STA 237 — Time Series Analysis
Homework Series 1

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- **Problem 1: [Stationarity]** Let $(Z_t :\ t \in \mathbb{Z})$ be a sequence of independent zero mean and normal random variables with variance σ^2 and let a, b, and c be constants. Which of the following processess are weakly and/or strictly stationary? For wach weakly stationary process specify the mean and ACVF.

  (a) $X_t = a + bZ_t + cZ_{t-1}$.

  Answer: Let $i, j \in \mathbb{Z}$ be any two integers. Then,

  $$E[X_i] = E[a + bZ_i + cZ_{i-1}] = a = E[a + bZ_j + cZ_{j-1}] = E[X_j],$$

  and

  $$Var[X_i] = Var[a + bZ_i + cZ_{i-1}] = 0 + b^2\sigma^2 + c^2\sigma^2 = Var[a + bZ_j + cZ_{j-1}] = Var[X_j],$$

  where we have implicitly used the fact that every pair of Z_t have zero covariance.

  Thus, with knowledge that linear combinations of normally distributed random variables are normal, we have

  $$X_i, X_j \sim \mathcal{N}(a, (b^2 + c^2)\sigma^2).$$

  Since i, j were arbitrary, we have that every element X_t has the same distribution, which implies that X_t is *strictly stationary.*

  Since we have explicitly shown that we have finite variance, then we may also conclude that X_t is *weakly stationary.* $E[X_t] = a\ \forall t$, and note that

  $$Cov(X_{t+1}, X_t) = Cov(a + bZ_{t+1} + cZ_t, a + bZ_t + cZ_{t-1}) = Cov(cZ_t, bZ_t) = bc\sigma^2.$$

  Thus,

  $$\gamma(h) = \begin{cases} (b^2 + c^2)\sigma^2 & h = 0 \\ bc\sigma^2 & h = \pm 1 \\ 0 & |h| > 1 \end{cases}.$$

  (b) $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

  Answer: Let $i, j \in \mathbb{Z}$ be any two integers. Then,

  $$E[X_i] = E[Z_i \cos(ci) + Z_{i-1} \sin(ci)] = 0 = E[Z_j \cos(cj) + Z_{j-1} \sin(cj)] = E[X_j],$$

  and

  $$Var[X_i] = Var[Z_i \cos(ci) + Z_{i-1} \sin(ci)] = \sigma^2 \cos^2(ci) + \sigma^2 \sin^2(ci) = \sigma^2,$$
  $$Var[X_j] = Var[Z_j \cos(cj) + Z_{j-1} \sin(cj)] = \sigma^2 \cos^2(cj) + \sigma^2 \sin^2(cj) = \sigma^2,$$

  where we have implicitly used the fact that every pair of Z_t have zero covariance. Thus, with knowledge that linear combinations of normally distributed random variables are normal, we have

  $$X_i, X_j \sim \mathcal{N}(0, \sigma^2).$$

  Since i, j were arbitrary, we have that every element X_t has the same distribution, which implies that X_t is *strictly stationary.*

Since we have explicitly shown that we have finite variance, then we may also conclude that X_t is *weakly stationary.* $E[X_t] = 0\ \forall t$, and note that

$$\begin{aligned} Cov(X_{t+1}, X_t) &= Cov(Z_{t+1}\cos(ct+c) + Z_t\sin(ct+c), Z_t\cos(ct) + Z_{t-1}\sin(ct)) \\ &= Cov(cZ_t, bZ_t) = \sin(ct+c)\cos(ct)\sigma^2 \end{aligned}$$

Thus,

$$\gamma(h) = \begin{cases} \sigma^2 & h = 0 \\ \sin(ct+c)\cos(ct)\sigma^2 & h = +1 \\ \cos(ct+c)\sin(ct)\sigma^2 & h = -1 \\ 0 & |h| > 1 \end{cases}.$$

(c) $X_t = a + bZ_0$

<u>*Answer:*</u> Let $i, j \in \mathbb{Z}$ be any two integers. Then,

$$E[X_i] = E[a + bZ_0] = a = E[a + bZ_0] = E[X_j],$$

and

$$Var[X_i] = Var[a + bZ_0] = 0 + b^2\sigma^2 = Var[a + bZ_0] = Var[X_j].$$

Thus, with knowledge that linear combinations of normally distributed random variables are normal, we have

$$X_i, X_j \sim \mathcal{N}(0, b^2\sigma^2).$$

Since i, j were arbitrary, we have that every element X_t has the same distribution, which implies that X_t is *strictly stationary.*

Since we have explicitly shown that we have finite variance, then we may also conclude that X_t is *weakly stationary.* $E[X_t] = 0\ \forall t$, and

$$\gamma(h) = b^2\sigma^2, \forall h.$$

(d) $X_t = Z_t Z_{t-1}$

<u>*Answer:*</u>

- **Problem 2: [U.S. Population]** Download the file `population.xls` from the course website. It contains the size of the population in the U.S.A. at ten-year intervals from 1790 to 2000.

  (a) Plot the data;

  (b) Assuming the model $X_t = m_t + Z_t$, $E[Z_t] = 0$, fit a polynomial trend $\widehat{m}_t$ to the data;

  (c) Plot the residuals $\hat{Z}_t = X_t - \hat{m}_t$. Comment on the quality of the fitted model;

  (d) Use the fitted model to predict the population size in 2010 and 2020 (using predicted noise values of zero).

- **Problem 3: [Projection Theorem]** If $\mathcal{M}$ is a closed subspace of a Hilbert Space $\mathcal{H}$ and $x \in \mathcal{H}$, prove that

$$\min_{y\in\mathcal{M}} ||x - y|| = \max\left\{ |\langle x, z\rangle| : z \in \mathcal{M}^{\perp}, ||z|| = 1 \right\},$$

where $\mathcal{M}^{\perp}$ is the orthogonal complement of $\mathcal{M}$.

<u>*Answer:*</u>

- **Problem 4: [Prediction Equations]** If $X_t = Z_t - \theta Z_{t-1}$, where $|\theta| < 1$ and $(Z_t : t \in \mathbb{Z})$ is a sequence of uncorrelated random variables, each with mean 0 and variance σ^2, show by checking the prediction equations that the best mean square predictor of X_{n+1} in $\overline{\mathrm{sp}}(X_j : j \leq n)$ is

$$\hat{X}_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j}.$$

What is the mean squared error of $\hat{X}_{n+1}$?

Answer:
