

# Cross Products

Most of the material presented here is taken from [1].

Given a finite dimensional vector space  $\mathcal{V}$  over a field  $\mathbb{F}$  with characteristic zero and with an inner product  $g : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{F}$  define a bilinear map  $E : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  with the following properties:

I. Bilinear property:  $E(ax + by, z) = aE(x, z) + bE(y, z)$ ,  $E(x, ay + bz) = aE(x, y) + bE(x, z)$  where  $x, y, z \in \mathcal{V}$  and  $a, b \in \mathbb{F}$ .

II. Antisymmetric property:

$$g(E(x, y), E(x, y)) = \begin{vmatrix} g(x, x) & g(x, y) \\ g(y, x) & g(y, y) \end{vmatrix}$$

for all  $x, y \in \mathcal{V}$ . This is an expression of the antisymmetric property since

$$\begin{aligned} g(E(x, y), E(y, x)) &= \begin{vmatrix} g(x, y) & g(x, x) \\ g(y, y) & g(y, x) \end{vmatrix} = - \begin{vmatrix} g(x, x) & g(x, y) \\ g(y, x) & g(y, y) \end{vmatrix} \\ &= -g(E(x, y), E(x, y)) = g(E(x, y), -E(x, y)) \end{aligned}$$

and so we must have  $E(x, y) = -E(y, x)$ .

III. Orthogonal property:  $g(E(x, y), x) = g(E(x, y), y) = 0$ .

We can use the bilinear map  $E$  to induce a linear map  $L_x : \mathcal{V} \rightarrow \mathcal{V}$  as follows:

$$L_x(y) = E(x, y), \text{ for all } y \in \mathcal{V}.$$

This follows from property I. We note that

$$L_x^2(y) = L_x E(x, y) = E(x, E(x, y)).$$

From property II we have

$$\begin{aligned} g(E(x, E(x, y)), E(x, E(x, y))) &= \begin{vmatrix} g(x, x) & g(x, E(x, y)) \\ g(E(x, y), x) & g(E(x, y), E(x, y)) \end{vmatrix} \\ &= \begin{vmatrix} g(x, x) & 0 \\ 0 & g(E(x, y), E(x, y)) \end{vmatrix} \\ &= g(x, x)g(E(x, y), E(x, y)) \\ &= g(x, x)(g(x, x)g(y, y) - g(x, y)^2) \\ &= g(x, x)^2g(y, y) - g(x, y)^2g(x, x) \\ &= g(x, x)^2g(y, y) + g(x, y)^2g(x, x) \\ &\quad - 2g(x, y)^2g(x, x) \end{aligned}$$

$$= g(g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}, g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}).$$

Therefore

$$E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}$$

and

$$L_x^2(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}.$$

The tensor product  $\mathbf{x} \otimes \mathbf{y}$  induces a linear map  $T_{x \otimes y} : \mathcal{V} \rightarrow \mathcal{V}$  defined as

$$T_{x \otimes y}(\mathbf{z}) = g(\mathbf{y}, \mathbf{z})\mathbf{x}, \text{ for all } \mathbf{z} \in \mathcal{V}.$$

From these definitions we can also generate the composite maps

$$T_{x \otimes y}L_z(\mathbf{u}) = T_{x \otimes y}(E(\mathbf{z}, \mathbf{u})) = g(\mathbf{y}, E(\mathbf{z}, \mathbf{u}))\mathbf{x}$$

and

$$L_xT_{y \otimes z}(\mathbf{u}) = L_x(g(\mathbf{z}, \mathbf{u})\mathbf{y}) = g(\mathbf{z}, \mathbf{u})E(\mathbf{x}, \mathbf{y}) = T_{E(\mathbf{x}, \mathbf{y}) \otimes z}\mathbf{u}. \quad (1)$$

Note that since

$$g(\mathbf{x} + \mathbf{y}, E(\mathbf{x} + \mathbf{y}, \mathbf{z})) = 0$$

(using property III) we have

$$\begin{aligned} g(\mathbf{x} + \mathbf{y}, E(\mathbf{x} + \mathbf{y}, \mathbf{z})) &= g(\mathbf{x} + \mathbf{y}, E(\mathbf{x}, \mathbf{z}) + E(\mathbf{y}, \mathbf{z})) \\ &= g(\mathbf{x}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{y}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) + g(\mathbf{y}, E(\mathbf{y}, \mathbf{z})) \\ &= g(\mathbf{y}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = 0. \end{aligned}$$

So we obtain

$$g(\mathbf{y}, E(\mathbf{z}, \mathbf{x})) = g(\mathbf{x}, E(\mathbf{y}, \mathbf{z}))$$

using the antisymmetric property of  $E$ . In the same way we can prove that this property is cyclic, i.e.

$$g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = g(\mathbf{z}, E(\mathbf{x}, \mathbf{y})) = g(\mathbf{y}, E(\mathbf{z}, \mathbf{x})). \quad (2)$$

Using this property  $g(\mathbf{y}, E(\mathbf{z}, \mathbf{u})) = g(\mathbf{u}, E(\mathbf{y}, \mathbf{z}))$  and so

$$T_{x \otimes y}L_z(\mathbf{u}) = g(\mathbf{u}, E(\mathbf{y}, \mathbf{z}))\mathbf{x} = T_{x \otimes E(\mathbf{y}, \mathbf{z})}(\mathbf{u}). \quad (3)$$

Note that

$$L_x^2(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y} = T_{x \otimes x}(\mathbf{y}) - g(\mathbf{x}, \mathbf{x})I(\mathbf{y})$$

where  $I : \mathcal{V} \rightarrow \mathcal{V}$  is the identity map. So

$$L_x^2 = T_{x \otimes x} - g(\mathbf{x}, \mathbf{x})I. \quad (4)$$

We can obtain the linearization of  $L_x^2$  by writing

$$\begin{aligned} L_{x+h}^2(\mathbf{y}) &= g(\mathbf{x} + \mathbf{h}, \mathbf{y})(\mathbf{x} + \mathbf{h}) - g(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h})\mathbf{y} \\ L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_xL_h + L_hL_x + L_h^2 \end{aligned}$$

The r.h.s. of the first equation expands to

$$\begin{aligned} L_{x+h}^2(\mathbf{y}) &= g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y} + g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \\ &\quad + g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y} + g(\mathbf{h}, \mathbf{y})\mathbf{h} - g(\mathbf{h}, \mathbf{h})\mathbf{y} \\ &= L_x^2(\mathbf{y}) + g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} + L_h^2(\mathbf{y}). \end{aligned}$$

We obtain

$$L_xL_h(\mathbf{y}) + L_hL_x(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} + g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y}.$$

If

$$L_xL_h(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \quad (5)$$

then

$$L_hL_x(\mathbf{y}) = g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y}.$$

Note that this linearization agrees with the property

$$\begin{aligned} L_xL_h(\mathbf{y}) &= g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \\ &= -(g(\mathbf{x}, \mathbf{h})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{h}) \\ &= -L_xL_y(\mathbf{h}). \end{aligned}$$

We can use the linearization of  $L_x^2$  to complete the following derivation:

$$\begin{aligned} (L_{E(\mathbf{x}, \mathbf{y})} + L_xL_y)(\mathbf{z}) &= E(E(\mathbf{x}, \mathbf{y}), \mathbf{z}) + L_xL_y(\mathbf{z}) \\ &= -E(\mathbf{z}, E(\mathbf{x}, \mathbf{y})) + L_xL_y(\mathbf{z}) \\ &= -L_zL_x(\mathbf{y}) + L_xL_y(\mathbf{z}) \\ &= -(g(\mathbf{z}, \mathbf{y})\mathbf{x} - g(\mathbf{z}, \mathbf{x})\mathbf{y}) + g(\mathbf{x}, \mathbf{z})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{z} \\ &= 2g(\mathbf{x}, \mathbf{z})\mathbf{y} - g(\mathbf{z}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{y})\mathbf{z}. \end{aligned}$$

This result can be used to write

$$L_{E(\mathbf{x}, \mathbf{y})} + L_xL_y = 2T_{y \otimes x} - T_{x \otimes y} - g(\mathbf{x}, \mathbf{y})I. \quad (6)$$

From this equation we get

$$\begin{aligned} L_{E(\mathbf{x}, \mathbf{y})}L_x + L_xL_yL_x &= 2T_{y \otimes x}L_x - T_{x \otimes y}L_x - g(\mathbf{x}, \mathbf{y})L_x \\ L_xL_yL_x &= -L_{E(\mathbf{x}, \mathbf{y})}L_x + 2T_{y \otimes x}L_x - T_{x \otimes y}L_x - g(\mathbf{x}, \mathbf{y})L_x \\ L_xL_yL_x &= -L_{E(\mathbf{x}, \mathbf{y})}L_x - T_{x \otimes E(\mathbf{y}, \mathbf{x})} - g(\mathbf{x}, \mathbf{y})L_x \end{aligned} \quad (7)$$

since using (3)

$$T_{y \otimes x} L_x(\mathbf{z}) = T_{y \otimes E(x, x)} = 0.$$

( $E(x, x) = 0$  follows from property II). From (6)

$$\begin{aligned} L_{E(x, y)} L_x &= -L_{E(E(x, y), x)} + 2T_{x \otimes E(x, y)} - T_{E(x, y) \otimes x} - g(E(\mathbf{x}, \mathbf{y}), \mathbf{y}) I \\ &= -L_{E(E(x, y), x)} + 2T_{x \otimes E(x, y)} - T_{E(x, y) \otimes x}; \end{aligned}$$

this produces a second equation for  $L_x L_y L_x$ :

$$\begin{aligned} L_x L_y L_x &= -\left(-L_{E(E(x, y), x)} + 2T_{x \otimes E(x, y)} - T_{E(x, y) \otimes x}\right) - T_{x \otimes E(y, x)} - g(\mathbf{x}, \mathbf{y}) L_x \\ &= L_{E(E(x, y), x)} - 2T_{x \otimes E(x, y)} + T_{E(x, y) \otimes x} - T_{x \otimes E(y, x)} - g(\mathbf{x}, \mathbf{y}) L_x \\ &= L_{E(E(x, y), x)} - 2T_{x \otimes E(x, y)} + T_{E(x, y) \otimes x} + T_{x \otimes E(x, y)} - g(\mathbf{x}, \mathbf{y}) L_x \\ &= L_{E(E(x, y), x)} - T_{x \otimes E(x, y)} + T_{E(x, y) \otimes x} - g(\mathbf{x}, \mathbf{y}) L_x. \end{aligned} \quad (8)$$

Next

$$E(E(\mathbf{x}, \mathbf{y}), \mathbf{x}) = -E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})) = -(g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}) = g(\mathbf{x}, \mathbf{x})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{x}.$$

Therefore

$$\begin{aligned} L_{E(E(x, y), x)}(\mathbf{z}) &= E(g(\mathbf{x}, \mathbf{x})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{x})E(\mathbf{y}, \mathbf{z}) - g(\mathbf{x}, \mathbf{y})E(\mathbf{x}, \mathbf{z}) \\ &= g(\mathbf{x}, \mathbf{x})L_y(\mathbf{z}) - g(\mathbf{x}, \mathbf{y})L_x(\mathbf{z}). \end{aligned}$$

The third equation for  $L_x L_y L_x$  is:

$$\begin{aligned} L_x L_y L_x &= g(\mathbf{x}, \mathbf{x})L_y - g(\mathbf{x}, \mathbf{y})L_x - T_{x \otimes E(x, y)} + T_{E(x, y) \otimes x} - g(\mathbf{x}, \mathbf{y})L_x \\ &= g(\mathbf{x}, \mathbf{x})L_y - 2g(\mathbf{x}, \mathbf{y})L_x - T_{x \otimes E(x, y)} + T_{E(x, y) \otimes x}. \end{aligned} \quad (9)$$

We choose an orthonormal basis  $\{\mathbf{e}_i\}_{i=1}^d$  where  $\dim \mathcal{V} = d$ . Define the following linear map  $S : \text{end}(\mathcal{V}) \rightarrow \text{end}(\mathcal{V})$

$$f \mapsto \sum_{i=1}^d L_{\mathbf{e}_i} \circ f \circ L_{\mathbf{e}_i}.$$

If  $f = I$  then using (4)

$$S(I) = \sum_{i=1}^d L_{\mathbf{e}_i}^2 = \sum_{i=1}^d T_{\mathbf{e}_i \otimes \mathbf{e}_i} - \sum_{i=1}^d I = \sum_{i=1}^d T_{\mathbf{e}_i \otimes \mathbf{e}_i} - dI.$$

Since

$$\begin{aligned} \sum_{i=1}^d T_{\mathbf{e}_i \otimes \mathbf{e}_i}(\mathbf{x}) &= \sum_{i=1}^d g(\mathbf{e}_i, \mathbf{x})\mathbf{e}_i = \mathbf{x}, \\ S(I) &= (1 - d)I. \end{aligned} \quad (10)$$

If  $f = T_{x \otimes y}$  then

$$\begin{aligned}
 S(T_{x \otimes y}) &= \sum_{i=1}^d L_{e_i} \circ T_{x \otimes y} \circ L_{e_i} \\
 &= \sum_{i=1}^d L_{e_i} \circ T_{x \otimes E(y, e_i)} && \text{using (3)} \\
 &= \sum_{i=1}^d T_{E(e_i, x) \otimes E(y, e_i)} && \text{using (1)}.
 \end{aligned}$$

Using (2)

$$\begin{aligned}
 T_{E(e_i, x) \otimes E(y, e_i)}(\mathbf{z}) &= g(E(\mathbf{y}, e_i), \mathbf{z})E(e_i, \mathbf{x}) \\
 &= g(\mathbf{z}, E(\mathbf{y}, e_i))E(e_i, \mathbf{x}) \\
 &= g(e_i, E(\mathbf{z}, \mathbf{y}))E(e_i, \mathbf{x}) \\
 &= g(e_i, E(\mathbf{y}, \mathbf{z}))E(\mathbf{x}, e_i) \\
 &= E(\mathbf{x}, g(e_i, E(\mathbf{y}, \mathbf{z})))e_i
 \end{aligned}$$

which substituted back in the summation gives

$$\begin{aligned}
 S(T_{x \otimes y}) &= \sum_{i=1}^d E(\mathbf{x}, g(e_i, E(\mathbf{y}, \mathbf{z})))e_i \\
 &= E\left(\mathbf{x}, \sum_{i=1}^d g(e_i, E(\mathbf{y}, \mathbf{z}))e_i\right) \\
 &= E(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = L_x L_y(\mathbf{z})
 \end{aligned}$$

and so

$$S(T_{x \otimes y}) = L_x L_y. \quad (11)$$

If  $f = L_y$  then using (9)

$$\begin{aligned}
 S(L_y) &= \sum_{i=1}^d L_{e_i} \circ L_y \circ L_{e_i} \\
 &= \sum_{i=1}^d \left( g(e_i, e_i)L_y - 2g(e_i, \mathbf{y})L_{e_i} - T_{e_i \otimes E(e_i, y)} + T_{E(e_i, y) \otimes e_i} \right).
 \end{aligned}$$

Since  $g(e_i, e_i) = 1$  the first term is simply  $dL_y$ ; for the second term write

$$\begin{aligned}
 \sum_{i=1}^d g(e_i, \mathbf{y})L_{e_i}(\mathbf{z}) &= \sum_{i=1}^d g(e_i, \mathbf{y})E(e_i, \mathbf{z}) = \sum_{i=1}^d E(g(e_i, \mathbf{y})e_i, \mathbf{z}) \\
 &= E\left(\sum_{i=1}^d g(e_i, \mathbf{y})e_i, \mathbf{z}\right) = E(\mathbf{y}, \mathbf{z}) = L_y(\mathbf{z}).
 \end{aligned}$$

For the third term

$$\begin{aligned}\sum_{i=1}^d T_{e_i \otimes E(e_i, \mathbf{y})}(\mathbf{z}) &= \sum_{i=1}^d g(\mathbf{z}, E(\mathbf{e}_i, \mathbf{y}))\mathbf{e}_i = \sum_{i=1}^d g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z}))\mathbf{e}_i \\ &= E(\mathbf{y}, \mathbf{z}) = L_y(\mathbf{z}).\end{aligned}$$

In a similar way the fourth term is

$$\begin{aligned}\sum_{i=1}^d T_{E(e_i, \mathbf{y}) \otimes e_i}(\mathbf{z}) &= \sum_{i=1}^d E(\mathbf{e}_i, \mathbf{y})g(\mathbf{z}, \mathbf{e}_i) \\ &= E\left(\sum_{i=1}^d g(\mathbf{z}, \mathbf{e}_i)\mathbf{e}_i, \mathbf{y}\right) = E(\mathbf{z}, \mathbf{y}) = -E(\mathbf{y}, \mathbf{z}) = -L_y(\mathbf{z}).\end{aligned}$$

Hence we conclude that

$$S(L_y) = (d - 2 - 1 - 1)L_y = (d - 4)L_y. \quad (12)$$

If  $f = L_x L_y$  then

$$\begin{aligned}S(L_x L_y) &= -S(L_{E(x, y)}) + 2S(T_{y \otimes x}) - S(T_{x \otimes y}) - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (6)} \\ &= -(d - 4)L_{E(x, y)} + 2S(T_{y \otimes x}) - S(T_{x \otimes y}) - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (12)} \\ &= -(d - 4)L_{E(x, y)} + 2L_y L_x - L_x L_y - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (11)} \\ &= -(d - 4)L_{E(x, y)} + 2L_y L_x - L_x L_y - (1 - d)g(\mathbf{x}, \mathbf{y})I && \text{using (10).} \quad (13)\end{aligned}$$

Using these derivations we can compute the following transformation:

$$g = \sum_{i=1}^d \sum_{j=1}^d L_{e_i} \circ L_x \circ L_{e_j} \circ L_{e_i} \circ L_{e_j}.$$

First using  $S(L_{e_i}) = \sum_{j=1}^d L_{e_j} \circ L_{e_i} \circ L_{e_j}$  and (12) we have

$$g = \sum_{i=1}^d L_{e_i} \circ L_x \circ (d - 4)L_{e_i} = (d - 4) \sum_{i=1}^d L_{e_i} \circ L_x \circ L_{e_i} = (d - 4)S(L_x) = (d - 4)^2 L_x.$$

Another way of computing the same transformation is to write

$$\begin{aligned}g &= \sum_{j=1}^d S(L_x L_{e_j}) \circ L_{e_j} \\ &= \sum_{j=1}^d \left( -(d - 4)L_{E(x, e_j)} + 2L_{e_j} L_x - L_x L_{e_j} - (1 - d)g(\mathbf{x}, \mathbf{e}_j)I \right) \circ L_{e_j}.\end{aligned}$$

using (13). Starting from the last term we have

$$g(\mathbf{x}, \mathbf{e}_j)I \circ L_{e_j}(\mathbf{z}) = g(\mathbf{x}, \mathbf{e}_j)L_{e_j}(\mathbf{z}) = g(\mathbf{x}, \mathbf{e}_j)E(\mathbf{e}_j, \mathbf{z})$$

which after applying the summation w.r.t.  $j$  becomes

$$\sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) I \circ L_{\mathbf{e}_j}(\mathbf{z}) = E\left(\sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) \mathbf{e}_j, \mathbf{z}\right) = E(\mathbf{x}, \mathbf{z}) = L_x(\mathbf{z}).$$

For  $L_{\mathbf{e}_j} L_x$  using (12) we have

$$2 \sum_{j=1}^d L_{\mathbf{e}_j} \circ L_x \circ L_{\mathbf{e}_j} = 2S(L_x) = 2(d-4)L_x$$

while for  $L_x L_{\mathbf{e}_j}$  we have using (10)

$$L_x \sum_{j=1}^d L_{\mathbf{e}_j} \circ L_{\mathbf{e}_j} = L_x \sum_{j=1}^d L_{\mathbf{e}_j} \circ I \circ L_{\mathbf{e}_j} = L_x S(I) = (1-d)L_x.$$

For  $L_{E(\mathbf{x}, \mathbf{e}_j)}$  write using (7)

$$L_{E(\mathbf{x}, \mathbf{e}_j)} L_{\mathbf{e}_j} = -L_{E(\mathbf{e}_j, \mathbf{x})} L_{\mathbf{e}_j} = -\left(-L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j} - L_{\mathbf{e}_j \otimes E(\mathbf{x}, \mathbf{e}_j)} - g(\mathbf{e}_j, \mathbf{x}) L_{\mathbf{e}_j}\right)$$

For the second term

$$L_{\mathbf{e}_j \otimes E(\mathbf{x}, \mathbf{e}_j)}(\mathbf{z}) = g(\mathbf{z}, E(\mathbf{x}, \mathbf{e}_j)) \mathbf{e}_j = g(\mathbf{e}_j, E(\mathbf{z}, \mathbf{x})) \mathbf{e}_j$$

which after summation w.r.t.  $j$  is equal to  $E(\mathbf{z}, \mathbf{x})$ . For the third term

$$g(\mathbf{e}_j, \mathbf{x}) L_{\mathbf{e}_j}(\mathbf{z}) = g(\mathbf{e}_j, \mathbf{x}) E(\mathbf{e}_j, \mathbf{z}) = E(g(\mathbf{e}_j, \mathbf{x}) \mathbf{e}_j, \mathbf{z})$$

which after summation w.r.t.  $j$  is equal to  $E(\mathbf{x}, \mathbf{z})$ . Therefore

$$\sum_{j=1}^d L_{E(\mathbf{x}, \mathbf{e}_j)} L_{\mathbf{e}_j} = \sum_{j=1}^d L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j} + E(\mathbf{z}, \mathbf{x}) + E(\mathbf{x}, \mathbf{z}) = S(L_x) = (d-4)L_x$$

and so computing the transformation using this method we have

$$\begin{aligned} g &= \sum_{j=1}^d \left( -(d-4)L_{E(\mathbf{x}, \mathbf{e}_j)} + 2L_{\mathbf{e}_j} L_x - L_x L_{\mathbf{e}_j} - (1-d)g(\mathbf{x}, \mathbf{e}_j)I \right) \circ L_{\mathbf{e}_j} \\ &= -(d-4) \underbrace{\sum_{j=1}^d L_{E(\mathbf{x}, \mathbf{e}_j)} L_{\mathbf{e}_j}}_{(d-4)L_x} + 2 \underbrace{\sum_{j=1}^d L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j}}_{(d-4)L_x} - \underbrace{L_x \sum_{j=1}^d L_{\mathbf{e}_j} L_{\mathbf{e}_j}}_{(1-d)L_x} - (1-d) \underbrace{\sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) I L_{\mathbf{e}_j}}_{L_x} \\ &= \left( -(d-4)^2 + 2(d-4) - (1-d) - (1-d) \right) L_x \\ &= \left( -(d-4)^2 + 2(d-4) - 2(1-d) \right) L_x \end{aligned}$$

Since both methods compute the same transformation we must have

$$-(d-4)^2 + 2(d-4) - 2(1-d) = (d-4)^2$$

or

$$(d-4)^2 - (d-4) - (d-1) = d^2 - 10d + 21 = (d-3)(d-7) = 0.$$

For a vector space  $\mathcal{V}$  with  $\dim \mathcal{V} = 3$ , given a linear map  $\phi : \mathcal{V} \rightarrow \mathcal{V}$ , define a linear map  $\bar{\phi} \in \text{end}(\mathcal{V})$

$$\bar{\phi}E(\mathbf{x}, \mathbf{y}) = E(\phi\mathbf{x}, \phi\mathbf{y}).$$

In this vector space the *volume form* is a (0,3) antisymmetric tensor

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \epsilon_\sigma \omega(\mathbf{x}_{\sigma(1)}, \mathbf{x}_{\sigma(2)}, \mathbf{x}_{\sigma(3)})$$

where  $\sigma \in S_3$  the symmetry group of permutations of  $\{1, 2, 3\}$  and  $\epsilon_\sigma = (-1)^{N(\sigma)}$  where  $N(\sigma)$  is the number of single transpositions that generate  $\sigma$ . Define the volume form:

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)); \quad (14)$$

We can use (2) to show that the definition is antisymmetric. For the symmetry group  $S_3$ , a cyclic transposition is even and this agrees with (2). We are left with  $f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ ,  $f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  and  $f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ . These are all odd permutations and,

$$\begin{aligned} \omega(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2) &= g(\mathbf{x}_1, E(\mathbf{x}_3, \mathbf{x}_2)) = -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) \\ &= -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \\ \omega(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) &= g(\mathbf{x}_3, E(\mathbf{x}_2, \mathbf{x}_1)) = -g(\mathbf{x}_3, E(\mathbf{x}_1, \mathbf{x}_2)) \\ &= -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) = -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \\ \omega(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) &= g(\mathbf{x}_2, E(\mathbf{x}_1, \mathbf{x}_3)) = -g(\mathbf{x}_2, E(\mathbf{x}_3, \mathbf{x}_1)) \\ &= -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) = -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \end{aligned}$$

Given a volume form the definition of the determinant is

$$\det(\phi) = \frac{\omega(\phi\mathbf{e}_1, \phi\mathbf{e}_2, \phi\mathbf{e}_3)}{\omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)}$$

where  $\{\mathbf{e}_i\}_{i=1}^3$  is a basis of  $\mathcal{V}$ . This is unique since the vector space of volume forms has dimension 1. It is also independent of the choice of basis. To prove this choose a basis  $\tilde{\mathbf{e}}_i = \Lambda^j_i \mathbf{e}_j$ . Write

$$\det(\phi) = \frac{\omega(\phi\tilde{\mathbf{e}}_1, \phi\tilde{\mathbf{e}}_2, \phi\tilde{\mathbf{e}}_3)}{\omega(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)} = \frac{\omega(\phi\Lambda^i_1 \mathbf{e}_i, \phi\Lambda^j_2 \mathbf{e}_j, \phi\Lambda^k_3 \mathbf{e}_k)}{\omega(\Lambda^i_1 \mathbf{e}_i, \Lambda^j_2 \mathbf{e}_j, \Lambda^k_3 \mathbf{e}_k)}.$$

Since the volume form is trilinear

$$\det(\phi) = \frac{\Lambda^i_1 \Lambda^j_2 \Lambda^k_3 \omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k)}{\Lambda^i_1 \Lambda^j_2 \Lambda^k_3 \omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)}.$$

The only combinations of indices  $i, j, k$  that do not set  $\omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k)$ ,  $\omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$  to zero are those in the symmetry group  $S_3$ . So

$$\Lambda^i_1 \Lambda^j_2 \Lambda^k_3 \omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k) = \sum_{\sigma} \Lambda^{\sigma(1)}_1 \Lambda^{\sigma(2)}_2 \Lambda^{\sigma(3)}_3 \omega(\phi\mathbf{e}_{\sigma(1)}, \phi\mathbf{e}_{\sigma(2)}, \phi\mathbf{e}_{\sigma(3)})$$

$$\begin{aligned}
&= \sum_{\sigma} \Lambda^{\sigma(1)}_1 \Lambda^{\sigma(2)}_2 \Lambda^{\sigma(3)}_3 \epsilon_{\sigma} \omega(\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3) \\
&= K_{\Lambda} \omega(\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3).
\end{aligned}$$

In the same way,

$$\Lambda^i_1 \Lambda^j_2 \Lambda^k_3 \omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k) = K_{\Lambda} \omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3).$$

Therefore,

$$\det(\phi) = \frac{\omega(\phi \tilde{\mathbf{e}}_1, \phi \tilde{\mathbf{e}}_2, \phi \tilde{\mathbf{e}}_3)}{\omega(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)} = \frac{K_{\Lambda} \omega(\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3)}{K_{\Lambda} \omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} = \frac{\omega(\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3)}{\omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)}.$$

This definition conforms with all the properties of a determinant. If  $\phi$  is singular then  $\{\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3\}$  are not independent vectors and so  $\det(\phi) = 0$  since  $\omega(\phi \mathbf{e}_1, \phi \mathbf{e}_2, \phi \mathbf{e}_3) = 0$ . For two non-singular endomorphisms  $\phi, \psi$

$$\begin{aligned}
\det(\phi\psi) &= \frac{\omega(\phi\psi \mathbf{e}_1, \phi\psi \mathbf{e}_2, \phi\psi \mathbf{e}_3)}{\omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} \\
&= \frac{\omega(\phi\psi \mathbf{e}_1, \phi\psi \mathbf{e}_2, \phi\psi \mathbf{e}_3)}{\omega(\psi \mathbf{e}_1, \psi \mathbf{e}_2, \psi \mathbf{e}_3)} \frac{\omega(\psi \mathbf{e}_1, \psi \mathbf{e}_2, \psi \mathbf{e}_3)}{\omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} = \det(\phi) \det(\psi)
\end{aligned}$$

since  $\{\psi \mathbf{e}_i\}_{i=1}^3$  is a basis of  $\mathcal{V}$ . Using the volume form definition from (14)

$$\det(\phi) = \frac{g(\phi \mathbf{e}_1, E(\phi \mathbf{e}_2, \phi \mathbf{e}_3))}{g(\mathbf{e}_1, E(\mathbf{e}_2, \mathbf{e}_3))}.$$

Wlog, to simplify the derivations assume that  $\{\mathbf{e}_i\}_{i=1}^3$  is the standard orthonormal basis with  $E(\mathbf{e}_1, \mathbf{e}_2) = \mathbf{e}_3$ ,  $E(\mathbf{e}_2, \mathbf{e}_3) = \mathbf{e}_1$  and  $E(\mathbf{e}_3, \mathbf{e}_1) = \mathbf{e}_2$ . So

$$g(\mathbf{e}_1, E(\mathbf{e}_2, \mathbf{e}_3)) = g(\mathbf{e}_1, \mathbf{e}_1) = 1.$$

From the definition of  $\bar{\phi}$

$$\det(\phi) = g(\phi \mathbf{e}_1, \bar{\phi} E(\mathbf{e}_2, \mathbf{e}_3)) = g(\phi \mathbf{e}_1, \bar{\phi} \mathbf{e}_1).$$

The adjoint of  $\phi$ ,  $\phi^*$ , is defined as the endomorphism with the property

$$g(\phi \mathbf{e}_1, \bar{\phi} \mathbf{e}_1) = g(\mathbf{e}_1, \phi^* \bar{\phi} \mathbf{e}_1).$$

So we must have

$$\det(\phi) = g(\mathbf{e}_1, \phi^* \bar{\phi} \mathbf{e}_1)$$

which holds only if

$$\phi^* \bar{\phi} = \det(\phi) I.$$

This can be rewritten as

$$\bar{\phi} = \det(\phi) (\phi^*)^{-1}.$$

For a vector space like  $\mathbb{R}^3$  we note that  $\phi^* \equiv \phi^T$  and so this equation can be rewritten as  $\bar{\phi} = \det(\phi) \phi^{-T}$ .

## References

- [1] Alberto Elduque. Vector cross products. *Electronic copy found at:* <http://www.unizar.es/matematicas/algebra/elduque/Talks/crossproducts.pdf>, 2004.