Most of the material presented here is taken from elduque2004vector [1].

Given a finite dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$ with characteristic zero and with an inner product $g : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ define a bilinear map $E : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with the following properties:

I Bilinear property: $E(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{z}) + bE(\boldsymbol{y}, \boldsymbol{z})$, $E(\boldsymbol{x}, a\boldsymbol{y} + b\boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{y}) + bE(\boldsymbol{x}, \boldsymbol{z})$ where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$ and $a, b \in \mathbb{F}$. num:bilinear

II Antisymmetric property: $g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. This is an expression of the antisymmetric property since num:antisymmetric

$$\begin{aligned} g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{y}, \boldsymbol{x})) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{y}) & g(\boldsymbol{x}, \boldsymbol{x}) \\ g(\boldsymbol{y}, \boldsymbol{y}) & g(\boldsymbol{y}, \boldsymbol{x}) \end{vmatrix} = - \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix} \\ &= -g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = g(E(\boldsymbol{x}, \boldsymbol{y}), -E(\boldsymbol{x}, \boldsymbol{y})) \end{aligned}$$

and so we must have $E(\boldsymbol{x}, \boldsymbol{y}) = -E(\boldsymbol{y}, \boldsymbol{x})$.

III Orthogonal property: $g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) = g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y}) = 0$. num:orthogonal

We can use the bilinear map E to induce a linear map $L_x : \mathcal{V} \to \mathcal{V}$ as follows:

$$L_x(\boldsymbol{y}) = E(\boldsymbol{x}, \boldsymbol{y}), \text{ for all } \boldsymbol{y} \in \mathcal{V}.$$

This follows from (num:bilinear I). We note that

$$L_x^2(\boldsymbol{y}) = L_x E(\boldsymbol{x}, \boldsymbol{y}) = E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})).$$

From (num:antisymmetric II) we have

$$\begin{aligned} g(E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})), E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y}))) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})) \\ g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & 0 \\ 0 & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= g(\boldsymbol{x}, \boldsymbol{x}) g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \\ &= g(\boldsymbol{x}, \boldsymbol{x}) \big(g(\boldsymbol{x}, \boldsymbol{x}) g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 \big) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) + g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &\quad - 2g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &= g(g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}, g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}). \end{aligned}$$

Therefore

$$E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}$$

and

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y}.$$

The tensor product $\boldsymbol{x} \otimes \boldsymbol{y}$ induces a linear map $T_{x\otimes y} : \mathcal{V} \to \mathcal{V}$ defined as

$$T_{x\otimes y}(\boldsymbol{z}) = g(\boldsymbol{y}, \boldsymbol{z})\boldsymbol{x}, \text{ for all } \boldsymbol{z} \in \mathcal{V}.$$

From these definitions we can also generate the composite maps

$$T_{x\otimes y} L_z(\boldsymbol{u}) = T_{x\otimes y}(E(\boldsymbol{z}, \boldsymbol{u})) = g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{u}))\boldsymbol{x}$$

and

$$L_x T_{y\otimes z}(\boldsymbol{u}) = L_x(g(\boldsymbol{z}, \boldsymbol{u})\boldsymbol{y}) = g(\boldsymbol{z}, \boldsymbol{u})E(\boldsymbol{x}, \boldsymbol{y}) = T_{E(x,y)\otimes z}\boldsymbol{u}. \quad (1)$$ eq:xcrossytensorz

Note that since

$$g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z})) = 0$$

(using property (III)) we have num:orthogonal

$$\begin{aligned} g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z})) &= g(\boldsymbol{x} + \boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z}) + E(\boldsymbol{y}, \boldsymbol{z})) \\ &= g(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{y}, \boldsymbol{z})) \\ &= g(\boldsymbol{y}, E(\boldsymbol{x}, \boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) = 0. \end{aligned}$$

So we obtain

$$g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{x})) = g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z}))$$

using the antisymmetric property of E. In the same way we can prove that this property is cyclic, i.e.

$$g(\boldsymbol{x}, E(\boldsymbol{y}, \boldsymbol{z})) = g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{x})) = g(\boldsymbol{z}, E(\boldsymbol{x}, \boldsymbol{y})).$$

This means that we can write

$$g(\boldsymbol{y}, E(\boldsymbol{z}, \boldsymbol{u})) = g(\boldsymbol{z}, E(\boldsymbol{u}, \boldsymbol{y})) = g(\boldsymbol{u}, E(\boldsymbol{y}, \boldsymbol{z})) \quad (2)$$ eq:cyclic

and so

$$T_{x\otimes y} L_z(\boldsymbol{u}) = g(\boldsymbol{u}, E(\boldsymbol{y}, \boldsymbol{z}))\boldsymbol{x} = T_{x\otimes E(y,z)}(\boldsymbol{u}). \quad (3)$$ eq:xtensorycrossz

Using these properties we can write

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x}, \boldsymbol{x})\boldsymbol{y} = T_{x\otimes x}(\boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{x})I(\boldsymbol{y})$$

where $I : \mathcal{V} \to \mathcal{V}$ is the identity map. So

$$L_x^2 = T_{x\otimes x} - g(\boldsymbol{x}, \boldsymbol{x})I.$$

We can obtain the linearization of L_x^2 by writing

$$\begin{aligned} L_{x+h}^2(\boldsymbol{y}) &= g(\boldsymbol{x} + \boldsymbol{h}, \boldsymbol{y})(\boldsymbol{x} + \boldsymbol{h}) - g(\boldsymbol{x} + \boldsymbol{h}, \boldsymbol{x} + \boldsymbol{h})\boldsymbol{y} \\ L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_x L_h + L_h L_x + L_h^2 \end{aligned}$$

The r.h.s. of the first equation expands to

$$\begin{aligned} L^2_{x+h}(\boldsymbol{y}) &= g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \\ &\quad + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{h},\boldsymbol{h})\boldsymbol{y} \\ &= L^2_x(\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + L^2_h(\boldsymbol{y}). \end{aligned}$$

We obtain

$$L_x L_h(\boldsymbol{y}) + L_h L_x(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

If

$$L_x L_h(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \tag{4}$$ eq:linearization

then

$$L_h L_x(\boldsymbol{y}) = g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

Note that this linearization agrees with the property

$$\begin{aligned} L_x L_h(\boldsymbol{y}) &= g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \\ &= -(g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h}) \\ &= -L_x L_y(\boldsymbol{h}). \end{aligned}$$

We can use the linearization of L^2_x to perform the following derivation:

$$\begin{aligned} \big(L_{E(x,y)} + L_x L_y\big)(\boldsymbol{z}) &= E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{z}) + L_x Ly(\boldsymbol{z}) \\ &= -E(\boldsymbol{z},E(\boldsymbol{x},\boldsymbol{y})) + L_x Ly(\boldsymbol{z}) \\ &= -L_z L_x(\boldsymbol{y}) + L_x Ly(\boldsymbol{z}) \\ &= -(g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{z},\boldsymbol{x})\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} \\ &= 2g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z}. \end{aligned}$$

This result can be used to write

$$L_{E(x,y)} + L_x L_y = 2T_{y\otimes x} - T_{x\otimes y} - g(\boldsymbol{x},\boldsymbol{y})I. \tag{5}$$ eq:Exy_xy

From this equation we get

$$\begin{aligned} L_{E(x,y)}L_x + L_x L_y L_x &= 2T_{y\otimes x}L_x - T_{x\otimes y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_x L_y L_x &= -L_{E(x,y)}L_x + 2T_{y\otimes x}L_x - T_{x\otimes y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_x L_y L_x &= -L_{E(x,y)}L_x - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \end{aligned} \tag{6}$$ eq:first

since

$$T_{y\otimes x}L_x(\boldsymbol{z}) = T_{y\otimes x}(E(\boldsymbol{x},\boldsymbol{z})) = g(\boldsymbol{x},E(\boldsymbol{x},\boldsymbol{z}))\boldsymbol{y} = 0,$$

and

$$T_{x\otimes y}L_x(\boldsymbol{z}) = T_{x\otimes y}E(\boldsymbol{x},\boldsymbol{z}) = g(\boldsymbol{y},E(\boldsymbol{x},\boldsymbol{z}))\boldsymbol{x} = g(\boldsymbol{z},E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{x} = T_{x\otimes E(y,x)}(\boldsymbol{z}).$$

From the same equation

$$\begin{aligned} L_{E(x,y)}L_x &= -L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x} - g(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y})I \\ &= -L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x}; \end{aligned}$$

this produces a second equation for $L_xL_yL_x$:

$$\begin{aligned} L_xL_yL_x &= -\big(-L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x}\big) - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} + T_{x\otimes E(x,y)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - g(\boldsymbol{x},\boldsymbol{y})L_x. \end{aligned} \tag{7}$$

eq:second

Next

$$E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{x}) = -E(\boldsymbol{x},E(\boldsymbol{x},\boldsymbol{y})) = -(g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}.$$

Therefore

$$\begin{aligned} L_{E(E(x,y),x)}(\boldsymbol{z}) &= E(g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x},\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{x})E(\boldsymbol{y},\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})E(\boldsymbol{x},\boldsymbol{z}) \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y(\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})L_x(\boldsymbol{z}). \end{aligned}$$

The third equation for $L_xL_yL_x$ is:

$$\begin{aligned} L_xL_yL_x &= g(\boldsymbol{x},\boldsymbol{x})L_y - g(\boldsymbol{x},\boldsymbol{y})L_x - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y - 2g(\boldsymbol{x},\boldsymbol{y})L_x - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x}. \end{aligned} \tag{8}$$

eq:third

We choose an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^d$ where $\dim\mathcal{V} = d$. Define the following linear map $S : \mathrm{end}(\mathcal{V}) \to \mathrm{end}(\mathcal{V})$

$$f \mapsto \sum_{i=1}^d L_{e_i} \circ f \circ L_{e_i}.$$

If $f = I$ then

$$S(I) = \sum_{i=1}^d L_{e_i}^2 = \sum_{i=1}^d T_{e_i\otimes e_i} - \sum_{i=1}^d I = \sum_{i=1}^d T_{e_i\otimes e_i} - dI.$$

Since

$$\sum_{i=1}^d T_{e_i\otimes e_i}(\boldsymbol{x}) = \sum_{i=1}^d g(\boldsymbol{e}_i,\boldsymbol{x})\boldsymbol{e}_i = \boldsymbol{x},$$

$$S(I) = (1-d)I. \tag{9}$$

eq:SI

If $f = T_{x\otimes y}$ then

$$\begin{aligned}
S(T_{x\otimes y}) &= \sum_{i=1}^{d} L_{e_i} \circ T_{x\otimes y} \circ L_{e_i} \\
&= \sum_{i=1}^{d} L_{e_i} \circ T_{x\otimes E(y,e_i)} && \text{using (3) eq:xtensorycrossz} \\
&= \sum_{i=1}^{d} T_{E(e_i,x)\otimes E(y,e_i)} && \text{using (1). eq:xcrossytensorz}
\end{aligned}$$

Using (2) eq:cyclic

$$\begin{aligned}
T_{E(e_i,x)\otimes E(y,e_i)}(\boldsymbol{z}) &= g(E(\boldsymbol{y},\boldsymbol{e}_i),\boldsymbol{z})E(\boldsymbol{e}_i,\boldsymbol{x}) \\
&= g(\boldsymbol{z},E(\boldsymbol{y},\boldsymbol{e}_i))E(\boldsymbol{e}_i,\boldsymbol{x}) \\
&= g(\boldsymbol{e}_i,E(\boldsymbol{z},\boldsymbol{y}))E(\boldsymbol{e}_i,\boldsymbol{x}) \\
&= g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))E(\boldsymbol{x},\boldsymbol{e}_i) \\
&= E(\boldsymbol{x},g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i)
\end{aligned}$$

which substituted back in the summation gives

$$\begin{aligned}
S(T_{x\otimes y}) &= \sum_{i=1}^{d} E(\boldsymbol{x},g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i) \\
&= E\left(\boldsymbol{x},\sum_{i=1}^{d} g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i\right) \\
&= E(\boldsymbol{x},E(\boldsymbol{y},\boldsymbol{z})) = L_x L_y(\boldsymbol{z})
\end{aligned}$$

and so

$$S(T_{x\otimes y}) = L_x L_y. \tag{10}$$

eq:Sxoy

If $f = L_y$ then using (8) eq:third

$$\begin{aligned}
S(L_y) &= \sum_{i=1}^{d} L_{e_i} \circ L_y \circ L(e_i) && (11) \\
&= \sum_{i=1}^{d} \big(g(\boldsymbol{e}_i,\boldsymbol{e}_i)L_y - 2g(\boldsymbol{e}_i,\boldsymbol{y})L_{e_i} - T_{e_i\otimes E(e_i,y)} + T_{E(e_i,y)\otimes e_i}\big). && (12)
\end{aligned}$$

Since $g(\boldsymbol{e}_i,\boldsymbol{e}_i) = 1$ the first term is simply dL_y; for the second term write

$$\sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{y})L_{e_i}(\boldsymbol{z}) = \sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{y})E(\boldsymbol{e}_i,\boldsymbol{z}) = \sum_{i=1}^{d} E(g(\boldsymbol{e}_i,\boldsymbol{y})\boldsymbol{e}_i,\boldsymbol{z})$$

$$= E\left(\sum_{i=1}^{d} g(\boldsymbol{e}_i, \boldsymbol{y})\boldsymbol{e}_i, \boldsymbol{z}\right) = E(\boldsymbol{y}, \boldsymbol{z}) = L_y(\boldsymbol{z}).$$

For the third term

$$\begin{aligned}\sum_{i=1}^{d} T_{e_i \otimes E(e_i, y)}(\boldsymbol{z}) &= \sum_{i=1}^{d} g(\boldsymbol{z}, E(\boldsymbol{e}_i, \boldsymbol{y}))\boldsymbol{e}_i = \sum_{i=1}^{d} g(\boldsymbol{e}_i, E(\boldsymbol{y}, \boldsymbol{z}))\boldsymbol{e}_i \\ &= E(\boldsymbol{y}, \boldsymbol{z}) = L_y(\boldsymbol{z}).\end{aligned}$$

In a similar way the fourth term is

$$\begin{aligned}\sum_{i=1}^{d} T_{E(e_i, y) \otimes e_i}(\boldsymbol{z}) &= \sum_{i=1}^{d} E(\boldsymbol{e}_i, \boldsymbol{y}) g(\boldsymbol{z}, \boldsymbol{e}_i) \\ &= E\left(\sum_{i=1}^{d} g(\boldsymbol{z}, \boldsymbol{e}_i)\boldsymbol{e}_i, \boldsymbol{y}\right) = E(\boldsymbol{z}, \boldsymbol{y}) = -E(\boldsymbol{y}, \boldsymbol{z}) = -L_y(\boldsymbol{z}).\end{aligned}$$

Hence we conclude that

$$S(L_y) = (d - 2 - 1 - 1)L_y = (d - 4)L_y. \tag{13}$$

eq:Sy

If $f = L_x L_y$ then

$$\begin{aligned}S(L_x L_y) &= -S(L_{E(x,y)}) + 2S(T_{y\otimes x}) - S(T_{x\otimes y}) - g(\boldsymbol{x}, \boldsymbol{y})S(I) && \text{using (5)} \; \texttt{eq:Exy\_xy} \\ &= -(d-4)L_{E(x,y)} + 2S(T_{y\otimes x}) - S(T_{x\otimes y}) - g(\boldsymbol{x}, \boldsymbol{y})S(I) && \text{using (13)} \; \texttt{eq:Sy} \\ &= -(d-4)L_{E(x,y)} + 2L_y L_x - L_x L_y - g(\boldsymbol{x}, \boldsymbol{y})S(I) && \text{using (10)} \; \texttt{eq:Sxoy} \\ &= -(d-4)L_{E(x,y)} + 2L_y L_x - L_x L_y - (1-d)g(\boldsymbol{x}, \boldsymbol{y})I && \text{using (9).} \; \texttt{eq:SI}\end{aligned} \tag{14}$$

eq:Sxy

Using this derivation we can compute the following transformation:

$$g = \sum_{i=1}^{d}\sum_{j=1}^{d} L_{e_i} \circ L_x \circ L_{e_j} \circ L_{e_i} \circ L_{e_j}.$$

First using $S(L_{e_i}) = \sum_{j=1}^{d} L_{e_j} \circ L_{e_i} \circ L_{e_j}$ and (13) eq:Sy we have

$$g = \sum_{i=1}^{d} L_{e_i} \circ L_x \circ (d-4)L_{e_i} = (d-4)S(L_x) = (d-4)^2 L_x.$$

Another way of computing the same transformation is to write

$$g = \sum_{j=1}^{d} S(L_x L_{e_j}) \circ L_{e_j}$$

$$= \sum_{j=1}^{d} \big(-(d-4)L_{E(x,\boldsymbol{e}_j)} + 2L_{e_j}L_x - L_xL_{e_j} - (1-d)g(\boldsymbol{x},\boldsymbol{e}_j)I\big) \circ L_{e_j}.$$

using (eq:Sxy 14). Starting from the last term we have

$$g(\boldsymbol{x},\boldsymbol{e}_j)I \circ L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{e}_j)L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{e}_j)E(\boldsymbol{e}_j,\boldsymbol{z})$$

which after applying the summation w.r.t. j becomes

$$\sum_{j=1}^{d} g(\boldsymbol{x},\boldsymbol{e}_j)I \circ L_{e_j}(\boldsymbol{z}) = E\left(\sum_{j=1}^{d} g(\boldsymbol{x},\boldsymbol{e}_j)\boldsymbol{e}_j, \boldsymbol{z}\right) = E(\boldsymbol{x},\boldsymbol{z}) = L_x(\boldsymbol{z}).$$

For $L_{e_j}L_x$ using (eq:Sy 13) we have

$$2\sum_{j=1}^{d} L_{e_j} \circ L_x \circ L_{e_j} = 2S(L_x) = 2(d-4)L_x$$

while for $L_xL_{e_j}$ we have using (eq:SI 9)

$$L_x\sum_{j=1}^{d} L_{e_j} \circ L_{e_j} = L_x\sum_{j=1}^{d} L_{e_j} \circ I \circ L_{e_j} = L_xS(I) = (1-d)L_x.$$

For $L_{E(x,\boldsymbol{e}_j)}$ write using (eq:first 6)

$$L_{E(x,e_j)}L_{e_j} = -L_{E(e_j,x)}L_{e_j} = -\big(-L_{e_j}L_xL_{e_j} - L_{e_j\otimes E(x,e_j)} - g(\boldsymbol{e}_j,\boldsymbol{x})L_{e_j}\big)$$

For the second term

$$L_{e_j\otimes E(x,e_j)}(\boldsymbol{z}) = g(\boldsymbol{z}, E(\boldsymbol{x},\boldsymbol{e}_j))\boldsymbol{e}_j = g(\boldsymbol{e}_j, E(\boldsymbol{z},\boldsymbol{x}))\boldsymbol{e}_j$$

which after summation w.r.t. j is equal to $E(\boldsymbol{z},\boldsymbol{x})$. For the third term

$$g(\boldsymbol{e}_j,\boldsymbol{x})L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{e}_j,\boldsymbol{x})E(\boldsymbol{e}_j,\boldsymbol{z}) = E(g(\boldsymbol{e}_j,\boldsymbol{x})\boldsymbol{e}_j,\boldsymbol{z})$$

which after summation w.r.t. j is equal to $E(\boldsymbol{x},\boldsymbol{z})$. Therefore

$$\sum_{j=1}^{d} L_{E(x,e_j)}L_{e_j} = \sum_{j=1}^{d} L_{e_j}L_xL_{e_j} + E(\boldsymbol{z},\boldsymbol{x}) + E(\boldsymbol{x},\boldsymbol{z}) = S(L_x) = (d-4)L_x$$

and so computing the transformation using this method we have

$$\begin{aligned} g &= \big(-(d-4)^2 + 2(d-4) - (1-d) - (1-d)\big)L_x \\ &= \big(-(d-4)^2 + 2(d-4) - 2(1-d)\big)L_x \end{aligned}$$

Since both methods compute the same transformation we must have

$$-(d-4)^2 + 2(d-4) - 2(1-d) = (d-4)^2$$

or

$$(d-4)^2 - (d-4) - (d-1) = d^2 - 10d + 21 = (d-3)(d-7) = 0.$$

References

[1] Alberto Elduque. Vector cross products. *Electronic copy found at: http://www. unizar. es/matematicas/algebra/elduque/Talks/crossproducts. pdf*, 2004.