Confirmatory Factor Analysis

Today . . .

- Variance of an item
- Identifiability
- Goodness of fit indices
- A CFA example

$$\sigma^2_{X_1X_1} = \lambda^2_{11} + \sigma^2_{\delta_1}$$

This reconstruction of the variance for X_1 has its origins in classical measurement theory. If X_1 and ξ_1 are standardized, then λ_{11} is the correlation of X_1 and ξ_1. The correlation between an observed score and a true score or latent variable is called the index of reliability. It is the square root of the reliability for X_1. So, λ^2_{11} is the reliability for X_1, which is the proportion of variance in X_1 due to ξ_1. With X_1 in standard score form, λ^2_{11} is also the true score variance. $\sigma^2_{\delta_1}$ is then the error variance in the classical measurement theory sense.

The "Identification Problem"

Identification means that there is sufficient information in the data and model to provide unique estimation of the free parameters. Identification is obtained by restricting the model in some way.

Example: There are an infinite number of solutions to the "model": X + Y = 15. A unique solution can only be found by restricting the model in some way. If X is restricted to equal 5, then the unique solution for Y is 10. Likewise an equality constraint, X = Y, provides for a unique solution.

Similarly, and with greater complexity, confirmatory factor models must have a sufficient number of restrictions before the parameters can be estimated uniquely.

Note how this is different compared to exploratory factor analysis. We impose no constraints in EFA and there are an infinite number of acceptable solutions (i.e., rotations).

In CFA, models are restricted by either forcing some parameters to specific values (usually 0 or 1) or constraining some parameters to be equal to other parameters.

One important condition for identification is called the order condition. This means that the number of parameters estimated in the model is less than or equal to the number of distinct values in the variance-covariance matrix (S).

The number of distinct elements in the S matrix with K measures is K(K+1)/2.

Depending on the number of estimated parameters relative to the number of known values (number of unique elements in the variance-covariance matrix):

- Over-identified models (degrees of freedom > 0)
- Just-identified models (0 degrees of freedom; trivial perfect fit)
- Under-identified models (cannot be estimated)

Goodness of fit

- Absolute indices
- Comparative indices
- Parsimony indices

Absolute indices provide a goodness of fit indication without reference to any other model.

Most common is the χ^2. If significant, the model does not do an adequate job of reproducing the original covariance matrix.

Very likely to reject good fitting models as sample size increases.

More useful in comparing nested models.

Root Mean Square Error of Approximation:

$$RMSEA = \frac{\sqrt{\chi^2_{proposed} - df_{proposed}}}{\sqrt{df_{proposed}(N-1)}}$$

A close fitting model $\leq .05$

Confidence intervals can be placed around RMSEA and provide an estimate of uncertainty.

"Probability of close fit" is often reported and should be greater than .05, preferably much greater.

Standardized Root Mean Residual (SRMR):

SRMR = mean difference between the observed correlations and the predicted correlations (Σ compared to S).

A close fitting model $\leq .08$.

Comparative fit indices provide a goodness of fit indication with reference to another model, most commonly (by default) the null model.

The null model is not usually a compelling alternative. It is usually defined as assuming all correlations among variables are zero.

Tucker-Lewis (Non-Normed Fit) Index:

$$TLI\ or\ NNFI = \frac{\dfrac{\chi^2_{null}}{df_{null}} - \dfrac{\chi^2_{proposed}}{df_{proposed}}}{\dfrac{\chi^2_{null}}{df_{null}} - 1}$$

A good fitting model $\geq$ .95

Comparative Fit Index:

$$CFI = \frac{(\chi^2_{null} - df_{null}) - (\chi^2_{proposed} - df_{proposed})}{\chi^2_{null} - df_{null}}$$

A good fitting model $\geq$ .95

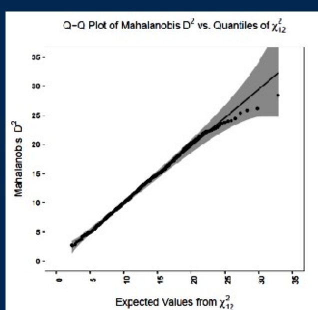
Parsimony fit indices take model complexity into account in much the same way as the adjusted R^2 in multiple regression.

These are less commonly reported.

The confirmatory approach begins with an explicit model that implies the elements of the key matrices. Some of these parameters are assumed to be fixed (usually at 0 or 1), some are freely estimated, and some may be constrained (e.g., to be equal to other parameters).

These expectations are then built into the model estimation.

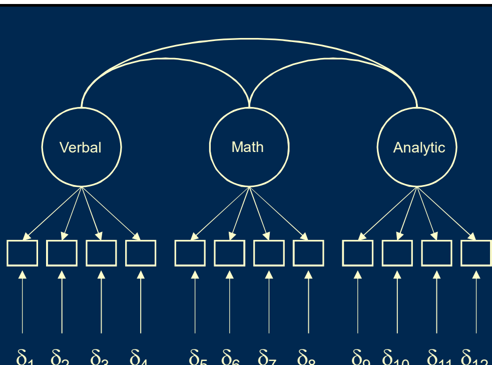
Hypothetical data (N = 500) were created for individuals completing a 12-section test of mental abilities. All variables are in standard form.



$multivariateNormality

	Test	Statistic	p value	Result
1	Nardia Skewness	367.600071004817	0.437353917013108	YES
2	Nardia Kurtosis	-0.866369530731211	0.386287547416061	YES
3	MVN	<NA>	<NA>	YES

No extreme outliers. The data are multivariate normal.

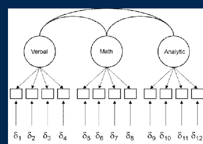


$$\Lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & 0 & 0 \\ \lambda_{41} & 0 & 0 \\ 0 & \lambda_{52} & 0 \\ 0 & \lambda_{62} & 0 \\ 0 & \lambda_{72} & 0 \\ 0 & \lambda_{82} & 0 \\ 0 & 0 & \lambda_{93} \\ 0 & 0 & \lambda_{10,3} \\ 0 & 0 & \lambda_{11,3} \\ 0 & 0 & \lambda_{12,3} \end{bmatrix}$$

$$\Theta_\delta = \begin{bmatrix} \theta^2_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^2_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^2_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^2_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta^2_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{77} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{12,12} \end{bmatrix}$$

The model implies quite a few fixed parameters.

$$\Phi = \begin{bmatrix} 1 & \phi_{12} & \phi_{13} \\ \phi_{21} & 1 & \phi_{23} \\ \phi_{31} & \phi_{32} & 1 \end{bmatrix}$$



The lavaan package can be used to conduct confirmatory factor analyses (in addition to SEM). It requires specifying a model using simple operators (e.f., =~, ~~) and a few simple rules and defaults.

```
mental.model.1 <- '
Verbal =~ Grammar + Paragraph_Comprehension + Vocabulary +
          Sentence_Completion
Math =~ Geometry + Algebra + Numerical_Puzzles +
        Series_Completion
Reasoning =~ Practical_Problem_Solving + Symbol_Manipulation
             + Analytical_Ability + Formal_Logic
Verbal ~~ Math
Verbal ~~ Reasoning
Math ~~ Reasoning
'

CFA_Fit_1 <- cfa(mental.model.1, data = Mental, missing="ML",
estimator="MLR",likelihood="wishart",representation="LISREL")
summary(CFA_Fit_1, standardized=TRUE,rsq=TRUE,fit.measures =
TRUE)
```

```
Number of observations                            500

Number of missing patterns                          1

Estimator                                          ML      Robust
Minimum Function Test Statistic                55.498      56.546
Degrees of freedom                                 51          51
P-value (Chi-square)                            0.309       0.276
Scaling correction factor                                   0.981
  for the Yuan-Bentler correction

User model versus baseline model:

  Comparative Fit Index (CFI)                   0.998       0.997
  Tucker-Lewis Index (TLI)                      0.997       0.996

Root Mean Square Error of Approximation:

  RMSEA                                         0.013       0.015
  90 Percent Confidence Interval          0.000  0.032       0.000  0.033
  P-value RMSEA <= 0.05                         1.000       1.000

Standardized Root Mean Square Residual:

  SRMR                                          0.026       0.026
```

```
Parameter estimates:

  Information                                 Observed
  Standard Errors                   Robust.huber.white

                   Estimate  Std.err  Z-value  P(>|z|)   Std.lv  Std.all
Latent variables:
  Verbal =~
    Grammar           1.000                               0.716    0.734
    Prgrph_Cmprhn     1.065    0.080   13.256    0.000    0.762    0.718
    Vocabulary        0.960    0.070   13.772    0.000    0.687    0.689
    Sentnc_Cmpltn     1.108    0.079   13.940    0.000    0.793    0.731
  Math =~
    Geometry          1.000                               0.669    0.684
    Algebra           1.002    0.082   12.225    0.000    0.670    0.698
    Numercl_Pzzls     0.994    0.080   12.357    0.000    0.664    0.672
    Series_Cmpltn     1.031    0.079   13.081    0.000    0.689    0.716
  Reasoning =~
    Prctcl_Prbl_S     1.000                               0.650    0.695
    Symbol_Mnpltn     1.030    0.081   12.652    0.000    0.670    0.690
    Anlytcl_Ablty     1.097    0.085   12.899    0.000    0.713    0.726
    Formal_Logic      1.143    0.090   12.647    0.000    0.744    0.732

Covariances:
  Verbal ~~
    Math              0.199    0.029    6.782    0.000    0.417    0.417
    Reasoning         0.215    0.029    7.442    0.000    0.461    0.461
  Math ~~
    Reasoning         0.176    0.026    6.759    0.000    0.405    0.405
```

```
R-Square:

    Grammar                  0.539
    Paragraph_Comprehension       0.515
    Vocabulary               0.474
    Sentence_Completion       0.534
    Geometry                 0.467
    Algebra                  0.487
    Numerical_Puzzles        0.452
    Series_Completion        0.512
    Practical_Problem_Solving      0.484
    Symbol_Manipulation       0.476
    Analytical_Ability       0.527
    Formal_Logic             0.536
```

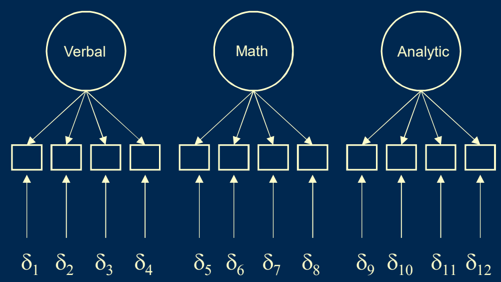
The R-Square reported by lavaan is the proportion of variance in each variable accounted for by the model. Sound familiar?

The chosen model fits the data quite well. How would other models do?

A complete confirmatory analysis would not only test the preferred model but also examine alternative models to assess how easily they could account for the data.

To the extent that reasonable alternatives exist, the preferred model must be considered with more caution.

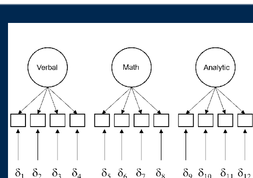
Other models can easily be tested.



$$\Lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & 0 & 0 \\ \lambda_{41} & 0 & 0 \\ 0 & \lambda_{52} & 0 \\ 0 & \lambda_{62} & 0 \\ 0 & \lambda_{72} & 0 \\ 0 & \lambda_{82} & 0 \\ 0 & 0 & \lambda_{93} \\ 0 & 0 & \lambda_{10,3} \\ 0 & 0 & \lambda_{11,3} \\ 0 & 0 & \lambda_{12,3} \end{bmatrix}$$

$$\Theta_\delta = \begin{bmatrix} \theta^2_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^2_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^2_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^2_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta^2_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{77} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{12,12} \end{bmatrix}$$

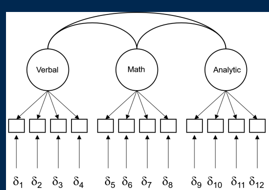
$$\Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

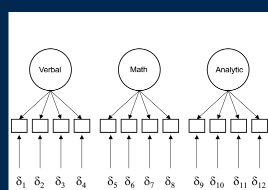



```
mental.model.2 <- '
Verbal =~ Grammar + Paragraph_Comprehension + Vocabulary +
          Sentence_Completion
Math =~ Geometry + Algebra + Numerical_Puzzles +
        Series_Completion
Reasoning =~ Practical_Problem_Solving + Symbol_Manipulation
             + Analytical_Ability + Formal_Logic

Verbal ~~ 0*Math
Verbal ~~ 0*Reasoning
Math ~~ 0*Reasoning
'
```

```
Number of observations                            500

Number of missing patterns                          1
Estimator                                          ML      Robust
Minimum Function Test Statistic               214.022     216.879
Degrees of freedom                                 54          54
P-value (Chi-square)                            0.000       0.000
Scaling correction factor                                   0.987
  for the Yuan-Bentler correction

User model versus baseline model:

  Comparative Fit Index (CFI)                   0.916       0.915
  Tucker-Lewis Index (TLI)                      0.897       0.896

Root Mean Square Error of Approximation:

  RMSEA                                         0.077       0.078
  90 Percent Confidence Interval         0.066  0.088       0.067  0.089
  P-value RMSEA <= 0.05                         0.000       0.000

Standardized Root Mean Square Residual:

  SRMR                                          0.158       0.158
```

Is one model a better fit than the other?

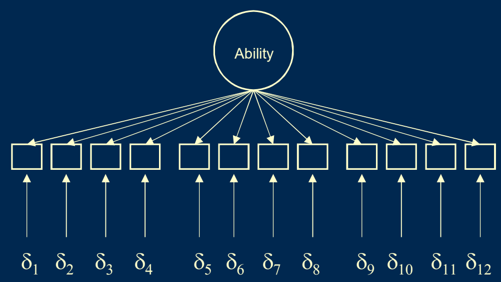




When models are nested, the difference in their chi-square tests is also chi-square distributed, with degrees of freedom equal to the difference in degrees of freedom. Models are nested when constraining one will produce the other.

```
anova(CFA_Fit_1, CFA_Fit_2)

## Scaled Chi Square Difference Test (method = "satorra.bentler.2001")
##
##           Df   AIC   BIC Chisq Chisq diff Df diff Pr(>Chisq)
## CFA_Fit_1 51 15099 15263  55.5
## CFA_Fit_2 54 15252 15403 214.0        147       3     <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

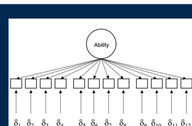
By constraining the latent variable correlations to 0, we produce a model that is a significantly worse fit than the original model.



$$\Lambda = \begin{bmatrix} \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \\ \lambda_{41} \\ \lambda_{51} \\ \lambda_{61} \\ \lambda_{71} \\ \lambda_{81} \\ \lambda_{91} \\ \lambda_{10,1} \\ \lambda_{11,1} \\ \lambda_{12,1} \end{bmatrix}$$

$$\Theta_\delta = \begin{bmatrix}
\theta^2_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \theta^2_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \theta^2_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta^2_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \theta^2_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \theta^2_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{77} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{88} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{99} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{10,10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{11,11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{12,12}
\end{bmatrix}$$

$$\Phi = [1]$$

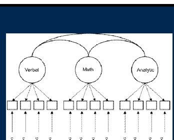


```
mental.model.3 <- '
Ability =~ Grammar + Paragraph_Comprehension Vocabulary +
           Sentence_Completion + Geometry + Algebra +
           Numerical_Puzzles+Series_Completion +
           Practical_Problem_Solving + Symbol_Manipulation +
           Analytical_Ability + Formal_Logic
'
```

```
mental.model.3b <- '
Verbal =~ NA*Grammar + Paragraph_Comprehension + Vocabulary
          + Sentence_Completion
Math =~ NA*Geometry + Algebra + Numerical_Puzzles +
        Series_Completion
Reasoning =~ NA*Practical_Problem_Solving +
             Symbol_Manipulation + Analytical_Ability +
             Formal_Logic
Verbal ~~ 1*Math
Verbal ~~ 1*Reasoning
Math ~~ 1*Reasoning
Verbal ~~ 1*Verbal
Math ~~ 1*Math
Reasoning ~~ 1*Reasoning
'
```

This model will produce identical results and shows that it is a constrained version of the first model and so a nested model comparison is possible.

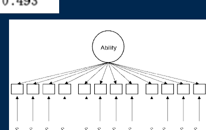
```
  Number of observations                           500

  Number of missing patterns                         1
  Estimator                                         ML      Robust
  Minimum Function Test Statistic              770.567     848.990
  Degrees of freedom                                54          54
  P-value (Chi-square)                           0.000       0.000
  Scaling correction factor                                  0.908
    for the Yuan-Bentler correction

User model versus baseline model:

  Comparative Fit Index (CFI)                    0.622       0.585
  Tucker-Lewis Index (TLI)                       0.538       0.493

Root Mean Square Error of Approximation:

  RMSEA                                          0.163       0.172
  90 Percent Confidence Interval          0.153  0.173       0.161  0.183
  P-value RMSEA <= 0.05                          0.000       0.000

Standardized Root Mean Square Residual:

  SRMR                                           0.112       0.112
```



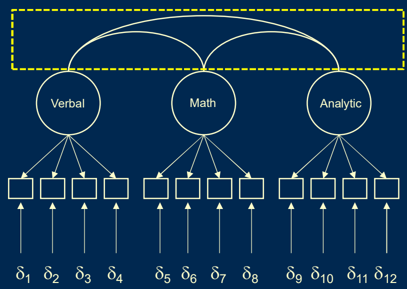
The substantially different fit is evident in the goodness-of-fit indices.

```
User model versus baseline model:

  Comparative Fit Index (CFI)        0.998        0.997
  Tucker-Lewis Index (TLI)           0.997        0.996
```

```
User model versus baseline model:

  Comparative Fit Index (CFI)        0.622        0.585
  Tucker-Lewis Index (TLI)           0.538        0.493
```



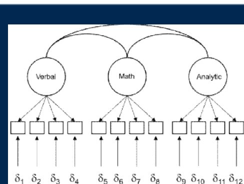
Can the latent variable correlations be set equal?

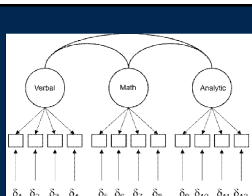


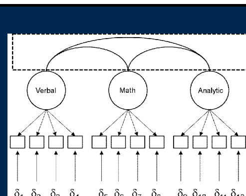
$$\Lambda = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ \lambda_{31} & 0 & 0 \\ \lambda_{41} & 0 & 0 \\ 0 & \lambda_{52} & 0 \\ 0 & \lambda_{62} & 0 \\ 0 & \lambda_{72} & 0 \\ 0 & \lambda_{82} & 0 \\ 0 & 0 & \lambda_{93} \\ 0 & 0 & \lambda_{10,3} \\ 0 & 0 & \lambda_{11,3} \\ 0 & 0 & \lambda_{12,3} \end{bmatrix}$$

$$\Theta_\delta = \begin{bmatrix} \theta^2_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^2_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^2_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^2_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^2_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta^2_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{77} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{10,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{11,11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta^2_{12,12} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & \phi & \phi \\ \phi & 1 & \phi \\ \phi & \phi & 1 \end{bmatrix}$$

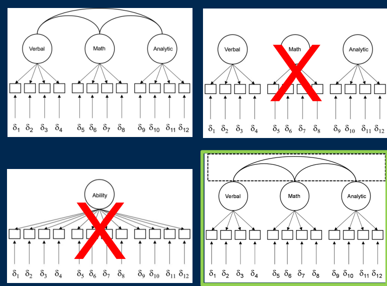







```
anova(CFA_Fit_1, CFA_Fit_4)

## Chi Square Difference Test
##
##           Df   AIC   BIC Chisq Chisq diff Df diff Pr(>Chisq)
## CFA_Fit_1 51 15099 15263  55.5
## CFA_Fit_4 53 15096 15252  56.5       1.02       2        0.6
```

This model is indistinguishable from the original.



Of the four alternatives, two are worse fits than the original. Of the two viable models, the last is the most parsimonious.

Next time . . .

- Scaling of latent variables
- Correlation versus covariance matrices
- Model modification ("exploratory" confirmatory factor analysis)
- Relation to measurement theory
- Measurement invariance (cross-validation)