Most of the material presented here is taken from [1].

Given a finite dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$ with characteristic zero and with an inner product $g : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ define a bilinear map $E : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with the following properties:

I Bilinear property: $E(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{z}) + bE(\boldsymbol{y}, \boldsymbol{z})$, $E(\boldsymbol{x}, a\boldsymbol{y} + b\boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{y}) + bE(\boldsymbol{x}, \boldsymbol{z})$ where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$ and $a, b \in \mathbb{F}$.

II Antisymmetric property: $g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. This is an expression of the antisymmetric property since

$$\begin{aligned} g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{y}, \boldsymbol{x})) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{y}) & g(\boldsymbol{x}, \boldsymbol{x}) \\ g(\boldsymbol{y}, \boldsymbol{y}) & g(\boldsymbol{y}, \boldsymbol{x}) \end{vmatrix} = -\begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix} \\ &= -g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = g(E(\boldsymbol{x}, \boldsymbol{y}), -E(\boldsymbol{x}, \boldsymbol{y})) \end{aligned}$$

and so we must have $E(\boldsymbol{x}, \boldsymbol{y}) = -E(\boldsymbol{y}, \boldsymbol{x})$.

III Orthogonal property: $g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) = g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y}) = 0$.

We can use the bilinear map E to induce a linear map $L_x : \mathcal{V} \to \mathcal{V}$ as follows:

$$L_x(\boldsymbol{y}) = E(\boldsymbol{x}, \boldsymbol{y}), \text{ for all } \boldsymbol{y} \in \mathcal{V}.$$

This follows from (I). We note that

$$L_x^2(\boldsymbol{y}) = L_x E(\boldsymbol{x}, \boldsymbol{y}) = E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})).$$

From (II) we have

$$\begin{aligned} g(E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})), E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y}))) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})) \\ g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & 0 \\ 0 & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= g(\boldsymbol{x}, \boldsymbol{x}) g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \\ &= g(\boldsymbol{x}, \boldsymbol{x}) \left(g(\boldsymbol{x}, \boldsymbol{x}) g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 \right) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) + g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &\quad - 2g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &= g(g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}, g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}). \end{aligned}$$

Therefore

$$E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}$$

and

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}.$$

The tensor product $\boldsymbol{x} \otimes \boldsymbol{y}$ induces a linear map $L_{x\otimes y} : \mathcal{V} \to \mathcal{V}$ defined as

$$\begin{aligned}(\boldsymbol{x} \otimes \boldsymbol{y})(\boldsymbol{z}) &= (x^i \boldsymbol{e}_i \otimes y^j \boldsymbol{e}_j)(\boldsymbol{z}) = x^i y^j (\boldsymbol{e}_i \otimes \boldsymbol{e}_j)(\boldsymbol{z}) \\ &= x^i y^j z_j \boldsymbol{e}_i = x^i y^j z^k g_{jk} \boldsymbol{e}_i \\ &= g(\boldsymbol{y}, \boldsymbol{z}) x^i \boldsymbol{e}_i = g(\boldsymbol{y}, \boldsymbol{z})\boldsymbol{x}, \text{ for all } \boldsymbol{z} \in \mathcal{V}.\end{aligned}$$

From these definitions we can also generate the composite maps

$$L_{x\otimes y} L_z(\boldsymbol{u}) = L_{x\otimes y}(E(\boldsymbol{z}, \boldsymbol{u})) = g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{u}))\boldsymbol{x}$$

and

$$L_x L_{y\otimes z}(\boldsymbol{u}) = L_x(g(\boldsymbol{z}, \boldsymbol{u})\boldsymbol{y}) = g(\boldsymbol{z}, \boldsymbol{u}) E(\boldsymbol{x}, \boldsymbol{y}) = L_{E(x,y)\otimes z}\boldsymbol{u}.$$

Note that since

$$g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z})) = 0$$

(using property (III)) we have

$$\begin{aligned}g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z})) &= g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z}) + E(\boldsymbol{y}, \boldsymbol{z})) \\ &= g(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{y}, \boldsymbol{z})) \\ &= g(\boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) = 0.\end{aligned}$$

So we obtain

$$g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{x})) = g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z}))$$

using the antisymmetric property of E. In the same way we can prove that this property is cyclic, i.e.

$$g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) = g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{x})) = g(\boldsymbol{z}, E(\boldsymbol{x}, \boldsymbol{y})).$$

This means that we can write

$$g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{u})) = g(\boldsymbol{z}, E(\boldsymbol{u}, \boldsymbol{y})) = g(\boldsymbol{u}, E(\boldsymbol{y}, \boldsymbol{z}))$$

and so

$$L_{x\otimes y} L_z(\boldsymbol{u}) = g(\boldsymbol{u}, E(\boldsymbol{y}, \boldsymbol{z}))\boldsymbol{x} = L_{x\otimes E(y,z)}(\boldsymbol{u}).$$

Using these properties we can write

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y} = L_{x\otimes x}(\boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{x}) I(\boldsymbol{y})$$

where $I : \mathcal{V} \to \mathcal{V}$ is the identity map. So

$$L_x^2 = L_{x\otimes x} - g(\boldsymbol{x}, \boldsymbol{x}) I.$$

We can obtain the linearization of L_x^2 by writing

$$\begin{aligned} L_{x+h}^2(\boldsymbol{y}) &= g(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{y})(\boldsymbol{x}+\boldsymbol{h}) - g(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{x}+\boldsymbol{h})\boldsymbol{y} \\ L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_xL_h + L_hL_x + L_h^2 \end{aligned}$$

The r.h.s. of the first equation expands to

$$\begin{aligned} L_{x+h}^2(\boldsymbol{y}) &= g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \\ &\quad + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{h},\boldsymbol{h})\boldsymbol{y} \\ &= L_x^2(\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + L_h^2(\boldsymbol{y}). \end{aligned}$$

We obtain

$$L_xL_h(\boldsymbol{y}) + L_hL_x(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

If

$$L_xL_h(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y}$$

then

$$L_hL_x(\boldsymbol{y}) = g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

We can use the linearization of L_x^2 to perform the following derivation:

$$\begin{aligned} (L_{E(x,y)} + L_xL_y)(\boldsymbol{z}) &= E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{z}) + L_xLy(\boldsymbol{z}) \\ &= -E(\boldsymbol{z},E(\boldsymbol{x},\boldsymbol{y})) + L_xLy(\boldsymbol{z}) \\ &= -L_zL_x(\boldsymbol{y}) + L_xLy(\boldsymbol{z}) \\ &= -(g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{z},\boldsymbol{x})\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} \\ &= 2g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z}. \end{aligned}$$

This result can be used to write

$$L_{E(x,y)} + L_xL_y = 2\,\boldsymbol{y}\otimes\boldsymbol{x} - \boldsymbol{x}\otimes\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})I.$$

From this equation we get

$$\begin{aligned} L_{E(x,y)}L_x + L_xL_yL_x &= 2\,\boldsymbol{y}\otimes\boldsymbol{x}L_x - \boldsymbol{x}\otimes\boldsymbol{y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x + 2\,\boldsymbol{y}\otimes\boldsymbol{x}L_x - \boldsymbol{x}\otimes\boldsymbol{y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x - \boldsymbol{x}\otimes E(\boldsymbol{y},\boldsymbol{x}) - g(\boldsymbol{x},\boldsymbol{y})L_x \end{aligned}$$

since

$$(\boldsymbol{y}\otimes\boldsymbol{x})L_x(\boldsymbol{z}) = (\boldsymbol{y}\otimes\boldsymbol{x})(E(\boldsymbol{x},\boldsymbol{z})) = g(\boldsymbol{x},E(\boldsymbol{x},\boldsymbol{z}))\boldsymbol{y} = 0,$$

and

$$(\boldsymbol{x}\otimes\boldsymbol{y})L_x(\boldsymbol{z}) = (\boldsymbol{x}\otimes\boldsymbol{y})E(\boldsymbol{x},\boldsymbol{z}) = g(\boldsymbol{y},E(\boldsymbol{x},\boldsymbol{z}))\boldsymbol{x} = (\boldsymbol{x}\otimes E(\boldsymbol{y},\boldsymbol{x}))(\boldsymbol{z}).$$

From the same equation

$$\begin{aligned} L_{E(x,y)}L_x &= -L_{E(E(x,y),x)} + 2\boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) - E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x} - g(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y})I \\ &= -L_{E(E(x,y),x)} + 2\boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) - E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x}; \end{aligned}$$

this produces a second equation for $L_xL_yL_x$:

$$\begin{aligned} L_xL_yL_x &= -(-L_{E(E(x,y),x)} + 2\boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) - E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x}) - \boldsymbol{x} \otimes E(\boldsymbol{y},\boldsymbol{x}) - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2\boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) + E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x} - \boldsymbol{x} \otimes E(\boldsymbol{y},\boldsymbol{x}) - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2\boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) + E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x} + \boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - \boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) + E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{y})L_x. \end{aligned}$$

Next

$$E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{x}) = -E(\boldsymbol{x},E(\boldsymbol{x},\boldsymbol{y})) = -(g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}.$$

Therefore

$$\begin{aligned} L_{E(E(x,y),x)}(\boldsymbol{z}) &= E(g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x},\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{x})E(\boldsymbol{y},\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})E(\boldsymbol{x},\boldsymbol{z}) \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y(\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})L_x(\boldsymbol{z}). \end{aligned}$$

The third equation for $L_xL_yL_x$ is:

$$\begin{aligned} L_xL_yL_x &= g(\boldsymbol{x},\boldsymbol{x})L_y - g(\boldsymbol{x},\boldsymbol{y})L_x - \boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) + E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y - 2g(\boldsymbol{x},\boldsymbol{y})L_x - \boldsymbol{x} \otimes E(\boldsymbol{x},\boldsymbol{y}) + E(\boldsymbol{x},\boldsymbol{y}) \otimes \boldsymbol{x}. \end{aligned}$$

References

[1] Alberto Elduque. Vector cross products. *Electronic copy found at: http://www. unizar. es/matematicas/algebra/elduque/Talks/crossproducts. pdf*, 2004.