Problem 5: [Linear Processes] Let $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$ and $|\phi| < 1$. Define the linear process $(X_t \colon t \in Z)$ by letting

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}, \qquad t \in \mathbb{Z}.$$

(a) Show that

$$\sqrt{n}\frac{\hat{\rho}(1) - \rho(1)}{\sqrt{1 - \rho^2(1)}} \xrightarrow{\mathcal{D}} Z \qquad (n \to \infty),$$

where Z denotes a standard normal random variable.

(b) If $\hat{\rho}(1) = .64$ has been obtained from a sample of size $n = 100$, construct a 95% (asymptotic) confidence interval for ϕ.

Problem 6: [Causality and Invertibility] Determine which of the following processes are causal and/or invertible:

(a) $X_t + .2X_{t-1} - .48X_{t-2} = Z_t$,

(b) $X_t + 1.9X_{t-1} + .88X_{t-2} = Z_t + .2Z_{t-1} + .7Z_{t-2}$,

(c) $X_t = 2 + 1.3X_{t-1} - .4X_{t-2} + Z_t + Z_{t-1}$,

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$. In the case of causality, determine the first 5 ψ-weights in the causal representation given in Theorem 2.2.1.

Problem 7: [Causality and Invertibility II] Find the mean and the ACVF of the ARMA(2,1) process given by the difference equations

$$X_t = 2 + 1.3X_{t-1} - .4X_{t-2} + Z_t + Z_{t-1}, \qquad t \in \mathbb{Z},$$

where $(Z_t \colon t \in \mathbb{Z}) \sim \mathrm{WN}(0, \sigma^2)$. Is the process causal and invertible?

Problem 8: [Forecasting] Let $(X_t \colon t \in \mathbb{Z})$ be a mean zero weakly stationary process. Denote by $\Gamma_n = (\gamma(k-j))_{j,k=1,\ldots,n}$ the $n \times n$ matrix with entries $\gamma(k-j) = E[X_{t-j}X_{t-k}]$. Show that Γ_n is nonsingular provided that $\gamma(0) > 0$ and $\gamma(h) \to 0$ as $h \to \infty$.