there does not exist any result showing that these behaviors will be held, it is shown in practice that an unstable iterative method on low-degree polynomials will be also unstable for other nonlinear functions. Also, when an iterative scheme shows a clear stable performance (as Newton's method does) then this good behavior will remain, up to some extent, when it is applied on more complicated functions.

The most used qualitative techniques are those of real and complex discrete dynamics, when the iterative processes under analysis are scalar. That is, when the methods are designed for solving nonlinear equations. Nevertheless, when the scheme or family of iterative methods are constructed for solving systems of nonlinear equations, the multidimensional real dynamics is needed to study its stability. In these cases, a system of second-degree polynomials is used and the resulting rational function is also multidimensional.

Let us also remark that, when the iterative scheme uses the evaluation of Jacobian matrices on the previous iterate, the Scaling Theorem is satisfied (see, for example, [31, 32]) and then the dynamics related to the rational function are equivalent, under conjugation, to that resulting from using any order second-degree polynomial system. Therefore, we work with the simplest nonlinear quadratic system.

In what follows, we use real multidimensional discrete dynamics tools to determine which elements of proposed classes of iterative methods $PM_4$ possess better performance, in terms of the dependence of their convergence on the initial estimations used. To get this aim, let us recall some concepts.

Let us denote by $R(x)$ the vectorial fixed-point rational function associated to an iterative method (or a parametric family of schemes) applied to $n$-variable polynomial system $p(x) = 0$, $p : \mathbb{R}^n \to \mathbb{R}^n$. Let us also remark that the majority of the definitions can be directly extended from those in complex dynamics (see, for example, [31, 33] for more explanations).

In the following, we denote by $M^4(x, a) = (m_1^4(x, a), m_2^4(x, a), \ldots, m_n^4(x, a))$, the fixed point function associated to (2) class, applied on a system of quadratic polynomials of size $n$, $p(x) = 0$, where:

$$p_i(x) = x_i^2 - 1, \; i = 1, 2, \ldots, n. \tag{27}$$

Let us remark that our system has separated variables, so all the coordinate functions $m_j^4(x, a)$ are similar, with the difference of the index $j = 1, 2, \ldots, n$. The expressions of these coordinate functions are:

$$\begin{aligned} &m_j^4(x, a) \\ &= \frac{1 - 6x_j^2 - 6x_j^4 - 6x_j^6 + x_j^8 + 2a^3 p_j(x)^3 \left(1 + x_j^2\right) - a^2 p_j(x)^3 \left(5 + 3x_j^2\right) - 2a\left(2 - 7x_j^2 + x_j^4 + 3x_j^6 + x_j^8\right)}{-8a^2 x_j p_j(x)^3 + 4a^3 x_j p_j(x)^3 - 8\left(x_j^3 + x_j^5\right) - 4a\left(x_j - 5x_j^3 + 3x_j^5 + x_j^7\right)}, \end{aligned} \tag{28}$$

for $j = 1, 2, \ldots, n$.

Let us remark that the components of this rational function can be highly simplified for specific values of parameter $a$; in particular, for $a = 1$, the resulting components are:

$$m_j^4(x, 1) = \frac{1 + 6x_j^2 + x_j^4}{4\left(x_j + x_j^3\right)}, \tag{29}$$

for $j = 1, 2, \ldots, n$.

A summary of the stability study of the fixed points of $M^4(x, a)$ appears in the following result.

**Theorem 3** *Rational function $M^4(x, a)$ associated to the family of iterative methods (2) has $2^n$ super attracting fixed points whose components are roots of $p(x)$. It also has some real fixed points different from the roots whose components are found combining the roots of polynomial $l(t) = 1 - 4a + 5a^2 - 2a^3 + (-5 + 14a - 15a^2 + 6a^3)t^2 + (-3 - 8a + 15a^2 - 6a^3)t^4 + (-1 - 2a - 5a^2 + 2a^3)t^6$, depending on a, and the roots of $p(x)$:*

- *(a) If $a < \frac{1}{2}$, there exist two real roots of $l(t)$, denoted by $l_i(a)$, $i = 1, 2$, whose respective eigenvalues are greater than 1 (in absolute value). So, the strange fixed point expressed as $(l_{\sigma_1}(a), l_{\sigma_2}(a), \ldots, l_{\sigma_n}(a))$ being $\sigma_i \in \{1, 2\}$, are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) is equal to 1 or $-1$, it will be a saddle fixed point.*
- *(b) If $\frac{1}{2} \leq a < c^*$, being $c^* \approx 2.90369$ the biggest real root of polynomial $-1 - 2t - 5t^2 + 2t^3$, then the roots of polynomial $l(t)$ are complex and there not exist any real strange fixed point.*
- *(c) If $a > c^*$, then $l_1(\alpha)$ and $l_2(a)$ are real. Moreover, the strange fixed point $(l_{\sigma_1}(a), l_{\sigma_2}(a), \ldots, l_{\sigma_n}(a))$ being $\sigma_i \in \{1, 2\}$, are attracting if $c^* < a < c^{**} \approx 3.864355$ and repulsive if $a > c^{**}$ ($c^{**}$ is the greatest real root of polynomial $-167 + 958t - 2031t^2 + 2666t^3 - 3716t^4 + 4152t^5 - 2224t^6 + 352t^7$). Indeed, if at least one of the components of the strange fixed point (but not all) are equal to $\pm 1$, it will be a saddle fixed point for $a > 3.864355$ and attracting in other cases.*

*Proof* Fixed points of $M^4(x, a)$ can be obtained by solving $m_j^4(x, a) = x_j, : -(x_j^2 - 1)(1 - 4a + 5a^2 - 2a^3 + (-5 + 14a - 15a^2 + 6a^3)x_j^2 + (-3 - 8a + 15a^2 - 6a^3)x_j^4 + (-1 - 2a - 5a^2 + 2a^3)x_j^6) = 0, \quad j = 1, 2, \ldots, n$, that is, the components of the fixed points are $x_j = \pm 1$ and also the roots of the polynomial $l(t)$, provided that $t \neq 0$.

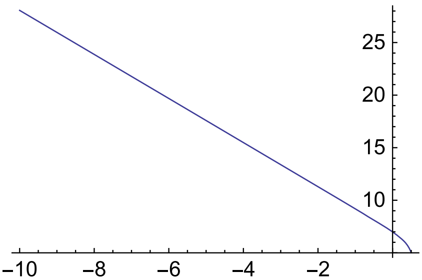
Let us denote the roots of $l(t)$ as $l_i(a)$, $i = 1, 2, \ldots, 6$. It can be checked that at most two of the roots of $l(t)$ are real, depending on the value of parameter $a$. The stability of the fixed points of $M^4(x, a)$ is given by the absolute value of the eigenvalues of the associated Jacobian matrix evaluated at the fixed points. Due to the nature of the polynomial system, these eigenvalues coincide with the coordinate function of the rational operator:

$$Eig_j(l_j(a),\ldots,l_j(a)) = (\mu_j(a))^3\left[\frac{2a^6(\mu_j(a))^4-a^5(\mu_j(a))^3\alpha_1(a)+2a^4(\mu_j(a))^2\alpha_2(a)}{4l_j(a)^2\left(2a^2(\mu_j(a))^3-a^3(\mu_j(a))^3+2\beta_1(a)+a\beta_2(a)\right)^2}\right.$$
$$\left.+\frac{-2l_j(a)^2\alpha_3(a)+2a^2\alpha_4(a)-a\alpha_5(a)+2a^3\alpha_6(a)}{4l_j(a)^2\left(2a^2(\mu_j(a))^3-a^3(\mu_j(a))^3+2\beta_1(a)+a\beta_2(a)\right)^2}\right],$$

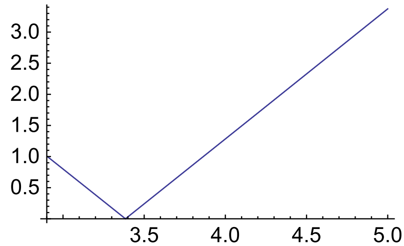
being $\mu_j(a) = -1+l_j(a)^2$, $\alpha_1(a) = -9+7l_j(a)^2$, $\alpha_2(a) = 8-17l_j(a)^2+l_j(a)^4$, $\alpha_3(a) = 3+8l_j(a)^2+3l_j(a)^4$, $\alpha_4(a) = 3-32l_j(a)^2-5l_j(a)^4+18l_j(a)^6$, $\alpha_5(a) = 1-30l_j(a)^2-42l_j(a)^4+6l_j(a)^6+l_j(a)^8$, $\alpha_6(a) = -7+40l_j(a)^2-33l_j(a)^4-4l_j(a)^6+4l_j(a)^8$, $\beta_1(a) = l_j(a)^2+l_j(a)^4$ and $\beta_2(a) = 1-5l_j(a)^2+3l_j(a)^4+l_j(a)^6$.

By taking into account the absolute value of these eigenvalues in the intervals where different fixed points are real, we state that those fixed points whose components are $\pm 1$ are superattracting. Roots of $l(t)$ are real for $a > c^*$, being $c^* \approx 2.90369$ the biggest real root of polynomial $-1-2t-5t^2+2t^3$. Then, it can be checked that the respective eigenvalues are bigger than 1 in absolute value, except in case $c^* < a < c^{**} \approx 3.864355$, as $c^{**}$ is the greatest real root of polynomial $-167+958t-2031t^2+2666t^3-3716t^4+4152t^5-2224t^6+352t^7$, where they are lower than 1. Therefore, combinations among the roots of $l(t)$ can give rise to attracting or repulsive strange fixed points depending on the value of $a$. Moreover, all the fixed points whose components are $\pm 1$ and real $l_j(\alpha)$ are classified as saddle. In Fig. 1, some of the eigenvalues are plotted; if $a < \frac{1}{2}$, the eigenvalues of the Jacobian matrix evaluated at all the real strange fixed points have the same performance (see Fig. 1a), being higher than 1 (so, points are repulsive). Moreover, for $a > c^*$, the possibility of attracting strange fixed point is deduced (see Fig. 1b), as the respective eigenvalues are lower than 1 when $c^* < a < c^{**} \approx 3.864355$. □

Once the existence of strange fixed points is studied and their stability is determined, it is necessary to analyze if it is possible to get any other attracting behavior, as attracting periodic orbits or even strange attractors. This can be made through the orbits of the free critical points, if they exist.




(a) $Eig_j(l_1(a),\ldots,l_1(a),a)$ for $a < \frac{1}{2}$




(b) $Eig_j(l_1(a),\ldots,l_1(a),a)$ for $a > c^*$

**Fig. 1** Eigenvalues associated to the fixed points

### 4.1 Bifurcation Analysis of Free Critical Points

Now, we analyze the Jacobian matrix $M^{4'}(x, a)$ of the rational function under analysis and its critical points. Let us recall that, in this context, the critical points are the solutions of $\det(M^{4'}(x, a)) = 0$. When the critical point is not a solution of $p(x) = 0$, it is called a free critical point.

**Theorem 4** *The free critical points of operator $M^4(x, a)$, denoted by $(cr_{\sigma_1}(a), cr_{\sigma_n}(a), \ldots, cr_{\sigma_n}(a))$, $\sigma_i \in \{1, 2, \ldots, m\}$ $m \leq 6$, are those making zero all the components of the Jacobian matrix, for $j = 1, 2, \ldots, n$, whose components are different from those of the roots of $p(x)$, that is:*

(a) *If $a \leq k^* \approx -0.744644$, $1 - \sqrt{2} < a < 0$, $\frac{1}{4}\left(4 - \sqrt{2}\right) < a < 0.710821$, $1.18627 < a < \frac{1}{4}\left(4 + \sqrt{2}\right)$, $\frac{1}{4}\left(4 + \sqrt{2}\right) < a < 1.88923$ or $a > 1 + \sqrt{2}$ (where $k^*$ is the lowest root of polynomial $1 - 2t - 3t^2 + 2t^3$) then $cr_1(a) = -\sqrt{s^*}$, $cr_2(a) = \sqrt{s^*}$, $cr_3(a) = -\sqrt{s^{**}}$, $cr_4(a) = \sqrt{s^{**}}$ are the different components of the free critical points, being $s^*$ and $s^{**}$ the two positive roots of polynomial $z(s) = -a + 6a^2 - 14a^3 + 16a^4 - 9a^5 + 2a^6 + (-6 + 30a - 64a^2 + 80a^3 - 66a^4 + 34a^5 - 8a^6)s + (-16 + 42a - 10a^2 - 66a^3 + 86a^4 - 48a^5 + 12a^6)s^2 + (-6 - 6a + 36a^2 - 8a^3 - 38a^4 + 30a^5 - 8a^6)s^3 + (-a + 8a^3 + 2a^4 - 7a^5 + 2a^6)s^4$.*

(b) *If $k^* < a < -0.578202$, $-0.464045 < a < -0.429149$, $-0.429149 < a < 1 - \sqrt{2}$ or $\frac{1}{2} < a < \frac{1}{4}\left(4 - \sqrt{2}\right)$, then $cr_1(a)$, $cr_2(a)$, $cr_3(a)$, $cr_4(a)$, $cr_5(a) = -\sqrt{s^{***}}$ and $cr_6(a) = \sqrt{s^{***}}$ are the different components of the free critical points, being $s^*$, $s^{**}$ and $s^{***}$ the three positive roots of polynomial $z(s)$ in this interval.*

(c) *If $a = -0.578202$ or $a = -0.464045$, the free critical points have as components $cr_1(a)$, $cr_3(a)$, $cr_4(a)$ and $cr_5(a)$.*

(d) *If $-0.578202 < a < -0.464045$ or $0.355416 < a < \dfrac{1}{2}$ or $1.88923 < a < 1 + \sqrt{2}$, then $cr_1(a)$ and $cr_2(a)$ are the only components of the free critical points.*

(e) *If $a = -0.429149$, then $cr_1(a)$, $cr_3(a)$, $cr_4(a)$, and $cr_6(a)$ are the different components of the free critical points.*

(f) *If $0 \leq a \leq 0.355416$ or $0.710821 < a < 1.18627$, then there are no free critical points.*

(g) *If $a = \dfrac{1}{2}$, then the components of the free critical points are $\pm - 2.54246$, the only real roots of polynomial $-3 - 6t^2 + t^4$.*

(h) *If $a = \frac{1}{4}\left(4 - \sqrt{2}\right)$ or $a = \frac{1}{4}\left(4 + \sqrt{2}\right)$, then the components of the free critical points are $cr_2(a)$ and $cr_3(a)$.*

(i) *If $a = 0.710821$ or $a = 1.18627$, then the components of the free critical points are $cr_1(a)$ and $cr_3(a)$.*

*Proof* The components of the Jacobian matrix $M^{4'}(x, a)$ different from zero are:

$$\frac{\partial m_j^4(x,a)}{\partial x_j} = (p_j(x))^3 \left[ \frac{2a^6(p_j(x))^4 - a^5(p_j(x))^3\left(-9+7x_j^2\right) + 2a^4(p_j(x))^2\left(8-17x_j^2+x_j^4\right)}{4x_j^2\left(2a^2(p_j(x))^3 - a^3(p_j(x))^3 + 2\left(x_j^2+x_j^4\right) + a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2} \right.$$
$$+ \frac{-2x_j^2\left(3+8x_j^2+3x_j^4\right) + 2a^2\left(3-32x_j^2-5x_j^4+18x_j^6\right)}{4x_j^2\left(2a^2(p_j(x))^3 - a^3(p_j(x))^3 + 2\left(x_j^2+x_j^4\right) + a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2}$$
$$\left. + \frac{-a\left(1-30x_j^2-42x_j^4+6x_j^6+x_j^8\right) + 2a^3\left(-7+40x_j^2-33x_j^4-4x_j^6+4x_j^8\right)}{4x_j^2\left(2a^2(p_j(x))^3 - a^3(p_j(x))^3 + 2\left(x_j^2+x_j^4\right) + a\left(1-5x_j^2+3x_j^4+x_j^6\right)\right)^2} \right], \quad j = 1, 2.$$

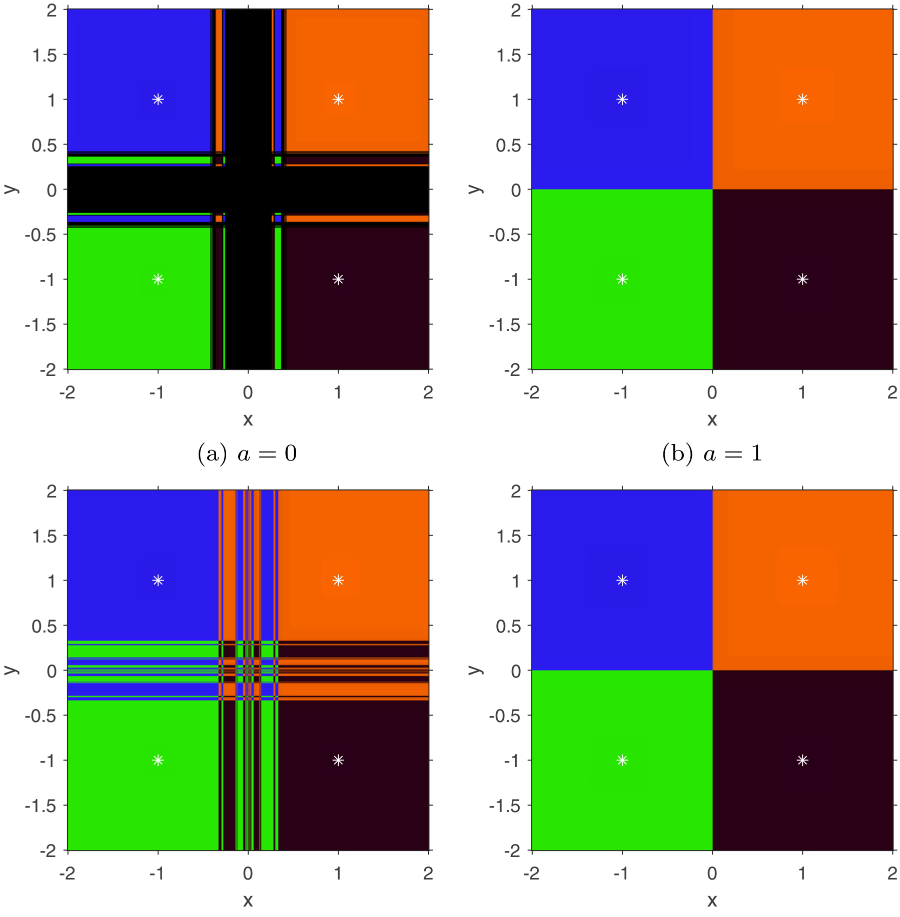
Therefore, it is straightforward that the roots of $-a + 6a^2 - 14a^3 + 16a^4 - 9a^5 + 2a^6 + (-6 + 30a - 64a^2 + 80a^3 - 66a^4 + 34a^5 - 8a^6)t^2 + (-16 + 42a - 10a^2 - 66a^3 + 86a^4 - 48a^5 + 12a^6)t^4 + (-6 - 6a + 36a^2 - 8a^3 - 38a^4 + 30a^5 - 8a^6)t^6 + (-a + 8a^3 + 2a^4 - 7a^5 + 2a^6)t^8$, when they are real, are the components of the critical points. A simple change $s = t^2$ yields to solve $z(s)$ in order to define the free critical points. $\square$ $\square$

Let us remark that the case $a = 1$, where rational function $M^4(x, a)$ is simplified, has no free critical points. The importance of this knowledge is in a classical result from Julia and Fatou (see, for example, [34]), stating that in the immediate basin of attraction of any attracting point (fixed or periodic) there exists at least one critical point. So, the existence of these free critical points states the possibility of another attracting behavior different from that of the roots and their absence means that no other behavior is possible other than convergence to the roots. We can check this performance by plotting the dynamical planes of $M^4(x, a)$ for different values of $a$ in these areas where free critical points do not exist.

Pictures in Fig. 2 have been obtained using the routines appearing in [23] in the following way: it has been used a mesh with $400 \times 400$ points, with 80 as the maximum number of iterations, being the tolerance of the stopping criterium $10^{-3}$. We have painted a point with different colors depending on where it converges to. This color is brighter when the number of iterations used is lower; moreover, it is colored in black if it reaches the maximum number of iterations without converging to any of the roots.

Figure 2a and c correspond to values of $a$ in the interval $0 \leq a \leq 0.355416$; it can be observed that the basins of attraction of the roots have an infinite number of connected components and in case of $a = 0$, there is a basin of attraction of infinity (in black in the figure), where all the initial estimations diverge. On the contrary, in Fig. 2b and d there exist only one connected component of each basin of attraction and there are no divergent behaviors; they correspond to values of $a$ in the interval $0.710821 < a < 1.18627$. They have a performance as stable as Newton's method but with fourth order of convergence.

Some examples of unstable behavior can be observed in Fig. 3: for $a = 3$ where many attracting strange fixed points appear (see Fig. 3a), combining in pairs $\{\pm 1, \pm 5.076757\}$. These points are marked with white stars and all of them have their own basin of attraction, although not all of them have a color assigned. In Fig. 3b,



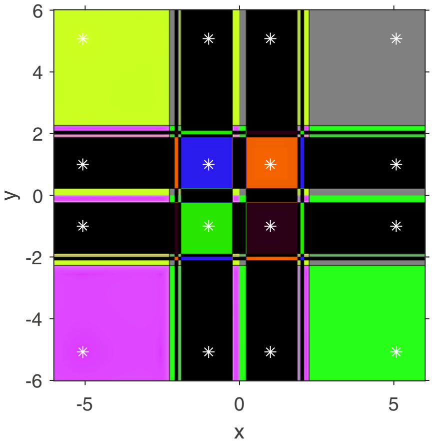

(a) $a = 0$ (b) $a = 1$ (c) $a = 0.2$ (d) $a = 0.8$

**Fig. 2** Stable dynamical planes of $M^4(x, a)$

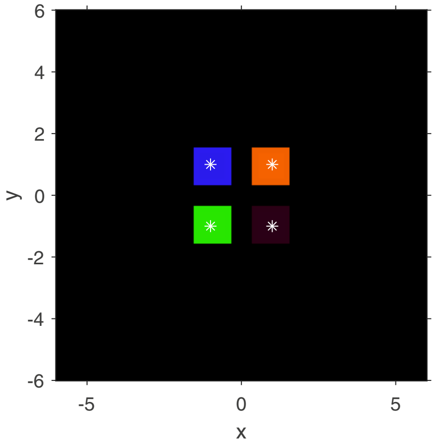
corresponding to $a = 3.864355$, the pairs obtained combining $\pm 1.86676$ are non-hyperbolic strange fixed points (the associate Jacobian matrix have eigenvalues equal to one) and they can behave as attracting points. So, the wideness of the basins of attraction of the roots is quite reduced.

## 5 Stability analysis of class $PM_6$

In order to analyze the performance of class $PM_6$ on the polynomial system $p(x) = 0$, we fix the value $a = 1$ that has shown to be the best in the study of the stability of class $PM_4$ (although close values of $a$ in $]0.710821, 1.18627[$ would lead to very similar results). Therefore, the third step of $PM_6$ depends only on parameter $r$. Let us




(a) $a = 3$




(b) $a = 3.864355$

**Fig. 3** Unstable dynamical planes of $M^4(x, a)$

denote by $M^6(x, r) = (m_1^6(x, r), m_2^6(x, r), \ldots, m_n^6(x, r))$, the fixed point function associated to (19) class on $p(x)$. Coordinate functions $m_j^6(x, r)$ can be expressed as:

$$m_j^6(x, r) = \frac{-2r\left(-1 + x_j^2\right)^5 + 3\left(1 - 7x_j^2 + 34x_j^4 + 90x_j^6 + 125x_j^8 + 13x_j^{10}\right)}{192x_j^5\left(1 + x_j^2\right)^2}, \tag{30}$$

for $j = 1, 2, \ldots, n$.

Let us remark that the components of this rational function can be highly simplified for specific values of parameter $r$; in particular, for $r = \frac{39}{2}$, the resulting components are:

$$m_j^6\left(x, \frac{39}{2}\right) = \frac{7 - 36x_j^2 + 82x_j^4 - 20x_j^6 + 95x_j^8}{32x_j^5\left(1 + x_j^2\right)^2}, \tag{31}$$

for $j = 1, 2, \ldots, n$.

In the following result, we summarize the information about the stability analysis of the fixed points of $M^6(x, r)$. The proof is similar to that of Theorem 3, so it is omitted.

**Theorem 5** *Rational function $M^6(x, r)$ associated to the family of iterative methods (19) has $2^n$ superattracting fixed points whose components are roots of $p(x)$. It has also several real strange fixed points whose components are found combining the roots of polynomial:*

$$h(t) = 3 + 2r + (-18 - 8r)t^2 + (84 + 12r)t^4 + (162 - 8r)t^6 + (153 + 2r)t^8,$$

*depending on $r$, and the roots of $p(x)$:*

*(a) If $r < -\frac{153}{2}$, four roots of $h(t)$, denoted by $h_i(r)$, $i = 1, 2, 3, 4$ are real, being their associate eigenvalues greater than 1 ( in absolute value). So, the strange*

*fixed point expressed as* $(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r))$ *being* $\sigma_i \in \{1, 2, 3, 4\}$*, are repulsive. Moreover, if at least one of the components of the strange fixed point (but not all) are equal to 1 or* $-1$*, it will be a saddle fixed point.*

*(b)* *If* $-\frac{153}{2} \leq r < -\dfrac{3}{2}$*, only* $h_1(r)$ *and* $h_2(r)$ *are real. The absolute value of the eigenvalues of their associate Jacobian matrix are all greater than 1. So, the strange fixed points expressed as* $(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r))$ *being* $\sigma_i \in \{1, 2\}$*, are repulsive. However, if at least one of the components of the strange fixed point (but not all) is equal to 1 or* $-1$*, it is a saddle fixed point.*

*(c)* *If* $r = -\dfrac{3}{2}$ *or* $r > \frac{3\left(-1237+41\sqrt{41}\right)}{2048} \approx -1.42745$*, then all the roots of polynomial* $h(t)$ *are complex and there does not exist any real strange fixed point.*

*(d)* *If* $-\frac{3}{2} < r < \frac{3\left(-1237+41\sqrt{41}\right)}{2048}$*, then* $l_i(r)$*,* $i = 1, 2, 3, 4$*, are real. Regarding the stability, strange fixed points:*

$$(h_{\sigma_1}(r), h_{\sigma_2}(r), \ldots, h_{\sigma_n}(r)) \text{ with } \sigma_i \in \{2, 3\}$$

*are attracting for* $r \in ]-1.427649, -1.427449[$*; in any other case, the strange fixed points are unstable repelling (if* $\sigma_i \in \{1, 4\}$*) or saddle.*

*(e)* *If* $r = \frac{3\left(-1237+41\sqrt{41}\right)}{2048}$*, then* $l_1(r)$ *and* $l_3(r)$ *are real and the strange fixed point* $(l_{\sigma_1}(r), l_{\sigma_2}(r), \ldots, l_{\sigma_n}(r))$ *being* $\sigma_i \in \{1, 3\}$ *are parabolic.*

From this result, it can be concluded that $r = -\dfrac{3}{2}$ or $r > \dfrac{3\left(-1237+41\sqrt{41}\right)}{2048} \approx -1.42745$ are the best areas for avoiding the existence of strange fixed points; however, the only area with attracting strange fixed points is extremely small, $]-1.427649, -1.427449[$. So, the existence of attracting periodic orbits would be the only undesirable behavior to avoid in practice. To get this aim, we study $M^{6'}(x, r)$ and its critical points.

**Theorem 6** *There exist* $2^n$ *real free critical points of* $M^6(x, r)$*, only if* $-\frac{3}{2} \leq r < \frac{39}{2}$*. In this case, their only components are* $\pm\sqrt{\frac{12+14r}{39-2r} + \sqrt{\frac{729+696r+176r^2}{(39-2r)^2}}}$*.*

*Proof* The eigenvalues of the Jacobian matrix $M^{6'}(x, r)$ are:

$$Eig_j(x, r) = \frac{\left(-1 + x_j^2\right)^4 \left(15 + 24x_j^2 - 39x_j^4 + 2r\left(5 + 14x_j^2 + x_j^4\right)\right)}{192x_j^6\left(1 + x_j^2\right)^3}, \quad j = 1, 2.$$

Then, to get the free critical points, it is only necessary to find the real roots of $15 + 10r + (24 + 28r)t + (-39 + 2r)t^2$, being $t = x_j^2$. $\square$

As there exists at least one critical point in each basin of attraction (see [34]), the orbits of these points give us relevant information about the stability of the rational function and, therefore, of the iterative method involved. We present in Fig. 4 the real parametric lines showing these orbits (see Theorem 6) for $n = 2$. In each one of these pictures, a different free critical point is used as starting point of each member of the family of iterative schemes, using $-\frac{3}{2} \leq r < \frac{39}{2}$ to ensure the existence of real critical points.

To plot these parameter lines, a mesh of $500 \times 500$ points is made in the region $[0, 1]\times] - \frac{3}{2}, \frac{39}{2}[$. We use $[0, 1]$ to fatten the interval where $r$ is defined, allowing a better visualization. So, the color corresponding to each value of $r$ is red if the corresponding critical point converges to one of the roots of the polynomial system, blue in case of divergence and black in other cases. This color is also assigned to all the values of $[0, 1]$ with the same value of the parameter. The maximum number of iterations used is 200 and the tolerance for the error estimation is $10^{-3}$, when the iterates tend to a fixed point.

All the pairs of free critical points have the same global performance, so only the case $(cr_1(r), cr_1(r))$ is presented in Fig. 4. In it, the parameter line is plotted for $-\dfrac{3}{2} < r < \dfrac{39}{2}$ as outside this interval all the components of the free critical points are complex. It is observed that there is convergence to the roots (red color) for almost all values of $r$, except a black small region around 18.5.

Bifurcation or Feigenbaum diagrams has been used in order to study the changes of each class of methods on $p(x)$ by using each of the free critical points of the rational function as a initial guess and noticing their performance for different ranges of the parameter. Therefore, a different behavior can be observed after 1000 iterations by using a mesh with 3000 subintervals.

Figure 5 corresponds to the bifurcation diagrams in the black area of the parameter line for $-\dfrac{3}{2} < r < \dfrac{39}{2}$ (see Fig. 4). In Fig. 5a and c, it can be observed as there is a general convergence to one of the roots, but in a small interval around $r = 18.5$ a period-doubling cascade appears, including not only periodic but also chaotic behavior (blue regions). In them, there can be found strange attractors. To represent it, we plot in Fig. 6 the $(x_1, x_2)$-space the orbit of $x^{(0)} = (0.35, 0.35)$ by $M^6((x_1, x_2), r)$, for $r = 18.659$, laying in the blue region. For each $r$, the number of different starting estimations is 1000 and, for each one, we do not plot the first 500 iterations, meanwhile following 400 are represented in blue color and the resting 100 in magenta. We observe it in Fig. 6 as a neutral strange fixed point, after bifurcating in periodic orbits with increasing periods, falls in a chaotic behavior where orbits are dense in a small area of $(x_1, x_2)$ space.

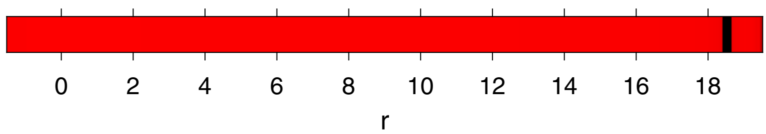
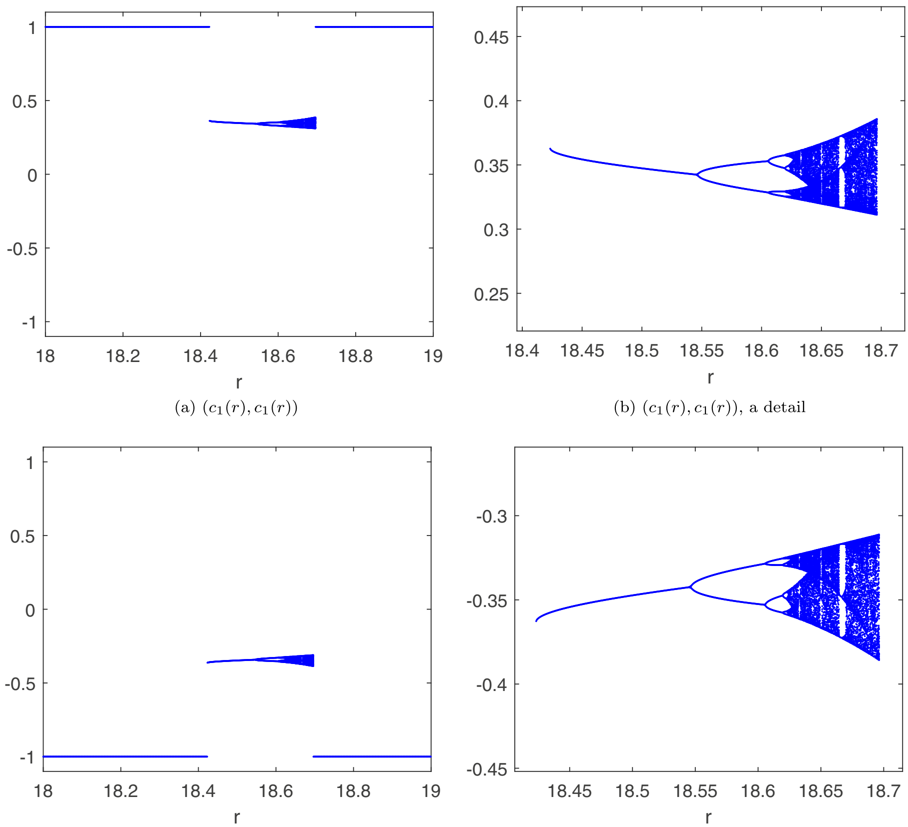



**Fig. 4** Parameter line of $M^6(x, r)$ for $-\dfrac{3}{2} < r < \dfrac{39}{2}$



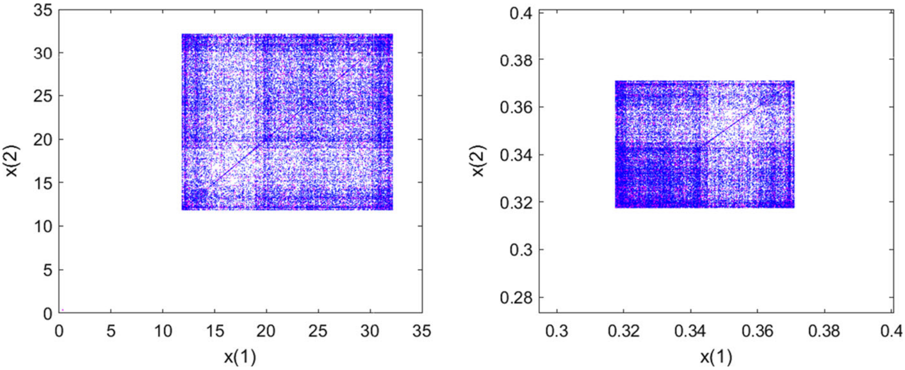

(a) $(c_1(r), c_1(r))$

(b) $(c_1(r), c_1(r))$, a detail

(c) $(c_2(r), c_2(r))$

(d) $(c_2(r), c_2(r))$, a detail

**Fig. 5** Feigenbaum diagrams of $M^6(x, r)$ for $-\dfrac{3}{2} < r < \dfrac{39}{2}$ from different critical points



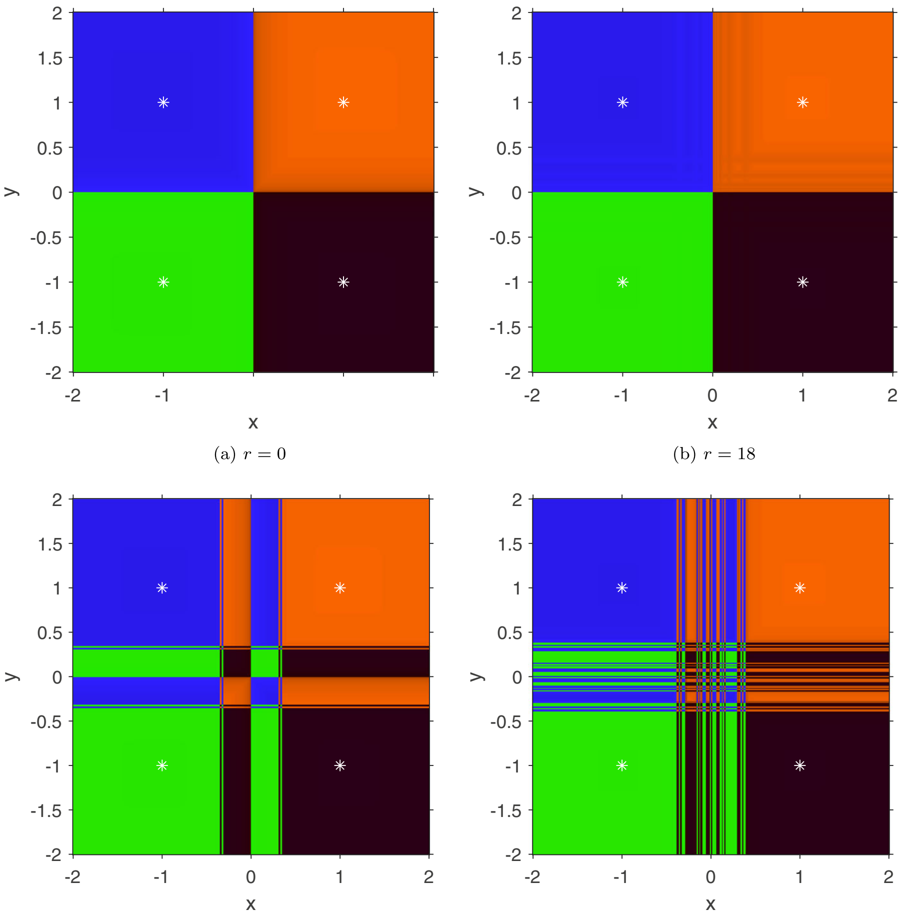

(a) $r = 18.659$

(b) $r = 18.659$, a detail

**Fig. 6** Strange attractors of $M^6(x, r)$ for $r$ in blue doubling-period cascade

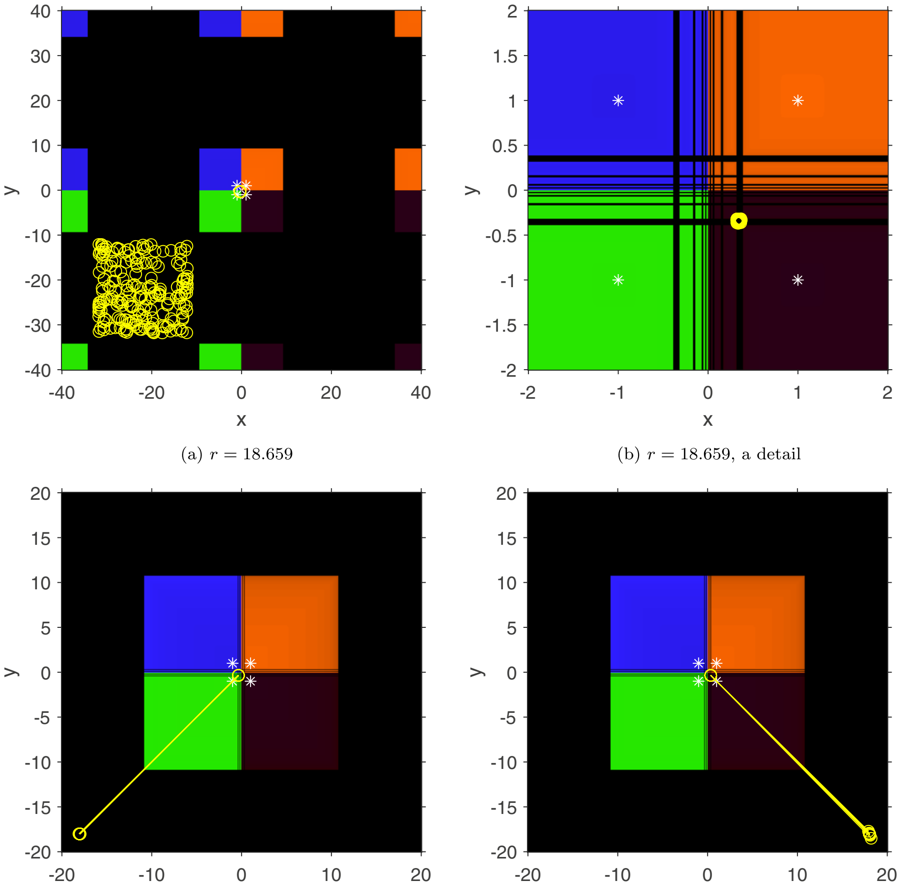
This unstable performance, as well as a stable one, can be checked by plotting the associated dynamical planes, where the value of the parameter is fixed and a mesh of initial estimations is used to see the performance of the methods. The first dynamical planes correspond to stable behavior, that is, there exists only convergence to the roots (see Fig. 7). For values of $r$ in $]-\frac{3}{2}, \frac{39}{2}[$, where there are free critical points, there is only one connected component per root in Fig. 7a and b if values of red color in the parameter line ( shown in Fig. 4) are selected. Outside this interval, there are no free critical point and the performance is also stable, but there exist infinite connected components of each basin of attraction (Fig. 7c and d).

Regarding the unstable behavior, it is limited to values of $r$ in the black region of the parameter line (Fig. 4). In Fig. 8a and b, two parts of the strange attractor found for $r = 18.659$ can be observed, that was plotted in Fig. 6; this dynamical plane has been




(a) $r = 0$ (b) $r = 18$ (c) $r = -2$ (d) $r = 20$

**Fig. 7** Stable dynamical planes of M6 on $p(x)$




(a) $r = 18.659$ (b) $r = 18.659$, a detail

(c) $r = 18.5$ (d) $r = 18.5$

**Fig. 8** Unstable dynamical planes of M4 on $p(x)$

obtained by avoiding the lines in the orbit of the initial estimation selected in the black area and plotting by yellow circles the elements of the orbit (this time 400 iterations have been used). Finally, in Fig. 8c and d, the phase space for $r = 18.5$ is represented. In them, the 2-period orbits $\{(-0.3473, -0.3473), (-18.0176, -18.0176)\}$ and $\{(0.3473, -0.3473), (18.0176, -18.0176)\}$ appear in yellow. In this case, two more attracting orbits exist, with symmetric coordinates.

In general, it can be concluded that the main performance of the members of M4 and M6 classes of iterative methods on this kind of polynomial systems is very stable. There are no attracting strange fixed points and the only attracting behavior different from the roots lies in a very small interval of $r$. These conclusions are numerically checked in the following numerical section.

## 6 Numerical results and discussions

In this section, some numerical problems are considered to demonstrate the convergence behavior and computational efficiency of the proposed methods. The proposed schemes (2) namely $PM_4^1$, $PM_4^2$, $PM_4^3$ for $a=\frac{1}{2}$, $a=\frac{5}{8}$, and $a=\frac{8}{10}$ respectively are considered and compared with existing fourth-order techniques namely $HM_4$, $JM_4$, $MM_4^1$, introduced by Sharma and Arora's method [35], Jarratt's method [36], and Narang's method [14], respectively. For $a=1$, Jarratt's method is the special case of proposed scheme (2). The proposed scheme (26) for $a=\frac{5}{8}, r=10$, for $a=\frac{1}{2}, r=\frac{1}{2}$, for $a=\frac{3}{2}, r=\frac{1}{4}$ and for $a=1, r=\frac{39}{2}$ are denoted as $PM_6^4$, $PM_6^5$, $PM_6^6$, and $PM_6^7$ respectively and compared with existing schemes of sixth order namely, $LM_6$, $MM_6^2$, $RM_6$ proposed by Lotfi et al. [15], Narang et al. [14], and Behl et al. [16], respectively. To verify the theoretical order of convergence, we have displayed the iterations $k$, $\|G(X^{(k)})\|$ and $||X^{(k+1)}-X^{(k)}||$, the approximation of the asymptotic error constant $\eta$, $e-time$ using TimeUsed[] command and computational order of convergence ($\rho$) using the following formulas, respectively:

$$\rho \approx \frac{\ln \frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|}}{\ln \frac{\|X^{(k)}-X^{(k-1)}\|}{\|X^{(k-1)}-X^{(k-2)}\|}}, \qquad \text{for each } k=2,\ 3,\ldots, \tag{32}$$

where $X^{(k-2)}$, $X^{(k-1)}$, $X^{(k)}$, and $X^{(k+1)}$ are four consecutive approximations in the iteration process;

$$\eta = \lim_{k\to\infty} \frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}, \ where\ (\rho = 4\ or\ 6). \tag{33}$$

All numerical computations have been done on Mathematica 11 with variable precision arithmetic using mantissa of 2000 digits and stopping criterion $||X^{k+1}-X^{k}||+||G(X^{k})|| < 10^{-400}$ to minimize rounding errors and in all tables, $b(\pm c)$ denotes $b\times 10^{\pm c}$.

### 6.1 Application of nonlinear optimization

This section reports the applicability of the designed methods on nonlinear optimization problems. A nonlinear optimization problem is a combination of an objective function, either minimized or maximized over a set of constraints. We deal with the unconstrained nonlinear optimization problem which includes an objective function without constraints. In the literature [37, 38], most of the optimization problems are based on the concept of nonlinear least squares. The nonlinear least squares optimization problem without constraints whose objective function is defined as:

$$\underset{X}{min}\ F(X),\ where \tag{34}$$

$$F(X) := \sum_{i=1}^{m} f_i(X)^2, \quad X\in\mathbb{R}^n. \tag{35}$$

If $m = n$, then $f_i(X) = 0,\ i = 1, 2, \ldots, n$ forms a nonlinear system of equations and for $m > n$, the system of nonlinear equations is given by:

$$\sum_{i=1}^{m} \frac{\partial f_i(X)}{\partial X_j} f_i(X) = 0,\ \ j = 1, \ldots, n. \tag{36}$$

Firstly, we test the methods on $m > n$ nonlinear system of equations to fit the exponential function $y = x_1 e^{-\tilde{t}x_2}$ [39] using data points in Table 1.

The parameters $(x_1, x_2) \in \mathbb{R}^2$ are estimated with proposed techniques and existing methods. The results shown in Table 2 demonstrate that the proposed methods are better than existing ones in terms of order of convergence. The exact curve fitted by the points shown in Table 1 and the approximated curve using parameters $(x_1, x_2)$ have been plotted in Fig. 9. It is clear that the approximated curve is close to the exact curve. Also the residual surface $||F(X)||^2$, (where $||.||$ denotes the usual $L_2$ norm) and the approximated parameters $(x_1, x_2)$ represented in red color are depicted in Fig. 10. Moreover, the comparisons of the number of iterations and residual errors of existing and proposed schemes have been shown in Figs. 11 and 12, respectively.

Now, we apply the methods on CUTE test set (https://github.com/ralna/CUTEst) namely, Freudenstein and Roth function, and Broyden tridiagonal function. These are zero-residual nonlinear least squares problem with $m = n$ variables. The performance evaluation is depicted in Tables 3 and 4, which shows less functional error and residue error.

### 6.2 Academic Problems

*Example 1* Consider the following small system of nonlinear equations:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1, \\ 2x_1^2 + x_2^2 - 4x_3 &= 0, \\ 3x_1^2 - 4x_2^2 + x_3^2 &= 0. \end{aligned}$$

The convergence of this system towards the solution $X^* = (0.6982886\ldots, 0.6285243, \ldots, 0.3425642\ldots)^T$ is tested and shown in Tables 5 and 6. In both tables, proposed methods depict better performance in terms of residual error compared with earlier schemes.

*Example 2* One of the most famous physical problems of radiative transfer theory which is the Chandrasekhar integral equation (see [40, pp. 251–252]) is considered for applicability and effectiveness of proposed schemes compared with existing methods, as follows:

$$F(P, c) = 0, \quad P : [0, 1] \to \mathbb{R}$$

**Table 1** Data points

| | | | | | | |
|---|---|---|---|---|---|---|
| $\tilde{t}_j$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $y_j$ | 10 | 5.9 | 0.89 | −0.14 | −1.07 | 0.84 |

**Table 2** Convergence behavior of different methods using initial value $X^0 = (22, \frac{8}{10}^T)$

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $e-time$ |
|---|---|---|---|---|---|
| | 3 | 1.6(−7) | 3.7(−7) | | |
| $HM_4$ | 4 | 9.3(−15) | 1.1(−16) | | |
| | 2 | 1.1(−2) | 2.1(−2) | | |
| $JM_4$ | 3 | 4.7(−11) | 6.0(−11) | | |
| | 4 | 9.3(−15) | 1.1(−16) | 0.6703 | 0.328 |
| | 2 | 1.4(−2) | 3.6(−2) | | |
| $MM_4^1$ | 3 | 1.2(−10) | 6.5(−10) | | |
| | 4 | 2.2(−14) | 3.5(−15) | 0.679 | 0.345 |
| | 2 | 6.1(−3) | 7.0(−2) | | |
| $PM_4^1$ | 3 | 5.9(−9) | 2.0(−8) | | |
| | 4 | 4.1(−35) | 1.3(−34) | 3.998 | 0.256 |
| | 2 | 2.7(−3) | 2.6(−2) | | |
| $PM_4^2$ | 3 | 2.1(−11) | 1.5(−10) | | |
| | 4 | 2.9(−44) | 1.8(−43) | 3.998 | 0.263 |
| | 1 | 2.1 | 3.5 | | |
| $PM_4^3$ | 2 | 9.0(−3) | 9.4(−3) | | |
| | 3 | 1.2(−12) | 8.9(−13) | 3.906 | 0.212 |
| | 1 | 2.14705 | 3.46764 | | |
| $LM_6$ | 2 | 6.9(−3) | 2.0(−2) | | |
| | 3 | 1.3(−14) | 7.1(−15) | 4.620 | 0.444 |
| | 1 | 2.14705 | 3.48138 | | |
| $MM_6^2$ | 2 | 2.3(−3) | 7.2(−3) | | |
| | 3 | 4.8(−15) | 1.2(−15) | 5.598 | 0.404 |
| | 1 | 2.14705 | 3.4763 | | |
| $RM_6$ | 2 | 3.9(−3) | 1.2(−2) | | |
| | 3 | 2.2(−14) | 3.5(−15) | 5.115 | 0.501 |
| | 1 | 2.1 | 3.5 | | |
| $PM_6^4$ | 2 | 1.2(−3) | 3.3(−3) | | |
| | 3 | 1.2(−20) | 3.3(−20) | 5.613 | 0.375 |
| | 1 | 2.1 | 3.5 | | |
| $PM_6^5$ | 2 | 3.8(−3) | 1.3(−2) | | |
| | 3 | 6.0(−17) | 1.3(−16) | 5.771 | 0.430 |
| | 1 | 2.1 | 3.5 | | |
| $PM_6^6$ | 2 | 2.2(−3) | 8.1(−3) | | |
| | 3 | 3.5(−18) | 7.7(−18) | 5.696 | 0.401 |
| | 2 | 2.4(−2) | 9.0(−2) | | |
| $PM_6^7$ | 3 | 7.7(−9) | 1.7(−8) | | |
| | 4 | 4.0(−49) | 8.4(−49) | 5.984 | 0.397 |

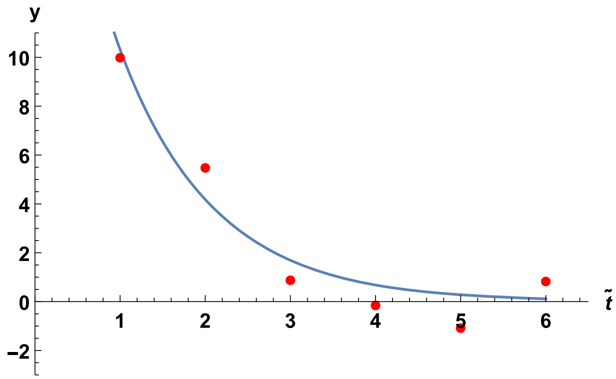



**Fig. 9** Exact curve fitting and approximate curve

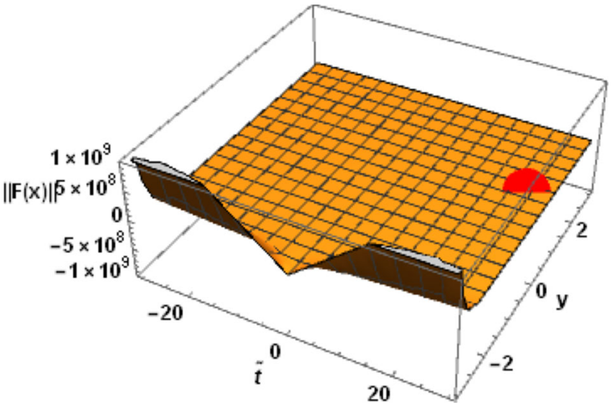



**Fig. 10** Residual graph

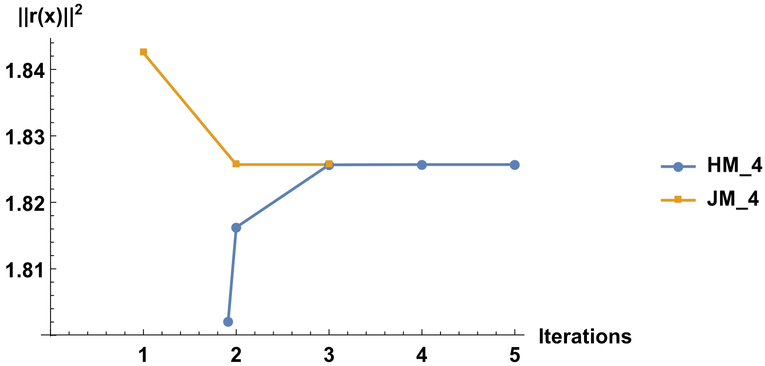



**Fig. 11** Iteration comparison of fourth-order schemes

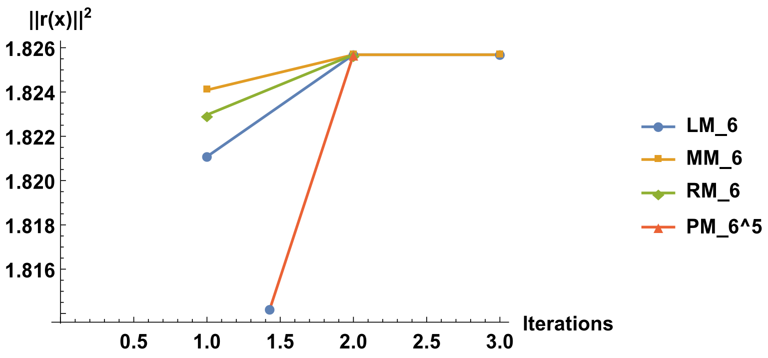



**Fig. 12** Iteration comparison of sixth-order schemes

with parameter $c$ and operation $F$ as:

$$F(P,c)(y) = P(y) - \left(1 - \frac{c}{2}\int_0^1 \frac{yP(v)}{y+v}dv\right)^{-1}.$$

The performance of the methods to estimate the solution $X^* = (1.021720\ldots, 1.073186\ldots, 1.125725\ldots, 1.169753\ldots, 1.203072\ldots, 1.226491\ldots, 1.241525\ldots, 1.249449\ldots)^T$ is checked in Tables 8 and 9.
Our methods present better computational time in Tables 8 and 9 in comparison with the existing schemes.

*Example 3* Consider the Frank-Kamenetskii problem [41] described by the following differential equation:

$$xy'' + y' + xe^y = 0, \;\; y'(0) = y(1) = 0. \tag{37}$$

We transform the Chandrasekhar equation in a nonlinear system by means of a Gauss-Legendre quadrature formula,
$\int_0^1 f(t)dt \simeq \sum_{j=1}^8 w_j f(t_j)$, where $t_j$ denotes the nodes and $w_j$ the weights of the Gauss-Legendre quadrature formula Table 7. Denoting the approximations of $P(t_i)$ by $y_i$ $(i = 1, 2, ..., 8)$, one gets the system of nonlinear equations:

$$y_i - \left(1 - \frac{c}{2}\sum_{j=1}^{8} a_{ij} y_j\right)^{-1} = 0, \;\; i = 1, 2, ..., 8,$$

where

$$a_{ij} = \frac{w_j \; t_i}{t_i + t_j}, \; and \; c = 0.5.$$

Nodes $t_j$ and weights $w_j$ are known and given in the following Table 7 for $t = 8$.

**Table 3** Convergence behavior of Freudenstein and Roth function using initial value $X^{(0)} = (6, 6, \ldots, 6)^T$

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $e-time$ |
|---|---|---|---|---|---|
| | 5 | 1.2(−51) | 2.4(−52) | | |
| $HM_4$ | 6 | 2.7(−212) | 5.5(−213) | | |
| | 7 | 6.9(−855) | 1.4(−855) | 4.000 | 2.688 |
| | 5 | 1.2(−87) | 1.7(−88) | | |
| $JM_4$ | 6 | 4.3(−357) | 6.0(−358) | | |
| | 7 | 6.3(−1435) | 8.8(−1436) | 4.000 | 2.452 |
| | 5 | 1.1(−87) | 3.8(−88) | | |
| $MM_4^1$ | 6 | 2.1(−357) | 7.4(−358) | | |
| | 7 | 3.0(−1436) | 1.0(−1436) | 4.000 | 2.609 |
| | 5 | 2.1(−83) | 5.5(−84) | | |
| $PM_4^1$ | 6 | 7.3(−340) | 1.9(−340) | | |
| | 7 | 1.0(−1365) | 2.7(−1366) | 4.000 | 2.155 |
| | 4 | 3.2(−37) | 1.2(−34) | | |
| $PM_4^2$ | 5 | 1.3(−155) | 4.9(−156) | | |
| | 6 | 4.0(−629) | 1.5(−629) | 4.000 | 2.094 |
| | 4 | 4.8(−23) | 4.1(−24) | | |
| $PM_4^3$ | 5 | 5.1(−99) | 4.4(−100) | | |
| | 6 | 6.9(−403) | 5.9(−404) | 4.000 | 2.032 |
| | 4 | 2.8(−39) | 5.8(−40) | | |
| $LM_6$ | 5 | 6.0(−245) | 1.2(−245) | | |
| | 6 | 5.4(−1479) | 1.1(−1479) | 6.000 | 2.922 |
| | 3 | 1.6(−7) | 3.3(−8) | | |
| $MM_6^2$ | 4 | 2.3(−55) | 4.7(−56) | | |
| | 5 | 2.1(−342) | 4.3(−343) | 5.999 | 2.312 |
| | 4 | 5.4(−44) | 1.1(−44) | | |
| $RM_6$ | 5 | 1.5(−273) | 3.1(−274) | | |
| | 6 | 8.1(−1651) | 1.6(−1651) | 6.000 | 2.031 |
| | 3 | 4.9(−16) | 1.1(−16) | | |
| $PM_6^4$ | 4 | 3.9(−106) | 8.5(−107) | | |
| | 5 | 8.7(−647) | 1.9(−647) | 6.000 | 1.906 |
| | 3 | 4.2(−7) | 8.5(−8) | | |
| $PM_6^5$ | 4 | 1.9(−52) | 3.8(−53) | | |
| | 5 | 1.5(−324) | 3.0(−325) | 6.000 | 1.863 |
| | 3 | 4.7(−13) | 9.5(−14) | | |
| $PM_6^6$ | 4 | 3.8(−88) | 7.8(−89) | | |
| | 5 | 1.1(−538) | 2.3(−539) | 6.000 | 1.892 |
| | 3 | 1.8(−8) | 3.8(−9) | | |
| $PM_6^7$ | 4 | 3.9(−60) | 8.4(−61) | | |
| | 5 | 4.2(−370) | 9.0(−371) | 6.003 | 1.767 |

**Table 4** Convergence behavior of Broyden tridiagonal function using initial guess $X^{(0)} = (-1, -1, \ldots, -1)^T$

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $e-time$ |
|---|---|---|---|---|---|
| | 4 | 4.6(−29) | 1.5(−29) | | |
| $HM_4$ | 5 | 4.7(−116) | 1.5(−116) | | |
| | 6 | 4.5(−464) | 1.4(−464) | 4.000 | 14.765 |
| | 4 | 2.8(−39) | 9.3(−40) | | |
| $JM_4$ | 5 | 1.4(−157) | 4.3(−158) | | |
| | 6 | 6.1(−631) | 1.9(−631) | 4.000 | 15.321 |
| | 4 | 1.2(−42) | 3.9(−43) | | |
| $MM_4^1$ | 5 | 6.7(−172) | 2.3(−172) | | |
| | 6 | 6.9(−689) | 2.4(−689) | 4.000 | 17.406 |
| | 4 | 6.6(−44) | 2.2(−44) | | |
| $PM_4^1$ | 5 | 4.0(−176) | 1.3(−176) | | |
| | 6 | 5.5(−705) | 1.8(−705) | 4.000 | 12.063 |
| | 4 | 5.9(−55) | 2.0(−55) | | |
| $PM_4^2$ | 5 | 3.0(−221) | 1.0(−221) | | |
| | 6 | 2.0(−886) | 6.8(−887) | 4.000 | 12.453 |
| | 4 | 1.0(−41) | 3.4(−42) | | |
| $PM_4^3$ | 5 | 1.6(−167) | 5.0(−168) | | |
| | 6 | 7.8(−671) | 2.4(−671) | 4.000 | 11.061 |
| | 4 | 1.3(−77) | 3.9(−78) | | |
| $LM_6$ | 5 | 2.7(−390) | 5.9(−391) | | |
| | 6 | 1.0(−1953) | 2.9(−1954) | 4.997 | 14.375 |
| | 3 | 1.7(−18) | 5.7(−19) | | |
| $MM_6^2$ | 4 | 8.4(−112) | 2.8(−112) | | |
| | 5 | 1.2(−671) | 3.8(−672) | 5.999 | 18.391 |
| | 3 | 1.9(−16) | 6.3(−17) | | |
| $RM_6$ | 4 | 2.1(−98) | 6.8(−99) | | |
| | 5 | 3.3(−590) | 1.0(−590) | 6.000 | 14.031 |
| | 3 | 9.7(−26) | 3.3(−26) | | |
| $PM_6^4$ | 4 | 4.0(−155) | 1.3(−155) | | |
| | 5 | 1.9(−931) | 6.4(−932) | 5.999 | 13.828 |
| | 3 | 4.8(−20) | 1.6(−20) | | |
| $PM_6^5$ | 4 | 2.3(−120) | 7.6(−121) | | |
| | 5 | 2.6(−722) | 8.3(−723) | 6.000 | 12.125 |
| | 3 | 4.1(−24) | 1.4(−24) | | |
| $PM_6^6$ | 4 | 9.0(−145) | 3.0(−145) | | |
| | 5 | 1.1(−868) | 0.1(−869) | 6.000 | 13.968 |
| | 3 | 3.9(−16) | 1.3(−16) | | |
| $PM_6^7$ | 4 | 3.8(−96) | 1.3(−96) | | |
| | 5 | 2.9(−576) | 9.3(−577) | 5.999 | 12.079 |

**Table 5** Convergence behavior of different methods using initial guess $X^{(0)} = (1, 1, 1)^T$ for Example 1

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 3.0(−24) | 6.0(−25) | | | | |
| $HM_4$ | 5 | 1.2(−96) | 2.3(−97) | | 1.81494 | | |
| | 6 | 2.6(−386) | 4.9(−387) | 4.000 | 1.83558 | 1.83558 | 0.0672 |
| | 4 | 2.9(−32) | 5.6(−33) | | | | |
| $JM_4$ | 5 | 1.9(−129) | 3.7(−130) | | 0.36709 | | |
| | 6 | 3.7(−518) | 7.1(−519) | 4.000 | 0.36711 | 0.36711 | 0.0648 |
| | 4 | 1.4(−39) | 3.1(−40) | | | | |
| $MM_4^1$ | 5 | 3.9(−159) | 7.3(−160) | | 0.07504 | | |
| | 6 | 1.4(−637) | 2.6(−638) | 3.999 | 0.09321 | 0.09321 | 0.0534 |
| | 4 | 1.2(−43) | 2.3(−44) | | | | |
| $PM_4^1$ | 5 | 5.4(−175) | 1.0(−175) | | 0.36708 | | |
| | 6 | 2.2(−700) | 4.3(−701) | 4.000 | 0.36712 | 0.36712 | 0.0456 |
| | 4 | 9.2(−41) | 1.8(−41) | | | | |
| $PM_4^2$ | 5 | 2.3(−164) | 4.4(−165) | | 0.04589 | | |
| | 6 | 8.9(−659) | 1.7(−659) | 4.000 | 0.04589 | 0.04589 | 0.0423 |
| | 4 | 6.5(−34) | 1.3(−34) | | | | |
| $PM_4^3$ | 5 | 3.2(−136) | 6.1(−137) | | 0.24964 | | |
| | 6 | 1.8(−545) | 3.5(−546) | 4.000 | 0.24964 | 0.24964 | 0.0488 |
| | 3 | 7.5(−13) | 1.6(−13) | | | | |
| $LM_6$ | 4 | 2.8(−76) | 5.4(−77) | | 3.38894 | | |
| | 5 | 4.8(−457) | 9.2(−458) | 5.999 | 3.57625 | 3.57625 | 0.0469 |
| | 3 | 4.4(−16) | 1.0(−16) | | | | |
| $MM_6^2$ | 4 | 9.4(−97) | 1.8(−97) | | 0.16095 | | |
| | 5 | 3.7(−581) | 7.1(−582) | 5.998 | 0.20913 | 0.20913 | 0.0437 |
| | 3 | 4.6(−14) | 1.1(−14) | | | | |
| $RM_6$ | 4 | 1.4(−83) | 2.7(−84) | | 1.93917 | | |
| | 5 | 5.5(−501) | 1.0(−501) | 5.998 | 2.43993 | 2.43993 | 0.0624 |
| | 3 | 6.1(−20) | 1.2(−20) | | | | |
| $PM_6^4$ | 4 | 3.8(−120) | 7.3(−121) | | 0.27710 | | |
| | 5 | 2.3(−721) | 4.3(−722) | 6.000 | 0.27710 | 0.27710 | 0.0321 |
| | 3 | 5.7(−20) | 1.1(−20) | | | | |
| $PM_6^5$ | 4 | 1.3(−119) | 2.5(−120) | | 1.62837 | | |
| | 5 | 2.2(−717) | 4.1(−718) | 6.000 | 1.62837 | 1.62837 | 0.0324 |
| | 3 | 1.6(−18) | 3.1(−19) | | | | |
| $PM_6^6$ | 4 | 5.0(−111) | 9.6(−112) | | 1.04568 | | |
| | 5 | 4.2(−666) | 8.0(−667) | 6.000 | 1.04568 | 1.04568 | 0.0374 |
| | 3 | 5.3(−15) | 1.0(−15) | | | | |
| $PM_6^7$ | 4 | 4.9(−89) | 9.3(−90) | | 9.11341 | | |
| | 5 | 3.1(−533) | 5.8(−534) | 6.000 | 9.13307 | 9.13307 | 0.0313 |

**Table 6** Convergence behavior of different methods using initial guess $X^{(0)} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ for Example 1

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\\|X^{(k+1)}-X^{(k)}\\|}{\\|X^{(k)}-X^{(k-1)}\\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 1.5(−28) | 2.8(−29) | | | | |
| $HM_4$ | 5 | 5.9(−114) | 1.1(−114) | | 1.85338 | | |
| | 6 | 1.5(−455) | 2.9(−456) | 4.000 | 1.85338 | 1.85338 | 0.0451 |
| | 4 | 4.3(−46) | 8.1(−47) | | | | |
| $JM_4$ | 5 | 8.5(−185) | 1.6(−185) | | 0.36711 | | |
| | 6 | 1.3(−739) | 2.5(−740) | 4.000 | 0.36711 | 0.36711 | 0.0715 |
| | 4 | 2.5(−37) | 4.7(−38) | | | | |
| $MM_4^1$ | 5 | 1.8(−150) | 3.5(−151) | | 0.06798 | | |
| | 6 | 5.1(−603) | 9.7(−604) | 4.000 | 0.06798 | 0.06798 | 0.0718 |
| | 4 | 2.2(−47) | 4.3(−48) | | | | |
| $PM_4^1$ | 5 | 6.3(−190) | 1.2(−190) | | 0.36711 | | |
| | 6 | 4.2(−760) | 7.9(−761) | 4.000 | 0.36711 | 0.36711 | 0.0401 |
| | 4 | 4.1(−69) | 7.8(−70) | | | | |
| $PM_4^2$ | 5 | 8.8(−278) | 1.7(−278) | | 0.04588 | | |
| | 6 | 1.9(−1112) | 3.6(−1113) | 4.000 | 0.04588 | 0.04588 | 0.0435 |
| | 4 | 1.2(−49) | 2.2(−50) | | | | |
| $PM_4^3$ | 5 | 3.2(−199) | 6.2(−200) | | 0.24963 | | |
| | 6 | 1.9(−797) | 3.6(−798) | 4.000 | 0.24963 | 0.24963 | 0.0427 |
| | 3 | 4.3(−16) | 8.3(−17) | | | | |
| $LM_6$ | 4 | 6.0(−96) | 1.1(−96) | | 3.57625 | | |
| | 5 | 4.2(−575) | 8.0(−576) | 6.000 | 3.57625 | 3.57625 | 0.0328 |
| | 3 | 8.2(−16) | 1.6(−16) | | | | |
| $MM_6^2$ | 4 | 1.6(−95) | 3.0(−96) | | 0.20913 | | |
| | 5 | 8.3(−574) | 1.6(−574) | 6.000 | 0.20913 | 0.20913 | 0.0343 |
| | 3 | 2.7(−17) | 5.2(−18) | | | | |
| $RM_6$ | 4 | 2.6(−103) | 5.0(−104) | | 2.43994 | | |
| | 5 | 2.0(−619) | 3.9(−620) | 6.000 | 2.43994 | 2.43994 | 0.0406 |
| | 3 | 2.1(−23) | 4.1(−24) | | | | |
| $PM_6^4$ | 4 | 6.5(−141) | 1.2(−141) | | 0.27710 | | |
| | 5 | 5.5(−846) | 1.0(−846) | 6.000 | 0.27710 | 0.27710 | 0.0222 |
| | 3 | 7.2(−21) | 1.4(−21) | | | | |
| $PM_6^5$ | 4 | 3.5(−125) | 6.7(−126) | | 0.96203 | | |
| | 5 | 4.4(−751) | 8.5(−752) | 6.000 | 0.96203 | 0.96203 | 0.0291 |
| | 3 | 3.7(−18) | 7.1(−19) | | | | |
| $PM_6^6$ | 4 | 7.3(−109) | 1.4(−109) | | 1.04568 | | |
| | 5 | 3.9(−653) | 7.5(−654) | 6.000 | 1.04568 | 1.04568 | 0.0316 |
| | 3 | 4.1(−13) | 7.9(−14) | | | | |
| $PM_6^7$ | 4 | 6.7(−78) | 1.3(−78) | | 5.39575 | | |
| | 5 | 1.3(−466) | 2.4(−467) | 6.000 | 5.39575 | 5.39575 | 0.0265 |

**Table 7** Nodes and weights of Gauss-Legendre quadrature formula

| $j$ | $t_j$ | $w_j$ |
|---|---|---|
| 1 | 0.019855071751231884158219 57... | 0.050614268145188129576265 67... |
| 2 | 0.101666761293186630204223 03... | 0.111190517226687235272178 00... |
| 3 | 0.237233795041835507091130 47... | 0.156853322938943643668981 10... |
| 4 | 0.408282678752175097530261 93... | 0.181341891689180991482575 22... |
| 5 | 0.591717321247824902469738 07... | 0.181341891689180991482575 22... |
| 6 | 0.762766204958164492908869 52... | 0.156853322938943643668981 10... |
| 7 | 0.898333238706813369795776 96... | 0.111190517226687235272178 00... |
| 8 | 0.980144928248768115841780 43... | 0.050614268145188129576265 67... |

To convert the above boundary value problem (37) into nonlinear system of size $50 \times 50$ with step size $h = \frac{1}{51}$, the finite difference discretization is used. For second derivative central difference has been used which is as follows:

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \ i = 1, \ 2, \ \ldots, \ 50.$$

The solution $(0.4742933\ldots, 0.4738815\ldots, 0.4730579\ldots, 0.4718232\ldots, 0.4701778\ldots, 0.4681227\ldots, 0.4656589\ldots, 0.4627878\ldots, 0.4595107\ldots, 0.4558294\ldots, 0.4517458\ldots, 0.4472620\ldots, 0.4423803\ldots, 0.4371031\ldots, 0.4314332\ldots, 0.4253735\ldots, 0.4189268\ldots, 0.4120966\ldots, 0.4048861\ldots, 0.3972989\ldots, 0.3893387\ldots, 0.3810094\ldots, 0.3723149\ldots, 0.3632593\ldots, 0.3538470\ldots, 0.3440823\ldots, 0.3339696\ldots, 0.3235137\ldots, 0.3127193\ldots, 0.3015910\ldots, 0.2901340\ldots, 0.2783530\ldots, 0.2662533\ldots, 0.2538399\ldots, 0.2411180\ldots, 0.2280928\ldots, 0.2147698\ldots, 0.2011542\ldots, 0.1872513\ldots, 0.1730667\ldots, 0.1586057\ldots, 0.1438739\ldots, 0.1288766\ldots, 0.1136194\ldots, 0.09810770\ldots, 0.08234704\ldots, 0.06634287\ldots, 0.05010065\ldots, 0.03362582\ldots, 0.01692381\ldots)^T$ of this problem is tested and shown in Tables 10 and 11. From these tables, proposed methods exhibit less functional and residual errors than the existing methods.

*Example 4* In this example, we want to check the effectiveness of our methods on a large system of nonlinear equations compared with the existing methods. Therefore, we consider the following $(100 \times 100)$ system of polynomial equations.

$$\begin{cases} x_i^2 \sin(x_{i+1}) - 1 = 0, & 1 \le i \le 99, \\ x_i^2 \sin(x_1) - 1 = 0, & i = 100. \end{cases}$$

The required solution of this system $X^* = (-1.114157\ldots, -1.114157\ldots, \ldots, -1.114157\ldots)^T$ is tested and depicted in Tables 12 and 13. The obtained results are equally competent with existing techniques.

*Example 5* Consider the Bratu Problem [42] that has a large variety of application areas such as the fuel ignition model of thermal combustion, radioactive heat transfer,

**Table 8** Convergence of different methods using initial value $X^{(0)} = (1, 1, \ldots, 1)^T$ for Example 2

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 3 | 2.1(−30) | 2.4(−30) | | | | |
| $HM_4$ | 4 | 9.0(−125) | 1.0(−124) | | 3.19792(−6) | | |
| | 5 | 2.9(−502) | 3.3(−502) | 4.000 | 3.16871(−6) | 3.16871(−6) | 0.4798 |
| | 3 | 5.4(−31) | 6.0(−31) | | | | |
| $JM_4$ | 4 | 2.4(−127) | 2.7(−127) | | 2.07422(−6) | | |
| | 5 | 9.9(−513) | 1.1(−512) | 4.000 | 2.04976(−6) | 2.04976(−6) | 0.2468 |
| | 3 | 3.4(−31) | 3.7(−31) | | | | |
| $MM_4^1$ | 4 | 3.1(−128) | 3.4(−128) | | 1.75506(−6) | | |
| | 5 | 2.1(−516) | 2.4(−516) | 4.000 | 1.73172(−6) | 1.73172(−6) | 0.2529 |
| | 3 | 2.3(−31) | 2.5(−31) | | | | |
| $PM_4^1$ | 4 | 5.6(−129 | 6.2(−129) | | 1.54003(−6) | | |
| | 5 | 2.1(−519) | 2.3(−519) | 4.000 | 1.51834(−6) | 1.51834(−6) | 0.2125 |
| | 3 | 3.4(−31) | 3.8(−31) | | | | |
| $PM_4^2$ | 4 | 3.2(−128) | 3.5(−128) | | 1.77057(−6) | | |
| | 5 | 2.5(−516) | 2.7(−516) | 4.000 | 1.74763(−6) | 1.74763(−6) | 0.2299 |
| | 3 | 4.8(−31) | 5.3(−31) | | | | |
| $PM_4^3$ | 4 | 1.4(−127) | 1.5(−127) | | 1.98712(−6) | | |
| | 5 | 1.0(−513) | 1.1(−513) | 4.000 | 1.96308(−6) | 1.96308(−6) | 0.2281 |
| | 2 | 8.8(−11) | 9.7(−11) | | | | |
| $LM_6$ | 3 | 9.3(−62) | 9.4(−62) | | 0.11436 | | |
| | 4 | 1.3(−319) | 1.4(−319) | 5.053 | 2.09430(47) | 2.09430(47) | 0.2355 |
| | 2 | 1.2(−10) | 1.3(−10) | | | | |
| $MM_6^2$ | 3 | 6.3(−69) | 7.1(−69) | | 1.38796(−9) | | |
| | 4 | 1.4(−418) | 1.5(−418) | 6.001 | 1.19766(−9) | 1.19766(−9) | 0.2296 |
| | 2 | 1.3(−10) | 1.5(−10) | | | | |
| $RM_6$ | 3 | 1.5(−68) | 1.7(−68) | | 1.52873(−9) | | |
| | 4 | 2.9(−416) | 3.3(−416) | 6.001 | 1.31247(−9) | 1.31247(−9) | 0.2752 |
| | 2 | 3.3(−11) | 3.9(−11) | | | | |
| $PM_6^4$ | 3 | 7.4(−72) | 8.7(−72) | | 2.58565(−9) | | |
| | 4 | 6.8(−436) | 8.0(−436) | 6.002 | 1.89554(−9) | 1.89554(−9) | 0.1922 |
| | 2 | 1.0(−10) | 1.1(−10) | | | | |
| $PM_6^5$ | 3 | 2.1(−69) | 2.4(−69) | | 1.15228(−9) | | |
| | 4 | 1.8(−421) | 2.0(−421) | 6.001 | 9.90919(−10) | 9.90919(−10) | 0.2079 |
| | 2 | 10(−10) | 1.2(−10) | | | | |
| $PM_6^6$ | 3 | 2.8(−64) | 3.2(−69) | | 1.21461(−9) | | |
| | 4 | 9.2(−421) | 1.0(−420) | 6.000 | 1.04639(−9) | 1.04639(−9) | 0.2000 |
| | 2 | 1.9(−10) | 2.2(−10) | | | | |
| $PM_6^7$ | 3 | 4.9(−67) | 5.7(−67) | | 5.21111(−9) | | |
| | 4 | 1.3(−406) | 1.5(−406) | 6.000 | 4.50911(−9) | 4.50911(−9) | 0.2094 |

**Table 9** Convergence of different methods using initial value $X^{(0)} = (\frac{1}{5}, \frac{1}{5}, \ldots, \frac{1}{5})^T$ for Example 2

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 3.9(−78) | 4.3(−78) | | | | |
| $HM_4$ | 5 | 9.9(−316) | 1.1(−315) | | 3.17128(−6) | | |
| | 6 | 4.3(−1266) | 4.8(−1266) | 4.000 | 3.16716(−6) | 3.16716(−6) | 0.5892 |
| | 4 | 3.6(−80) | 4.0(−80) | | | | |
| $JM_4$ | 5 | 4.8(−324) | 5.3(−324) | | 2.05204(−6) | | |
| | 6 | 1.5(−1299) | 1.6(−1299) | 4.000 | 2.04842(−6) | 2.04842(−6) | 0.3172 |
| | 4 | 8.7(−81) | 9.6(−81) | | | | |
| $MM_4^1$ | 5 | 1.3(−326) | 1.5(−326) | | 1.73396(−6) | | |
| | 6 | 7.7(−1310) | 8.5(−1310) | 4.000 | 1.73043(−6) | 1.73043(−6) | 0.2751 |
| | 4 | 1.9(−81) | 2.1(−81) | | | | |
| $PM_4^1$ | 5 | 2.8(−329) | 3.1(−329) | | 1.52045(−6) | | |
| | 6 | 1.3(−1320) | 1.4(−1320) | 4.000 | 1.51715(−6) | 1.51715(−6) | 0.1719 |
| | 4 | 7.3(−81) | 8.1(−81) | | | | |
| $PM_4^2$ | 5 | 7.0(−327) | 7.7(−327) | | 1.74981(−6) | | |
| | 6 | 5.6(−1311) | 6.2(−1311) | 4.000 | 1.74637(−6) | 1.74637(−6) | 0.1576 |
| | 4 | 2.3(−80) | 2.6(−80) | | | | |
| $PM_4^3$ | 5 | 7.9(−325) | 8.8(−325) | | 1.96533(−6) | | |
| | 6 | 1.1(−1302) | 1.2(−1302) | 4.000 | 1.96177(−6) | 1.96177(−6) | 0.1579 |
| | 3 | 6.4(−40) | 6.4(−40) | | | | |
| $LM_6$ | 4 | 4.9(−211) | 5.5(−211) | | 8.00530(24) | | |
| | 5 | 1.1(−1062) | 1.1(−1062) | 4.978 | 3.81124(199) | 3.81124(199) | 0.2374 |
| | 3 | 1.4(−42) | 1.5(−42) | | | | |
| $MM_6^2$ | 4 | 1.4(−260) | 1.5(−260) | | 1.21142(−9) | | |
| | 5 | 1.4(−1568) | 1.6(−1568) | 6.000 | 1.19014(−9) | 1.19014(−9) | 0.2908 |
| | 3 | 2.4(−92) | 2.7(−42) | | | | |
| $RM_6$ | 4 | 4.6(−259) | 5.2(−259) | | 1.32794(−9) | | |
| | 5 | 2.2(−1559) | 2.5(−1559) | 6.000 | 1.30395(−9) | 1.30395(−9) | 0.2345 |
| | 3 | 2.3(−50) | 2.7(−50) | | | | |
| $PM_6^4$ | 4 | 7.2(−307) | 8.5(−307) | | 2.14285(−9) | | |
| | 5 | 6.1(−1846) | 7.2(−1846) | 6.000 | 1.89464(−9) | 1.89464(−9) | 0.2268 |
| | 3 | 5.5(−43) | 6.2(−43) | | | | |
| $PM_6^5$ | 4 | 5.2(−263) | 5.8(−263) | | 1.00250(−9) | | |
| | 5 | 3.4(−1583) | 3.8(−1583) | 6.000 | 9.84567(−10) | 9.84567(−10) | 0.2207 |
| | 3 | 6.9(−43) | 7.8(−43) | | | | |
| $PM_6^6$ | 4 | 2.2(−262) | 2.5(−262) | | 1.05851(−9) | | |
| | 5 | 2.1(−1579) | 2.3(−1579) | 6.000 | 1.03976(−9) | 1.03976(−9) | 0.2155 |
| | 3 | 5.1(−42) | 5.9(−42) | | | | |
| $PM_6^7$ | 4 | 1.6(−256) | 1.9(−256) | | 4.61147(−9) | | |
| | 5 | 1.6(−1543) | 1.8(−1543) | 6.000 | 4.48715(−9) | 4.48715(−9) | 0.2036 |

**Table 10** Convergence behavior of different methods using initial value $X^{(0)} = (\frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10})^T$ for Example 3

| $Cases$ | $k$ | $\lVert G(X^{(k+1)})\rVert$ | $\lVert X^{(k+1)} - X^{(k)}\rVert$ | $\rho$ | $\frac{\lVert X^{(k+1)}-X^{(k)}\rVert}{\lVert X^{(k)}-X^{(k-1)}\rVert^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 2.6(−61) | 4.5(−58) | | | | |
| $HM_4$ | 5 | 1.5(−236) | 2.6(−233) | | 0.62376(−3) | | |
| | 6 | 1.6(−937) | 2.9(−934) | 4.000 | 0.62376(−3) | 0.62376(−3) | 10.498 |
| | 4 | 3.2(−68) | 5.8(−65) | | | | |
| $JM_4$ | 5 | 1.5(−264) | 2.7(−261) | | 0.23422(−3) | | |
| | 6 | 6.6(−1050) | 1.2(−1046) | 4.000 | 0.23422(−3) | 0.23422(−3) | 8.973 |
| | 4 | 7.6(−71) | 1.5(−67) | | | | |
| $MM_4^1$ | 5 | 2.8(−275) | 5.3(−272) | | 0.11937(−3) | | |
| | 6 | 4.9(−1093) | 9.4(−1090) | 4.000 | 0.11937(−3) | 0.11937(−3) | 8.6002 |
| | 4 | 3.5(−78) | 6.5(−75) | | | | |
| $PM_4^1$ | 5 | 4.2(−305) | 7.8(−302) | | 0.42972(−4) | | |
| | 6 | 8.6(−1213) | 1.6(−1209) | 4.000 | 0.42972(−4) | 0.42972(−4) | 3.767 |
| | 4 | 2.4(−72) | 4.6(−69) | | | | |
| $PM_4^2$ | 5 | 3.0(−281) | 5.6(−278) | | 0.12517(−3) | | |
| | 6 | 6.4(−1117) | 1.2(−1113) | 4.000 | 0.12517(−3) | 0.12517(−3) | 3.835 |
| | 4 | 3.4(−69) | 6.2(−66) | | | | |
| $PM_4^3$ | 5 | 1.7(−268) | 3.1(−265) | | 0.20313(−3) | | |
| | 6 | 9.7(−1066) | 1.8(−1062) | 4.000 | 0.20313(−3) | 0.20313(−3) | 3.871 |
| | 3 | 3.8(−16) | 5.2(−34) | | | | |
| $LM_6$ | 4 | 1.7(−177) | 2.7(−174) | | 1.30020(26) | | |
| | 5 | 5.9(−879) | 7.7(−877) | 5.000 | 1.89387(165) | 1.89387(165) | 5.679 |
| | 3 | 1.0(−36) | 1.8(−33) | | | | |
| $MM_6^2$ | 4 | 3.5(−206) | 6.2(−203) | | 1.82531(−6) | | |
| | 5 | 6.0(−1223) | 1.1(−1219) | 6.000 | 1.82531(−6) | 1.82531(−6) | 7.789 |
| | 3 | 3.3(−35) | 5.8(−32) | | | | |
| $RM_6$ | 4 | 1.2(−196) | 2.0(−193) | | 5.11568(−6) | | |
| | 5 | 2.1(−1165) | 3.7(−1162) | 6.000 | 5.11548(−6) | 5.11548(−6) | 5.139 |
| | 3 | 4.1(−38) | 7.0(−35) | | | | |
| $PM_6^4$ | 4 | 1.9(−214) | 3.2(−211) | | 2.76658(−6) | | |
| | 5 | 1.9(−1272) | 3.3(−1269) | 6.000 | 2.76667(−6) | 2.76667(−6) | 4.124 |
| | 3 | 2.2(−39) | 4.0(−36) | | | | |
| | 4 | 1.2(−222) | 2.2(−219) | | 5.32026(−7) | | |
| $PM_6^5$ | 5 | 3.1(−1322) | 5.5(−1319) | 6.000 | 5.32139(−7) | 5.32139(−7) | 4.115 |
| | 3 | 7.3(−39) | 1.3(−35) | | | | |
| $PM_6^6$ | 4 | 1.5(−219) | 2.7(−216) | | 5.68749(−7) | | |
| | 5 | 1.2(−1303) | 2.1(−1300) | 6.000 | 5.68874(−7) | 5.68874(−7) | 3.967 |
| | 3 | 3.0(−33) | 5.1(−30) | | | | |
| $PM_6^7$ | 4 | 1.4(−184) | 2.4(−181) | | 1.35828(−5) | | |
| | 5 | 1.5(−1092) | 2.6(−1089) | 6.000 | 1.35818(−5) | 1.35818(−5) | 4.838 |

**Table 11** Convergence behavior of different methods using initial value $X^{(0)} = (\frac{2}{5}, \frac{2}{5}, \ldots, \frac{2}{5})^T$ for Example 3

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\\|X^{(k+1)}-X^{(k)}\\|}{\\|X^{(k)}-X^{(k-1)}\\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 8.0(−86) | 1.4(−82) | | | | |
| $HM_4$ | 5 | 1.4(−334) | 2.4(−331) | | 0.62387(−3) | | |
| | 6 | 1.3(−1329) | 2.2(−1326) | 4.000 | 0.62377(−3) | 0.62377(−3) | 9.779 |
| | 4 | 4.2(−89) | 7.7(−86) | | | | |
| $JM_4$ | 5 | 4.5(−348) | 8.1(−345) | | 0.23424(−3) | | |
| | 6 | 5.6(−1384) | 1.0(−1380) | 4.000 | 0.23422(−3) | 0.23422(−3) | 8.646 |
| | 4 | 5.0(−91) | 9.5(−88) | | | | |
| $MM_4^1$ | 5 | 5.1(−356) | 9.6(−353) | | 0.11936(−3) | | |
| | 6 | 5.4(−1416) | 1.0(−1412) | 4.000 | 0.11937(−3) | 0.11937(−3) | 8.187 |
| | 4 | 9.9(−94) | 1.8(−90) | | | | |
| $PM_4^1$ | 5 | 2.6(−367) | 4.9(−364) | | 0.42918(−4) | | |
| | 6 | 1.3(−1461) | 2.4(−1458) | 4.000 | 0.42988(−4) | 0.42988(−4) | 3.4813 |
| | 4 | 7.5(−91) | 1.4(−87) | | | | |
| $PM_4^2$ | 5 | 2.6(−355) | 4.9(−352) | | 0.12516(−3) | | |
| | 6 | 3.8(−1413) | 7.3(−1410) | 4.000 | 0.12517(−3) | 0.12517(−3) | 3.446 |
| | 4 | 1.6(−89) | 3.0(−86) | | | | |
| $PM_4^3$ | 5 | 8.8(−350) | 1.6(−346) | | 0.20314(−3) | | |
| | 6 | 7.5(−1391) | 1.4(−1387) | 4.000 | 0.20313(−3) | 0.20313(−3) | 3.468 |
| | 3 | 2.8(−40) | 3.8(−38) | | | | |
| $LM_6$ | 4 | 1.2(−197) | 2.0(−194) | | 6.506998(30) | | |
| | 5 | 1.6(−979) | 2.2(−977) | 5.001 | 3.05404(185) | 3.05404(185) | 5.575 |
| | 3 | 9.8(−50) | 1.7(−46) | | | | |
| $MM_6^2$ | 4 | 2.4(−284) | 4.2(−281) | | 1.82247(−6) | | |
| | 5 | 6.1(−1692) | 1.1(−1688) | 6.000 | 1.82519(−6) | 1.82519(−6) | 7.395 |
| | 3 | 5.3(−49) | 9.2(−46) | | | | |
| $RM_6$ | 4 | 1.7(−279) | 3.1(−276) | | 5.11768(−6) | | |
| | 5 | 2.5(−1662) | 4.4(−1659) | 6.000 | 5.11588(−6) | 5.11588(−6) | 5.503 |
| | 3 | 3.8(−51) | 6.8(−48) | | | | |
| $PM_6^4$ | 4 | 1.6(−292) | 2.8(−289) | | 2.77129(−6) | | |
| | 5 | 8.3(−1741) | 1.4(−1737) | 6.000 | 2.76693(−6) | 2.76693(−6) | 4.025 |
| | 3 | 1.9(−50) | 3.3(−47) | | | | |
| $PM_6^5$ | 4 | 3.7(−289) | 6.6(−286) | | 5.64945(−7) | | |
| | 5 | 2.4(−1721) | 4.3(−1718) | 6.000 | 5.68625(−7) | 5.68625(−7) | 3.984 |
| | 3 | 2.1(−50) | 3.7(−47) | | | | |
| $PM_6^6$ | 4 | 8.2(−289) | 1.5(−285) | | 5.64945(−7) | | |
| | 5 | 3.1(−1719) | 5.6(−1716) | 6.000 | 5.68625(−7) | 5.68625(−7) | 3.901 |
| | 3 | 1.2(−53) | 9.2(−51) | | | | |
| $PM_6^7$ | 4 | 4.3(−309) | 8.1(−306) | | 1.29409(−5) | | |
| | 5 | 2.1(−1839) | 3.7(−1836) | 6.000 | 1.35510(−5) | 1.35510(−5) | 3.951 |

**Table 12** Convergence behavior of different methods using initial value $X^{(0)} = (-1, -1, \dots, -1)^T$ for Example 4

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 1.6(−98) | 1.1(−98) | | | | |
| $HM_4$ | 5 | 4.4(−397) | 3.2(−397) | | 0.19180(−4) | | |
| | 6 | 2.7(−1591) | 2.0(−1591) | 4.000 | 0.19180(−4) | 0.19180(−4) | 75.640 |
| | 4 | 9.6(−99) | 6.9(−99) | | | | |
| $JM_4$ | 5 | 6.1(−398) | 4.4(−398) | | 0.18873(−4) | | |
| | 6 | 9.5(−1595) | 6.8(−1595) | 4.000 | 0.18873(−4) | 0.18873(−4) | 48.234 |
| | 4 | 8.3(−99) | 5.9(−99) | | | | |
| $MM_4^1$ | 5 | 3.3(−398) | 2.3(−398) | | 0.18781(−4) | | |
| | 6 | 7.9(−1596) | 5.7(−1596) | 4.000 | 0.18781(−4) | 0.18781(−4) | 63.938 |
| | 4 | 7.5(−99) | 5.4(−99) | | | | |
| $PM_4^1$ | 5 | 2.2(−398) | 1.6(−398) | | 0.18719(−4) | | |
| | 6 | 1.7(−1596) | 1.2(−1596) | 4.000 | 0.18719(−4) | 0.18719(−4) | 40.797 |
| | 4 | 8.4(−99) | 6.0(−99) | | | | |
| $PM_4^2$ | 5 | 3.5(−398) | 2.5(−398) | | 0.18786(−4) | | |
| | 6 | 1.0(−1595) | 7.2(−1596) | 4.000 | 0.18786(−4) | 0.18786(−4) | 40.452 |
| | 4 | 9.3(−99) | 6.7(−99) | | | | |
| $PM_4^3$ | 5 | 5.2(−398) | 3.7(−398) | | 0.18848(−4) | | |
| | 6 | 5.0(−1595) | 3.6(−1595) | 4.000 | 0.18848(−4) | 0.18848(−4) | 40.202 |
| | 3 | 8.7(−49) | 6.3(−49) | | | | |
| $LM_6$ | 4 | 6.7(−297) | 4.8(−297) | | 7.96435(−8) | | |
| | 5 | 1.4(−1785) | 9.8(−1786) | 6.000 | 7.96435(−8) | 7.96435(−8) | 31.407 |
| | 3 | 1.7(−49) | 1.2(−49) | | | | |
| $MM_6^2$ | 4 | 3.1(−301) | 2.2(−301) | | 7.38016(−8) | | |
| | 5 | 1.3(−1811) | 9.3(−1812) | 6.000 | 7.38016(−8) | 7.38016(−8) | 58.031 |
| | 3 | 1.8(−49) | 1.3(−49) | | | | |
| $RM_6$ | 4 | 5.1(−301) | 3.7(−301) | | 7.42388(−8) | | |
| | 5 | 2.6(−1810) | 1.9(−1810) | 6.000 | 7.42388(−8) | 7.42388(−8) | 37.892 |
| | 3 | 6.4(−46) | 4.6(−46) | | | | |
| $PM_6^4$ | 4 | 1.6(−279) | 1.1(−279) | | 6.77788(−8) | | |
| | 5 | 3.8(−1681) | 2.8(−1681) | 6.000 | 6.77788(−8) | 6.77788(−8) | 27.751 |
| | 3 | 1.6(−49) | 1.1(−49) | | | | |
| $PM_6^5$ | 4 | 2.1(−301) | 1.5(−301) | | 7.32543(−8) | | |
| | 5 | 1.2(−1812) | 8.9(−1813) | 6.000 | 7.32543(−8) | 7.32543(−8) | 28.798 |
| | 3 | 1.6(−49) | 1.1(−49) | | | | |
| $PM_6^6$ | 4 | 2.3(−301) | 1.7(−301) | | 7.34048(−8) | | |
| | 5 | 2.1(−1812) | 1.5(−1812) | 6.000 | 7.34048(−8) | 7.34048(−8) | 26.658 |
| | 3 | 4.6(−50) | 3.3(−50) | | | | |
| $PM_6^7$ | 4 | 1.2(−304) | 8.8(−305) | | 6.23256(−8) | | |
| | 5 | 3.9(−1832) | 2.8(−1832) | 6.000 | 6.23256(−8) | 6.23256(−8) | 27.421 |

**Table 13** Convergence behavior of different methods using initial value $X^{(0)} = (-\frac{8}{10}, -\frac{8}{10}, \ldots, -\frac{8}{10})^T$ for Example 4

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\\|X^{(k+1)}-X^{(k)}\\|}{\\|X^{(k)}-X^{(k-1)}\\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 7.3(−56) | 5.2(−56) | | | | |
| $HM_4$ | 5 | 2.0(−226) | 1.4(−226) | | 0.19180(−4) | | |
| | 6 | 1.1(−998) | 8.1(−909) | 4.000 | 0.19180(−4) | 0.19180(−4) | 69.673 |
| | 4 | 1.3(−58) | 9.3(−59) | | | | |
| $JM_4$ | 5 | 1.9(−237) | 1.4(−237) | | 0.18873(−3) | | |
| | 6 | 9.9(−953) | 7.1(−953) | 4.000 | 0.18873(−3) | 0.18873(−3) | 55.203 |
| | 4 | 2.2(−60) | 1.6(−60) | | | | |
| $MM_4^1$ | 5 | 1.7(−244) | 1.3(−244) | | 0.18719(−4) | | |
| | 6 | 6.4(−981) | 4.6(−981) | 4.000 | 0.18719(−4) | 0.18719(−4) | 68.624 |
| | 4 | 3.6(−60) | 2.6(−60) | | | | |
| $PM_4^1$ | 5 | 1.1(−243) | 8.3(−244) | | 0.18719(−4) | | |
| | 6 | 1.2(−977) | 8.7(−978) | 4.000 | 0.18719(−4) | 0.18719(−4) | 44.344 |
| | 4 | 1.8(−59) | 1.3(−59) | | | | |
| $PM_4^2$ | 5 | 8.0(−241) | 5.8(−241) | | 0.18786(−4) | | |
| | 6 | 2.9(−966) | 2.1(−966) | 4.000 | 0.18786(−4) | 0.18786(−4) | 43.516 |
| | 4 | 7.5(−59) | 5.4(−59) | | | | |
| $PM_4^3$ | 5 | 2.2(−238) | 1.6(−238) | | 0.18848(−4) | | |
| | 6 | 1.7(−956) | 1.2(−956) | 4.000 | 0.18848(−4) | 0.18848(−4) | 45.046 |
| | 3 | 5.5(−28) | 4.0(−28) | | | | |
| $LM_6$ | 4 | 4.3(−172) | 3.1(−172) | | 7.96435(−8) | | |
| | 5 | 1.0(−1036) | 7.4(−1037) | 6.000 | 7.96435(−8) | 7.96435(−8) | 33.126 |
| | 3 | 6.7(−29) | 4.8(−29) | | | | |
| $MM_6^2$ | 4 | 1.3(−177) | 9.4(−178) | | 7.38016(−8) | | |
| | 5 | 6.9(−1070) | 5.0(−1070) | 6.000 | 7.38016(−8) | 7.38016(−8) | 62.501 |
| | 3 | 4.3(−28) | 3.1(−28) | | | | |
| $RM_6$ | 4 | 9.7(−173) | 7.0(−173) | | 7.42388(−8) | | |
| | 5 | 1.2(−1040) | 8.5(−1041) | 6.000 | 7.42388(−8) | 7.42388(−8) | 31.624 |
| | 3 | 5.5(−23) | 4.0(−23) | | | | |
| $PM_6^4$ | 4 | 6.5(−142) | 4.7(−142) | | 6.77788(−8) | | |
| | 5 | 1.7(−855) | 1.3(−855) | 6.000 | 6.77788(−8) | 6.77788(−8) | 29.468 |
| | 3 | 4.9(−29) | 3.5(−29) | | | | |
| $PM_6^5$ | 4 | 2.0(−178) | 1.4(−178) | | 7.32543(−8) | | |
| | 5 | 9.2(−1075) | 6.6(−1075) | 6.000 | 7.32543(−8) | 7.32543(−8) | 28.421 |
| | 3 | 5.1(−29) | 3.7(−29) | | | | |
| $PM_6^6$ | 4 | 2.5(−178) | 1.8(−178) | | 7.34048(−8) | | |
| | 5 | 3.6(−1074) | 2.6(−1074) | 6.000 | 7.34048(−8) | 7.34048(−8) | 28.657 |
| | 3 | 1.8(−26) | 1.3(−26) | | | | |
| $PM_6^7$ | 4 | 4.4(−163) | 3.1(−163) | | 6.23256(−8) | | |
| | 5 | 8.5(−983) | 6.1(−983) | 6.000 | 6.23256(−8) | 6.23256(−8) | 28.234 |

**Table 14** Convergence results of different methods at initial value $X^{(0)} = (sin(\pi h), \ldots, sin(40\pi h))^T$ for Example 5

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 3.1(−19) | 1.1(−16) | | | | |
| $HM_4$ | 5 | 2.7(−69) | 9.4(−67) | | 0.73888(−2) | | |
| | 6 | 1.6(−269) | 5.7(−267) | 4.000 | 0.73888(−2) | 0.73888(−2) | 2.504 |
| | 4 | 1.2(−37) | 4.2(−35) | | | | |
| $JM_4$ | 5 | 1.9(−143) | 6.6(−141) | | 0.21560(−2) | | |
| | 6 | 1.2(−566) | 4.0(−564) | 4.000 | 0.21560(−2) | 0.21560(−2) | 4.396 |
| | 4 | 6.3(−67) | 2.1(−64) | | | | |
| $MM_4^1$ | 5 | 3.5(−261) | 1.1(−258) | | 0.60708(−3) | | |
| | 6 | 3.1(−1038) | 1.0(−1035) | 4.000 | 0.60708(−3) | 0.60708(−3) | 6.640 |
| | 4 | 3.2(−42) | 1.2(−39) | | | | |
| $PM_4^1$ | 5 | 2.3(−162) | 8.4(−160) | | 0.46394(−3) | | |
| | 6 | 6.3(−643) | 2.3(−640) | 4.000 | 0.46394(−3) | 0.46394(−3) | 2.104 |
| | 4 | 1.0(−57) | 3.4(−55) | | | | |
| $PM_4^2$ | 5 | 2.8(−224) | 9.3(−222) | | 0.68556(−3) | | |
| | 6 | 1.5(−890) | 5.1(−888) | 4.000 | 0.68556(−3) | 0.68556(−3) | 2.051 |
| | 4 | 5.3(−41) | 1.8(−38) | | | | |
| $PM_4^3$ | 5 | 5.7(−157) | 1.9(−154) | | 0.17375(−2) | | |
| | 6 | 7.1(−621) | 2.4(−618) | 4.000 | 0.17375(−2) | 0.17375(−2) | 2.875 |
| | 3 | 1.5(−57) | 2.7(−56) | | | | |
| $LM_6$ | 4 | 5.1(−286) | 1.8(−283) | | 4.55456(50) | | |
| | 5 | 1.5(−1423) | 2.8(−1422) | 5.012 | 8.52956(274) | 8.52956(274) | 3.6514 |
| | 3 | 3.4(−34) | 1.2(−31) | | | | |
| $MM_6^2$ | 4 | 3.7(−193) | 1.3(−190) | | 0.47527(−4) | | |
| | 5 | 6.1(−1147) | 2.1(−1144) | 6.000 | 0.47527(−4) | 0.47527(−4) | 6.073 |
| | 3 | 4.0(−13) | 1.4(−10) | | | | |
| $RM_6$ | 4 | 5.9(−66) | 2.1(−63) | | 2.82988(−4) | | |
| | 5 | 6.3(−383) | 2.2(−380) | 6.000 | 2.82990(−4) | 2.82990(−4) | 3.149 |
| | 3 | 1.5(−36) | 5.1(−34) | | | | |
| $PM_6^4$ | 4 | 2.8(−204) | 4.8(−202) | | 9.05968(−5) | | |
| | 5 | 1.4(−1210) | 5.0(−1208) | 6.000 | 9.05881(−5) | 9.05881(−5) | 2.784 |
| | 3 | 1.5(−17) | 5.3(−15) | | | | |
| $PM_6^5$ | 4 | 2.0(−93) | 6.8(−91) | | 3.19132(−5) | | |
| | 5 | 9.1(−549) | 3.2(−546) | 6.000 | 3.19116(−5) | 3.19116(−5) | 2.339 |
| | 3 | 8.6(−14) | 3.0(−11) | | | | |
| $PM_6^6$ | 4 | 7.3(−71) | 2.5(−68) | | 3.43933(−5) | | |
| | 5 | 2.7(−413) | 9.4(−411) | 6.000 | 3.43892(−5) | 3.43892(−5) | 2.424 |
| | 3 | 1.5(−29) | 5.3(−27) | | | | |
| $PM_6^7$ | 4 | 4.7(−164) | 1.6(−161) | | 7.21596(−4) | | |
| | 5 | 4.2(−971) | 1.4(−968) | 6.000 | 7.21563(−4) | 7.21563(−4) | 2.824 |

**Table 15** Convergence behavior of different methods using initial value $X^{(0)} = (\frac{1}{5}, \frac{1}{5}, \ldots, \frac{1}{5})^T$ for Example 5

| $Cases$ | $k$ | $\|\|G(X^{(k+1)})\|\|$ | $\|\|X^{(k+1)} - X^{(k)}\|\|$ | $\rho$ | $\frac{\|X^{(k+1)}-X^{(k)}\|}{\|X^{(k)}-X^{(k-1)}\|^{\rho}}$ | $\eta$ | $e-time$ |
|---|---|---|---|---|---|---|---|
| | 4 | 7.9(−42) | 2.7(−39) | | | | |
| $HM_4$ | 5 | 1.2(−159) | 4.2(−159) | | 0.73888(−2) | | |
| | 6 | 6.4(−631) | 2.2(−628) | 4.000 | 0.73888(−2) | 0.73888(−2) | 2.304 |
| | 4 | 2.1(−49) | 7.4(−47) | | | | |
| $JM_4$ | 5 | 1.8(−190) | 6.3(−188) | | 0.21560(−2) | | |
| | 6 | 1.0(−754) | 3.5(−752) | 4.000 | 0.21560(−2) | 0.21560(−2) | 4.373 |
| | 4 | 2.7(−52) | 8.8(−50) | | | | |
| $MM_4^1$ | 5 | 1.1(−202) | 3.6(−200) | | 0.60708(−3) | | |
| | 6 | 3.1(−804) | 1.0(−801) | 4.000 | 0.60708(−3) | 0.60708(−3) | 5.002 |
| | 4 | 5.9(−62) | 2.1(−59) | | | | |
| $PM_4^1$ | 5 | 2.7(−241) | 9.6(−239) | | 0.46394(−3) | | |
| | 6 | 1.1(−958) | 4.0(−956) | 4.000 | 0.46394(−3) | 0.46394(−3) | 2.076 |
| | 4 | 2.1(−55) | 7.0(−53) | | | | |
| $PM_4^2$ | 5 | 4.9(−215) | 1.6(−212) | | 0.68556(−3) | | |
| | 6 | 1.4(−853) | 4.7(−851) | 4.000 | 0.68556(−3) | 0.68556(−3) | 2.212 |
| | 4 | 1.2(−50) | 4.0(−48) | | | | |
| $PM_4^3$ | 5 | 1.3(−195) | 4.5(−193) | | 0.17375(−2) | | |
| | 6 | 2.0(−775) | 6.9(−773) | 4.000 | 0.17375(−2) | 0.17375(−2) | 2.028 |
| | 3 | 2.1(−6) | 3.8(−23) | | | | |
| $LM_6$ | 4 | 2.7(−120) | 8.7(−119) | | 3.03871(16) | | |
| | 5 | 1.1(−597) | 1.9(−595) | 4.984 | 4.47150(113) | 4.47150(113) | 3.476 |
| | 3 | 5.8(−26) | 2.0(−23) | | | | |
| $MM_6^2$ | 4 | 9.5(−144) | 3.3(−141) | | 4.75311(−5) | | |
| | 5 | 1.8(−850) | 6.3(−848) | 6.000 | 4.75274(−5) | 4.75274(−5) | 4.878 |
| | 3 | 2.5(−24) | 8.6(−22) | | | | |
| $RM_6$ | 4 | 3.4(−133) | 1.2(−130) | | 2.83008(−4) | | |
| | 5 | 2.2(−786) | 7.7(−784) | 6.000 | 2.82991(−4) | 2.82991(−4) | 3.134 |
| | 3 | 2.5(−29) | 8.8(−27) | | | | |
| $PM_6^4$ | 4 | 1.2(−163) | 4.3(−161) | | 9.05904(−5) | | |
| | 5 | 1.7(−969) | 6.1(−967) | 6.000 | 9.05877(−5) | 9.05877(−5) | 2.417 |
| | 3 | 1.1(−29) | 3.9(−27) | | | | |
| $PM_6^5$ | 4 | 3.1(−166) | 1.1(−163) | | 3.19135(−5) | | |
| | 5 | 1.3(−985) | 4.6(−983) | 6.000 | 3.19116(−5) | 3.19116(−5) | 2.337 |
| | 3 | 2.9(−23) | 1.0(−20) | | | | |
| $PM_6^6$ | 4 | 2.2(−126) | 7.8(−124) | | 3.43911(−5) | | |
| | 5 | 4.7(−745) | 1.6(−742) | 6.000 | 3.43891(−5) | 3.43891(−5) | 2.275 |
| | 3 | 1.8(−28) | 6.3(−26) | | | | |
| $PM_6^7$ | 4 | 6.2(−159) | 2.2(−156) | | 7.21631(−4) | | |
| | 5 | 1.0(−941) | 3.5(−939) | 6.000 | 7.21565(−4) | 7.21565(−4) | 2.254 |

thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory, and nanotechnology. The problem is defined as:

$$y'' + C_1 e^y = 0, \; y(0) = y(1) = 0. \tag{38}$$

The finite difference discretization is used to convert this boundary value problem into nonlinear system of size $40 \times 40$ with $C_1$=1 and step size $h = \frac{1}{41}$. For second derivative central difference has been used which is as follows:

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}, \; i = 1, \; 2, \; \ldots, \; 40.$$

The required solution of this system $X^* = (0.055685\ldots, 0.109484\ldots, 0.161292\ldots, 0.211002\ldots, 0.258509\ldots, 0.303705\ldots, 0.346483\ldots, 0.386737\ldots, 0.424363\ldots, 0.459262\ldots, 0.491336\ldots, 0.520492\ldots, 0.546646\ldots, 0.569716\ldots, 0.589632\ldots, 0.606329\ldots, 0.619754\ldots, 0.629862\ldots, 0.636620\ldots, 0.640005\ldots, 0.640005\ldots, 0.636620\ldots, 0.629862\ldots, 0.619754\ldots, 0.606329\ldots, 0.589632\ldots, 0.569716\ldots, 0.546646\ldots, 0.520492\ldots, 0.491336\ldots, 0.459262\ldots, 0.424363\ldots, 0.386737\ldots, 0.346483\ldots, 0.303705\ldots, 0.258509\ldots, 0.211002\ldots, 0.161292\ldots, 0.109484\ldots, 0.0556856\ldots)^T$ is tested and shown in Tables 14 and 15. The computational results of our techniques are effective than earlier schemes in terms of computational time.

Numerical experiments are carried out on both academic and unconstrained optimization problems. The performance of proposed schemes in Tables 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, and 15 reveals the promising results. Notice that in every table error at each iteration and functional residual error of proposed methods reduce much faster than existing ones. Clearly, in Table 2, the proposed methods achieve the order of convergence fourth and sixth respectively with the achievement of approximated parameters whereas the existing four- and six-order schemes fail to do so. Also, in each table, computational time of proposed techniques depicts less time than the compared methods, since the computational time plays a crucial role in scientific problems. So, our methods are more preferable than the existing ones.

For evaluating and comparing the performance of different iterative techniques, we have used the concept of performance ratio [43]. The performance ratio denoted as $r_{p,s}$ is the ratio of the computing time of the proposed technique versus the best

**Table 16** Performance ratio of fourth-order different methods

| $EX\backslash IT$ | $HM_4$ | $JM_4$ | $MM_4$ | $PM_4^1$ | $PM_4^2$ | $PM_4^3$ |
|---|---|---|---|---|---|---|
| Example 1 | 1.125 | 1.783 | 1.791 | 1 | 1.085 | 1.065 |
| Example 2 | 3.739 | 2.013 | 1.746 | 1.091 | 1 | 1.002 |
| Example 3 | 2.838 | 2.509 | 2.376 | 1.010 | 1 | 1.006 |
| Example 4 | 1.881 | 1.199 | 1.590 | 1.015 | 1.006 | 1 |
| Example 5 | 1.136 | 2.156 | 2.467 | 1.024 | 1.091 | 1 |

**Table 17** Performance ratio of sixth-order different methods

| $EX\backslash IT$ | $LM_6$ | $MM_6$ | $RM_6$ | $PM_6^4$ | $PM_6^5$ | $PM_6^6$ | $PM_6^7$ |
|---|---|---|---|---|---|---|---|
| Example 1 | 1.478 | 1.545 | 1.829 | 1 | 1.311 | 1.423 | 1.194 |
| Example 2 | 1.166 | 1.429 | 1.152 | 1.114 | 1.084 | 1.058 | 1 |
| Example 3 | 1.429 | 1.896 | 1.411 | 1.032 | 1.021 | 1 | 1.013 |
| Example 4 | 1.178 | 2.177 | 1.421 | 1.041 | 1.080 | 1 | 1.029 |
| Example 5 | 1.542 | 2.164 | 1.390 | 1.072 | 1.037 | 1.009 | 1 |

time of all of the techniques. Let $P$ be set of all problems and $S$ be set of all iterative techniques. The performance ratio is defined as:

$$r_{p,s} = \frac{t_{p,s}}{min\{t_{p,s} : s \in S\}}, \tag{39}$$

where $t_{p,s}$ is computing time required to solve problem $p$ by solver $s$. Tables 16 and 17 demonstrate the performance ratio of all iterative techniques, where $IT$ and $EX$ stand for iterative techniques and examples, respectively. The results show that proposed methods have optimized performance ratio in comparison with the existing techniques.

## 7 Concluding remarks

In this work, we have developed new families of fourth- and sixth-order iterative methods for solving systems of nonlinear equations numerically. As these classes depend on parameters, a stability analysis has been performed, by using tools from multidimensional discrete dynamics, in order to select those values of the parameters with better properties. Then, some specific elements of both families are chosen. In order to check their effectiveness, the proposed schemes are applied to some large-scale systems arising from various academic problems. Furthermore, the numerical results show that the proposed techniques perform better than the existing methods of the same order in terms of residual error, the difference between two consecutive approximations, and asymptotic error constant. The performance of the proposed schemes is verified on optimization nonlinear least square problem and it is observed that the proposed techniques are better than existing ones.

**Acknowledgments** The authors would like to express their gratitude to the anonymous reviewers for their valuable comments and suggestions which have greatly improved the presentation of this paper

## References

1. Cordero, A., Torregrosa, J.R.: Variants of Newton's method for functions of several variables. Appl. Math. Comput. **183**, 199–208 (2006)

2. Frontini, M., Sormani, E.: Third-order methods from quadrature formulae for solving systems of nonlinear equations. Appl. Math. Comput. **149**, 771–782 (2004)
3. Grau-Sánchez, M., Grau, À., Noguera, M.: On the computational efficiency index and some iterative methods for solving systems of non-linear equations. Comput. Appl. Math. **236**, 1259–1266 (2011)
4. Homeier, H.H.H.: A modified Newton method with cubic convergence:the multivariable case. Comput Appl. Math. **169**, 161–169 (2004)
5. Cordero, A., Torregrosa, J.R.: Variants of Newton's method using fifth-order quadrature formulas. Comput. Appl. Math. **190**, 686–698 (2007)
6. Darvishi, M.T., Barati, A.: Super cubic iterative methods to solve systems of nonlinear equations. Appl. Math. Comput. **188**, 1678–1685 (2007)
7. Darvishi, M.T., Barati, A.: A third-order Newton-type method to solve systems of non-linear equations. Appl.Math. Comput. **187**, 630–635 (2007)
8. Potra, F.A., Pták, V.: Nondiscrete Induction and Iterarive Processes Pitman Publishing Boston (1984)
9. Babajee, D.K.R., Madhu, K., Jayaraman, J.: On some improved harmonic mean Newton-like methods for solving systems of nonlinear equations. Algor. **8**, 895–909 (2015)
10. Cordero, A., Torregrosa, J.R.: Iterative methods of order four and five for systems of nonlinear equations. Comput. Appl. Math. **231**, 541–551 (2009)
11. Cordero, A., Hueso, J.L., Martínez, E., Torregrosa, J.R.: Efficient high-order methods based on golden ratio for nonlinear systems. Appl. Math. Comput. **217**, 4548–4556 (2011)
12. Arroyo, V., Cordero, A., Torregrosa, J.R.: Approximation of artificial satellites' preliminary orbits: the efficiency challenge. Math. Comput. Modelling **54**, 1802–1807 (2011)
13. Cordero, A., Torregrosa, J.R.: Variants of Newton's method using fifth-order quadrature formulas. Appl. Math. Comput. **190**, 686–698 (2007)
14. Narang, M., Bhatia, S., Kanwar, V.: New two-parameter Chebyshev Halley-like family of fourth and sixth-order methods for systems of nonlinear equations. Appl. Math. Comput. **275**, 394–403 (2016)
15. Lotfi, T., Bakhtiari, P., Cordero, A., Mahdiani, K., Torregrosa, J.R.: Some new efficient multipoint iterative methods for solving nonlinear systems of equations. Int. J. Comput. Math. **92**, 1921–1934 (2015)
16. Alzahrani, A.K.H., Behl, R., Alshomrani, A.: Some higher-order iteration functions for solving nonlinear models. Appl. Math. Comput. **334**, 80–93 (2018)
17. Babajee, D.K.R.: On a two-parameter Chebyshev-Halley like family of optimal two-point fourth order methods free from second derivatives. Afrika Matematika. **26**, 689–695 (2015)
18. Behl, R., Kanwar, V.: Highly efficient classes of Chebyshev-Halley type methods free from second-order derivative (2014)
19. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables Academic Press New York (1970)
20. Cordero, A., Hueso, J.L., Torregrosa, J.R.: A modified Newton-Jarratt's composition. Numer. Algor. **55**, 87–99 (2010)
21. Fatou, P.: Sur les équations fonctionelles. Bull. Soc. Mat. Fr. 47 (1919) 161–271 **48**, 33–94 (1920)
22. Julia, G.: Mémoire sur l'iteration des fonctions rationnelles. J. Mat. Pur. Appl. **8**, 47–245 (1918)
23. Chicharro, F.I., Cordero, A., Torregrosa, J.R.: Drawing dynamical and parameters planes of iterative families and methods, The Scientific World Journal, Volume 2013, Article ID 780153, 11 pages
24. Cordero, A., García, J., Torregrosa, J.R., Vassileva, M.P., Vindel, P.: Chaos in King's iterative family. Appl. Math. Letters **26**, 842–848 (2013)
25. Amat, S., Busquier, S., Bermúdez, C., Plaza, S.: On two families of high order Newton type methods. Appl. Math. Letters **25**, 2209–2217 (2012)
26. Neta, B., Chun, C., Scott, M.: Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations. Appl. Math. Comput. **227**, 567–592 (2014)
27. Amat, S., Busquier, S., Plaza, S.: Chaotic dynamics of a third-order Newton-type method. J. Math. Anal. Appl. **366**, 24–32 (2010)
28. Cordero, A., Torregrosa, J.R., Vindel, P.: Dynamics of a family of Chebyshev-Halley type methods. Appl. Math. Comput. **219**, 8568–8583 (2013)
29. Geum, Y.H., Kim, Y.I., Neta, B.: A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points. Appl. Math. Comput. **283**, 120–140 (2016)
30. Cordero, A., Giménez-Palacios, I., Torregrosa, J.R.: Avoiding strange attractors in efficient parametric families of iterative methods for solving nonlinear problems. Appl. Num. Math. **137**, 1–18 (2019)

31. Cordero, A., Soleymani, F., Torregrosa, J.R.: Dynamical analysis of iterative methods for nonlinear systems or how to deal with the dimension? Appl. Math. Comput. **244**, 398–412 (2014)
32. Chicharro, F.I., Cordero, A., Torregrosa, J.T.: Real stability of an efficient family of iterative methods for solving nonlinear systems. In: Proceedings of XXV Congreso de ecuaciones diferenciales y aplicaciones CEDYA 2017, ISBN 978-84-944402-1-2, 206–212 Chicharro, F.I., Cordero, A., Torregrosa, J.T.: Real stability of an efficient family of iterative methods for solving nonlinear systems. Proceedings of XXV Congreso de ecuaciones diferenciales y aplicaciones CEDYA 2017, ISBN 978-84-944402-1-2, 206–212 (2017)
33. Robinson, R.C.: An introduction to dynamical systems, continous and discrete, americal mathematical society, providence, RI USA (2012)
34. Devaney, R.L.: An Introduction to Chaotic Dynamical Systems Advances in Mathematics and Engineering CRC Press (2003)
35. Sharma, J.R., Arora, H.: Efficient Jarratt-like methods for solving systems of nonlinear equations. Calcolo **51**, 193–210 (2014)
36. Jarratt, P.: Some fourth order multipoint iterative methods for solving equations. Math. Comput. **20**, 434–437 (1966)
37. Hillstrom, K.E.: A stmulatlon test approach to the evaluation of nonlinear optimization algorithms. ACM Trans. Math Softw. **3**(4), 305–315 (1977)
38. Box, M.J.: A comparison of several current optimization methods, and the use of transformations in constrained problems. Comput. J **9**, 67–77 (1966)
39. Recktenwald, G.: Least squares fitting of data to a curve, department of mechanical engineering portland state university (2001)
40. Argyros, I.K., Hilout, S.: Computational Methods In Nonlinear Analysis: Efficient Algorithms, Fixed Point Theory And Applications, World Scientific Publications (2013)
41. Kamenetskii, F., Al'bertovich, D.: Diffusion and heat transfer in chemical kinetics Plenum Press (1969)
42. Alaidarous, E.S., Ullah, M.Z., Ahmad, F., Al-Fhaid, A.S.: An Efficient Higher-Order Quasilinearization Method for Solving Nonlinear BVPs J Appl Math Hindawi pulisher (2013)
43. Dolan, E.D., Moŕe, J.J.: Benchmarking optimization software with performance profiles. Math. Program. **91**, 201–213 (2002)



## Affiliations

**Munish Kansal[1] · Alicia Cordero[2] · Sonia Bhalla[3] · Juan R. Torregrosa[2]**

Munish Kansal
mkmaths@gmail.com

Alicia Cordero
acordero@mat.upv.es

Sonia Bhalla
soniamaths5@gmail.com

[1] School of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147004, India

[2] Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, 46022 València, Spain

[3] Department of Mathematics, Chandigarh University, Mohali 140413, India