ASV proportions response $\mathbf{y}_i = (y_{i1}, \ldots, y_{ij}, \ldots, y_{ip})^T \sim \text{Dirichlet}(\alpha_{i1}, \ldots, \alpha_{ip})$

$$E(y_{ij}) = \mu_{ij} = \frac{\alpha_{ij}}{\alpha_{i1}}$$

where $\alpha_{i0} = \sum_{j=1}^{p} \alpha_{i0}$.

Link covarites to α's:

$$\log(\alpha_{ij}) = \mathbf{x_i}^T \boldsymbol{\beta}_j$$

and

$$\alpha_{ij} = e^{\mathbf{x_i}^T \boldsymbol{\beta}_j}$$

where $\boldsymbol{\beta}_j = (\beta_1, \ldots, \beta_q)$ depending on the number of covariates in the model.

Then the GEE equations are

$$\sum_{i=1}^{n} \left(\frac{\partial \boldsymbol{\mu}_i}{\partial \boldsymbol{\beta}} \right)^T \mathbf{V}_i^{-1} (\mathbf{Y_i} - \boldsymbol{\alpha}_i) = 0$$

Where $\boldsymbol{V}_i = A_i^{\frac{1}{2}} R_i A_i^{\frac{1}{2}}$

Then,

$$\begin{aligned}
\frac{\partial \boldsymbol{\mu}_i}{\partial \boldsymbol{\beta}} &= \frac{\partial}{\partial \boldsymbol{\beta}} \frac{\boldsymbol{\alpha}_i}{\alpha_{i0}} \\
&= \frac{\partial}{\partial \boldsymbol{\beta}} \frac{e^{\mathbf{x_i}^T \boldsymbol{\beta}}}{\sum_{j=1}^{p} e^{\mathbf{x_i}^T \boldsymbol{\beta}_j}} \\
&= \begin{pmatrix} \frac{\partial}{\partial \boldsymbol{\beta}_1} \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}_1}}{\sum_{j=1}^{p} e^{\mathbf{x_i}^T \boldsymbol{\beta}_j}} & \cdots & \frac{\partial}{\partial \boldsymbol{\beta}_1} \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}_p}}{\sum_{j=1}^{p} e^{\mathbf{x}_i^T \boldsymbol{\beta}_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \boldsymbol{\beta}_p} \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}_1}}{\sum_{j=1}^{p} e^{\mathbf{x}_i^T \boldsymbol{\beta}_j}} & \cdots & \frac{\partial}{\partial \boldsymbol{\beta}_p} \frac{e^{\mathbf{x}_i^T \boldsymbol{\beta}_p}}{\sum_{j=1}^{p} e^{\mathbf{x}_i^T \boldsymbol{\beta}_j}} \end{pmatrix} \\
&= \frac{1}{\left(\sum_{j=1}^{p} e^{\mathbf{x}_i^T \boldsymbol{\beta}_j} \right)^2} \begin{pmatrix} \mathbf{x}_i \alpha_{i1} \alpha_{i0} - \mathbf{x}_i \alpha_{i1} \alpha_{i1} & \cdots & -\mathbf{x}_i \alpha_{i1} \alpha_{ip} \\ -\mathbf{x}_i \alpha_{i1} \alpha_{i2} & \mathbf{x}_i \alpha_{i0} \alpha_{i2} - \mathbf{x}_i \alpha_{i2} \alpha_{i2} & \cdots \\ \vdots & \ddots & \vdots \\ -\mathbf{x}_i \alpha_{i1} \alpha_{ip} & \cdots & \mathbf{x}_i \alpha_{ip} \alpha_{i0} - \mathbf{x}_i \alpha_{ip} \alpha_{ip} \end{pmatrix} \\
&= \frac{1}{\alpha_{i0}^2} \left(\alpha_{i0} \mathbf{x}_i \text{diag}(\boldsymbol{\alpha}_i) - \mathbf{x}_i \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_i \right)
\end{aligned}$$