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Dynamical analysis of iterative methods for nonlinear systems or how to deal with the dimension? ☆

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ABSTRACT

This paper deals with the real dynamical analysis of iterative methods for solving nonlinear systems on vectorial quadratic polynomials. We use the extended concept of critical point and propose an easy test to determine the stability of fixed points to multivariate rational functions. Moreover, an Scaling Theorem for different known methods is satisfied. We use these tools to analyze the dynamics of the operator associated to known iterative methods on vectorial quadratic polynomials of two variables. The dynamical behavior of Newton's method is very similar to the obtained in the scalar case, but this is not the case for other schemes. Some procedures of different orders of convergence have been analyzed under this point of view and some "dangerous" numerical behavior have been found, as attracting strange fixed points or periodic orbits.

1. Introduction

Recently, the dynamical behavior of the rational operator associated to an iterative method for solving nonlinear equations applied to low-degree polynomials has shown to be an efficient tool for analyzing the stability and reliability of the methods, see for example [1–10] an the references therein. In this work, we propose a generalization of these dynamical tools in order to be applied on iterative schemes for solving nonlinear systems.

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution $\bar{x} \in D$ of the nonlinear system $F(x) = 0$, of n equations with n variables, being $f_i,\ i = 1, 2, \ldots, n$ the coordinate functions of F. This solution can be obtained as a fixed point of some function $\bar{G} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of the fixed-point iteration method

$$x^{(k+1)} = \bar{G}(x^{(k)}), \quad k = 0, 1, \ldots, \tag{1}$$

where $x^{(0)}$ is the initial estimation.

A basic result in order to analyze the convergence of an iterative method for solving nonlinear systems is the following, that can be found in [11].

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Theorem 1. *Let* $D = \{(x_1, x_2, \ldots, x_n)^T$ such that $a_i \leqslant x_i \leqslant b_i$, for each $i = 1, 2, \ldots, n\}$ *for some collection of constants* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$. *Suppose* $\bar{G} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function with the property that* $\bar{G}(x) \in D$ *whenever* $x \in D$. *Then* $\bar{G}$ *has a fixed point in D. Moreover, suppose that all the component functions of* $\bar{G}$ *have continuous partial derivatives and a constant* $K < 1$ *exists with*

$$\left|\frac{\partial \bar{g}_i(x)}{\partial x_j}\right| \leqslant \frac{K}{n} < 1, \quad x \in D,$$

for each $j = 1, 2, \ldots, n$ *and each component function* $\bar{g}_i$ *of* $\bar{G}$. *Then, the sequence* $\{x^{(k)}\}_{k=0}^{\infty}$ *defined by an arbitrarily selected* $x^{(0)} \in D$ *and generated by (1) converges to the unique fixed point* $\bar{x} \in D$ *and*

$$\|x^{(k)} - \bar{x}\| \leqslant \frac{K^k}{1-K}\|x^{(1)} - x^{(0)}\|.$$

In order to analyze the dynamical behavior of a fixed-point iterative method for nonlinear systems when is applied to n-variable polynomial $p(x), p : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, it is necessary to recall some basic dynamical concepts.

Let us denote by $G(x)$ the vectorial fixed-point function associated to the iterative method on polynomial $p(x)$. Let us note that the next concepts and results are also valid when the iterative method is applied on a general function $F(x)$.

Definition 1. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a vectorial rational function. The orbit of a point $x^{(0)} \in \mathbb{R}^n$ is defined as the set of successive images of $x^{(0)}$ by the vectorial rational function, $\{x^{(0)}, G(x^{(0)}), \ldots, G^m(x^{(0)}), \ldots\}$.

The dynamical behavior of the orbit of a point of $\mathbb{R}^n$ can be classified depending on its asymptotic behavior. In this way, a point $x^* \in \mathbb{R}^n$ is a fixed point of G if $G(x^*) = x^*$.

We recall a known result in Discrete Dynamics that gives the stability of fixed points for nonlinear operators.

Theorem 2 [12, page 558]. *Let G from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *be* $\mathcal{C}^2$. *Assume* x^* *is a period-k point. Let* $\lambda_1, \lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $G'(x^*)$.

(a) *If all the eigenvalues* λ_j *have* $|\lambda_j| < 1$, *then* x^* *is attracting.*
(b) *If one eigenvalue* λ_{j_0} *has* $|\lambda_{j_0}| > 1$, *then* x^* *is unstable, that is, repelling or saddle.*
(c) *If all the eigenvalues* λ_j *have* $|\lambda_j| > 1$, *then* x^* *is repelling.*

In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue λ_i such that $|\lambda_i| < 1$ and an eigenvalue λ_j such that $|\lambda_j| > 1$, the hyperbolic point is called saddle point.

Let us note that, the entries of $G'(x^*)$ are the partial derivatives of each coordinate function of the vectorial rational operator that defines the iterative scheme. To avoid the calculation of spectrum of $G'(x^*)$ we propose the following result that, being consistent with the previous theorem, gives us a practical tool for classifying the stability of fixed points in many cases.

Proposition 1. *Let* x^* *be a fixed point of G. Then,*

(a) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| < \frac{1}{n}$ *for all* $i, j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is attracting.*
(b) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| = 0$, *for all* $i, j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is superattracting.*
(c) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| > \frac{1}{n}$, *for all* $i, j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is unstable and lies at the Julia set.*

being $g_i(x)$, $i = 1, 2, \ldots, n$, *the coordinate functions of the fixed point multivariate function G.*

The proof of this result is based in Theorem 2 and on the facts that $\rho(G'(x^*)) \leqslant \|G'(x^*)\|$, where $\rho(A)$ denotes the spectral radius of matrix A and the unstable points (repelling and saddle) are always on Julia set.

It is obvious that, if the order of the iterative method is at least two, then the roots of the nonlinear function are superattracting fixed points of the vectorial rational function associated to the iterative method. If a fixed point is not a root of the nonlinear function, it is called strange fixed point and its character can be analyzed in the same manner.

Then, if x^* is an attracting fixed point of the rational function G, its basin of attraction $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \{x^{(0)} \in \mathbb{R}^n : G^m(x^{(0)}) \to x^*, m \to \infty\}.$$

In the same way as in the scalar case, the set of points whose orbits tend to an attracting fixed point x^* is defined as the Fatou set, $\mathcal{F}(G)$. The complementary set, the Julia set $\mathcal{J}(G)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

The concept of critical point can be defined following the idea of multivariate convergence of iterative methods.

Definition 2. A fixed point $x \in \mathbb{R}^n$ is a critical point of G if its coordinate functions g_i satisfy $\frac{\partial g_i(x)}{\partial x_j} = 0$ for all $i, j \in \{1, \ldots, n\}$.

In this terms, a superattracting fixed point will be also a critical point and, from the numerical point of view, the iterative method involved will be, at least, of second order of convergence. A critical point that is not root of the polynomial $p(x)$ will be called free critical point.

In this paper we apply these dynamical concepts, that have been extended to nonlinear systems, to some known iterative schemes of different orders of convergence: Newton's scheme is analyzed in Section 2 and its third-order extension, due to Traub, in Section 3. In addition, a fourth-order method with a more complicated iterative expression (with two Jacobian matrices involved) is analyzed in Section 4. Finally, a parametric family of third- and fourth-order methods is studied, finding several interesting behaviors, like attracting strange fixed points or periodical orbits, of different periods. Finally, some conclusions are stated.

2. Newton's method

In this section we will apply the extended dynamical concepts to the best known iterative scheme, Newton's method, whose iterative expression is

$$x^{(k+1)} = x^{(k)} - \left[F'(x^{(k)})\right]^{-1} F(x^{(k)}), \quad k \geqslant 0.$$

This method converges quadratically in a neighborhood of the solution where the Jacobian matrix is non-singular.

In the following, we will denote by N_F the fixed point function associated to Newton's method. In order to establish a Scaling Theorem for Newton's method, we recall that a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation if it can written as $T(x) = Ax + b$, where $x = (x_1, x_2, \ldots, x_n)^T$, $b \in \mathbb{R}^n$ and A is a real invertible matrix of size $n \times n$.

Theorem 3. *Let $T(x) = Ax + b$ be an affine transformation on $\mathbb{R}^n$. Let also $F(x)$ be an n-dimensional vectorial function and $G(x) = (F \circ T)(x)$. Then, operators N_F and N_G are affinely conjugated by T. In other words,*

$$\left(T \circ N_G \circ T^{-1}\right)(x) = N_F(x),$$

for all $x \in \mathbb{R}^n$.

Proof. The thesis of this theorem is equivalent to prove

$$(T \circ N_G)(x) = (N_F \circ T)(x),$$

for all $x \in \mathbb{R}^n$.

Now,

$$(T \circ N_G)(x) = T(N_G(x)) = A \cdot N_G(x) + b = Ax - A \cdot [G'(x)]^{-1} G(x) + b.$$

On the other hand,

$$(N_F \circ T)(x) = N_G(T(x)) = Ax + b - \left[F'(Ax + b)\right]^{-1} F(Ax + b).$$

So, we need to prove that

$$[F'(T(x))]^{-1} (F \circ T)(x) = A \cdot [G'(x)]^{-1} (F \circ T)(x).$$

However, by applying the string rule to $G(x) = (F \circ T)(x)$, we obtain $G'(x) = F'(T(x)) \cdot A$, and the thesis is satisfied. □

This result allows us to reduce the dynamical study of Newton's method on n-dimensional quadratic polynomials to simpler ones. In particular, we will analyze the dynamical behavior of Newton's method acting on the polynomial systems $p(x) = 0$ and $q(x) = 0$, denoted by (S1) and (S2), respectively, where

$$\left.\begin{array}{l} p_1(x) = x_1^2 - 1 \\ p_2(x) = x_2^2 - 1 \end{array}\right\}, \quad \left.\begin{array}{l} q_1(x) = x_1x_2 + x_1 - x_2 - 1 \\ q_2(x) = x_1x_2 - x_1 + x_2 - 1 \end{array}\right\}.$$

Let us note that the dynamical properties of an iterative method on a two-dimensional polynomial can be extended to a similar n-dimensional one. The only difference is that dimension two allows us to paint dynamical planes and to visualize graphically the analytical results obtained.

At this point, it will be useful to establish the notation to be used in the rest of the paper. This notation is firstly defined in the text of Traub (see [13]), for proving the order of convergence of iterative methods for nonlinear systems.

In the following, we will denote by $H_{ij}(x)$ the (i,j)-entry of the inverse matrix of the Jacobian one, whose (i,j) entry is $F'_{ij}(x)$. Then,

$$\sum_{j=1}^{n} H_{ij}(x) F'_{jm}(x) = \delta_{im}, \tag{2}$$

is satisfied, where δ_{im} is the Kronecker symbol.

The next result was proved in [13]. Let us note that, by using expressions (4) and (5) in Lemma 1, it can be concluded that the convergence order of Newton's method is $p = 2$.

Lemma 1. *Let $\bar{\lambda}(x)$ be the iteration function of classical Newton's method, whose coordinates are:*

$$\bar{\lambda}_j(x) = x_j - \sum_{i=1}^{n} H_{ji}(x) f_i(x), \tag{3}$$

for $j = 1, \ldots, n$. Then,

$$\frac{\partial \bar{\lambda}_j(\bar{x})}{\partial x_l} = 0, \tag{4}$$

$$\frac{\partial^2 \bar{\lambda}_j(\bar{x})}{\partial x_r \partial x_l} = \sum_{i=1}^{n} H_{ji}(\bar{x}) \frac{\partial^2 f_i(\bar{x})}{\partial x_r \partial x_l}. \tag{5}$$

This formulation help us to analyze the stability of Newton's method on quadratic polynomials of two variables. Taking into account that the jth-coordinate of its fixed point function on $p(x)$ is

$$\lambda_j(x) = x_j - \sum_{i=1}^{2} H_{ji}(x) p_i(x) = x_j - \frac{x_j^2 - 1}{2x_j}, \quad j = 1, 2$$

and the Jacobian matrix must be nonsingular at the solution, the only fixed points are the roots of $p(x)$, that is, ((1,1), (1,−1), (−1,−1) and (−1,1)). These points are superattracting, due to the order of convergence of the method.

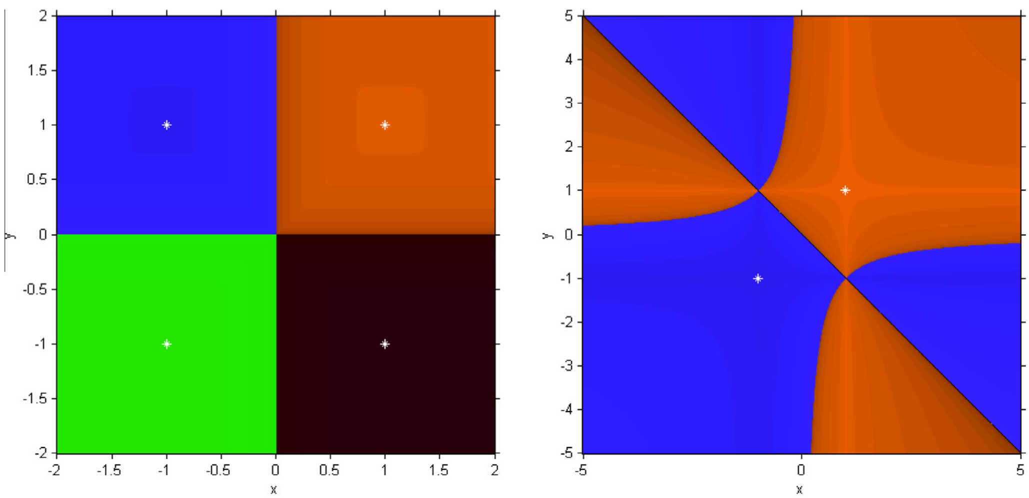
In Fig. 1, the dynamical planes associated with Newton's method on $p(x)$ and $q(x)$ are showed. These planes has been generated by slightly modifying the routines described in [14]. In them, a mesh of 400×400 points has been used, 40 has been the maximum number of iterations involved and 10^{-3} the tolerance used as a stopping criterium. Then, if an starting point of this mesh converges to one of the roots of the polynomial, it is painted in the color assigned to the root which has converged to. The color used is brighter when the number of iterations is lower. If it reaches the maximum number of iterations without converging to any of the roots, it is painted in black.

It can be observed in Fig. 1(a) that the four roots of the vectorial polynomial have their respective basins of attraction forming a balanced division of the real plane into four parts (with only a connected component for each root), separated by the Julia set, in a similar way as happens in the scalar case.

On the other hand, the jth-coordinate of the vectorial rational function associated to Newton's scheme on polynomial $q(x)$ is

$$\lambda_j(x) = \frac{x_1 x_2 + 1}{x_1 + x_2}, \quad j = 1, 2.$$

From this expression, it can be easily deduced the following result.



(a) $p(x)$ (b) $q(x)$

Fig. 1. Dynamical plane of multivariate Newton's method on $p(x)$ and $q(x)$.

Theorem 4. *The 2-dimensional rational function associated to Newton's scheme on the polynomial $q(x)$ has, as the only fixed and critical points, the roots of the polynomial, (−1,−1) and (1,1).*

Proof. The solution of

$$x_1 = \frac{x_1x_2+1}{x_1+x_2}, \quad x_2 = \frac{x_1x_2+1}{x_1+x_2},$$

yields to $(-1,-1)$ and $(1,1)$ as the unique fixed points. Moreover,

$$\frac{\partial\lambda_j(x_1,x_2)}{\partial x_1} = \frac{x_2^2-1}{(x_1+x_2)^2}, \quad j=1,2 \quad \frac{\partial\lambda_j(x_1,x_2)}{\partial x_2} = \frac{x_1^2-1}{(x_1+x_2)^2}, \quad j=1,2.$$

So, the only points that make null all the partial derivatives of the coordinate functions λ_j, $j=1,2$, are the solutions of $q(x)=0$. By definition, they are the critical points. □

In Fig. 1(b) the unique basins of attraction corresponding to $q(x)$ are shown. The stability of Newton's scheme is observed in the clean basins and the simple Julia set.

3. Traub's method

This third-order scheme, appearing in [13], is also known as Potra–Ptak's method in scalar case and also appears (as well as Newton's one) as a particular case in the families described in [15,16] or [17]. It's iterative expression is

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}\left(F(x^{(k)}) + F(y^{(k)})\right),$$

where $y^{(k)}$ is Newton's iteration. So, its fixed-point formulation is defined by its jth-coordinate of the iteration function,

$$\bar{g}_j(x) = x_j - \sum_{i=1}^{n} H_{ji}(x)\left(f_i(x) + f_i(\bar{\lambda}(x))\right), \quad j=1,\dots,n. \tag{6}$$

In a similar way as in case of Newton's method, an Scaling Theorem can be proved. Let us assume now that Traub's scheme acts on the polynomial $p(x)$. Then, the following result can be established.

Theorem 5. *There exist four strange fixed points of the operator associated to Traub's iterative method on quadratic polynomial $p(x)$. These points are denoted by s^i, with $i=1,\ 2,\ 3,\ 4$, and their components are $\pm\frac{\sqrt{5}}{5}$. Indeed, these points are repulsive, so they remain in the Julia set.*

Proof. As we have seen in the previous section, the jth-coordinate function of the fixed-point operator associated to Newton's scheme on $p(x)$ is

$$\lambda_j(x) = x_j - \frac{x_j^2-1}{2x_j}, \quad \text{for} \quad j=1,2.$$

Then, jth-coordinate of the iteration function associated to Traub's method is

$$g_j(x) = x_j - H_{jj}(x)\left(p_j(x) + p_j(\lambda(x))\right) = \frac{3x_j^4+6x_j^2-1}{8x_j^3},$$

for $j=1,\ 2$. By imposing the condition $x_j = g_j(x)$, for $j=1,\ 2$, and using that the Jacobian matrix must be nonsingular, we have

$$0 = p_j(x) + p_j(\lambda(x)) = x_j^2 - 1 - \left(x_j - \frac{x_j^2-1}{2x_j}\right)^2 - 1 = (x_j^2-1)(5x_j^2-1).$$

So, the fixed points are the roots of $p(x)$, but also there exist four strange fixed points, s^i with $i=1,\ 2,\ 3,\ 4$ whose jth-components are $\pm\frac{\sqrt{5}}{5}$, $j=1,2$. Respect to the stability of these four strange fixed points, let us calculate the corresponding partial derivatives of the components of the iteration function:

$$\frac{\partial g_j(x)}{\partial x_j} = \frac{3\left(-1+x_j^2\right)^2}{8x_j^4}, \quad j=1,2$$
$$\frac{\partial g_j(x)}{\partial x_i} = 0, \quad i \neq j.$$

Then,

$$\left|\frac{\partial g_j(s^i)}{\partial x_j}\right| = 6 > \frac{1}{2}$$

and the character of the strange fixed points can not be determined by using Proposition 1. So, we are going to analyze the eigenvalues of $G'(s^i)$. In all cases, the eigenvalues are $\lambda_i = 6,\ i = 1,2$. Therefore, the strange fixed points s^i are repulsive. □

These results are graphically showed in Fig. 2(a), where the dynamical plane associated to Traub's method acting on $p(x)$ is presented. In it, the roots of $p(x)$ are showed with white stars and the strange fixed points are white circles. It can be seen that only the roots of the polynomial have their own basins of attraction, as the strange fixed points belong to the Julia set.

Moreover, there are no free critical points, as is stated in the following result.

Proposition 2. *The unique solution of the system of equations* $\frac{\partial g_j(x)}{\partial x_i} = 0$*, for* $i,\ j = 1,2$*, are the four roots of* $p(x)$*. So, there are no free critical points.*

On the other hand, the multivariate rational function associated to Traub's scheme on $q(x)$ is

$$g_i(x) = \frac{-1 + (2 + x_1x_2)\left(x_1^2 + x_1x_2 + x_2^2\right)}{(x_1 + x_2)^3}, \quad i = 1,2.$$

The basins of attraction, as the behavior of Traub's method on $q(x)$, with stable (white stars) and unstable (white circles) fixed points, can be seen at Fig. 2(b). In a similar way as in the case of Newton's, the following result can be proved.

Theorem 6. *The 2-dimensional rational function associated to Traub's scheme on the polynomial* $q(x)$ *has four fixed points,*

(i) *two of them are the roots of the polynomial,* $(-1,-1)$ *and* $(1,1)$*, which are superattracting, and*
(ii) *another two points* $s^1 = \left(-\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right)$ *and* $s^2 = \left(\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)$ *are strange fixed points, and they are repulsive.*

Moreover, the only critical points are the roots of $q(x)$*.*

Proof. In this case, the coordinate functions of the fixed-point operator associated to Traub's scheme on $q(x)$ is

$$g_1(x) = g_2(x) = \frac{-1 + (2 + x_1x_2)\left(x_1^2 + x_1x_2 + x_2^2\right)}{(x_1 + x_2)^3}.$$

By solving the system of equations $x_j = g_j(x)$, for $j = 1,\ 2$, we have that the fixed points are $(-1,-1)$, and $(1,1)$ (roots of $q(x)$), but also there exist two strange fixed points, s^1 and s^2. To analyze the stability of these strange fixed points, we calculate the corresponding partial derivatives of the components of the iteration function:

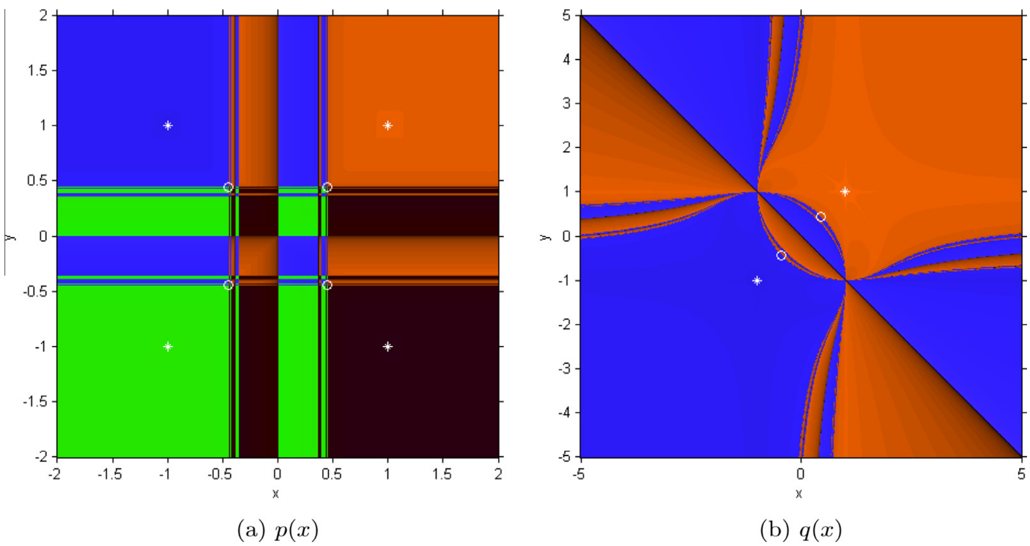


Fig. 2. Dynamical plane of multivariate Traub's method on $p(x)$ and $q(x)$.

$$\frac{\partial g_1(x)}{\partial x_1} = \frac{\partial g_2(x)}{\partial x_1} = \frac{\left(-1+x_2^2\right)\left(-3+2x_1^2+x_2^2\right)}{\left(x_1+x_2\right)^4},$$
$$\frac{\partial g_1(x)}{\partial x_2} = \frac{\partial g_2(x)}{\partial x_2} = \frac{\left(-1+x_1^2\right)\left(-3+x_1^2+2x_2^2\right)}{\left(x_1+x_2\right)^4}.$$

Then,

$$\left|\frac{\partial g_i(s^1)}{\partial x_j}\right| = \left|\frac{\partial g_i(s^2)}{\partial x_j}\right| = 3 > \frac{1}{2}, \quad i, \quad j = 1,2$$

and the character of the strange fixed points can be determined (by using Proposition 1) as repulsive. □

4. Fourth-order NAd1 method

In this section, we are going to consider a fourth-order method which is called NAd1 (see [15]), designed as a variant of Newton's method by using Adomian polynomials of sub-index 1. Nevertheless, the iterative expression of NAd1 can be used with no knowledge of Adomian polynomials, only in terms of previous estimations and Newton's approximation. So, being $x^{(0)}$ the initial estimation of the iterative process and being

$$y^{(k)} = x^{(k)} - \left[F'(x^{(k)})\right]^{-1}F(x^{(k)})$$

the kth-approximation of Newton's method, a new estimation $x^{(k+1)}$ can be obtained by means of the following expression:

$$x^{(k+1)} = y^{(k)} - \left[2[F'(x^{(k)})]^{-1} - [F'(x^{(k)})]^{-1}F'(y^{(k)})[F'(x^{(k)})]^{-1}\right]F(y^{(k)}). \tag{7}$$

We denote by $M_{ij}(x)$ the (i,j) entry of the matrix

$$M(x) = F'(x)^{-1}F'(\lambda(x))F'(x)^{-1},$$

where $\lambda(x) = x - H(x)F(x)$. Thus, the jth component of the iteration function corresponding to method NAd1 is

$$\bar{g}_j(x) = \lambda_j(x) - 2\sum_{i=1}^{n} H_{ji}(x)f_i(\lambda(x)) + \sum_{i=1}^{n} M_{ji}(x)f_i(\lambda(x)). \tag{8}$$

In a similar way as in case of Newton's method, an Scaling Theorem can be proved. The following result shows the behavior of NAd1 method on the polynomial $p(x)$.

Theorem 7. *There exist one real strange fixed point of the operator associated to NAd1 iterative method on the quadratic polynomial $p(x)$. Indeed, the character of this point is repulsive, so it remains in the Julia set.*

Proof. The jth-coordinate function of the fixed point operator associated to NAd1 method is

$$g_j(x) = x_j - 2\frac{x_j^2-1}{2x_j} - 2\frac{1}{2x_j}\left[\left(x_j - \frac{x_j^2-1}{2x_j}\right)^2 - 1\right] + \frac{x_j^4-1}{8x_j^5}\left[\left(x_j - \frac{x_j^2-1}{2x_j}\right)^2 - 1\right]$$
$$= \frac{8x_j^7 + x_j^6 + 32x_j^5 - 3x_j^4 - 8x_j^3 + 3x_j^2 - 1}{32x_j^6}, \quad \text{for} \quad j = 1,2.$$

By imposing $x_j = g_j(x)$, for $j = 1,\ 2$, we obtain

$$0 = -\frac{(x_j-1)(x_j+1)(24x_j^5 - x_j^4 + 2x_j^2 - 1)}{32x_j^6}.$$

So, the fixed points are the roots of $p(x)$, but also there is one real strange fixed point, $s \approx (0.628361, 0.628361)$. In order to analyze the stability of s, we calculate:

$$\frac{\partial g_j(x)}{\partial x_j} = \frac{(-1+x_j)(1+x_j)\left(-3+3x_j^2-12x_j^3+4x_j^5\right)}{16x_j^7}, \quad j = 1,2,$$
$$\frac{\partial g_j(x)}{\partial x_i} = 0, \quad i \neq j$$

and

$$\left|\frac{\partial g_j(s)}{\partial x_j}\right| \approx 4.30353 > \frac{1}{2}.$$

So, the character of the strange fixed point s can not be determined by using Proposition 1. The eigenvalues of $G'(s)$ satisfy $|\lambda_i| \approx 4.30353,\ i = 1,2$. Therefore, the strange fixed point s is repulsive. □

The previous result is graphically showed in Fig. 3(a), where the dynamical plane of method NAd1 on $p(x)$ is presented. All the roots are marked with white stars and their basins of attraction can be seen with different colors. Also the Julia set can be seen as black lines of unstable behavior. It is observed that, in spite of the complexity of its iterative expression, its dynamical behavior is more stable in the real plane than Traub's scheme, and its order of convergence is higher.

Respect to NAd1 method on $q(x)$, the coordinate functions of the associate rational function are

$$\begin{aligned} g_i(x) = {} & \frac{-4 + x_1^7 x_2 + 6x_2^2 - 4x_2^4 + 3x_2^6 + 3x_1^5 x_2(4 + 3x_2^2) + x_1^6(3 + 4x_2^2) + x_1^4(4 + x_2^2)(-1 + 8x_2^2)}{(x_1 + x_2)^7} \\ & + \frac{x_1^3 x_2(-2 + 28x_2^2 + 9x_2^4) + x_1 x_2(-4 - 2x_2^2 + 12x_2^4 + x_2^6) + x_1^2(6 - 20x_2^2 + 31x_2^4 + 4x_2^6)}{(x_1 + x_2)^7}, \quad i = 1,2 \end{aligned}$$

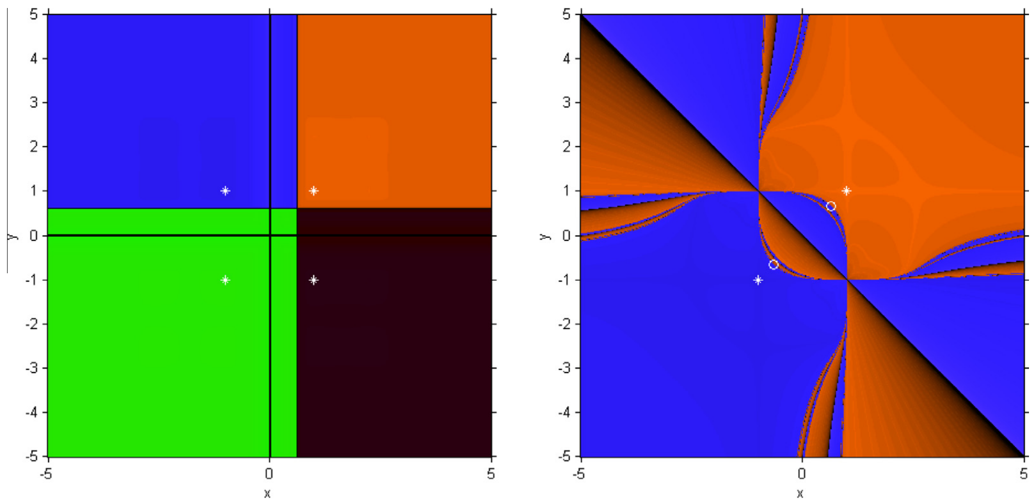
and the respective partial derivatives can be expressed as

$$\begin{aligned} \frac{\partial g_i(x_1,x_2)}{\partial x_1} = {} & \frac{(-1 + x_2^2)(-28 + 30x_1^2 - 12x_1^4 + 3x_1^6 - 36x_1x_2 + 8x_1^3x_2 + 6x_1^5x_2 + 18x_2^2 - 64x_1^2x_2^2)}{(x_1 + x_2)^8} \\ & + \frac{(-1 + x_2^2)(21x_1^4x_2^2 - 8x_1x_2^3 - 4x_1^3x_2^3 - 8x_2^4 + 7x_1^2x_2^4 + 2x_1x_2^5 + x_2^6)}{(x_1 + x_2)^8}, \quad i = 1,2 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial g_i(x_1,x_2)}{\partial x_2} = {} & \frac{(-1 + x_1^2)(-28 + 18x_1^2 - 8x_1^4 + x_1^6 - 36x_1x_2 - 8x_1^3x_2 + 2x_1^5x_2 + 30x_2^2 - 64x_1^2x_2^2)}{(x_1 + x_2)^8} \\ & + \frac{(-1 + x_1^2)(7x_1^4x_2^2 + 8x_1x_2^3 - 4x_1^3x_2^3 - 12x_2^4 + 21x_1^2x_2^4 + 6x_1x_2^5 + 3x_2^6)}{(x_1 + x_2)^8}, \quad i = 1,2. \end{aligned}$$

Then, the following result can be stated, by using Proposition 1.



(a) $p(x)$

(b) $q(x)$

Fig. 3. Dynamical plane of multivariate NAd1 method on $p(x)$ and $q(x)$.

Theorem 8. *The 2-dimensional rational function associated to fourth-order NAd1 scheme on the polynomial $q(x)$ has four fixed points,*

(i) *two of them are the roots of the polynomial $(-1,-1)$ and $(1,1)$, which are superattracting, and*
(ii) *two strange fixed points $(-0.65920017,-0.65920017)$ and $(0.65920017, 0.65920017)$ that are repulsive, as*

$$\left|\frac{\partial g_i(\pm 0.65920017,\pm 0.65920017)}{\partial x_j}\right| = 2.211080, \quad \text{for any} \quad i, \quad j \in \{1,2\}.$$

Moreover, the only critical points are the roots of $q(x)$.

The dynamical behavior of NAd1 is presented in Fig. 3(b), where the stable and unstable fixed points lie in Fatou and Julia sets, respectively. A very stable behavior can be observed, as in $p(x)$ case.

5. Third- and fourth-order HMT family of iterative methods

Let us consider now a parametric family of iterative methods which involves only a functional evaluation of the Jacobian matrix. In [18] the authors prove that its members have, in general, third-order of convergence although there are two elements of the class with fourth-order of convergence, corresponding to the values ± 1 of the parameter. Its iterative expression is

$$\begin{aligned}
y^{(k)} &= x^{(k)} - \theta[F'(x^{(k)})]^{-1}F(x^{(k)}),\\
z^{(k)} &= x^{(k)} - [F'(x^{(k)})]^{-1}\left[F(y^{(k)}) + \theta F(x^{(k)})\right],\\
x^{(k+1)} &= x^{(k)} - [F'(x^{(k)})]^{-1}\left[F(z^{(k)}) + F(y^{(k)}) + \theta F(x^{(k)})\right].
\end{aligned}$$

We denote again by $H_{ij}(x)$ the (i,j) entry of $H(x) = F'(x)^{-1}$, by $\lambda(x) = x - \theta H(x)F(x)$ the iteration function for the first step and by $\mu(x) = x - H(x)(F(\lambda(x)) + \theta F(x))$ the iteration function for the second step. Thus, the jth-component of the iteration function corresponding to a method of the class HMT is

$$\bar{g}_j(x) = x_j - \sum_{i=1}^{n} H_{ji}(x)(f_i(\mu(x)) + f_i(\lambda(x)) + \theta f_i(x)). \tag{9}$$

As in the previous cases, an Scaling Theorem can be established. When method HMT is applied on polynomial $p(x)$, components of the iteration function become

$$\begin{aligned}
g_1(x) &= -\frac{8\theta^2x_1^2(-1+x_1^2)^3 + \theta^4(-1+x_1^2)^4 - 16x_1^4(-1+6x_1^2+3x_1^4)}{128x_1^7},\\
g_2(x) &= -\frac{8\theta^2x_2^2(-1+x_2^2)^3 + \theta^4(-1+x_2^2)^4 - 16x_2^4(-1+6x_2^2+3x_2^4)}{128x_2^7}.
\end{aligned}$$

The following properties of the rational function associated to the family have been proved. In Fig. 4 dynamical planes for different values of the parameter *theta* are showed. Let us note that the most stable behavior is found when $|\theta| = 1$ and they become more unstable when $|\theta|$ increases.

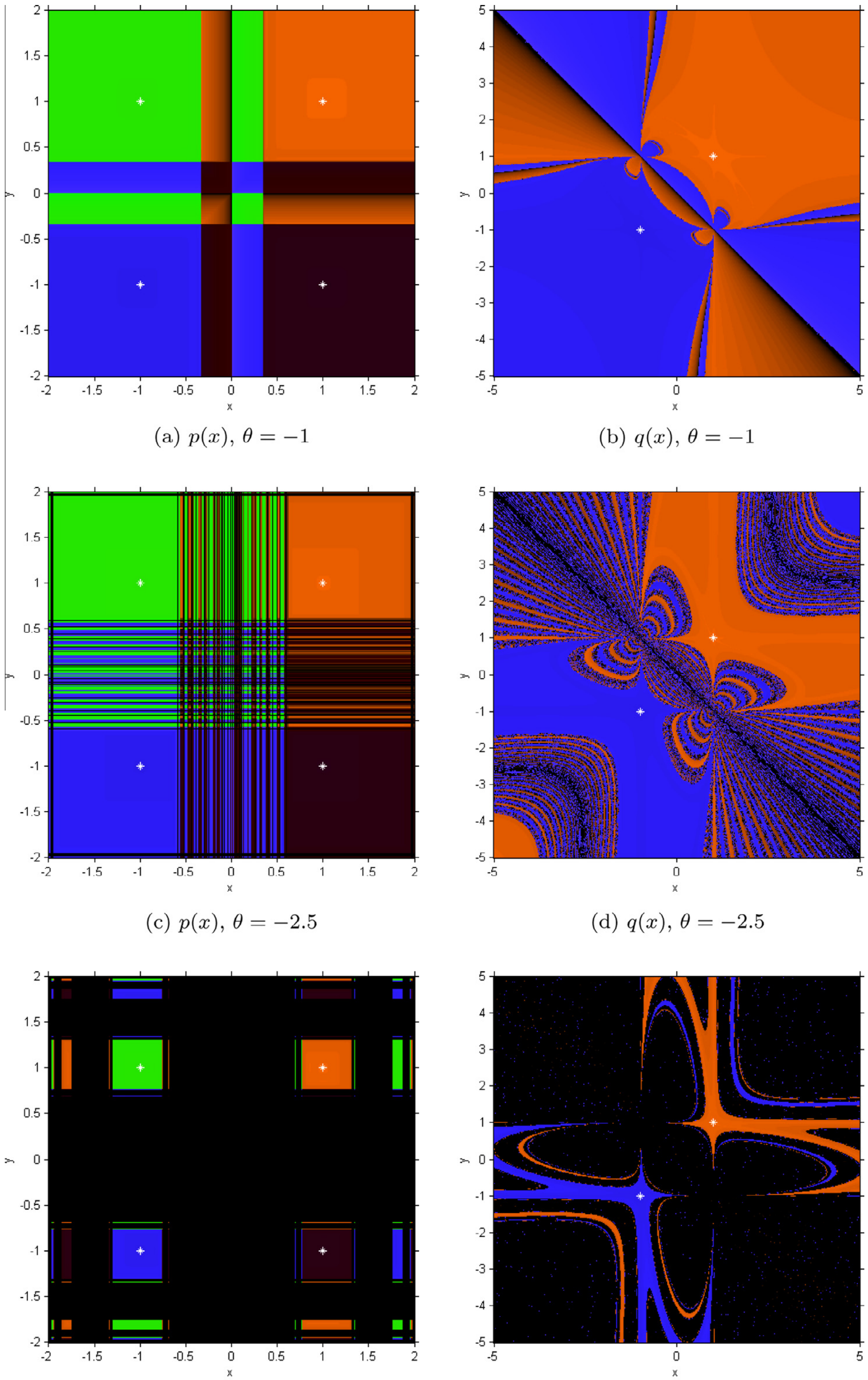
Theorem 9. *The number of fixed points of the vectorial rational function associated to HMT iterative method on polynomial $p(x)$ is 64, and*

(i) *The four fixed points corresponding to the roots of $p(x)$ are superattractive, for all values of θ.*
(ii) *There exist 48 strange fixed points that are unstable, for all values of θ.*
(iii) *There exist 12 real strange fixed points that are simultaneously attractive for two ranges of values of θ, $[-0.3847551,-0.3838109]$ and $[0.3838109, 0.3847551]$. Moreover, they are superattractive if $\theta \approx -0.3843718$ or $\theta \approx 0.3843718$.*

In Fig. 5 the dynamical plane of HMT method for $\theta \approx -0.3843718$ can be seen. Let us note that twelve strange fixed points appear as white circles, with their own basins of attraction.

We show in Fig. 6 the stability function $\left|\frac{\partial g_1(s^1)}{\partial x_1}\right|$ associated to s^1, one of these 12 strange fixed points that can be attractive, of components

$$\begin{aligned}
s_1^1 &\approx -\sqrt{\frac{(-13.44+23.28i) - (0.53+0.92i)A^2 + (73.92-128.03i)\theta^2 + (29.40-50.92i)\theta^4 + A\left(\frac{16}{3}(1+\theta^2)+\theta^4\right)}{A(80+8\theta^2+\theta^4)}},\\
s_2^1 &= -1,
\end{aligned} \tag{10}$$



(a) $p(x)$, $\theta = -1$ (b) $q(x)$, $\theta = -1$

(c) $p(x)$, $\theta = -2.5$ (d) $q(x)$, $\theta = -2.5$

(e) $p(x)$, $\theta = -4.5$ (f) $q(x)$, $\theta = -4.5$

Fig. 4. Dynamical plane of multivariate HMT method on $p(x)$ and $q(x)$.

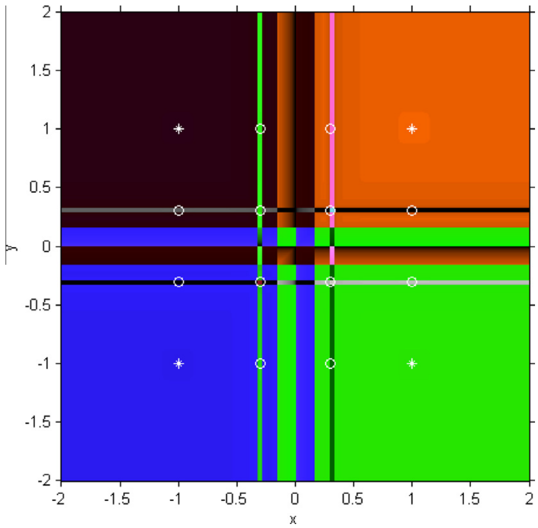


Fig. 5. Dynamical plane of multivariate HMT method on $p(x)$ for $\theta \approx -0.3843718$.

where $A \approx \left(128 - 1056\theta^2 + 1032\theta^4 - 214\theta^6 - 27\theta^8 + 5.20\sqrt{\theta^6(-4+27\theta^2)(80+8\theta^2+\theta^4)^2}\right)^{\frac{1}{3}}$.

In this case, $\left|\frac{\partial g_2(s^1)}{\partial x_2}\right| = 0$ for all real values of θ. Let us remark that, by the nature of the system we are solving, the other two partial derivatives are always null. Moreover, the intervals where the point is attractive can be observed in Fig. 6.

The stability functions of other strange fixed points have been also analyzed: in some cases, only one of the partial derivatives satisfied, $\left|\frac{\partial g_i(s^h)}{\partial x_i}\right| < \frac{1}{2}$, satisfying the other one $\left|\frac{\partial g_j(s^h)}{\partial x_j}\right| > 1$. Taking into account that only the elements of the diagonal are different from zero, these points are saddle points, being the rest of them repulsive.

Respect to the critical points, the following statements can be proved.

Theorem 10. *The number of critical points of the vectorial rational function associated to HMT iterative method on the quadratic polynomial $p(x)$ is 20, 16 of them are free critical points, whose components are one of the following:*

$$c_1 = -\sqrt{\frac{\gamma - 4\sqrt{\alpha}\theta}{\beta}}, \quad c_2 = \sqrt{\frac{\gamma - 4\sqrt{\alpha}\theta}{\beta}}, \quad c_3 = -\sqrt{\frac{\gamma + 4\sqrt{\alpha}\theta}{\beta}}, \quad c_4 = \sqrt{\frac{\gamma + 4\sqrt{\alpha}\theta}{\beta}},$$

where $\alpha = 4\theta^2 + 11\theta^3 + \theta^4$, $\beta = -48 + 8\theta^2 + \theta^4$ and $\gamma = -20\theta^2 - 3\theta^4$.

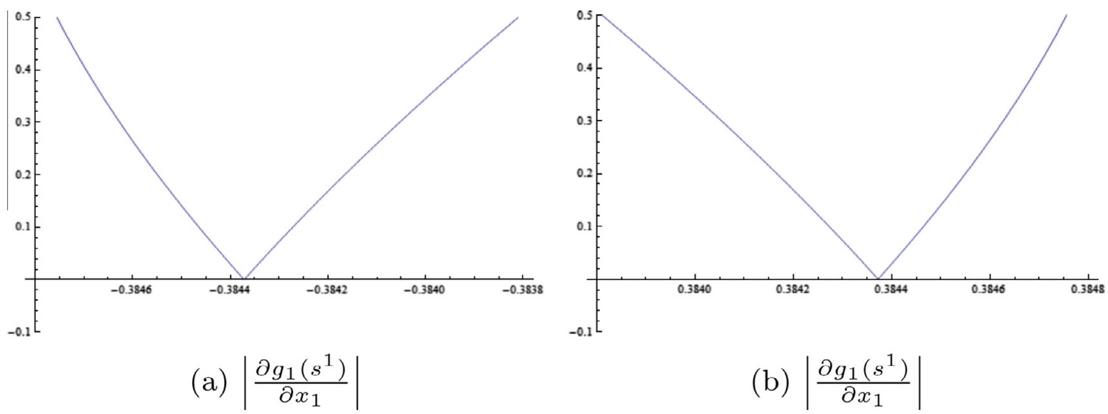


Fig. 6. One of the stability functions of HMT family on $p(x)$.

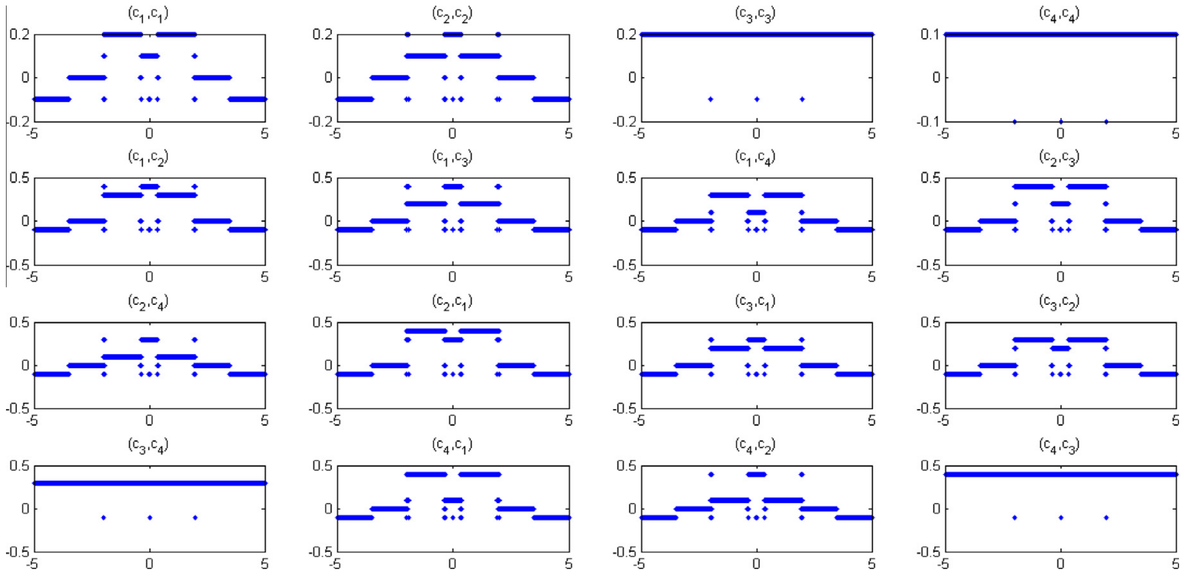
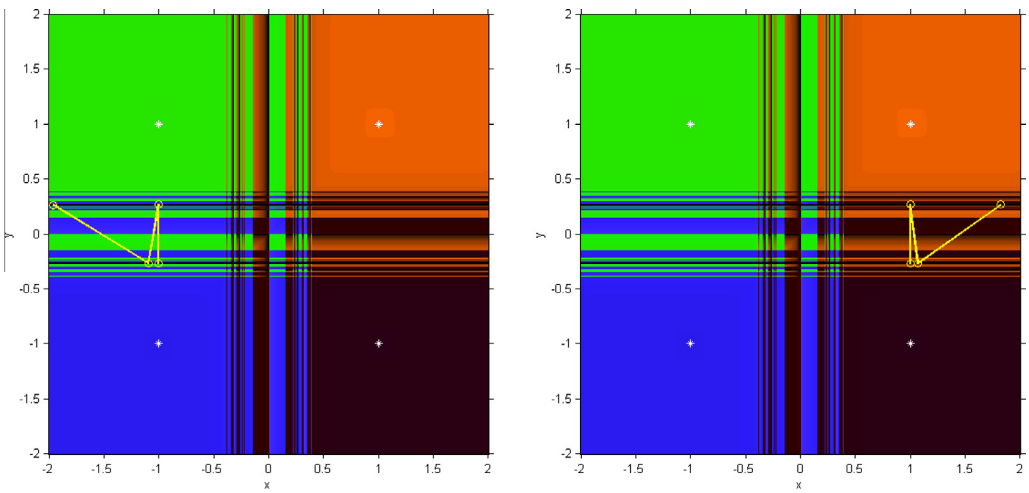


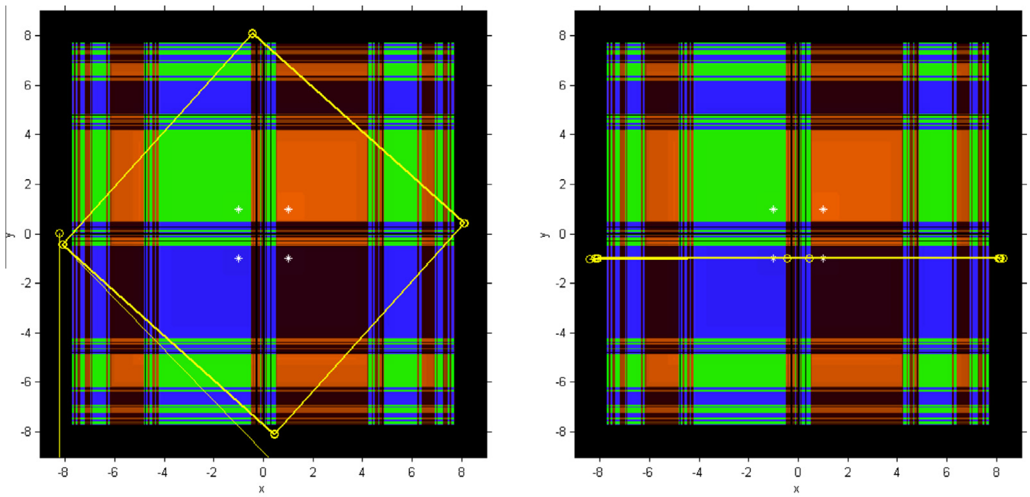
Fig. 7. Different parameter plots of HMT family on $p(x)$.

By using the information that gives us the iteration of the elements of the family on the different free critical points, we can know the global behavior of the family depending on the value of the parameter. When the complex dynamics of iterative methods on polynomials is analyzed, the parameter plane plays an important role (see, for example, [14] or [19]). In Fig. 7 we show the orbits of each one of the free critical points (see Theorem 10). In each one of these pictures, a different free critical point is used as starting point of each member of the family of iterative schemes, taking values of the parameter θ in $[-5,5]$. These values (and the associated members of the family) correspond to the different values of the abscissa, and the ordinate corresponds to 0.1, 0.2, 0.3 or 0.4 if the iterative process has converged to $(1,1)$, $(-1,-1)$, $(-1,1)$ or $(1,-1)$, respectively. Moreover, the ordinate of a point is -0.1 if the process diverges and it is null in other cases. Indeed, when $|\theta| > 2$, the components c_1 and c_2 of the critical points become complex, so in these cases the method can lead us to a complex result. Therefore, the aim of these graphics is to observe the values of the parameter θ (it is, the elements of the family HMT) whose iterations yields to unstable behaviors (attracting strange fixed points, periodic orbits,...). These plots have been obtained by using 20,000 subintervals, a maximum of 40 iterations and an error estimation of 10^{-3}, when the iterates tend to a fixed point.



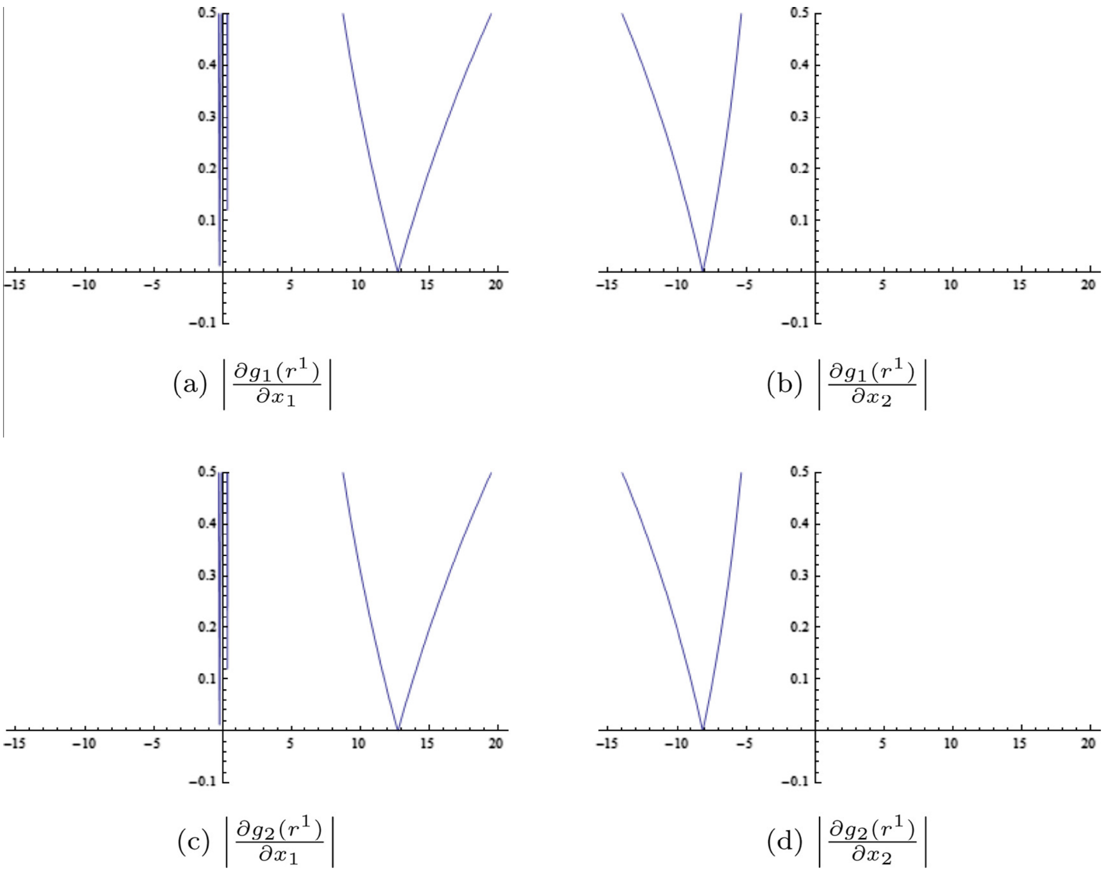
(a) 2-Orbit 1 (b) 2-Orbit 2

Fig. 8. 2-Periodic orbits for $\theta = -0.3355$ in HMT family.



(a) 4-Orbit 1 (b) 4-Orbit 2

Fig. 9. 4-Periodic orbits for $\theta = -1.9490$ in HMT family.



(a) $\left|\dfrac{\partial g_1(r^1)}{\partial x_1}\right|$ (b) $\left|\dfrac{\partial g_1(r^1)}{\partial x_2}\right|$

(c) $\left|\dfrac{\partial g_2(r^1)}{\partial x_1}\right|$ (d) $\left|\dfrac{\partial g_2(r^1)}{\partial x_2}\right|$

Fig. 10. Stability functions of HMT family on $q(x)$.

It can be observed that four free critical points ((c_3, c_3), (c_4, c_4), (c_3, c_4) and (c_4, c_3)) converge almost always to one of the roots of the polynomial $p(x)$, or diverge for θ in small intervals. In the rest of plots in Fig. 7, it can be noted that the intervals where exist 12 attracting strange fixed points appear (see Theorem 9), but also do different very small regions (all of them with ordinate zero). These regions correspond to values of the parameter whose respective vectorial rational functions (associated to the iterative methods) have basins of attraction of periodic orbits. For example, Fig. 8 shows two 2-periodic orbits for $\theta = -0.3355$, $\{(-1.0000, -0.2665), (-1.0000, 0.2665)\}$ and $\{(1.0000, -0.2665), (1.0000, 0.2665)\}$ in an small interval near the one of attracting strange fixed points.

Indeed, in Fig. 9, some periodic orbits of period 4 appear, which correspond to the value of the parameter $\theta = -1.9490$, which is very close to the values of the parameter where the method has problems of convergence. The 4-periodic orbits appearing in Fig. 9 are only two of the seven periodic orbits of period four that exist for this value of the parameter θ.

Finally, let us analyze the behavior of the parametric family HMT on $q(x)$. The coordinate functions of the associated rational iteration function are calculated. Then, the different partial derivatives must be obtained and evaluated at each one of the fixed strange points. In Fig. 10, the partial derivatives of the coordinate functions at one of the strange fixed points r^1 is showed for values of the parameter in $[-15,20]$. From them, and by using Proposition 1, it can be deduced that this point is repulsive, as the intersection between the intervals where the partial derivatives are lower than $\frac{1}{2}$ is empty.

Similar statements can be deduced for the rest of strange fixed points. The analysis of fixed and critical points can be summarized in the following result.

Theorem 11. *The rational function associated to fourth-order parametric HMT family of iterative methods on the polynomial $q(x)$ has six fixed points,*

(i) *two of them are the roots of the polynomial $(-1, -1)$ and $(1, 1)$, which are superattracting, and*
(ii) *four strange points r^i, $i = 1, \ldots, 4$ depending on the parameter θ. The character of these strange fixed points is repulsive for all the values of the parameter, as*

$$\left|\frac{\partial g_j(r^i)}{\partial x_k}\right| > \frac{1}{2} \quad \text{for all} \quad j, \quad k \in \{1,2\}, \quad i \in \{1,2,3,4\}.$$

Moreover, the only critical points are the roots of $q(x)$.

6. Conclusions

In this paper, we have introduced a new tool to analyze the stability of the fixed points of vectorial rational functions corresponding to iterative schemes for solving nonlinear systems. We have also checked its consistence by applying it, not only on known methods as Newton' and Traub's schemes, but also on a parametric family of iterative procedures. In all cases, these methods are applied on quadratic polynomial systems. The parametric plots, as the extension of parameter planes in multivariable case, has revealed to be an interesting procedure that allows us to detect the most stable and unstable elements of a family of iterative methods.

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