ICSI 401 – Numerical Methods

Instructor: Abram Magner Date: Fall 2020

Instructions: Please answer the following questions in complete sentences, showing all work (including code and output of programs, when applicable). I prefer that you type your solutions (e.g., using LaTeX, with Overleaf, TeXworks, etc., or Word), but will accept handwritten notes. If the grader cannot read your handwriting, then they cannot award you points.

Due date: Monday, 9/14/2020, 11:59 p.m., on Blackboard.

1.1 Calculus, Taylor series

Consider the function $f(x) = \frac{\sin(x)}{x}$.

1. Compute $\lim_{x\to 0} f(x)$ using l'Hôpital's rule.
2. Use Taylor's remainder theorem to get the same result:
   (a) Write down $P_1(x)$, the first-order Taylor polynomial for $\sin(x)$ centered at $a = 0$.
   (b) Write down a good upper bound on the absolute value of the remainder $R_1(x) = \sin(x) - P_1(x)$, using your knowledge about the derivatives of $\sin(x)$. The goal here is to show that $R_1(x)/x$ is negligible.
   (c) Express $f(x)$ as $f(x) = \frac{P_1(x)}{x} + \frac{R_1(x)}{x}$, and compute the limits of the two terms as $x \to 0$.

1.2 Asymptotic notation

Recall the definitions of the asymptotic notations. We will say that $f(x)$ has "order of growth x^α as $x \to x_0$" (where x_0 is either some fixed real number or $\pm\infty$) if $f(x) = \Theta(x^\alpha)$ as $x \to x_0$.

1. Consider the functions $f(x) = x \sin x$ and $g(x) = x$. Is $f(x) = \Theta(g(x))$ as $x \to \infty$? Why or why not? (Hint: As always, you should refer back carefully to the definition of $\Theta(\cdot)$.)
2. Suppose that we know that $f(x) = x + \Theta(x^2)$ and $g(x) = \Theta(x) > 0$ as $x \to 0$. Determine the order of growth of $f(x) + g(x)$.

   (This problem is meant to get you comfortable with manipulating asymptotic notation when it appears in expressions. When I say something like "$f(x) = x + \Theta(x^2)$", this means that there is some function $h(x) = \Theta(x^2)$, and $f(x) = x + h(x)$. That is, the fact that $h(x) = \Theta(x^2)$ is the *only* thing you know about $h(x)$.)
3. Suppose that we know that $f(x) = e^{\Theta(x)}$ as $x \to \infty$. Does this imply that $f(x) = \Theta(e^x)$? (Hint: Think carefully about the definition of $\Theta(\cdot)$, and consider $f(x) = e^{2x}$.)

1.3 Relative versus absolute error

1. Suppose that you are approximating a function $g(n)$ by some function $f(n)$. Suppose, further, that you know that the absolute error in approximating $g(n)$ by $f(n)$ satisfies $|f(n) - g(n)| = o(1)$ as $n \to \infty$ (that is, $\lim_{n\to\infty} |f(n) - g(n)| = 0$). Is it true that the *relative* error also decays to 0? If not, come up with functions $f(n)$ and $g(n)$ for which this is not true. (Hint: Come up with some $g(n)$ and $f(n)$ satisfying $g(n) = o(1)$ and $f(n)/g(n) = \Theta(1)$.)

1.4 Matlab warmup/Gentle linear algebra review

1. Complete G&C Chapter 2, Exercise 2.
2. Complete G&C Chapter 2, Exercise 3.