Vector Spaces

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1 Answers to selected exercises

Problem 1: Suppose W is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that ST is the identity map on V.

Solution: If $ST : \mathcal{L}(V,V)$ is the identity map then for any $v \in V$, $STv = v$. This can be rewritten as $S(Tv)$; now assume that T is not injective in which case for $v_1 \neq v_2$, $Tv_1 = Tv_2 = w$. Since $STv_1 \neq STv_2$ we must have $Sw \neq Sw$ which is not possible. Therefore T must be injective.
Next assume that T is injective and define a linear map $S = \mathcal{L}(W,V)$ with the property $Sw = v$ if $Tv = w$. This map is defined since T is injective and there is a unique v for which $Tv = w$. Therefore $STv = v$ and ST is the identity map on V.

Problem 2: Suppose V is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W,V)$ such that TS is the identity map on W.

Solution: If $TS \in \mathcal{L}(W,W)$ is the identity map then for all $w \in W$, $TSw = w$. For the identity map to exist S must map all $w \in W$ to some $\mathsf{range}\, S \subseteq V$ and then T must map $\mathsf{range}\, S$ to the whole of W. But if T is not surjective, $\mathsf{range}\, T \subset W$ and therefore for any $\mathsf{range}\, S$ it is not possible for $\mathsf{range}\, TS = W$. Therefore T must be surjective.
If T is surjective then $\mathsf{range}\, T = W$; therefore for any $w \in W$ there is one or more $v \in V$ such that $Tv = w$. Define a map $S : \mathcal{L}(W,V)$ which maps $w \in W$ to $v \in V$ such that $Tv = w$ (if more than one v have this property select one and keep this choice fixed). So if we start with $w \in W$ then we can map it to $v = Sw$. From the definition of S then it follows that $Tv = w$ or $TSw = w$ and therefore TS is the identity map.

Problem 3: Suppose V is finite dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that $\mathsf{range}\, T_1 \subset \mathsf{range}\, T_2$ if and only if there exists $S \in \mathcal{L}(V,V)$ such that $T_1 = T_2 S$.

Solution: If $\mathsf{range}\, T_1 \subset \mathsf{range}\, T_2$ then if $w = T_1 v$ there exists $v' \in V$ for which $w = T_2 v'$. We can define $S \in \mathcal{L}(V,V)$ such that given $v \in \mathsf{range}\, T_1$ we find $v' \in \mathsf{range}\, T_2$ such that $T_1 v = T_2 v'$. Therefore if $v' = Sv$ then $T_2 v' = T_2 S v$ and since $T_1 v = T_2 v'$ we have $T_1 v = T_2 S v$ i.e. $T_1 = T_2 S$.
Next assume that there exists $S \in \mathcal{L}(V,V)$ such that $T_1 = T_2 S$. If $w \in \mathsf{range}\, T_1$ we have $w = T_1 v = T_2 S v$ and since $Sv \in V$, $T_2 v \in \mathsf{range}\, T_2$, $w \in \mathsf{range}\, T_2$. If $w \in \mathsf{range}\, T_2$ we have $w = T_2 v$;

however there is no linear map $S' \in \mathcal{L}(V,V)$ for which $T_2 = T_1 S'$ and therefore $w \in \operatorname{range} T_1$. Therefore it is not always true that if $w \in \operatorname{range} T_2$ it is also the case that $w \in \operatorname{range} T_1$ and therefore $\operatorname{range} T_1 \subset \operatorname{range} T_2$.

Problem 4: Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that $\operatorname{null} T_1 \subset \operatorname{null} T_2$ if and only if there exists $S \in \mathcal{L}(W,W)$ such that $T_2 = ST_1$.

Solution: Start with $\operatorname{null} T_1 \subset \operatorname{null} T_2$; this means that for some $v \in \operatorname{null} T_2$, $T_1 v \neq 0$. We can define a linear map $S \in \mathcal{L}(W,W)$ with the following properties: if $T_1 v = T_2 v = 0$ then $ST_1 v = S0 = 0 = T_2 v$ which is a property of all linear maps; if $T_2 v = 0$ and $T_1 v \neq 0$ then $T_1 v \in \operatorname{null} S$ and $S(T_1 v) = 0 = T_2 v$; in all other cases S maps $T_1 v$ to $T_2 v$, i.e. $ST_1 v = T_2 v$. Therefore we can define a linear map with the property $ST_1 = T_2$.
Next assume the existence of a linear map $S \in \mathcal{L}(W,W)$ such that $T_2 = ST_1$. For $v \in \operatorname{null} T_1$ we have $ST_1 v = 0 = T_2 v$ and therefore $v \in \operatorname{null} T_2$. For $v \in \operatorname{null} T_2$ there is no linear map $S' \in \mathcal{L}(W,W)$ such that $T_1 = S' T_2$. Therefore if $T_2 v = 0$ it is not always the case that $T_1 v = 0$. Therefore $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

Problem 5: Suppose $\phi \in \mathcal{L}(V,\mathbb{F})$. Suppose $u \in V$ is not in $\operatorname{null} \phi$. Prove that,

$$V = \operatorname{null} \phi \oplus \{au : a \in \mathbb{F}\}$$

Solution: To simplify the expressions denote $\{au : a \in \mathbb{F}\}$ as U. First we must prove that,

$$\operatorname{null} \phi \cap U = \{0\}$$

Since $u \notin \operatorname{null} \phi$ it is also the case that for $a \in \mathbb{F}$, $au \notin \operatorname{null} \phi$ except from the trivial case $u = 0$ for which $U = \{0\}$ or $a = 0, u \neq 0$ for which $au = 0$. Therefore,

$$\operatorname{null} \phi \cap U = \{0\}$$

Next we must prove that any $v \in V$ can be written as the sum of two vectors, $w \in \operatorname{null} \phi$ and $u \in U$. Since $\operatorname{dim} \operatorname{range} \phi \leq \operatorname{dim} \mathbb{F}$ we have,

$$\operatorname{dim} V - \operatorname{dim} \operatorname{null} \phi \leq 1 \Rightarrow \operatorname{dim} V - 1 \leq \operatorname{dim} \operatorname{null} \phi$$

For the case $\operatorname{dim} \operatorname{null} \phi = \operatorname{dim} V = n$ then all $v \in V$ are also $v \in \operatorname{null} \phi$. In this case $u = 0$ and since,

$$v = v + 0$$

any $v \in V$ can be written as the sum of $v \in \operatorname{null} \phi$ and $u \in U$. If $\operatorname{dim} \operatorname{null} \phi = \operatorname{dim} V - 1 = n - 1$, then any $w \in \operatorname{null} \phi$ can be written as,

$$w = w^1 e_1 + \ldots + w^{n-1} e_{n-1}$$

i.e. a linear combination of $n-1$ basis vectors. Since $u \notin \operatorname{null} \phi$,

$$au = au^n e_n$$

For $v \in V$,

$$v = v^1 e_1 + \ldots + v^{n-1} e_{n-1} + v^n e_n$$

Set $w^1 = v^1, \ldots, w^{n-1} = v^{n-1}$ and $au^n = v^n$ by setting $a = v^n / u^n$ (note that if $\operatorname{dim} \operatorname{null} \phi = n-1$, $u^n \neq 0$ and $au = 0$ only for $a = 0$). Then any $v \in V$ can be expressed as the sum of $w \in \operatorname{null} \phi$ and au where $u \in U$ and $a \in \mathbb{F}$.

Problem 6: Suppose ϕ_1 and ϕ_2 are linear maps from V to $\mathbb{F}$ that have the same null space. Show that there exists a constant $c \in \mathbb{F}$ such that $\phi_1 = c\phi_2$.

Solution: Denote $\operatorname{null} \phi_1 = \operatorname{null} \phi_2$ as N. For any linear map $\operatorname{dim} V = \operatorname{dim} \operatorname{null} + \operatorname{dim} \operatorname{range}$ so $\operatorname{dim} V - \operatorname{dim} N = \operatorname{dim} \operatorname{range} \phi_1 \leq \operatorname{dim} \mathbb{F} = 1$ (same is true for ϕ_2). If $\operatorname{dim} V = n$ then $\operatorname{dim} V - \operatorname{dim} N \leq 1$ or $n - 1 \leq \operatorname{dim} N$.
If $\operatorname{dim} N = n$ then $\operatorname{dim} \operatorname{range} \phi_1 = 0$ (same is true for ϕ_2) and therefore $\operatorname{range} \phi_1 = \operatorname{range} \phi_2 = \{0\}$. This is only possible if $\phi_1 v = \phi_2 v = 0v = 0$ for all $v \in V$. So if $\operatorname{dim} \operatorname{null} \phi_1 = \operatorname{dim} \operatorname{null} \phi_2 = n$ for any $c \in \mathbb{F}$, $\phi_1 = c\phi_2$.
If $\operatorname{dim} N = n-1$ then $\operatorname{dim} \operatorname{range} \phi_1 = \operatorname{dim} \operatorname{range} \phi_2 = 1$ so for $v \notin N$,

$$\begin{aligned} v &= v^1 e_1 + \ldots + v^n e_n \\ \phi_1 v &= v^n \phi_1 e_n \\ \phi_2 v &= v^n \phi_2 e_n \end{aligned}$$

Define $c = \phi_1 e_n / \phi_2 e_n$ (note that $\phi_2 e_n \neq 0$ since if this is not the case $\phi_2 v = 0$ for all v); then

$$\phi_1 v = v^n (c\phi_2 e_n) = c\phi_2 v$$

and therefore $\phi_1 = c\phi_2$.

Problem 7: Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution: Consider,

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix}$$

where A_i are 2×1 independent vectors. For $i = 3, 4, 5$,

$$A_i = a_{i1} A_1 + a_{i2} A_2$$

and the null space of A is composed of vectors x with the property,

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5 = 0$$

or,

$$(x_1 + a_{31} x_3 + a_{41} x_4 + a_{51} x_5) A_1 + (x_2 + a_{32} x_3 + a_{42} x_4 + a_{52} x_5) A_2 = 0$$

Since A_1, A_2 are independent vectors the last condition holds only if,

$$\begin{aligned} x_1 &= -a_{31} x_3 - a_{41} x_4 - a_{51} x_5 \\ x_2 &= -a_{32} x_3 - a_{42} x_4 - a_{52} x_5 \end{aligned}$$

therefore any $x \in \mathsf{null}A$,

$$x = \begin{pmatrix} -a_{31}x_3 - a_{41}x_4 - a_{51}x_5 \\ -a_{32}x_3 - a_{42}x_4 - a_{52}x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -a_{31} \\ -a_{32} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -a_{41} \\ -a_{42} \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -a_{51} \\ -a_{52} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Since the three vectors in the last equation are independent they form the basis of $\mathsf{null}A$.
Now define another 5×2 matrix,

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix}$$

where B_i are 2×1 independent vectors. For $i = 3,4,5$,

$$B_i = a_{i1}B_1 + a_{i2}B_2$$

and using the same procedure as above the same three vectors form the basis of $\mathsf{null}B$. Therefore $\mathsf{null}A = \mathsf{null}B$; since B_1, B_2 are different from A_1, A_2, A is not a scalar multiple of B. As an example,

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 5 \\ 1 & 2 & 3 & -1 & 0 \end{bmatrix}$$

and,

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

have null space,

$$\mathsf{null}A = \mathsf{null}B = \mathsf{Span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. A vector $x \in V$,

$$x = x^1 v_1 + \ldots + x^n v_n$$

is mapped to a vector $y \in W$ using a $m \times n$ matrix A as,

$$\begin{pmatrix} y^1 \\ \vdots \\ y^r \\ \vdots \\ y^m \end{pmatrix} = x^1 \begin{pmatrix} A_{1,1} \\ \vdots \\ A_{r,1} \\ \vdots \\ A_{m,1} \end{pmatrix} + \ldots + x^n \begin{pmatrix} A_{1,n} \\ \vdots \\ A_{r,n} \\ \vdots \\ A_{m,n} \end{pmatrix}$$

and therefore,

$$y^r = x^1 A_{r,1} + \ldots + x^n A_{r,n} = \sum_{i=1}^{n} A_{r,i} x^i \tag{1}$$

We also have,

$$y = Tx = x^1 T v_1 + \ldots + x^n T v_n \tag{2}$$

The linear map T transforms each basis vector v_k to a linear combination of basis vectors of W,

$$T v_k = C_{1,k} w_1 + \ldots + C_{m,k} w_m = \sum_{j=1}^{m} C_{j,k} w_j$$

Combining (1) and (2) we get,

$$y = x^1 \sum_{j=1}^{m} C_{j,1} w_j + \ldots + x^n \sum_{j=1}^{m} C_{j,n} w_j \tag{3}$$

Since $y \in W$,

$$y = y^1 w_1 + \ldots + y^m w_m \tag{4}$$

it follows from (3) that,

$$y^r = \sum_{i=1}^{n} C_{r,i} x^i \tag{5}$$

If we compare (1) and (5) we must have $C_{r,p} = A_{r,p}$. Therefore,

$$T v_k = A_{1,k} w_1 + \ldots + A_{m,k} w_m = \sum_{i=1}^{m} A_{i,k} w_i \tag{6}$$

i.e. the linear map T transforms the k^{th} basis vector of V into a linear combination of the basis vectors of W whose coefficients are the elements of the k^{th} column of matrix $A = \mathcal{M}(T)$.
Now suppose we want to use a new set of basis vectors $\hat{w}_1, \ldots, \hat{w}_m$. We can express,

$$w_r = \sum_{j=1}^{m} C_{j,r} \hat{w}_j \tag{7}$$

Substituting (7) into (6) we get,

$$T v_k = \sum_{i=1}^{m} A_{i,k} w_i \left(\sum_{j=1}^{m} C_{j,i} \hat{w}_j \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} C_{j,i} A_{i,k} \right) \hat{w}_j \tag{8}$$

Therefore starting from (7) we can derive a matrix $B = CA$ such that,

$$T v_k = \sum_{j=1}^{m} B_{j,k} \hat{w}_j \tag{9}$$

Problem 8: Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution: If we express,

$$\mathcal{P}_3(x) = \sum_{i=1}^{4} b_i f_i(x)$$

where f_i is the i$^{\text{th}}$ basis of $\mathcal{P}_3$ and,

$$\mathcal{P}_2(x) = \sum_{i=1}^{3} b_i g_i(x)$$

where g_i is the i$^{\text{th}}$ basis of $\mathcal{P}_2$ then from the structure of the matrix of D we have,

$$\sum_{i=1}^{3} b_i g_i(x) = \sum_{i=1}^{4} b_i f_i'(x)$$

Since $f_4'(x) = 0$, $f_4(x) = \text{constant}$ and we can set $f_4(x) = 1$. For $i = 3$, $f_3'(x) = g_3(x)$ so one choice is $f_3(x) = x$ and $g_3(x) = 1$; similarly $f_2(x) = x^2$, $g_2(x) = 2x$ and $f_1(x) = x^3$, $g_1(x) = 3x^2$. So starting from the polynomial,

$$\mathcal{P}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

we can write it as,

$$\mathcal{P}_3(x) = a_3 f_1(x) + a_2 f_2(x) + a_1 f_3(x) + a_0 f_4(x)$$

which when differentiated gives,

$$D\mathcal{P}_3(x) = \mathcal{P}_2(x) = a_3 g_1(x) + a_2 g_2(x) + a_1 g_3(x)$$

We can rewrite the last equation in matrix form as,

$$\begin{pmatrix} a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix}$$

Problem 9: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \operatorname{dim} \operatorname{range} T$.

Solution: Denote $\dim V$ as n and $\dim \operatorname{null} T$ as r; from the Fundamental Theorem of Linear Maps we have, $\dim \operatorname{range} T = n - r$, $\operatorname{range} T = \operatorname{Span}\{e_1, \ldots, e_{n-r}\}$ and $\operatorname{null} T = \operatorname{Span}\{e_{n-r+1}, \ldots, e_n\}$. Any $v \in V$ can be written as,

$$v = v^1 e_1 + \ldots + v^{n-r} e_{n-r} + v^{n-r+1} e_{n-r+1} + \ldots + v^n e_n$$

where $\{e_1, \ldots, e_n\}$ is a basis of V. If $v \in \operatorname{null} T$,

$$v = 0e_1 + \ldots + 0e_{n-r} + v^{n-r+1} e_{n-r+1} + \ldots + v^n e_n$$

and since $Te_{n-r+1} = \ldots = Te_n = 0$ we have,

$$Tv = 0Te_1 + \ldots + 0Te_{n-r} + v^{n-r+1}0 + \ldots + v^n 0 = 0$$

For $w \in \operatorname{range} T$ with $w \neq 0$,

$$w = Tv = v^1 Te_1 + \ldots + v^{n-r} Te_{n-r} + v^{n-r+1} Te_{n-r+1} + \ldots + v^n Te_n$$

or,

$$w = Tv = v^1 Te_1 + \ldots + v^{n-r} Te_{n-r} + v^{n-r+1}0 + \ldots + v^n 0$$

Note that vectors $Te_1, \ldots, Te_{n-r}$ are independent since $w = 0$ only if $v = 0$ if $v \notin \operatorname{null} T$. Therefore vectors $Te_1, \ldots, Te_{n-r}$ can be the basis vectors for $\operatorname{range} T$. From the definition of $\mathcal{M}(T)$ the entries of the matrix satisfy the condition,

$$Te_k = A_{1,k} w_1 + \ldots + A_{n-r,k} w_{n-r}$$

where $w_1, \ldots, w_{n-r}$ are the basis vectors of $\operatorname{range} T$. If $w_k = Te_k$ for $j = 1, \ldots, n-r$ then $A_{k,k} = 1$ and $A_{k,k'} = 0$ for $k \neq k'$. Therefore A is,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

or,

$$A = \begin{bmatrix} I & O \end{bmatrix}$$

where I is the $n-r \times n-r$ identity matrix and O is the $n-r \times r$ zero matrix.

Problem 10: Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove there exists a basis $w_1, \ldots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$) are 0 except from possibly a 1 in the first row, first column.

Solution: The format of $\mathcal{M}(T)$ is as follows,

$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_m \end{array} \\ \begin{array}{c} w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{array} & \begin{bmatrix} A_{1,1} & \cdots & A_{1,j} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i,1} & \cdots & A_{i,j} & \cdots & A_{i,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,j} & \cdots & A_{n,m} \end{bmatrix} \end{array}$$

and the matrix coefficients are defined by,

$$Tv_j = \sum_{i=1}^{n} A_{i,j} w_i$$

We want a basis $\hat{w}_1, \ldots, \hat{w}_n$ for which the linear map $T' : V \to \hat{W}$,

$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_m \end{array} \\ \begin{array}{c} \hat{w}_1 \\ \vdots \\ \hat{w}_i \\ \vdots \\ \hat{w}_n \end{array} & \begin{bmatrix} 1 \text{ or } 0 & \cdots & B_{1,j} & \cdots & B_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_{i,j} & \cdots & B_{i,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_{n,j} & \cdots & B_{n,m} \end{bmatrix} \end{array}$$

If we start with the linear map $T : V \to W$ and then add a linear map $S : W \to \hat{W}$ then since $T' : V \to \hat{W}$, $\mathcal{M}(T') = \mathcal{M}(S)\mathcal{M}(T)$ or $B = CA$.
If $B_{1,1} = \ldots = B_{n,1} = 0$ then the equation,

$$T'v_1 = 0\hat{w}_1 + \ldots + 0\hat{w}_n$$

holds for any basis $\{\hat{w}_1, \ldots, \hat{w}_n\}$. If $B_{1,1} = 1$ and $B_{i,1} = 0$ for $i > 1$ then we must have,

$$Tv_1 = 1\hat{w}_1 + \ldots + 0\hat{w}_n$$

and therefore $\hat{w}_1 = T v_1$.
Define a $n \times n$ matrix with the following properties,

$$
\begin{aligned}
C_{i,i} &= A_{i,1}^{-1} \\
C_{i,1} &= A_{1,1}^{-1} \\
C_{i,j} &= 0 \text{ for } i \neq j \text{ , } j > 1
\end{aligned}
$$

so that,

$$
CA = \begin{bmatrix}
A_{1,1}^{-1} & 0 & 0 & 0 & \cdots & 0 \\
A_{1,1}^{-1} & -A_{2,1}^{-1} & 0 & 0 & \cdots & 0 \\
A_{1,1}^{-1} & 0 & -A_{3,1}^{-1} & 0 & \cdots & 0 \\
A_{1,1}^{-1} & 0 & 0 & -A_{4,1}^{-1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{1,1}^{-1} & 0 & 0 & 0 & \cdots & -A_{n,1}^{-1}
\end{bmatrix}
\begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & A_{1,m} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \cdots & A_{2,m} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \cdots & A_{3,m} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & \cdots & A_{4,m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n,1} & A_{n,2} & A_{n,3} & A_{n,4} & \cdots & A_{n,m}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & \frac{A_{1,2}}{A_{1,1}} & \frac{A_{1,3}}{A_{1,1}} & \frac{A_{1,4}}{A_{1,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} \\
0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{2,2}}{A_{2,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{2,3}}{A_{2,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{2,4}}{A_{2,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{2,m}}{A_{2,1}} \\
0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{3,2}}{A_{3,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{3,3}}{A_{3,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{3,4}}{A_{3,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{3,m}}{A_{3,1}} \\
0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{4,2}}{A_{4,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{4,3}}{A_{4,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{4,4}}{A_{4,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{4,m}}{A_{4,1}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{n,2}}{A_{n,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{n,3}}{A_{n,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{n,4}}{A_{n,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{n,m}}{A_{n,1}}
\end{bmatrix}
$$

C is admissible as a basis transformation matrix only if $A_{i,1} \neq 0$ for all i. Since,

$$
w_r = \sum_{j=1}^{n} C_{j,r} \hat{w}_j
$$

and both w_r and $\hat{w}_j$ are $n \times 1$ vectors, if W is the matrix whose columns are the w basis vectors and $\hat{W}$ is the matrix whose columns are the $\hat{w}$ basis vectors then,

$$W_{i,r} = \sum_{j=1}^{n} C_{j,r} \hat{W}_{i,j}$$

and therefore,

$$W = \hat{W} C$$

To express the new basis vectors in terms of the original basis we can right multiply both sides with C^{-1} to get,

$$\hat{W} = W C^{-1}$$

It is easy to show that,

$$C^{-1} = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 & \cdots & \hat{w}_n \end{array} \\ \begin{bmatrix} A_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ A_{2,1} & -A_{2,1} & 0 & 0 & \cdots & 0 \\ A_{3,1} & 0 & -A_{3,1} & 0 & \cdots & 0 \\ A_{4,1} & 0 & 0 & -A_{4,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ A_{n,1} & 0 & 0 & 0 & \cdots & -A_{n,1} \end{bmatrix} \end{array}$$

and therefore,

$$\hat{w}_1 = \sum_{i=1}^{n} A_{i,1} w_i = T v_1$$
$$\hat{w}_i = -A_{i,1} w_i \quad \text{for } i > 1$$

Why can't we go further and derive a basis of W such that the first two entries in the first column of $\mathcal{M}(T)$ are equal to 1? Let's assume that a matrix C exists with the property,

$$\sum_{r=1}^{n} C_{1,r} A_{r,1} = 1$$
$$\sum_{r=1}^{n} C_{2,r} A_{r,1} = 1$$

This is only possible if $C_{1,r} = C_{2,r}$ i.e. if the first two rows of C are identical. A matrix with two identical rows is not admissible as a basis transformation matrix. In order to prove this lets assume that a matrix C exists such that,

$$
\begin{array}{c c}
 & \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & \cdots & w_n \end{array} \\
\begin{array}{c} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \hat{w}_4 \\ \vdots \\ \hat{w}_n \end{array} &
\begin{bmatrix}
C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & \cdots & C_{1,n} \\
C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & \cdots & C_{1,n} \\
C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & \cdots & C_{2,n} \\
C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & \cdots & C_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
C_{n-1,1} & C_{n-1,2} & C_{n-1,3} & A = C_{n-1,4} & \cdots & C_{n-1,n}
\end{bmatrix}
\end{array}
$$

Given that vectors $\hat{w}_i$ are independent we want to show that we can choose a set of constants $a_1, \ldots, a_n$ that are not all zero but for which,

$$\sum_{i=1}^{n} a_i w_i = 0$$

Since,

$$w_i = C_{1,i}(\hat{w}_1 + \hat{w}_2) + \sum_{j=2}^{n} C_{j,i}\hat{w}_j$$

we must have,

$$\sum_{i=1}^{n} a_i \left(C_{1,i}(\hat{w}_1 + \hat{w}_2) + \sum_{j=2}^{n} C_{j,i}\hat{w}_j \right) = 0$$

The last equation holds only if the coefficients of the $\hat{w}_j$ vectors are all zero, i.e,

$$\sum_{i=1}^{n} C_{j,i} a_i = 0$$

for all j. Since C is the matrix of a linear map from a vector space with dimension n to vector space with dimension $n-1$ this linear map cannot be injective; therefore the null set of this transformation contains non-zero vectors. It follows that $w_1, w_2, \ldots, w_n$ are not independent and a basis transformation matrix with two identical rows is not admissible.