ICSI 401 Midterm Exam (70 points)
October 5, 2020

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> This exam is open book, open note. Read all directions carefully and write your answers in the space provided (you can use additional space if you need it). To receive full credit, you must show all of your work. When you are finished, submit your scan, Word document, or PDF document to Blackboard.
> **The due date is Friday, October 9, 2020, at 11:59 p.m.**

1. (10 points) Which of the following is a valid expression for the function $f(x) = e^x x - x$ from Taylor's remainder theorem?

   A. $x^2 + \frac{x^3}{2} + (4e^0 + 0e^0)\xi^4/4!$, where ξ is some real number between 0 and x.

   B. $x^2 + \frac{x^3}{2} + (4e^0 + 0e^0)x^4/4!$.

   C. $x^2 + \frac{x^3}{2} + (4e^\xi + \xi e^\xi)x^4/4!$ where ξ is some real number between 0 and x.

   D. None of the above. Taylor's remainder theorem doesn't apply because $f(x)$ isn't differentiable everywhere.

   Hint: I've listed some derivatives of f: $f'(x) = -1 + xe^x + e^x$; $f''(x) = 2e^x + xe^x$; $f'''(x) = 3e^x + xe^x$; $f^{(4)}(x) = 4e^x + xe^x$.

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$= 0 + (-1+1) + \frac{\cancel{2}x^2}{\cancel{2!}} + \frac{\cancel{3}x^3}{\cancel{3!}}$$

$$P_3(x) = x^2 + \frac{x^3}{2!}$$

$$R_3(x) = f(x) - P_3(x) = \frac{f^{(4)}(\xi)x^4}{4!}$$

2. (10 points) The following code is meant to find a root of the input function F inside the interval $[a, b]$.

```
%
% Implements multisection search to find a root
% of F in the interval [a, b].  Runs for k iterations.
%
function x = multisection_search(F, a, b, k):
    if (F(a) <= 0 && F(b) > 0)
        direction = 1
    elseif (F(a) > 0 && F(b) <= 0)
        direction = -1
```

```
    end
    for i = 1:k
        z = a/10 + 9*b/10
        if (F(z) == 0)
            x = z
            return(x)
        end
        if (F(z) * direction < 0)
            a = z
        elseif (F(z) * direction > 0)
           b = z
        end
    end
    x = z
end
```

Assume infinite precision, and assume that F is a continuous function.

(a) Is z guaranteed to converge to a root of F in the interval $[a, b]$ if $F(a)$ and $F(b)$ have opposite signs and F is a continuous function? If so, what theorem guarantees this?

Yes this is guarenteed by the Intermediate value theorem

(b) What is the maximum possible length of the new interval, in terms of $b - a$, after a single iteration of the i loop?

$\frac{b-a}{10}$ or $\frac{9(b-a)}{10}$

so max poss length is $\frac{9}{10}(b-a)$

3. (10 points) (a) Derive the Newton update equation (i.e., x_{k+1} in terms of x_k) for the function $F(x) = \sin(x)$.

$$p_1(x) = \sin(x_0) + \cos(x_0)(x - x_0) = 0$$

$$\Rightarrow \frac{-\sin(x_0)}{\cos(x_0)} = x - x_0$$

$$\Rightarrow x = x_0 - \frac{\sin(x_0)}{\cos(x_0)}$$

$$\Rightarrow \boxed{x_{k+1} = x_k - \frac{\sin(x_k)}{\cos(x_k)}}$$

(b) Suppose that you choose an initial point $x_0 \in (\pi, 3\pi/2)$ sufficiently close to π. Can convergence of Newton's method be guaranteed? Why, or why not?

$$g(x) = x - \frac{\sin x}{\cos x}$$

Also $|g(\pi) - g(x)| \leq C \cdot |\pi - x|$ where $C < 1$

$$g'(x) = 1 - \sec^2 x$$

if $x_0 \to \pi$, $|g'(x_0)| \to 0$

$$\lim_{x_0 \to \pi} |g'(x_0)| = |1 - \lim_{x_0 \to \pi} \sec^2 x|$$
$$= |1 - 1|$$
$$= |0| = 0$$

So for x_0 sufficiently close to π, $|g'(x_0)| < 1$

so g is a contraction on some interal around π

$\Rightarrow$ By Banach's fixed point theorem, convergence is guaranteed

4. (10 points) Consider finding fixed points of the function $F(x) = 2xe^x + x$.

(a) For which values of x is $F(x)$ a contraction?

$|F'(x)| = |2xe^x + 2e^x + 1|$ where is this < 0?

$= |1 + 2e^x(x+1)| \Rightarrow x + 1 < 0$

$x < -1$

$\boxed{\text{So it is a contraction when } x < -1}$

(b) Will fixed point iteration starting with some point $x_0 \in [-1/2, 1]$ converge to a fixed point of F in that interval? If so, why? If not, why doesn't Banach's fixed point theorem apply?

For x_0 in $[-\frac{1}{2}, 1]$ $\quad f(x_0) = x_0(2e^{x_0} + 1)$

$2e^{x_0} + 1 > 1 \Rightarrow f(x_0) > x_0$

$f(1) > 1 \Rightarrow f(1) \notin [-\frac{1}{2}, 1]$

so no it does not converge within that interval $[-\frac{1}{2}, 1]$

5. (10 points) Consider the function $F(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$.

(a) Derive an expression for the relative condition number $\kappa(x)$ of $F(x)$.

$$K(x) = \left|\frac{x F'(x)}{F(x)}\right| \qquad F'(x) = -\csc^2 x$$

$$= \left|\frac{x \cdot (-\csc^2 x)}{\cot x}\right| = \left|\frac{-x \frac{1}{\sin^2 x}}{\frac{\cos x}{\sin x}}\right| = \boxed{\left|-x \cdot \sec(x) \cdot \csc(x)\right|}$$

(b) Determine all points $x_* \in \mathbb{R}$ near which $F(x)$ is ill-conditioned, in the sense that $\lim_{x \to x_*} \kappa(x) = \infty$.

$$K(x) = \left|-x \sec x \cdot \csc x\right| = \left|-x \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x}\right| \to \infty ?$$

$\frac{1}{\sin x} \to \infty$ when $x \Rightarrow k\pi$

$\frac{1}{\cos x} \to \infty$ when $x \to \frac{\pi}{2} + k\pi$

$$\lim_{x \to 0} \frac{-x}{\sin x} = -1$$

$\boxed{\text{F(x) is ill conditioned near } x = k\pi + \frac{\pi}{2} \text{ and } k\pi \text{ ; where } k \neq 0}$

6. (10 points) If a function $F(x) = \Theta(x^2)$ as $x \to \infty$, is it always true that $3F(x) = \Theta(x^2)$? If so, explain why. If not, give a counterexample.

   Hint: $F(x) = \Theta(G(x))$ as $x \to \infty$ means that there exist positive constants C_1, C_2 such that $0 < C_1 \le \left|\frac{F(x)}{G(x)}\right| < C_2$ for all large enough x.

Yes bc $3x^2$ is $\theta(x^2)$ as proven by the properties of asymptotic notations

$$\lim_{x\to\infty} \left|\frac{3x^2}{x^2}\right| = 3$$

$C \cdot \theta(g(x)) = \theta(g(x))$

7. (10 points) Consider a floating point number system with 8 mantissa bits and 8 exponent bits that works as follows: if a number N is $+(1.x_1x_2x_3...x_8)_2 \times 2^{(y_7y_6y_5...y_0)_2}$, then its floating point representation is given by $\boxed{1}\boxed{x_1x_2...x_8}\boxed{y_7y_6...y_0}$ where the first bit represents the positive sign. Note that the initial 1 in the mantissa is not represented explicitly, so we are using a *hidden bit representation.*

   (a) What is the floating point representation of the number 3.75? Write your answer in binary, clearly giving the mantissa and exponent bit strings.

$3.75 \le 2^1 \quad j = 1$

$3.75 - 2^1 = 1.75 \le 2^0 \quad j = 0$

$1.75 - 2^0 = .75 \le 2^{-1} \quad j = -1$

$.75 - 2^{-1} = .25 \le 2^{-2} \quad j = -2$

$(3.75)_{10} = (11.11)_2$

$= (1.111)_2 \cdot 2^1$

$\boxed{1}\boxed{11100000}\boxed{00000001}$

mantissa exponent

(b) What is the smallest number larger than 3.75 that is representable in the floating point number system described above? Write your answer in the binary floating point format, clearly giving the mantissa and exponent bit strings.

1	1110 0001	0000 0001