Most of the material presented here is taken from [1].

Given a finite dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$ with characteristic zero and with an inner product $g : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ define a bilinear map $E : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with the following properties:

I Bilinear property: $E(ax+by,z) = aE(x,z)+bE(y,z)$, $E(x,ay+bz) = aE(x,y)+bE(x,z)$ where $x,y,z \in \mathcal{V}$ and $a,b \in \mathbb{F}$.

II Antisymmetric property: $g(E(x,y),E(x,y)) = \begin{vmatrix} g(x,x) & g(x,y) \\ g(y,x) & g(y,y) \end{vmatrix}$ for all $x,y \in \mathcal{V}$. This is an expression of the antisymmetric property since

$$\begin{aligned} g(E(x,y),E(y,x)) &= \begin{vmatrix} g(x,y) & g(x,x) \\ g(y,y) & g(y,x) \end{vmatrix} = -\begin{vmatrix} g(x,x) & g(x,y) \\ g(y,x) & g(y,y) \end{vmatrix} \\ &= -g(E(x,y),E(x,y)) = g(E(x,y),-E(x,y)) \end{aligned}$$

and so we must have $E(x,y) = -E(y,x)$.

III Orthogonal property: $g(E(x,y),x) = g(E(x,y),y) = 0$.

We can use the bilinear map E to induce a linear map $L_x : \mathcal{V} \to \mathcal{V}$ as follows:

$$L_x(y) = E(x,y), \text{ for all } y \in \mathcal{V}.$$

This follows from (I). We note that

$$L_x^2(y) = L_x E(x,y) = E(x,E(x,y)).$$

From (II) we have

$$\begin{aligned} g(E(x,E(x,y)),E(x,E(x,y))) &= \begin{vmatrix} g(x,x) & g(x,E(x,y)) \\ g(E(x,y),x) & g(E(x,y),E(x,y)) \end{vmatrix} \\ &= \begin{vmatrix} g(x,x) & 0 \\ 0 & g(E(x,y),E(x,y)) \end{vmatrix} \\ &= g(x,x)g(E(x,y),E(x,y)) \\ &= g(x,x)\left(g(x,x)g(y,y) - g(x,y)^2\right) \\ &= g(x,x)^2 g(y,y) - g(x,y)^2 g(x,x) \\ &= g(x,x)^2 g(y,y) + g(x,y)^2 g(x,x) \\ &\quad - 2g(x,y)^2 g(x,x) \\ &= g(g(x,y)x - g(x,x)y, g(x,y)x - g(x,x)y). \end{aligned}$$

Therefore

$$E(x,E(x,y)) = g(x,y)x - g(x,x)y$$

and

$$L_x^2(y) = g(x,y)x - g(x,x)y.$$

The tensor product $x \otimes y$ induces a linear map $L_{x\otimes y} : \mathcal{V} \to \mathcal{V}$ defined as

$$\begin{aligned}(x\otimes y)(z) &= (x^i e_i \otimes y^j e_j)(z) = x^i y^j (e_i \otimes e_j)(z)\\ &= x^i y^j z_j e_i = x^i y^j z^k g_{jk} e_i\\ &= g(y,z) x^i e_i = g(y,z)x, \text{ for all } z \in \mathcal{V}.\end{aligned}$$

From these definitions we can also generate the composite maps

$$L_{x\otimes y} L_z(u) = L_{x\otimes y}(E(z,u)) = g(y, E(z,u))x$$

and

$$L_x L_{y\otimes z}(u) = L_x(g(z,u)y) = g(z,u)E(x,y) = L_{E(x,y)\otimes z} u. \tag{1}$$

Note that since

$$g(x+y, E(x+y,z)) = 0$$

(using property (III)) we have

$$\begin{aligned}g(x+y, E(x+y,z)) &= g(x+y, E(x,z)+E(y,z))\\ &= g(x,E(x,z)) + g(y,E(x,z)) + g(x,E(y,z)) + g(y,E(y,z))\\ &= g(y,E(x,z)) + g(x,E(y,z)) = 0.\end{aligned}$$

So we obtain

$$g(y,E(z,x)) = g(x,E(y,z))$$

using the antisymmetric property of E. In the same way we can prove that this property is cyclic, i.e.

$$g(x,E(y,z)) = g(y,E(z,x)) = g(z,E(x,y)).$$

This means that we can write

$$g(y,E(z,u)) = g(z,E(u,y)) = g(u,E(y,z)) \tag{2}$$

and so

$$L_{x\otimes y} L_z(u) = g(u,E(y,z))x = L_{x\otimes E(y,z)}(u). \tag{3}$$

Using these properties we can write

$$L_x^2(y) = g(x,y)x - g(x,x)y = L_{x\otimes x}(y) - g(x,x)I(y)$$

where $I : \mathcal{V} \to \mathcal{V}$ is the identity map. So

$$L_x^2 = L_{x\otimes x} - g(x,x)I.$$

We can obtain the linearization of L_x^2 by writing

$$
\begin{aligned}
L_{x+h}^2(y) &= g(x+h,y)(x+h) - g(x+h,x+h)y \\
L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_xL_h + L_hL_x + L_h^2
\end{aligned}
$$

The r.h.s. of the first equation expands to

$$
\begin{aligned}
L_{x+h}^2(y) &= g(x,y)x - g(x,x)y + g(x,y)h - g(x,h)y \\
&\quad + g(h,y)x - g(h,x)y + g(h,y)h - g(h,h)y \\
&= L_x^2(y) + g(x,y)h - g(x,h)y + L_h^2(y).
\end{aligned}
$$

We obtain

$$
L_xL_h(y) + L_hL_x(y) = g(x,y)h - g(x,h)y + g(h,y)x - g(h,x)y.
$$

If

$$
L_xL_h(y) = g(x,y)h - g(x,h)y \tag{4}
$$

then

$$
L_hL_x(y) = g(h,y)x - g(h,x)y.
$$

Note that this linearization agrees with the property

$$
\begin{aligned}
L_xL_h(y) &= g(x,y)h - g(x,h)y \\
&= -(g(x,h)y - g(x,y)h) \\
&= -L_xL_y(h).
\end{aligned}
$$

We can use the linearization of L_x^2 to perform the following derivation:

$$
\begin{aligned}
\left(L_{E(x,y)} + L_xL_y\right)(z) &= E(E(x,y),z) + L_xLy(z) \\
&= -E(z,E(x,y)) + L_xLy(z) \\
&= -L_zL_x(y) + L_xLy(z) \\
&= -(g(z,y)x - g(z,x)y) + g(x,z)y - g(x,y)z \\
&= 2g(x,z)y - g(z,y)x - g(x,y)z.
\end{aligned}
$$

This result can be used to write

$$
L_{E(x,y)} + L_xL_y = 2y \otimes x - x \otimes y - g(x,y)I. \tag{5}
$$

From this equation we get

$$
\begin{aligned}
L_{E(x,y)}L_x + L_xL_yL_x &= 2y \otimes xL_x - x \otimes yL_x - g(x,y)L_x \\
L_xL_yL_x &= -L_{E(x,y)}L_x + 2y \otimes xL_x - x \otimes yL_x - g(x,y)L_x \\
L_xL_yL_x &= -L_{E(x,y)}L_x - x \otimes E(y,x) - g(x,y)L_x
\end{aligned} \tag{6}
$$

since
$$(y\otimes x)L_x(z)=(y\otimes x)(E(x,z))=g(x,E(x,z))y=0,$$
and
$$(x\otimes y)L_x(z)=(x\otimes y)E(x,z)=g(y,E(x,z))x=(x\otimes E(y,x))(z).$$
From the same equation

$$\begin{aligned}L_{E(x,y)}L_x&=-L_{E(E(x,y),x)}+2x\otimes E(x,y)-E(x,y)\otimes x-g(E(x,y),y)I\\&=-L_{E(E(x,y),x)}+2x\otimes E(x,y)-E(x,y)\otimes x;\end{aligned}$$

this produces a second equation for $L_xL_yL_x$:

$$\begin{aligned}L_xL_yL_x&=-\big(-L_{E(E(x,y),x)}+2x\otimes E(x,y)-E(x,y)\otimes x\big)-x\otimes E(y,x)-g(x,y)L_x\\&=L_{E(E(x,y),x)}-2x\otimes E(x,y)+E(x,y)\otimes x-x\otimes E(y,x)-g(x,y)L_x\\&=L_{E(E(x,y),x)}-2x\otimes E(x,y)+E(x,y)\otimes x+x\otimes E(x,y)-g(x,y)L_x\\&=L_{E(E(x,y),x)}-x\otimes E(x,y)+E(x,y)\otimes x-g(x,y)L_x.\end{aligned}\tag{7}$$

Next

$$E(E(x,y),x)=-E(x,E(x,y))=-(g(x,y)x-g(x,x)y)=g(x,x)y-g(x,y)x.$$

Therefore

$$\begin{aligned}L_{E(E(x,y),x)}(z)&=E(g(x,x)y-g(x,y)x,z)=g(x,x)E(y,z)-g(x,y)E(x,z)\\&=g(x,x)L_y(z)-g(x,y)L_x(z).\end{aligned}$$

The third equation for $L_xL_yL_x$ is:

$$\begin{aligned}L_xL_yL_x&=g(x,x)L_y-g(x,y)L_x-x\otimes E(x,y)+E(x,y)\otimes x-g(x,y)L_x\\&=g(x,x)L_y-2g(x,y)L_x-x\otimes E(x,y)+E(x,y)\otimes x.\end{aligned}\tag{8}$$

We choose an orthonormal basis $\{e_i\}_{i=1}^d$ where $\dim\mathcal{V}=d$. Define the following linear map $S:\mathrm{end}(\mathcal{V})\to\mathrm{end}(\mathcal{V})$
$$f\mapsto\sum_{i=1}^d L_{e_i}\circ f\circ L_{e_i}.$$
If $f=I$ then
$$S(I)=\sum_{i=1}^d L_{e_i}^2=\sum_{i=1}^d L_{e_i\otimes e_i}-\sum_{i=1}^d I=\sum_{i=1}^d L_{e_i\otimes e_i}-dI.$$
Since
$$\sum_{i=1}^d L_{e_i\otimes e_i}(x)=\sum_{i=1}^d g(e_i,x)e_i=x,$$
$$S(I)=(1-d)I.\tag{9}$$

If $f = L_{x\otimes y}$ then using (1) and (3)

$$\begin{aligned} S(L_{x\otimes y}) &= \sum_{i=1}^{d} L_{e_i} \circ L_{x\otimes y} \circ L_{e_i} \\ &= \sum_{i=1}^{d} L_{e_i} \circ L_{x\otimes E(y,e_i)} \\ &= \sum_{i=1}^{d} L_{E(e_i,x)\otimes E(y,e_i)}. \end{aligned}$$

Using (2)

$$\begin{aligned} L_{E(e_i,x)\otimes E(y,e_i)}(z) &= g(E(y,e_i),z)E(e_i,x) \\ &= g(z,E(y,e_i))E(e_i,x) \\ &= g(e_i,E(z,y))E(e_i,x) \\ &= g(e_i,E(y,z))E(x,e_i) \\ &= E(x,g(e_i,E(y,z))e_i) \end{aligned}$$

which substituted back in the summation gives

$$\begin{aligned} S(L_{x\otimes y}) &= \sum_{i=1}^{d} E(x,g(e_i,E(y,z))e_i) \\ &= E\left(x,\sum_{i=1}^{d} g(e_i,E(y,z))e_i\right) \\ &= E(x,E(y,z)) = L_x L_y(z) \end{aligned}$$

and so

$$S(L_{x\otimes y}) = L_x L_y. \tag{10}$$

If $f = L_y$ then using (8)

$$\begin{aligned} S(L_y) &= \sum_{i=1}^{d} L_{e_i} \circ L_y \circ L(e_i) && (11) \\ &= \sum_{i=1}^{d} (g(e_i,e_i)L_y - 2g(e_i,y)L_{e_i} - e_i \otimes E(e_i,y) + E(e_i,y)\otimes e_i). && (12) \end{aligned}$$

Since $g(e_i,e_i) = 1$ the first term is simply dL_y; for the second term write

$$\begin{aligned} \sum_{i=1}^{d} g(e_i,y)L_{e_i}(z) = \sum_{i=1}^{d} g(e_i,y)E(e_i,z) &= \sum_{i=1}^{d} E(g(e_i,y)e_i,z) \\ &= E\left(\sum_{i=1}^{d} g(e_i,y)e_i,z\right) = E(y,z) = L_y(z). \end{aligned}$$

For the third term

$$\sum_{i=1}^{d} e_i \otimes E(e_i,y)(z) = \sum_{i=1}^{d} g(z,E(e_i,y))e_i = \sum_{i=1}^{d} g(e_i,E(y,z))e_i$$
$$= E(y,z) = L_y(z).$$

In a similar way the fourth term is

$$\sum_{i=1}^{d} E(e_i,y) \otimes e_i(z) = \sum_{i=1}^{d} E(e_i,y)g(z,e_i)$$
$$= E\left(\sum_{i=1}^{d} g(z,e_i)e_i, y\right) = E(z,y) = -E(y,z) = -L_y(z).$$

Hence we conclude that

$$S(L_y) = (d-4)L_y. \tag{13}$$

If $f = L_x L_y$ then

$$\begin{aligned} S(L_x L_y) &= -S(L_{E(x,y)}) + 2S(L_{y\otimes x}) - S(L_{x\otimes y}) - g(x,y)S(I) && \text{using (5)} \\ &= -(d-4)L_{E(x,y)} + 2S(L_{y\otimes x}) - S(L_{x\otimes y}) - g(x,y)S(I) && \text{using (13)} \\ &= -(d-4)L_{E(x,y)} + 2L_y L_x - LxLy - g(x,y)S(I) && \text{using (10)} \\ &= -(d-4)L_{E(x,y)} + 2L_y L_x - L_x L_y - (1-d)g(x,y)I && \text{using (9).} \end{aligned} \tag{14}$$

Using this derivation we can compute the following transformation:

$$g = \sum_{i=1}^{d} \sum_{j=1}^{d} L_{e_i} \circ L_x \circ L_{e_j} \circ L_{e_i} \circ L_{e_j}.$$

First using $S(L_{e_i}) = \sum_{j=1}^{d} L_{e_j} \circ L_{e_i} \circ L_{e_j}$ and (13) we have

$$g = \sum_{i=1}^{d} L_{e_i} \circ L_x \circ (d-4)L_{e_i} = (d-4)S(L_x) = (d-4)^2 L_x.$$

Another way of computing the same transformation is to write

$$g = \sum_{j=1}^{d} S(L_x L_{e_j}) \circ L_{e_j}$$
$$= \sum_{j=1}^{d} \Big(-(d-4)L_{E(x,e_j)} + 2L_{e_j}L_x - L_x L_{e_j} - (1-d)g(x,e_j)I\Big) \circ L_{e_j}.$$

Starting from the last term we have

$$g(x,e_j)I \circ L_{e_j}(z) = g(x,e_j)L_{e_j}(z) = g(x,e_j)E(e_j,z)$$

which after applying the summation w.r.t. j becomes

$$\sum_{j=1}^{d} g(x,e_j)I \circ L_{e_j}(z) = E\left(\sum_{j=1}^{d} g(x,e_j)e_j, z\right) = E(x,z) = L_x(z).$$

For $L_{e_j}L_x$ we have

$$2\sum_{j=1}^{d} L_{e_j}L_x \circ L_{e_j} = 2S(L_x) = 2(d-4)L_x$$

while for $L_xL_{e_j}$ we have

$$L_x \sum_{j=1}^{d} L_{e_j} \circ L_{e_j} = L_xS(I) = (1-d)L_x.$$

For $L_{E(x,e_j)}$ write using (6)

$$L_{E(x,e_j)}L_{e_j} = -L_{E(e_j,x)}L_{e_j} = -\Big(-L_{e_j}L_xL_{e_j} - L_{e_j \otimes E(x,e_j)} - g(e_j,x)L_{e_j}\Big)$$

For the second term

$$L_{e_j \otimes E(x,e_j)}(z) = g(z,E(x,e_j))e_j = g(e_j,E(z,x))e_j$$

which after summation w.r.t. j is equal to $E(z,x)$. For the third term

$$g(e_j,x)L_{e_j}(z) = g(e_j,x)E(e_j,z) = E\big(g(e_j,x)e_j,z\big)$$

which after summation w.r.t. j is equal to $E(x,z)$. Therefore

$$\sum_{j=1}^{d} L_{E(x,e_j)}L_{e_j} = \sum_{j=1}^{d} L_{e_j}L_xL_{e_j} + E(z,x) + E(x,z) = S(L_x) = (d-4)L_x$$

and so computing the transformation using this method we have

$$\begin{aligned} g &= \big(-(d-4)^2 + 2(d-4) - (1-d) - (1-d)\big)L_x \\ &= \big(-(d-4)^2 + 2(d-4) - 2(1-d)\big)L_x \end{aligned}$$

Since both methods compute the same transformation we must have

$$-(d-4)^2 + 2(d-4) - 2(1-d) = (d-4)^2$$

or

$$(d-4)^2 - (d-4) - (d-1) = d^2 - 10d + 21 = (d-3)(d-7) = 0.$$

References

[1] Alberto Elduque. Vector cross products. *Electronic copy found at: http://www. unizar. es/matematicas/algebra/elduque/Talks/crossproducts. pdf*, 2004.