Sta 237 a— Solutions to Homework Series 1

Problem 1: Observe that the sequences in (a)–(d) are either both weakly and strictly stationary or neither stationarity notion holds true. This is due to the fact that, by assumption, $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of independent zero mean normal random variables with variance σ^2.

(a) (Strictly and weakly) stationary, since $E[X_t] = a$ and

$$\mathrm{Cov}(X_{t+h}, X_t) = \begin{cases} (b^2+c^2)\sigma^2, & h = 0, \\ bc\sigma^2, & h = \pm 1, \\ 0, & |h| > 1, \end{cases}$$

does not depend on t.

(b) Not (strictly and weakly) stationary, since (only $h \geq 0$ is considered)

$$\mathrm{Cov}(X_{t+h}, X_t) = \begin{cases} \sigma^2, & h = 0, \\ \sigma^2 \sin[c(t+1)]\cos(ct), & h = 1, \\ 0, & h > 1, \end{cases}$$

depends on t unless c is a multiple of 2π.

(c) (Strictly and weakly) stationary, since $E[X_t] = a$ and $\mathrm{Cov}(X_{t+h}, X_t) = b^2\sigma^2$ is independent of t.

(d) (Strictly and weakly) stationary, since $E[X_t] = 0$ and

$$\mathrm{Cov}(X_{t+h}, X_t) = \begin{cases} \sigma^4, & h = 0, \\ 0, & h \neq 0, \end{cases}$$

is independent of t.

Problem 2: The left part of Figure 1 shows the graph of the population data including the fitted polynomial $\hat{m}_t = \hat{b}_0 + \hat{b}_1 t + \hat{b}_2 t^2$ for $t = 1790, \ldots, 2000$, where

$$\hat{b}_0 = 2.162 \times 10^{10}, \quad \hat{b}_1 = -2.403 \times 10^7, \quad \hat{b}_2 = 6.681 \times 10^3.$$

The corresponding residuals can be found in the right panel of the same figure. They display an apparent trough for the time of WW II. The predicted values for the years 2010 and 2020 are therefore computed as $\hat{m}_{2010} = 311,608,100$ and $\hat{m}_{2020} = 340,552,400$.

Problem 3: Suppose $z \in \mathcal{M}^\perp$ and $\|z\| = 1$. Then by Cauchy-Schwarz

$$|\langle x, z\rangle| = |\langle x - \hat{x}, z\rangle + \langle \hat{x}, z\rangle| = |\langle x - \hat{x}, z\rangle| \leq \|x - \hat{x}\|$$

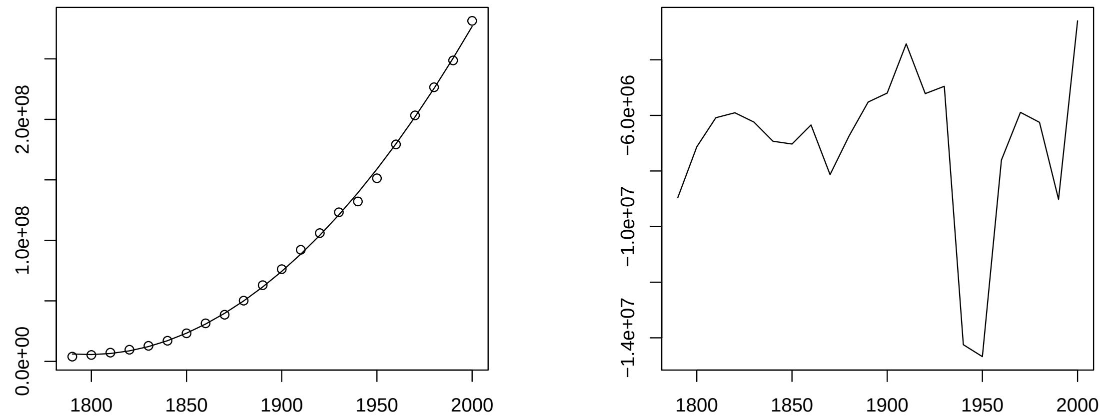

Figure 1: The U.S. population data and the quadratic polynomial fit (left) and the corresponding residuals (right).

and so $\max\{|\langle x, z\rangle| \colon z \in \mathcal{M}^{\perp}, \|z\| = 1\} \le \|x - \hat{x}\|$. Now let $z_0 = (x - \hat{x})/\|x - \hat{x}\|$. Then $z_0 \in \mathcal{M}^{\perp}$, $\|z_0\| = 1$ and

$$\langle x, z_0\rangle = \langle x - \hat{x}, z_0\rangle + \langle \hat{x}, z_0\rangle = \|x - \hat{x}\|$$

which implies $\|x - \hat{x}\| \le \max\{|\langle x, z\rangle| \colon z \in \mathcal{M}^{\perp}, \|z\| = 1\}$. This together with the inequality in the reverse direction gives the desired result.

Problem 4: Show first that $-\sum_{j=1}^{\infty} \theta^j X_{n+1-j}$ converges in norm (that is, in mean square). It is enough to show that, for fixed n, $(Y_m \colon m \in \mathbb{N})$ given by

$$Y_m = \sum_{j=1}^{m} \theta^j X_{n+1-j}, \qquad m \in \mathbb{N},$$

is a Cauchy sequence in L^2. Clearly $E[X_t] = 0$ and $|\langle X_{n+1-k}, X_{n+1-j}\rangle| \le E[X_1^2] = \gamma(0)$, so for $s > r$,

$$\begin{aligned}
\|Y_s - Y_r\|^2 &= \sum_{j=r+1}^{s} \sum_{k=r+1}^{s} \theta^k \theta^j \langle X_{n+1-k}, X_{n+1-j}\rangle \\
&\le \gamma(0) \left(\sum_{j=r+1}^{s} |\theta|^j \right)^2 \\
&\le \gamma(0) \left(\sum_{j=r+1}^{\infty} |\theta|^j \right)^2 \to 0
\end{aligned}$$

as $r \to \infty$ and so the series $-\sum_{j=1}^{\infty} \theta^j X_{n+1-j}$ is convergent in L^2. It is obviously a member of $\overline{\mathrm{sp}}(X_j \colon j \le n)$. Hence it suffices to show that

$$\Big\langle X_{n+1} + \sum_{j=1}^{\infty} \theta^j X_{n+1-j}, X_{n+1-k} \Big\rangle = 0 \qquad \text{for all } k \in \mathbb{N}.$$

But for $k \in \mathbb{N}$, utilizing that $\gamma(0) = \sigma^2(1+\theta^2)$ and $\gamma(1) = -\theta\sigma^2$, we have that

$$\Big\langle X_{n+1} + \sum_{j=1}^{\infty} \theta^j X_{n+1-j}, X_{n+1-k} \Big\rangle = \theta^{k-1}[\gamma(1) + \theta\gamma(0) + \theta^2\gamma(1)] = 0.$$

The mean squared error of prediction is

$$\begin{aligned} E\Big[\sum_{j=0}^{\infty} \theta^j X_{n+1-j} \sum_{k=0}^{\infty} \theta^k X_{n+1-k}\Big] &= \gamma(0)(1+\theta^2+\theta^4+\dots) + 2\gamma(1)\theta(1+\theta^2+\theta^4+\dots) \\ &= \frac{\sigma^2(1+\theta^2)}{1-\theta^2} - \frac{2\sigma^2\theta^2}{1-\theta^2} = \sigma^2. \end{aligned}$$

It is worthwhile remarking that

$$\begin{aligned} \hat{X}_n &= -\sum_{j=1}^{\infty} \theta^j (Z_{n+1-j} - \theta Z_{n-j}) \\ &= -\theta Z_n - \sum_{j=2}^{\infty} \theta^j Z_{n+1-j} + \sum_{j=1}^{\infty} \theta^{j+1} Z_{n-j} \\ &= -\theta Z_n \end{aligned}$$

and hence $E[(X_{n+1} - \hat{X}_{n+1})^2] = E[Z_{n+1}^2] = \sigma^2$.