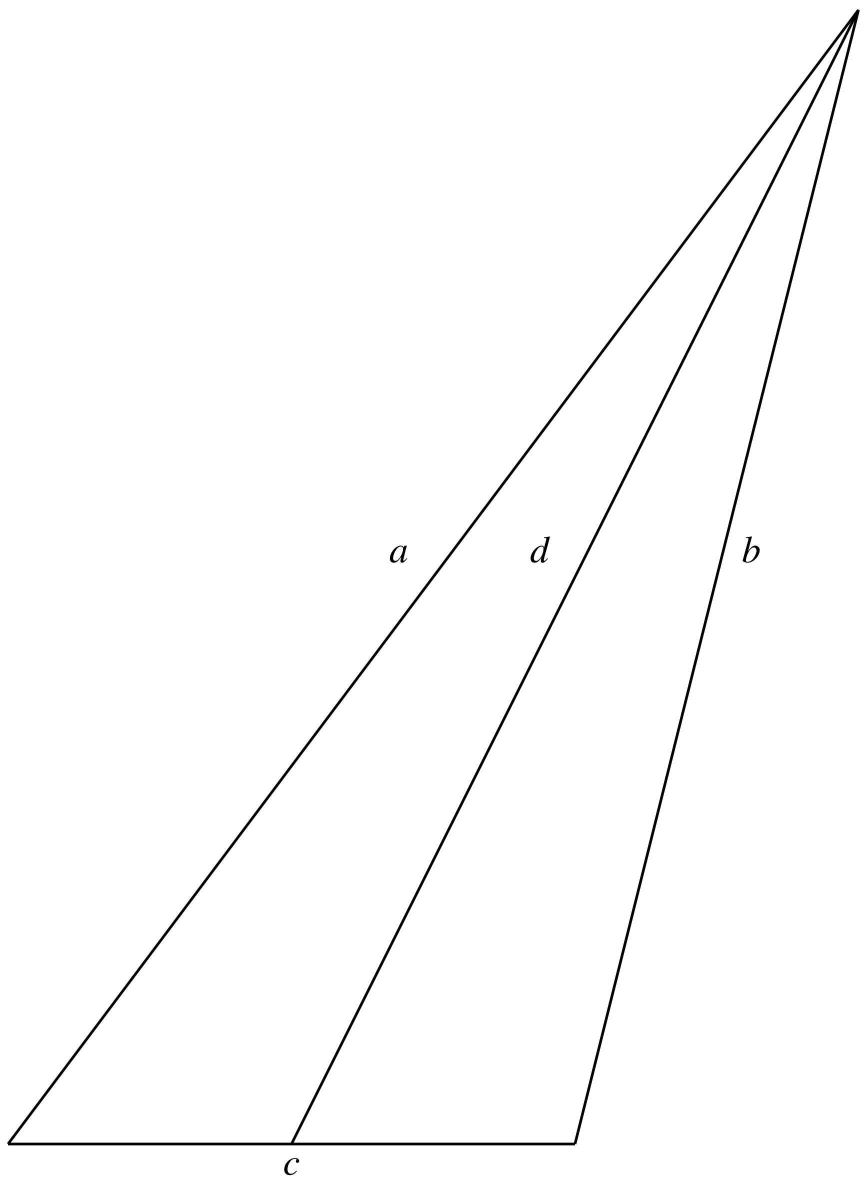
Solutions to selected exercises



Chapter 3

Problem 1 (3.B.1): Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution: One example is $T : \mathbb{R}^5 \to \mathbb{R}^5$ using a 5×5 matrix with only two independent columns:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} {\scriptstyle A_1} & {\scriptstyle A_2} & {\scriptstyle A_3 = A_1 + A_2} & {\scriptstyle A_4 = A_1 - A_2} & {\scriptstyle A_5 = 2A_1 - A_2} \end{array} \\
A = & \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 2 & 1 & 3 & 1 & 3 \\ 0 & -1 & -1 & 1 & 1 \\ 1 & 3 & 4 & -2 & -1 \\ -1 & 1 & 0 & -2 & -3 \end{bmatrix}
\end{array}
$$

We can check that the first two columns are independent. If,

$$a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} = 0$$

we must have $a \cdot 0 + b \cdot (-1) = 0 \Rightarrow b = 0$; also $a \cdot 1 + b \cdot 2 = 0 \Rightarrow a = 0$.

The product of matrix A with a vector x can be written as,

$$y = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5$$

i.e. a linear combination of the five columns of A. Since,

$$\begin{aligned} A_3 &= A_1 + A_2 \\ A_4 &= A_1 - A_2 \\ A_5 &= 2A_1 - A2 \end{aligned}$$

we have,

$$y = (x_1 + x_3 + x_4 + 2x_5)A_1 + (x_2 + x_3 - x_4 - x_5)A_2$$

Since all $y = Ax$ can be written a a linear combination of two independent vectors $\dim \operatorname{range} T = 2$. For the null space of T we must have,

$$\begin{aligned} x_1 + x_3 + x_4 + 2x_5 &= 0 \\ x_2 + x_3 - x_4 - x_5 &= 0 \end{aligned}$$

since A_1, A_2 are independent vectors. So,

$$x_1 = -x_3 - x_4 - 2x_5$$

and,

$$x_2 = -x_3 + x_4 + x_5$$

Therefore all $x \in \operatorname{null} T$ can be written as,

$$\begin{pmatrix} -x_3 - x_4 - 2x_5 \\ -x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

i.e. a linear combination of three vectors. It is easy to show that these three vectors are independent and therefore $\dim \operatorname{null} T = 3$.

Problem 2 (3.B.1): Give an example of a linear map T such that $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

Solution: One example is $T : \mathbb{R}^5 \to \mathbb{R}^5$ using a 5×5 matrix with only two independent columns:

$$\begin{array}{c} \begin{array}{ccccc} \scriptstyle A_1 & \scriptstyle A_2 & \scriptstyle A_3 = A_1 + A_2 & \scriptstyle A_4 = A_1 - A_2 & \scriptstyle A_5 = 2A_1 - A_2 \end{array} \\ A = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 2 & 1 & 3 & 1 & 3 \\ 0 & -1 & -1 & 1 & 1 \\ 1 & 3 & 4 & -2 & -1 \\ -1 & 1 & 0 & -2 & -3 \end{bmatrix} \end{array}$$

We can check that the first two columns are independent. If,

$$a \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} = 0$$

we must have $a \cdot 0 + b \cdot (-1) = 0 \Rightarrow b = 0$; also $a \cdot 1 + b \cdot 2 = 0 \Rightarrow a = 0$.

The product of matrix A with a vector x can be written as,

$$y = x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5$$

i.e. a linear combination of the five columns of A. Since,

$$\begin{aligned} A_3 &= A_1 + A_2 \\ A_4 &= A_1 - A_2 \\ A_5 &= 2A_1 - A2 \end{aligned}$$

we have,

$$y = (x_1 + x_3 + x_4 + 2x_5)A_1 + (x_2 + x_3 - x_4 - x_5)A_2$$

Since all $y = Ax$ can be written a a linear combination of two independent vectors $\dim \operatorname{range} T = 2$. For the null space of T we must have,

$$\begin{aligned} x_1 + x_3 + x_4 + 2x_5 &= 0 \\ x_2 + x_3 - x_4 - x_5 &= 0 \end{aligned}$$

since A_1, A_2 are independent vectors. So,

$$x_1 = -x_3 - x_4 - 2x_5$$

and,

$$x_2 = -x_3 + x_4 + x_5$$

Therefore all $x \in \operatorname{null} T$ can be written as,

$$\begin{pmatrix} -x_3 - x_4 - 2x_5 \\ -x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

i.e. a linear combination of three vectors. It is easy to show that these three vectors are independent and therefore $\dim \operatorname{null} T = 3$.

Problem 3 (3.B.4): Show that,

$$\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Solution: $\dim \operatorname{null} T > 2$ implies that any $v \in \operatorname{null} T$ can be written as a linear combination of 3 or more independent vectors. For a 4×5 matrix,

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix}$$

this means that a maximum of two columns can be independent. So lets assume that columns A_3, A_4, A_5 can be written as linear combinations of A_1, A_2.

Now consider another 4×5 matrix,

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix}$$

with B_3 and B_4 two independent vectors and all other columns linear combinations of these two vectors. If we add the two matrices we have,

$$C = A + B = \begin{bmatrix} A_1 + B_1 & A_2 + B_2 & A_3 + B_3 & A_4 + B_4 & A_5 + B_5 \end{bmatrix}$$

Since A_1, A_2, B_3, B_4 are independent vectors only $A_5 + B_5$ can be written as a linear combination of these vectors. We have,

$$\begin{aligned}
C_1 &= A_1 + B_1 = A_1 + b_{13}B_3 + b_{14}B_4 \\
C_2 &= A_2 + B_2 = A_2 + b_{23}B_3 + b_{24}B_4 \\
C_3 &= A_3 + B_3 = a_{31}A_1 + a_{32}A_2 + B_3 \\
C_4 &= A_4 + B_4 = a_{41}A_1 + a_{42}A_2 + B_4
\end{aligned}$$

which are independent. We must show that the following equation,

$$c_1C_1 + c_2C_2 + c_3C_3 + c_4C_4 = 0$$

holds only if $c_1 = c_2 = c_3 = c_4 = 0$. Rewrite it as,

$$(c_1 + c_3a_{31} + c_4a_{41})A_1 + (c_2 + c_3a_{32} + c_4a_{42})A_2 + (c_1b_{13} + c_2b_{23} + c_3)B_3 + (c_1b_{14} + c_2b_{24} + c_4)B_4 = 0$$

Since A_1, A_2, B_3, B_4 are independent we must have,

$$\begin{aligned}
c_1 + c_3a_{31} + c_4a_{41} &= 0 \\
c_2 + c_3a_{32} + c_4a_{42} &= 0 \\
c_1b_{13} + c_2b_{23} + c_3 &= 0 \\
c_1b_{14} + c_2b_{24} + c_4 &= 0
\end{aligned}$$

or,

$$c_1 \begin{pmatrix} 1 \\ 0 \\ b_{13} \\ b_{14} \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ b_{23} \\ b_{24} \end{pmatrix} + c_3 \begin{pmatrix} a_{31} \\ a_{32} \\ 1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} a_{41} \\ a_{42} \\ 0 \\ 1 \end{pmatrix} = 0$$

This equation has non-zero solutions only if the following special condition is satisfied:

$$1 - a_{32}b_{23} - a_{42}b_{24} - a_{31}b_{13} - a_{41}b_{14} + (b_{13}b_{24} - b_{23}b_{14})(a_{31}a_{42} - a_{41}a_{32}) = 0$$

In general these four vectors are independent and therefore $\operatorname{dim} \operatorname{range} C = 4$ and $\operatorname{dim} \operatorname{null} C = 1$. One example is with all the a and b coefficients equal to 0. So we have shown that the set of linear maps from $\mathbb{R}^5$ to $\mathbb{R}^4$ with $\operatorname{dim} \operatorname{null} T > 2$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Problem 4 (3.B.5): Given an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that,

$$\mathsf{range}\,T = \mathsf{null}\,T$$

Solution: We know that a matrix A,

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \end{bmatrix} \tag{1}$$

with A_1, A_2 two independent vectors and $A_3 = aA_1 + bA_2$, $A_4 = cA_1 + dA_2$ has a range equal to $\mathsf{Span}\{A_1, A_2\}$. $\mathsf{null}A$ contains vectors x with the property,

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0 \tag{2}$$

or,

$$(x_1 + ax_3 + cx_4)A_1 + (x_2 + bx_3 + dx_4)A_2 = 0 \tag{3}$$

Since A_1, A_2 are independent vectors (3) holds only if,

$$x_1 = -ax_3 - cx_4 \tag{4}$$

$$x_2 = -bx_3 - dx_4 \tag{5}$$

Therefore any $x \in \mathsf{null}A$ can be written as,

$$x = x_3 \begin{pmatrix} -a \\ -b \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -c \\ -d \\ 0 \\ 1 \end{pmatrix} \tag{6}$$

If,

$$A_1 = \begin{pmatrix} -a \\ -b \\ 1 \\ 0 \end{pmatrix} \tag{7}$$

and,

$$A_2 = \begin{pmatrix} -c \\ -d \\ 0 \\ 1 \end{pmatrix} \tag{8}$$

then $\mathsf{null}A = \mathsf{Span}\{A_1, A_2\} = \mathsf{range}A$. Note that A_1, A_2 are independent since,

$$x \begin{pmatrix} -a \\ -b \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -c \\ -d \\ 0 \\ 1 \end{pmatrix} = 0 \tag{9}$$

holds only for $x = y = 0$. Therefore we can write,

$$A = \begin{bmatrix} -a & -c & -a^2 - bc & -(a+d)c \\ -b & -d & -(a+d)b & -d^2 - bc \\ 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} = \begin{bmatrix} -\tilde{A} & -\tilde{A}^2 \\ I & \tilde{A} \end{bmatrix} \tag{10}$$

where,

$$\tilde{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{11}$$

The form of (10) can be extended to any $2n \times 2n$ matrix so that the first n columns are independent vectors and the last n columns are linear combinations of these vectors. All matrices of this form have $\mathsf{range} A = \mathsf{null} A$. Note that this is not possible for $T : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ since $\dim\ \mathsf{range} A + \dim \mathsf{null} A = 2n+1$ and therefore $\dim\ \mathsf{range} A \neq \dim \mathsf{null} A$ so $\mathsf{range} A \neq \mathsf{null} A$.

Problem 5 (3.B.7): Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not injective } \}$ is not a subspace of $\mathcal{L}(V,W)$.

Solution: This is equivalent to showing that although two linear maps T_1, T_2 have $\dim \mathsf{null} > 0$, $\dim \mathsf{null}\, T_1 + T_2 = 0$. We can choose two linear maps with the property $\mathsf{null}\, T_1 \cap \mathsf{null}\, T_2 = \{0\}$. Then if $v_1 \in \mathsf{null}\, T_1$,

$$(T_1 + T_2)v_1 = T_1 v_1 + T_2 v_1 = T_2 v_1 \neq 0$$

and therefore $v_1 \notin \mathsf{null}\, T_1 + T_2$. In the same way if $v_2 \in \mathsf{null}\, T_2$, $v_2 \notin \mathsf{null}\, T_1 + T_2$. Therefore $\mathsf{null}\, T_1 + T_2 = \{0\}$ and $T_1 + T_2$ is an injective linear map.

Note that if $\dim V = 1$ a linear map will always be injective since $\dim \mathsf{null}\, T < 1$ (excluding the trivial case of $Tv = 0$ for all v) and therefore $\mathsf{null}\, T = \{0\}$ (as an example consider the linear map $T : x \to ax$). Also if $\dim W < \dim V$ then it is not possible for any linear map to be injective, so in this case,

$$\{T \in \mathcal{L}(V,W) : T \text{ is not injective } \} = \mathcal{L}(V,W)$$

Problem 6 (3.B.8): Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not surjective } \}$ is not a subspace of $\mathcal{L}(V,W)$.

Solution: The restriction $\dim W \geq 2$ is necessary to avoid dealing with trivial cases. If $\dim W = 1$ then if T is not surjective $\dim \mathsf{range}\, T = 0$ which is only possible if $TV = 0v$ and the set of non-surjective linear maps has a single member. If $\dim W = 0$ then $\mathcal{L}(V,\{0\}) = \{T : Tv = 0v\}$ and the set of non-surjective linear maps is the empty set.

Now start with two linear maps T_1, T_2 which are not surjective with the property $\mathsf{range}\, T_1 \subset W$, $\mathsf{range}\, T_2 \subset W$ and $\mathsf{range}\, T_1 \cup \mathsf{range}\, T_2 = W$. Then $\mathsf{range}\, T_1 + T_2 = W$ and therefore $T_1 + T_2$ is surjective.

Problem 7 (3.B.12): Suppose V is finite-dimensional and that $T \in \mathcal{L}(V,W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

Solution: Think of the domain of T as composed of two sets: the null set for which $v \in \operatorname{null} T$, $Tv = 0$ and U for which $u \in U$, $Tu \neq 0$ unless $u = 0$. Then since $\operatorname{domain} T = U \cup \operatorname{null} T$, $\operatorname{range} T = \{Tu : u \in U\}$. For U to be a subspace of V it must have the following properties:

i) $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$: since $T(u_1 + u_2) = 0$ only if $Tu_1 + Tu_2 = 0 \Rightarrow Tu_1 = -Tu_2$ and this is true only if $u_1 = -u_2$ (in which case $u_1 + u_2 = 0 \in U$) this property holds.
ii) $u \in U, a \in \mathbb{F} \Rightarrow au \in U$: unless $u = 0$ or $a = 0$ or both are zero $au \neq 0$ so $au \in U$.
iii) $0 \in U$ from the definition of U so this completes the proof.

Problem 8 (3.B.20): Suppose W is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W,V)$ such that ST is the identity map on V.

Solution: If $ST : \mathcal{L}(V,V)$ is the identity map then for any $v \in V$, $STv = v$. This can be rewritten as $S(Tv)$; now assume that T is not injective in which case for $v_1 \neq v_2$, $Tv_1 = Tv_2 = w$. Since $STv_1 \neq STv_2$ we must have $Sw \neq Sw$ which is not possible. Therefore T must be injective.

Next assume that T is injective and define a linear map $S = \mathcal{L}(W,V)$ with the property $Sw = v$ if $Tv = w$. This map is defined since T is injective and there is a unique v for which $Tv = w$. Therefore $STv = v$ and ST is the identity map on V.

Problem 9 (3.B.21): Suppose V is finite dimensional and $T \in \mathcal{L}(V,W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W,V)$ such that TS is the identity map on W.

Solution: If $TS \in \mathcal{L}(W,W)$ is the identity map then for all $w \in W$, $TSw = w$. For the identity map to exist S must map all $w \in W$ to some $\operatorname{range} S \subseteq V$ and then T must map $\operatorname{range} S$ to the whole of W. But if T is not surjective, $\operatorname{range} T \subset W$ and therefore for any $\operatorname{range} S$ it is not possible for $\operatorname{range} TS = W$. Therefore T must be surjective.

If T is surjective then $\operatorname{range} T = W$; therefore for any $w \in W$ there is one or more $v \in V$ such that $Tv = w$. Define a map $S : \mathcal{L}(W,V)$ which maps $w \in W$ to $v \in V$ such that $Tv = w$ (if more than one v have this property select one and keep this choice fixed). So if we start with $w \in W$ then we can map it to $v = Sw$. From the definition of S then it follows that $Tv = w$ or $TSw = w$ and therefore TS is the identity map.

Problem 10 (3.B.24): Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that $\operatorname{null} T_1 \subset \operatorname{null} T_2$ if and only if there exists $S \in \mathcal{L}(W,W)$ such that $T_2 = ST_1$.

Solution: Start with $\operatorname{null} T_1 \subset \operatorname{null} T_2$; this means that for some $v \in \operatorname{null} T_2$, $T_1 v \neq 0$. We can define a linear map $S \in \mathcal{L}(W,W)$ with the following properties: if $T_1 v = T_2 v = 0$ then $ST_1 v = S0 = 0 = T_2 v$ which is a property of all linear maps; if $T_2 v = 0$ and $T_1 v \neq 0$ then $T_1 v \in \operatorname{null} S$ and $S(T_1 v) = 0 = T_2 v$; in all other cases S maps $T_1 v$ to $T_2 v$, i.e. $ST_1 v = T_2 v$. Therefore we can define a linear map with the property $ST_1 = T_2$.

Next assume the existence of a linear map $S \in \mathcal{L}(W,W)$ such that $T_2 = ST_1$. For $v \in \operatorname{null} T_1$ we have $ST_1 v = 0 = T_2 v$ and therefore $v \in \operatorname{null} T_2$. For $v \in \operatorname{null} T_2$ there is no linear map $S' \in \mathcal{L}(W,W)$ such that $T_1 = S'T_2$. Therefore if $T_2 v = 0$ it is not always the case that $T_1 v = 0$. Therefore $\operatorname{null} T_1 \subset \operatorname{null} T_2$.

Problem 11 (3.B.25): Suppose V is finite dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that $\operatorname{range} T_1 \subset \operatorname{range} T_2$ if and only if there exists $S \in \mathcal{L}(V,V)$ such that $T_1 = T_2 S$.

Solution: If $\operatorname{range} T_1 \subset \operatorname{range} T_2$ then if $w = T_1 v$ there exists $v' \in V$ for which $w = T_2 v'$. We can define $S \in \mathcal{L}(V,V)$ such that given $v \in \operatorname{range} T_1$ we find $v' \in \operatorname{range} T_2$ such that $T_1 v = T_2 v'$. Therefore if $v' = Sv$ then $T_2 v' = T_2 Sv$ and since $T_1 v = T_2 v'$ we have $T_1 v = T_2 Sv$ i.e. $T_1 = T_2 S$.

Next assume that there exists $S \in \mathcal{L}(V,V)$ such that $T_1 = T_2 S$. If $w \in \operatorname{range} T_1$ we have $w = T_1 v = T_2 Sv$ and since $Sv \in V$, $T_2 v \in \operatorname{range} T_2$, $w \in \operatorname{range} T_2$. If $w \in \operatorname{range} T_2$ we have $w = T_2 v$; however there is no linear map $S' \in \mathcal{L}(V,V)$ for which $T_2 = T_1 S'$ and therefore $w \in \operatorname{range} T_1$. Therefore it is not always true that if $w \in \operatorname{range} T_2$ it is also the case that $w \in \operatorname{range} T_1$ and therefore $\operatorname{range} T_1 \subset \operatorname{range} T_2$.

Problem 12 (3.B.29): Suppose $\phi \in \mathcal{L}(V,\mathbb{F})$. Suppose $u \in V$ is not in $\operatorname{null} \phi$. Prove that,

$$V = \operatorname{null} \phi \oplus \{au : a \in \mathbb{F}\}$$

Solution: To simplify the expressions denote $\{au : a \in \mathbb{F}\}$ as U. First we must prove that,

$$\operatorname{null} \phi \cap U = \{0\}$$

Since $u \notin \operatorname{null} \phi$ it is also the case that for $a \in \mathbb{F}$, $au \notin \operatorname{null} \phi$ except from the trivial case $u = 0$ for which $U = \{0\}$ or $a = 0, u \neq 0$ for which $au = 0$. Therefore,

$$\operatorname{null} \phi \cap U = \{0\}$$

Next we must prove that any $v \in V$ can be written as the sum of two vectors, $w \in \operatorname{null} \phi$ and $u \in U$. Since $\dim \operatorname{range} \phi \leq \dim \mathbb{F}$ we have,

$$\dim V - \dim \operatorname{null} \phi \leq 1 \Rightarrow \dim V - 1 \leq \dim \operatorname{null} \phi$$

For the case $\dim \operatorname{null} \phi = \dim V = n$ then all $v \in V$ are also $v \in \operatorname{null} \phi$. In this case $u = 0$ and since,

$$v = v + 0$$

any $v \in V$ can be written as the sum of $v \in \operatorname{null} \phi$ and $u \in U$. If $\dim \operatorname{null} \phi = \dim V - 1 = n - 1$, then any $w \in \operatorname{null} \phi$ can be written as,

$$w = w^1 e_1 + \ldots + w^{n-1} e_{n-1}$$

i.e. a linear combination of $n-1$ basis vectors. Since $u \notin \operatorname{null} \phi$,

$$au = au^n e_n$$

For $v \in V$,

$$v = v^1 e_1 + \ldots + v^{n-1} e_{n-1} + v^n e_n$$

Set $w^1 = v^1, \ldots, w^{n-1} = v^{n-1}$ and $au^n = v^n$ by setting $a = v^n / u^n$ (note that if $\dim \operatorname{null} \phi = n - 1$, $u^n \neq 0$ and $au = 0$ only for $a = 0$). Then any $v \in V$ can be expressed as the sum of $w \in \operatorname{null} \phi$ and au where $u \in U$ and $a \in \mathbb{F}$.

Problem 13 (3.B.30): Suppose ϕ_1 and ϕ_2 are linear maps from V to $\mathbb{F}$ that have the same null space. Show that there exists a constant $c \in \mathbb{F}$ such that $\phi_1 = c\phi_2$.

Solution: Denote $\operatorname{null} \phi_1 = \operatorname{null} \phi_2$ as N. For any linear map $\dim V = \dim \operatorname{null} + \dim \operatorname{range}$ so $\dim V - \dim N = \dim \operatorname{range} \phi_1 \leq \dim \mathbb{F} = 1$ (same is true for ϕ_2). If $\dim V = n$ then $\dim V - \dim N \leq 1$ or $n - 1 \leq \dim N$.

If $\dim N = n$ then $\dim \operatorname{range} \phi_1 = 0$ (same is true for ϕ_2) and therefore $\operatorname{range} \phi_1 = \operatorname{range} \phi_2 = \{0\}$. This is only possible if $\phi_1 v = \phi_2 v = 0v = 0$ for all $v \in V$. So if $\dim \operatorname{null} \phi_1 = \dim \operatorname{null} \phi_2 = n$ for any $c \in \mathbb{F}$, $\phi_1 = c\phi_2$.

If $\dim N = n - 1$ then $\dim \operatorname{range} \phi_1 = \dim \operatorname{range} \phi_2 = 1$ so for $v \notin N$,

$$\begin{aligned} v &= v^1 e_1 + \ldots + v^n e_n \\ \phi_1 v &= v^n \phi_1 e_n \\ \phi_2 v &= v^n \phi_2 e_n \end{aligned}$$

Define $c = \phi_1 e_n / \phi_2 e_n$ (note that $\phi_2 e_n \neq 0$ since if this is not the case $\phi_2 v = 0$ for all v); then

$$\phi_1 v = v^n (c\phi_2 e_n) = c\phi_2 v$$

and therefore $\phi_1 = c\phi_2$.

Problem 14 (3.B.31): Give an example of two linear maps T_1 and T_2 from $\mathbb{R}^5$ to $\mathbb{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.

Solution: Consider,

$$A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{bmatrix}$$

where A_i are 2×1 independent vectors. For $i = 3, 4, 5$,

$$A_i = a_{i1} A_1 + a_{i2} A_2$$

and the null space of A is composed of vectors x with the property,

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 + x_5 A_5 = 0$$

or,

$$(x_1 + a_{31} x_3 + a_{41} x_4 + a_{51} x_5) A_1 + (x_2 + a_{32} x_3 + a_{42} x_4 + a_{52} x_5) A_2 = 0$$

Since A_1, A_2 are independent vectors the last condition holds only if,

$$\begin{aligned} x_1 &= -a_{31} x_3 - a_{41} x_4 - a_{51} x_5 \\ x_2 &= -a_{32} x_3 - a_{42} x_4 - a_{52} x_5 \end{aligned}$$

therefore any $x \in \operatorname{null} A$,

$$x = \begin{pmatrix} -a_{31} x_3 - a_{41} x_4 - a_{51} x_5 \\ -a_{32} x_3 - a_{42} x_4 - a_{52} x_5 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -a_{31} \\ -a_{32} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -a_{41} \\ -a_{42} \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -a_{51} \\ -a_{52} \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Since the three vectors in the last equation are independent they form the basis of $\operatorname{null} A$.

Now define another 5×2 matrix,

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix}$$

where B_i are 2×1 independent vectors. For $i = 3, 4, 5$,

$$B_i = a_{i1} B_1 + a_{i2} B_2$$

and using the same procedure as above the same three vectors form the basis of $\operatorname{null} B$. Therefore $\operatorname{null} A = \operatorname{null} B$; since B_1, B_2 are different from A_1, A_2, A is not a scalar multiple of B. As an example,

$$A = \begin{bmatrix} 2 & -1 & 1 & 3 & 5 \\ 1 & 2 & 3 & -1 & 0 \end{bmatrix}$$

and,

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

have null space,

$$\text{null}A = \text{null}B = \text{Span}\left\{\begin{pmatrix}-1\\-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\0\\0\\1\end{pmatrix}\right\}$$

Problem 15 (3.C.2): Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is,

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

Solution: If we express,

$$\mathcal{P}_3(x) = \sum_{i=1}^{4} b_i f_i(x)$$

where f_i is the i^{th} basis of $\mathcal{P}_3$ and,

$$\mathcal{P}_2(x) = \sum_{i=1}^{3} b_i g_i(x)$$

where g_i is the i^{th} basis of $\mathcal{P}_2$ then from the structure of the matrix of D we have,

$$\sum_{i=1}^{3} b_i g_i(x) = \sum_{i=1}^{4} b_i f_i'(x)$$

Since $f_4'(x) = 0$, $f_4(x) =$ constant and we can set $f_4(x) = 1$. For $i = 3$, $f_3'(x) = g_3(x)$ so one choice is $f_3(x) = x$ and $g_3(x) = 1$; similarly $f_2(x) = x^2$, $g_2(x) = 2x$ and $f_1(x) = x^3$, $g_1(x) = 3x^2$. So starting from the polynomial,

$$\mathcal{P}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

we can write it as,

$$\mathcal{P}_3(x) = a_3 f_1(x) + a_2 f_2(x) + a_1 f_3(x) + a_0 f_4(x)$$

which when differentiated gives,

$$D\mathcal{P}_3(x) = \mathcal{P}_2(x) = a_3 g_1(x) + a_2 g_2(x) + a_1 g_3(x)$$

We can rewrite the last equation in matrix form as,

$$\begin{pmatrix}a_3\\a_2\\a_1\end{pmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{pmatrix}a_3\\a_2\\a_1\\a_0\end{pmatrix}$$

Problem 16 (3.C.3): Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Prove that there exist a basis of V and W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Solution: Denote $\dim V$ as n and $\dim \operatorname{null} T$ as r; from the Fundamental Theorem of Linear Maps we have, $\dim \operatorname{range} T = n - r$, $\operatorname{range} T = \operatorname{Span}\{e_1, \ldots, e_{n-r}\}$ and $\operatorname{null} T = \operatorname{Span}\{e_{n-r+1}, \ldots, e_n\}$. Any $v \in V$ can be written as,

$$v = v^1 e_1 + \ldots + v^{n-r} e_{n-r} + v^{n-r+1} e_{n-r+1} + \ldots + v^n e_n$$

where $\{e_1, \ldots, e_n\}$ is a basis of V. If $v \in \operatorname{null} T$,

$$v = 0e_1 + \ldots + 0e_{n-r} + v^{n-r+1} e_{n-r+1} + \ldots + v^n e_n$$

and since $Te_{n-r+1} = \ldots = Te_n = 0$ we have,

$$Tv = 0Te_1 + \ldots + 0Te_{n-r} + v^{n-r+1} 0 + \ldots + v^n 0 = 0$$

For $w \in \operatorname{range} T$ with $w \neq 0$,

$$w = Tv = v^1 Te_1 + \ldots + v^{n-r} Te_{n-r} + v^{n-r+1} Te_{n-r+1} + \ldots + v^n Te_n$$

or,

$$w = Tv = v^1 Te_1 + \ldots + v^{n-r} Te_{n-r} + v^{n-r+1} 0 + \ldots + v^n 0$$

Note that vectors $Te_1, \ldots, Te_{n-r}$ are independent since $w = 0$ only if $v = 0$ if $v \notin \operatorname{null} T$. Therefore vectors $Te_1, \ldots, Te_{n-r}$ can be the basis vectors for $\operatorname{range} T$. From the definition of $\mathcal{M}(T)$ the entries of the matrix satisfy the condition,

$$Te_k = A_{1,k} w_1 + \ldots + A_{n-r,k} w_{n-r}$$

where $w_1, \ldots, w_{n-r}$ are the basis vectors of $\operatorname{range} T$. If $w_k = Te_k$ for $j = 1, \ldots, n-r$ then $A_{k,k} = 1$ and $A_{k,k'} = 0$ for $k \neq k'$. Therefore A is,

$$\begin{pmatrix} w^1 \\ \vdots \\ w^{n-r} \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}$$

or,

$$A = \begin{bmatrix} I & O \end{bmatrix}$$

where I is the $n - r \times n - r$ identity matrix and O is the $n - r \times r$ zero matrix.

Problem 17 (3.C.4): Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove there exists a basis $w_1, \dots, w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$) are 0 except from possibly a 1 in the first row, first column.

Solution: The format of $\mathcal{M}(T)$ is as follows,

$$
\begin{array}{c} \\ w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{array}
\begin{array}{c}
\begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_m \end{array} \\
\begin{bmatrix}
A_{1,1} & \cdots & A_{1,j} & \cdots & A_{1,m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_{i,1} & \cdots & A_{i,j} & \cdots & A_{i,m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,j} & \cdots & A_{n,m}
\end{bmatrix}
\end{array}
$$

and the matrix coefficients are defined by,

$$Tv_j = \sum_{i=1}^{n} A_{i,j} w_i$$

We want a basis $\hat{w}_1, \dots, \hat{w}_n$ for which the linear map $T' : V \to \hat{W}$,

$$
\begin{array}{c} \\ \hat{w}_1 \\ \vdots \\ \hat{w}_i \\ \vdots \\ \hat{w}_n \end{array}
\begin{array}{c}
\begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_m \end{array} \\
\begin{bmatrix}
1 \text{ or } 0 & \cdots & B_{1,j} & \cdots & B_{1,m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & B_{i,j} & \cdots & B_{i,m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & B_{n,j} & \cdots & B_{n,m}
\end{bmatrix}
\end{array}
$$

If we start with the linear map $T : V \to W$ and then add a linear map $S : W \to \hat{W}$ then since $T' : V \to \hat{W}$, $\mathcal{M}(T') = \mathcal{M}(S)\mathcal{M}(T)$ or $B = CA$.
If $B_{1,1} = \dots = B_{n,1} = 0$ then the equation,

$$T'v_1 = 0\hat{w}_1 + \dots + 0\hat{w}_n$$

holds for any basis $\{\hat{w}_1, \ldots, \hat{w}_n\}$. If $B_{1,1} = 1$ and $B_{i,1} = 0$ for $i > 1$ then we must have,

$$Tv_1 = 1\hat{w}_1 + \ldots + 0\hat{w}_n$$

and therefore $\hat{w}_1 = Tv_1$.
Define a $n \times n$ matrix with the following properties,

$$\begin{aligned} &C_{i,i} = A_{i,1}^{-1} \\ &C_{i,1} = A_{1,1}^{-1} \\ &C_{i,j} = 0 \text{ for } i \neq j \text{ , } j > 1 \end{aligned}$$

so that,

$$CA = \begin{bmatrix} A_{1,1}^{-1} & 0 & 0 & 0 & \cdots & 0 \\ A_{1,1}^{-1} & -A_{2,1}^{-1} & 0 & 0 & \cdots & 0 \\ A_{1,1}^{-1} & 0 & -A_{3,1}^{-1} & 0 & \cdots & 0 \\ A_{1,1}^{-1} & 0 & 0 & -A_{4,1}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{1,1}^{-1} & 0 & 0 & 0 & \cdots & -A_{n,1}^{-1} \end{bmatrix} \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots & A_{1,m} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & \cdots & A_{2,m} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & \cdots & A_{3,m} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} & \cdots & A_{4,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & A_{n,3} & A_{n,4} & \cdots & A_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{A_{1,2}}{A_{1,1}} & \frac{A_{1,3}}{A_{1,1}} & \frac{A_{1,4}}{A_{1,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} \\ 0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{2,2}}{A_{2,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{2,3}}{A_{2,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{2,4}}{A_{2,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{2,m}}{A_{2,1}} \\ 0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{3,2}}{A_{3,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{3,3}}{A_{3,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{3,4}}{A_{3,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{3,m}}{A_{3,1}} \\ 0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{4,2}}{A_{4,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{4,3}}{A_{4,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{4,4}}{A_{4,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{4,m}}{A_{4,1}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{A_{1,2}}{A_{1,1}} - \frac{A_{n,2}}{A_{n,1}} & \frac{A_{1,3}}{A_{1,1}} - \frac{A_{n,3}}{A_{n,1}} & \frac{A_{1,4}}{A_{1,1}} - \frac{A_{n,4}}{A_{n,1}} & \cdots & \frac{A_{1,m}}{A_{1,1}} - \frac{A_{n,m}}{A_{n,1}} \end{bmatrix}$$

C is admissible as a basis transformation matrix only if $A_{i,1} \neq 0$ for all i. Since,

$$w_r = \sum_{j=1}^{n} C_{j,r} \hat{w}_j$$

and both w_r and $\hat{w}_j$ are $n \times 1$ vectors, if W is the matrix whose columns are the w basis vectors and $\hat{W}$ is the matrix whose columns are the $\hat{w}$ basis vectors then,

$$W_{i,r} = \sum_{j=1}^{n} C_{j,r} \hat{W}_{i,j}$$

and therefore,

$$W = \hat{W} C$$

To express the new basis vectors in terms of the original basis we can right multiply both sides with C^{-1} to get,

$$\hat{W} = W C^{-1}$$

It is easy to show that,

$$C^{-1} = \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{cccccc} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 & \cdots & \hat{w}_n \end{array} \\ \begin{bmatrix} A_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ A_{2,1} & -A_{2,1} & 0 & 0 & \cdots & 0 \\ A_{3,1} & 0 & -A_{3,1} & 0 & \cdots & 0 \\ A_{4,1} & 0 & 0 & -A_{4,1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & 0 & 0 & 0 & \cdots & -A_{n,1} \end{bmatrix} \end{array}$$

and therefore,

$$\hat{w}_1 = \sum_{i=1}^{n} A_{i,1} w_i = T v_1$$
$$\hat{w}_i = -A_{i,1} w_i \quad \text{for } i > 1$$

Why can't we go further and derive a basis of W such that the first two entries in the first column of $\mathcal{M}(T)$ are equal to 1? Let's assume that a matrix C exists with the property,

$$\sum_{r=1}^{n} C_{1,r} A_{r,1} = 1$$
$$\sum_{r=1}^{n} C_{2,r} A_{r,1} = 1$$

This is only possible if $C_{1,r} = C_{2,r}$ i.e. if the first two rows of C are identical. A matrix with two identical rows is not admissible as a basis transformation matrix. In order to prove this lets assume that a matrix C exists such that,

$$
\begin{array}{c c}
 & \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & \cdots & w_n \end{array} \\
\begin{array}{c} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \\ \hat{w}_4 \\ \vdots \\ \hat{w}_n \end{array} &
\begin{bmatrix}
C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & \cdots & C_{1,n} \\
C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & \cdots & C_{1,n} \\
C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & \cdots & C_{2,n} \\
C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & \cdots & C_{3,n} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
C_{n-1,1} & C_{n-1,2} & C_{n-1,3} & C_{n-1,4} & \cdots & C_{n-1,n}
\end{bmatrix}
\end{array}
$$

Given that vectors $\hat{w}_i$ are independent we want to show that we can choose a set of constants $a_1, \ldots, a_n$ that are not all zero but for which,

$$\sum_{i=1}^{n} a_i w_i = 0$$

Since,

$$w_i = C_{1,i}(\hat{w}_1 + \hat{w}_2) + \sum_{j=2}^{n} C_{j,i} \hat{w}_j$$

we must have,

$$\sum_{i=1}^{n} a_i \left(C_{1,i}(\hat{w}_1 + \hat{w}_2) + \sum_{j=2}^{n} C_{j,i} \hat{w}_j \right) = 0$$

The last equation holds only if the coefficients of the $\hat{w}_j$ vectors are all zero, i.e,

$$\sum_{i=1}^{n} C_{j,i} a_i = 0$$

for all j. Since C is the matrix of a linear map from a vector space with dimension n to vector space with dimension $n-1$ this linear map cannot be injective; therefore the null set of this transformation contains non-zero vectors. It follows that $w_1, w_2, \ldots, w_n$ are not independent and a basis transformation matrix with two identical rows is not admissible.

Problem 18 (3.D.2): Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

Solution: If an operator T is noninvertible it is also noninjective and nonsurjective since V is finite-dimensional. If an operator is nonsurjective then $\dim \operatorname{range} T < \dim V$ (this explains the condition $\dim V > 1$; if $\dim V = 1$ then $\dim rangeT = 0$, T is an operator which maps all $v \in V$ to 0 and the set of operators with this property is a subspace).

We can choose two operators T_1 and T_2 with the following property: $\operatorname{range} T_1 = \operatorname{Span}\{v_1, \dots, v_r\}$ and $\operatorname{range} T_2 = \operatorname{Span}\{v_{r+1}, \dots, V_n\}$ where $n = \dim V$. Then $T_1 + T_2$ has the property $\operatorname{range} T_1 + T_2 = \operatorname{Span}\{v_1, \dots, v_n\} = V$. Since $T_1 + T_2$ is surjective it is also the case that it is invertible. Therefore the set of noninvertible operators is not closed under addition and is not a subspace of $\mathcal{L}(V)$.

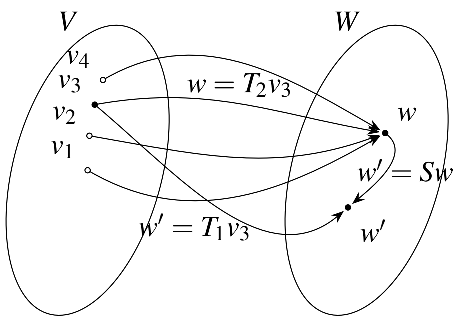
Problem 19 (3.D.3): Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution: Since U is a subspace of V we can construct an operator $T \in \mathcal{L}(V)$ with the property $Tu = Su$ for every $u \in U$. It is clear that since S is injective in U, T has the same property. For $v \in V \setminus U$, we can use the identity operator so that $Tv = Iv = v$ which is also injective. Therefore, T is injective for all $v \in V$; since an injective operator in finite dimensional space is also invertible it follows that T is invertible.

Now start with the condition that T is an invertible operator in $\mathcal{L}(V)$; this implies it is also injective. Since $Tu = Su$ for every $u \in U$, S must also be injective.

Problem 20 (3.D.4): Suppose W is finite dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in \mathcal{L}(W)$ such that $T_1 = ST_2$.

Solution: Start with $\operatorname{null} T_1 = \operatorname{null} T_2$; $T_1 v \neq 0 \Leftrightarrow Tv_2 \neq 0$; define an operator S using a construction as in the following graph:



For any $w \in \operatorname{range} T_2$ we can find a v such that $w = T_2 v$ (in the example above there are four different vs with this property so we choose v_3). For the same v, $w' = T_1 v$ (it is possible that in some cases $w' = w$). The linear operator S has the property $w' = Sw$. Note that if $v \in \operatorname{null} T_2$, $T_2 v = 0$ and since $\operatorname{null} T_1 = \operatorname{null} T_2$, $S0 = 0$ which is a property of any linear operator. From the definition of S, if $T_2 v \neq 0$ then $ST_2 v = T_1 v \neq 0$ and therefore $\operatorname{null} S = \{0\}$. This means S is injective and using the properties of linear operators it is also invertible.

Now assume that S is an invertible linear operator with the property $T_1 = ST_2$. For $v \in \operatorname{null} T_2$, $T_1 v = ST_2 v = S0 = 0$ and therefore $v \in \operatorname{null} T_1$. Since S is invertible, $T_2 = S^{-1}T_1$; therefore for $v \in \operatorname{null} T_1$, $T_2 v = S^{-1}T_1 v = S^{-1}0 = 0$ and $v \in \operatorname{null} T_2$. We conclude that $\operatorname{null} T_1 = \operatorname{null} T_2$.

Problem 21 (3.D.7): Suppose V and W are finite-dimensional. Let $v \in V$. Let, $E = \{T \in \mathcal{L}(V,W) : Tv = 0\}$. Show that,

a) E is a subspace of $\mathcal{L}(V,W)$.

b) Suppose $v \neq 0$. What is $\dim E$?

Solution: E is a subspace of $\mathcal{L}(V,W)$ if the following conditions hold:

i) Linear map $0 \in E$: since by definition $0v = 0$ for all v this condition holds.

ii) $T_1, T_2 \in E \Rightarrow T_1 + T_2 \in E$: since $T_1 v = 0$ and $T_2 v = 0$ we have $T_1 v + T_2 v = (T_1 + T_2)v = 0$.

iii) $T \in E$, $a \in \mathbb{F} \Rightarrow aT \in E$: since $Tv = 0$, $aTv = 0$ and therefore $aT \in E$.

Any linear map $T \in \mathcal{L}(V,W)$ can be represented by a matrix,

$$\begin{array}{c} \\ w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \cdots & v_j & \cdots & v_m \end{array} \\ \begin{bmatrix} A_{1,1} & \cdots & A_{1,j} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i,1} & \cdots & A_{i,j} & \cdots & A_{i,m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,j} & \cdots & A_{n,m} \end{bmatrix} \end{array}$$

whose elements are defined by,

$$Tv_i = \sum_{j=1}^{n} A_{j,i} w_j$$

We can set the basis of V to have $v_1 = v$ without any loss of generality. Since $T \in E \Rightarrow Tv = 0$ a linear map $T \in E$ can be represented by the following matrix,

$$
\begin{array}{c|ccccc|}
 & v_1 & \cdots & v_j & \cdots & v_m \\
\hline
w_1 & 0 & \cdots & A_{1,j} & \cdots & A_{1,m} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
w_i & 0 & \cdots & A_{i,j} & \cdots & A_{i,m} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
w_n & 0 & \cdots & A_{n,j} & \cdots & A_{n,m}
\end{array}
$$

The matrix representing $T \in \mathcal{L}(V,W)$ is the linear combination of $n \times m$ basis matrices each with zero for all elements except from $A_{i,j} = 1$ i.e.,

$$\mathcal{M}(T) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_{i,j} \mathcal{M}_{i,j}$$

where,

$$
\mathcal{M}_{i,j} = \begin{array}{c|ccccc|}
 & v_1 & \cdots & v_j & \cdots & v_m \\
\hline
w_1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
w_i & 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
w_n & 0 & \cdots & 0 & \cdots & 0
\end{array}
$$

For $T \in E$, $A_{i,1} = 0$ for all i and therefore,

$$\mathcal{M}(T) = \sum_{i=1}^{n} \sum_{j=2}^{m} A_{i,j} \mathcal{M}_{i,j}$$

i.e. the basis of $\mathcal{M}(T)$ consists of $(m-1) \times n$ matrices and therefore,

$$\dim E = (m-1)n = (\dim V - 1) \dim W$$

If $v \in V$ and U is a subspace of V the affine subset,

$$v + U = \{v + u : u \in U\}$$

is said to be parallel to U. Two affine subsets parallel to U are either equal or disjoint. To show that this is the case start with the assumption that $v+U \subset w+U$ and $w+U \not\subset v+U$. Then there exists $u \in U$ s.t. $v+u \in w+U$; this means we can find $u' \in U$ s.t. $v+u = w+u'$. Rearanging terms we have $v-w+u = u'$; since U is a subspace of V this is only possible if $v-w \in U$ since U is closed under addition.

Next we want to show that there exists $u \in U$ s.t. $w+u \notin v+U$. This means we cannot find any $u' \in U$ s.t. $w+u = v+u'$; but this is not the case since $w+u = v+(w-v)+u$ and since U is a subspace it is closed under scalar multiplication and addition, therefore, since $w-v, u \in U$, we have found $u' = (w-v)+u$ so that $w+u = v+u'$. So $w+u \subset v+U$ and $v+U = w+U$.

For the disjoint case, start with $v+u \notin w+U$; there is no $u' \in U$ s.t. $v+u = w+u'$. Lets assume that $w+u \in v+U$, i.e. there is $u' \in U$ s.t. $w+u = v+u'$; since U is a subspace, this means that $w-v \in U$. If this is the case $v+u = w+(v-w)+u = w+u'$ and therefore $v+U \in w+U$ which contradicts our assumption. Therefore $v+U \cap w+U = \varnothing$.