

# Cross Products

Most of the material presented here is taken from [1].

Given a finite dimensional vector space  $\mathcal{V}$  over a field  $\mathbb{F}$  with characteristic zero and with an inner product  $g : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{F}$  define a bilinear map  $E : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$  with the following properties:

I. Bilinear property:  $E(ax + by, z) = aE(x, z) + bE(y, z)$ ,  $E(x, ay + bz) = aE(x, y) + bE(x, z)$  where  $x, y, z \in \mathcal{V}$  and  $a, b \in \mathbb{F}$ .

II. Antisymmetric property:

$$g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) = \begin{vmatrix} g(\mathbf{x}, \mathbf{x}) & g(\mathbf{x}, \mathbf{y}) \\ g(\mathbf{y}, \mathbf{x}) & g(\mathbf{y}, \mathbf{y}) \end{vmatrix}$$

for all  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ . This is an expression of the antisymmetric property since

$$\begin{aligned} g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{y}, \mathbf{x})) &= \begin{vmatrix} g(\mathbf{x}, \mathbf{y}) & g(\mathbf{x}, \mathbf{x}) \\ g(\mathbf{y}, \mathbf{y}) & g(\mathbf{y}, \mathbf{x}) \end{vmatrix} = -\begin{vmatrix} g(\mathbf{x}, \mathbf{x}) & g(\mathbf{x}, \mathbf{y}) \\ g(\mathbf{y}, \mathbf{x}) & g(\mathbf{y}, \mathbf{y}) \end{vmatrix} \\ &= -g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) = g(E(\mathbf{x}, \mathbf{y}), -E(\mathbf{x}, \mathbf{y})) \end{aligned}$$

and so we must have  $E(\mathbf{x}, \mathbf{y}) = -E(\mathbf{y}, \mathbf{x})$ .

III. Orthogonal property:  $g(E(\mathbf{x}, \mathbf{y}), \mathbf{x}) = g(E(\mathbf{x}, \mathbf{y}), \mathbf{y}) = 0$ .

We can use the bilinear map  $E$  to induce a linear map  $L_x : \mathcal{V} \rightarrow \mathcal{V}$  as follows:

$$L_x(\mathbf{y}) = E(\mathbf{x}, \mathbf{y}), \text{ for all } \mathbf{y} \in \mathcal{V}.$$

This follows from property I. We note that

$$L_x^2(\mathbf{y}) = L_x E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})).$$

From property II we have

$$\begin{aligned} g(E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})), E(\mathbf{x}, E(\mathbf{x}, \mathbf{y}))) &= \begin{vmatrix} g(\mathbf{x}, \mathbf{x}) & g(\mathbf{x}, E(\mathbf{x}, \mathbf{y})) \\ g(E(\mathbf{x}, \mathbf{y}), \mathbf{x}) & g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) \end{vmatrix} \\ &= \begin{vmatrix} g(\mathbf{x}, \mathbf{x}) & 0 \\ 0 & g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) \end{vmatrix} \\ &= g(\mathbf{x}, \mathbf{x})g(E(\mathbf{x}, \mathbf{y}), E(\mathbf{x}, \mathbf{y})) \\ &= g(\mathbf{x}, \mathbf{x})(g(\mathbf{x}, \mathbf{x})g(\mathbf{y}, \mathbf{y}) - g(\mathbf{x}, \mathbf{y})^2) \\ &= g(\mathbf{x}, \mathbf{x})^2g(\mathbf{y}, \mathbf{y}) - g(\mathbf{x}, \mathbf{y})^2g(\mathbf{x}, \mathbf{x}) \\ &= g(\mathbf{x}, \mathbf{x})^2g(\mathbf{y}, \mathbf{y}) + g(\mathbf{x}, \mathbf{y})^2g(\mathbf{x}, \mathbf{x}) \\ &\quad - 2g(\mathbf{x}, \mathbf{y})^2g(\mathbf{x}, \mathbf{x}) \end{aligned}$$

$$= g(g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}, g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}).$$

Therefore

$$E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}$$

and

$$L_x^2(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}.$$

The tensor product  $\mathbf{x} \otimes \mathbf{y}$  induces a linear map  $T_{x \otimes y} : \mathcal{V} \rightarrow \mathcal{V}$  defined as

$$T_{x \otimes y}(\mathbf{z}) = g(\mathbf{y}, \mathbf{z})\mathbf{x}, \text{ for all } \mathbf{z} \in \mathcal{V}.$$

From these definitions we can also generate the composite maps

$$T_{x \otimes y} L_z(\mathbf{u}) = T_{x \otimes y}(E(\mathbf{z}, \mathbf{u})) = g(\mathbf{y}, E(\mathbf{z}, \mathbf{u}))\mathbf{x}$$

and

$$L_x T_{y \otimes z}(\mathbf{u}) = L_x(g(\mathbf{z}, \mathbf{u})\mathbf{y}) = g(\mathbf{z}, \mathbf{u})E(\mathbf{x}, \mathbf{y}) = T_{E(x,y) \otimes z}\mathbf{u}. \quad (1)$$

Note that since

$$g(\mathbf{x} + \mathbf{y}, E(\mathbf{x} + \mathbf{y}, \mathbf{z})) = 0$$

(using property III) we have

$$\begin{aligned} g(\mathbf{x} + \mathbf{y}, E(\mathbf{x} + \mathbf{y}, \mathbf{z})) &= g(\mathbf{x} + \mathbf{y}, E(\mathbf{x}, \mathbf{z}) + E(\mathbf{y}, \mathbf{z})) \\ &= g(\mathbf{x}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{y}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) + g(\mathbf{y}, E(\mathbf{y}, \mathbf{z})) \\ &= g(\mathbf{y}, E(\mathbf{x}, \mathbf{z})) + g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = 0. \end{aligned}$$

So we obtain

$$g(\mathbf{y}, E(\mathbf{z}, \mathbf{x})) = g(\mathbf{x}, E(\mathbf{y}, \mathbf{z}))$$

using the antisymmetric property of  $E$ . In the same way we can prove that this property is cyclic, i.e.

$$g(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = g(\mathbf{z}, E(\mathbf{x}, \mathbf{y})) = g(\mathbf{y}, E(\mathbf{z}, \mathbf{x})). \quad (2)$$

Using this property  $g(\mathbf{y}, E(\mathbf{z}, \mathbf{u})) = g(\mathbf{u}, E(\mathbf{y}, \mathbf{z}))$  and so

$$T_{x \otimes y} L_z(\mathbf{u}) = g(\mathbf{u}, E(\mathbf{y}, \mathbf{z}))\mathbf{x} = T_{x \otimes E(y,z)}(\mathbf{u}). \quad (3)$$

Note that

$$L_x^2(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y} = T_{x \otimes x}(\mathbf{y}) - g(\mathbf{x}, \mathbf{x})I(\mathbf{y})$$

where  $I : \mathcal{V} \rightarrow \mathcal{V}$  is the identity map. So

$$L_x^2 = T_{x \otimes x} - g(\mathbf{x}, \mathbf{x})I. \quad (4)$$

We can obtain the linearization of  $L_x^2$  by writing

$$\begin{aligned} L_{x+h}^2(\mathbf{y}) &= g(\mathbf{x} + \mathbf{h}, \mathbf{y})(\mathbf{x} + \mathbf{h}) - g(\mathbf{x} + \mathbf{h}, \mathbf{x} + \mathbf{h})\mathbf{y} \\ L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_xL_h + L_hL_x + L_h^2 \end{aligned}$$

The r.h.s. of the first equation expands to

$$\begin{aligned} L_{x+h}^2(\mathbf{y}) &= g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y} + g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \\ &\quad + g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y} + g(\mathbf{h}, \mathbf{y})\mathbf{h} - g(\mathbf{h}, \mathbf{h})\mathbf{y} \\ &= L_x^2(\mathbf{y}) + g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} + L_h^2(\mathbf{y}). \end{aligned}$$

We obtain

$$L_xL_h(\mathbf{y}) + L_hL_x(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} + g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y}.$$

If

$$L_xL_h(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \quad (5)$$

then

$$L_hL_x(\mathbf{y}) = g(\mathbf{h}, \mathbf{y})\mathbf{x} - g(\mathbf{h}, \mathbf{x})\mathbf{y}.$$

Note that this linearization agrees with the property

$$\begin{aligned} L_xL_h(\mathbf{y}) &= g(\mathbf{x}, \mathbf{y})\mathbf{h} - g(\mathbf{x}, \mathbf{h})\mathbf{y} \\ &= -(g(\mathbf{x}, \mathbf{h})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{h}) \\ &= -L_xL_y(\mathbf{h}). \end{aligned}$$

We can use the linearization of  $L_x^2$  to complete the following derivation:

$$\begin{aligned} (L_{E(x,y)} + L_xL_y)(\mathbf{z}) &= E(E(\mathbf{x}, \mathbf{y}), \mathbf{z}) + L_xL_y(\mathbf{z}) \\ &= -E(\mathbf{z}, E(\mathbf{x}, \mathbf{y})) + L_xL_y(\mathbf{z}) \\ &= -L_zL_x(\mathbf{y}) + L_xL_y(\mathbf{z}) \\ &= -(g(\mathbf{z}, \mathbf{y})\mathbf{x} - g(\mathbf{z}, \mathbf{x})\mathbf{y}) + g(\mathbf{x}, \mathbf{z})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{z} \\ &= 2g(\mathbf{x}, \mathbf{z})\mathbf{y} - g(\mathbf{z}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{y})\mathbf{z}. \end{aligned}$$

This result can be used to write

$$L_{E(x,y)} + L_xL_y = 2T_{y \otimes x} - T_{x \otimes y} - g(\mathbf{x}, \mathbf{y})I. \quad (6)$$

From this equation we get

$$\begin{aligned} L_{E(x,y)}L_x + L_xL_yL_x &= 2T_{y \otimes x}L_x - T_{x \otimes y}L_x - g(\mathbf{x}, \mathbf{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x + 2T_{y \otimes x}L_x - T_{x \otimes y}L_x - g(\mathbf{x}, \mathbf{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x - T_{x \otimes E(y,x)} - g(\mathbf{x}, \mathbf{y})L_x \end{aligned} \quad (7)$$

since using (3)

$$T_{y \otimes x} L_x(\mathbf{z}) = T_{y \otimes E(x,x)} = 0.$$

$(E(x,x) = 0)$  follows from property II). From (6)

$$\begin{aligned} L_{E(x,y)} L_x &= -L_{E(E(x,y),x)} + 2T_{x \otimes E(x,y)} - T_{E(x,y) \otimes x} - g(E(\mathbf{x}, \mathbf{y}), \mathbf{y})I \\ &= -L_{E(E(x,y),x)} + 2T_{x \otimes E(x,y)} - T_{E(x,y) \otimes x}; \end{aligned}$$

this produces a second equation for  $L_x L_y L_x$ :

$$\begin{aligned} L_x L_y L_x &= -(-L_{E(E(x,y),x)} + 2T_{x \otimes E(x,y)} - T_{E(x,y) \otimes x}) - T_{x \otimes E(y,x)} - g(\mathbf{x}, \mathbf{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x \otimes E(x,y)} + T_{E(x,y) \otimes x} - T_{x \otimes E(y,x)} - g(\mathbf{x}, \mathbf{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x \otimes E(x,y)} + T_{E(x,y) \otimes x} + T_{x \otimes E(x,y)} - g(\mathbf{x}, \mathbf{y})L_x \\ &= L_{E(E(x,y),x)} - T_{x \otimes E(x,y)} + T_{E(x,y) \otimes x} - g(\mathbf{x}, \mathbf{y})L_x. \end{aligned} \quad (8)$$

Next

$$E(E(\mathbf{x}, \mathbf{y}), \mathbf{x}) = -E(\mathbf{x}, E(\mathbf{x}, \mathbf{y})) = -(g(\mathbf{x}, \mathbf{y})\mathbf{x} - g(\mathbf{x}, \mathbf{x})\mathbf{y}) = g(\mathbf{x}, \mathbf{x})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{x}.$$

Therefore

$$\begin{aligned} L_{E(E(x,y),x)}(\mathbf{z}) &= E(g(\mathbf{x}, \mathbf{x})\mathbf{y} - g(\mathbf{x}, \mathbf{y})\mathbf{x}, \mathbf{z}) = g(\mathbf{x}, \mathbf{x})E(\mathbf{y}, \mathbf{z}) - g(\mathbf{x}, \mathbf{y})E(\mathbf{x}, \mathbf{z}) \\ &= g(\mathbf{x}, \mathbf{x})L_y(\mathbf{z}) - g(\mathbf{x}, \mathbf{y})L_x(\mathbf{z}). \end{aligned}$$

The third equation for  $L_x L_y L_x$  is:

$$\begin{aligned} L_x L_y L_x &= g(\mathbf{x}, \mathbf{x})L_y - g(\mathbf{x}, \mathbf{y})L_x - T_{x \otimes E(x,y)} + T_{E(x,y) \otimes x} - g(\mathbf{x}, \mathbf{y})L_x \\ &= g(\mathbf{x}, \mathbf{x})L_y - 2g(\mathbf{x}, \mathbf{y})L_x - T_{x \otimes E(x,y)} + T_{E(x,y) \otimes x}. \end{aligned} \quad (9)$$

We choose an orthonormal basis  $\{\mathbf{e}_i\}_{i=1}^d$  where  $\dim \mathcal{V} = d$ . Define the following linear map  $S : \text{end}(\mathcal{V}) \rightarrow \text{end}(\mathcal{V})$

$$f \mapsto \sum_{i=1}^d L_{e_i} \circ f \circ L_{e_i}.$$

If  $f = I$  then using (4)

$$S(I) = \sum_{i=1}^d L_{e_i}^2 = \sum_{i=1}^d T_{e_i \otimes e_i} - \sum_{i=1}^d I = \sum_{i=1}^d T_{e_i \otimes e_i} - dI.$$

Since

$$\sum_{i=1}^d T_{e_i \otimes e_i}(\mathbf{x}) = \sum_{i=1}^d g(\mathbf{e}_i, \mathbf{x})\mathbf{e}_i = \mathbf{x},$$

$$S(I) = (1 - d)I. \quad (10)$$

If  $f = T_{x \otimes y}$  then

$$\begin{aligned}
S(T_{x \otimes y}) &= \sum_{i=1}^d L_{e_i} \circ T_{x \otimes y} \circ L_{e_i} \\
&= \sum_{i=1}^d L_{e_i} \circ T_{x \otimes E(y, e_i)} && \text{using (3)} \\
&= \sum_{i=1}^d T_{E(e_i, x) \otimes E(y, e_i)} && \text{using (1).}
\end{aligned}$$

Using (2)

$$\begin{aligned}
T_{E(e_i, x) \otimes E(y, e_i)}(\mathbf{z}) &= g(E(\mathbf{y}, \mathbf{e}_i), \mathbf{z}) E(\mathbf{e}_i, \mathbf{x}) \\
&= g(\mathbf{z}, E(\mathbf{y}, \mathbf{e}_i)) E(\mathbf{e}_i, \mathbf{x}) \\
&= g(\mathbf{e}_i, E(\mathbf{z}, \mathbf{y})) E(\mathbf{e}_i, \mathbf{x}) \\
&= g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z})) E(\mathbf{x}, \mathbf{e}_i) \\
&= E(\mathbf{x}, g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z}))) \mathbf{e}_i
\end{aligned}$$

which substituted back in the summation gives

$$\begin{aligned}
S(T_{x \otimes y}) &= \sum_{i=1}^d E(\mathbf{x}, g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z}))) \mathbf{e}_i \\
&= E\left(\mathbf{x}, \sum_{i=1}^d g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z})) \mathbf{e}_i\right) \\
&= E(\mathbf{x}, E(\mathbf{y}, \mathbf{z})) = L_x L_y(\mathbf{z})
\end{aligned}$$

and so

$$S(T_{x \otimes y}) = L_x L_y. \quad (11)$$

If  $f = L_y$  then using (9)

$$\begin{aligned}
S(L_y) &= \sum_{i=1}^d L_{e_i} \circ L_y \circ L_{e_i} \\
&= \sum_{i=1}^d \left( g(\mathbf{e}_i, \mathbf{e}_i) L_y - 2g(\mathbf{e}_i, \mathbf{y}) L_{e_i} - T_{e_i \otimes E(e_i, y)} + T_{E(e_i, y) \otimes e_i} \right).
\end{aligned}$$

Since  $g(\mathbf{e}_i, \mathbf{e}_i) = 1$  the first term is simply  $dL_y$ ; for the second term write

$$\begin{aligned}
\sum_{i=1}^d g(\mathbf{e}_i, \mathbf{y}) L_{e_i}(\mathbf{z}) &= \sum_{i=1}^d g(\mathbf{e}_i, \mathbf{y}) E(\mathbf{e}_i, \mathbf{z}) = \sum_{i=1}^d E(g(\mathbf{e}_i, \mathbf{y}) \mathbf{e}_i, \mathbf{z}) \\
&= E\left(\sum_{i=1}^d g(\mathbf{e}_i, \mathbf{y}) \mathbf{e}_i, \mathbf{z}\right) = E(\mathbf{y}, \mathbf{z}) = L_y(\mathbf{z}).
\end{aligned}$$

For the third term

$$\begin{aligned}\sum_{i=1}^d T_{e_i \otimes E(e_i, y)}(\mathbf{z}) &= \sum_{i=1}^d g(\mathbf{z}, E(\mathbf{e}_i, \mathbf{y})) \mathbf{e}_i = \sum_{i=1}^d g(\mathbf{e}_i, E(\mathbf{y}, \mathbf{z})) \mathbf{e}_i \\ &= E(\mathbf{y}, \mathbf{z}) = L_y(\mathbf{z}).\end{aligned}$$

In a similar way the fourth term is

$$\begin{aligned}\sum_{i=1}^d T_{E(e_i, y) \otimes e_i}(\mathbf{z}) &= \sum_{i=1}^d E(\mathbf{e}_i, \mathbf{y}) g(\mathbf{z}, \mathbf{e}_i) \\ &= E\left(\sum_{i=1}^d g(\mathbf{z}, \mathbf{e}_i) \mathbf{e}_i, \mathbf{y}\right) = E(\mathbf{z}, \mathbf{y}) = -E(\mathbf{y}, \mathbf{z}) = -L_y(\mathbf{z}).\end{aligned}$$

Hence we conclude that

$$S(L_y) = (d - 2 - 1 - 1)L_y = (d - 4)L_y. \quad (12)$$

If  $f = L_x L_y$  then

$$\begin{aligned}S(L_x L_y) &= -S(L_{E(x, y)}) + 2S(T_{y \otimes x}) - S(T_{x \otimes y}) - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (6)} \\ &= -(d - 4)L_{E(x, y)} + 2S(T_{y \otimes x}) - S(T_{x \otimes y}) - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (12)} \\ &= -(d - 4)L_{E(x, y)} + 2L_y L_x - L_x L_y - g(\mathbf{x}, \mathbf{y})S(I) && \text{using (11)} \\ &= -(d - 4)L_{E(x, y)} + 2L_y L_x - L_x L_y - (1 - d)g(\mathbf{x}, \mathbf{y})I && \text{using (10).} \quad (13)\end{aligned}$$

Using these derivations we can compute the following transformation:

$$g = \sum_{i=1}^d \sum_{j=1}^d L_{e_i} \circ L_x \circ L_{e_j} \circ L_{e_i} \circ L_{e_j}.$$

First using  $S(L_{e_i}) = \sum_{j=1}^d L_{e_j} \circ L_{e_i} \circ L_{e_j}$  and (12) we have

$$g = \sum_{i=1}^d L_{e_i} \circ L_x \circ (d - 4)L_{e_i} = (d - 4) \sum_{i=1}^d L_{e_i} \circ L_x \circ L_{e_i} = (d - 4)S(L_x) = (d - 4)^2 L_x.$$

Another way of computing the same transformation is to write

$$\begin{aligned}g &= \sum_{j=1}^d S(L_x L_{e_j}) \circ L_{e_j} \\ &= \sum_{j=1}^d \left( -(d - 4)L_{E(x, e_j)} + 2L_{e_j} L_x - L_x L_{e_j} - (1 - d)g(\mathbf{x}, \mathbf{e}_j)I \right) \circ L_{e_j}.\end{aligned}$$

using (13). Starting from the last term we have

$$g(\mathbf{x}, \mathbf{e}_j)I \circ L_{e_j}(\mathbf{z}) = g(\mathbf{x}, \mathbf{e}_j)L_{e_j}(\mathbf{z}) = g(\mathbf{x}, \mathbf{e}_j)E(\mathbf{e}_j, \mathbf{z})$$

which after applying the summation w.r.t.  $j$  becomes

$$\sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) I \circ L_{\mathbf{e}_j}(\mathbf{z}) = E\left(\sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) \mathbf{e}_j, \mathbf{z}\right) = E(\mathbf{x}, \mathbf{z}) = L_x(\mathbf{z}).$$

For  $L_{\mathbf{e}_j} L_x$  using (12) we have

$$2 \sum_{j=1}^d L_{\mathbf{e}_j} \circ L_x \circ L_{\mathbf{e}_j} = 2S(L_x) = 2(d-4)L_x$$

while for  $L_x L_{\mathbf{e}_j}$  we have using (10)

$$L_x \sum_{j=1}^d L_{\mathbf{e}_j} \circ L_{\mathbf{e}_j} = L_x \sum_{j=1}^d L_{\mathbf{e}_j} \circ I \circ L_{\mathbf{e}_j} = L_x S(I) = (1-d)L_x.$$

For  $L_{E(x, \mathbf{e}_j)}$  write using (7)

$$L_{E(x, \mathbf{e}_j)} L_{\mathbf{e}_j} = -L_{E(e_j, x)} L_{\mathbf{e}_j} = -(-L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j} - L_{\mathbf{e}_j \otimes E(x, \mathbf{e}_j)} - g(\mathbf{e}_j, \mathbf{x}) L_{\mathbf{e}_j})$$

For the second term

$$L_{\mathbf{e}_j \otimes E(x, \mathbf{e}_j)}(\mathbf{z}) = g(\mathbf{z}, E(\mathbf{x}, \mathbf{e}_j)) \mathbf{e}_j = g(\mathbf{e}_j, E(\mathbf{z}, \mathbf{x})) \mathbf{e}_j$$

which after summation w.r.t.  $j$  is equal to  $E(\mathbf{z}, \mathbf{x})$ . For the third term

$$g(\mathbf{e}_j, \mathbf{x}) L_{\mathbf{e}_j}(\mathbf{z}) = g(\mathbf{e}_j, \mathbf{x}) E(\mathbf{e}_j, \mathbf{z}) = E(g(\mathbf{e}_j, \mathbf{x}) \mathbf{e}_j, \mathbf{z})$$

which after summation w.r.t.  $j$  is equal to  $E(\mathbf{x}, \mathbf{z})$ . Therefore

$$\sum_{j=1}^d L_{E(x, \mathbf{e}_j)} L_{\mathbf{e}_j} = \sum_{j=1}^d L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j} + E(\mathbf{z}, \mathbf{x}) + E(\mathbf{x}, \mathbf{z}) = S(L_x) = (d-4)L_x$$

and so computing the transformation using this method we have

$$\begin{aligned} g &= \sum_{j=1}^d \left( -(d-4)L_{E(x, \mathbf{e}_j)} + 2L_{\mathbf{e}_j} L_x - L_x L_{\mathbf{e}_j} - (1-d)g(\mathbf{x}, \mathbf{e}_j)I \right) \circ L_{\mathbf{e}_j} \\ &= -(d-4) \underbrace{\sum_{j=1}^d L_{E(x, \mathbf{e}_j)} L_{\mathbf{e}_j}}_{(d-4)L_x} + 2 \underbrace{\sum_{j=1}^d L_{\mathbf{e}_j} L_x L_{\mathbf{e}_j}}_{(d-4)L_x} - \underbrace{L_x \sum_{j=1}^d L_{\mathbf{e}_j} L_{\mathbf{e}_j}}_{(1-d)L_x} - \underbrace{(1-d) \sum_{j=1}^d g(\mathbf{x}, \mathbf{e}_j) I L_{\mathbf{e}_j}}_{L_x} \\ &= \left( -(d-4)^2 + 2(d-4) - (1-d) - (1-d) \right) L_x \\ &= \left( -(d-4)^2 + 2(d-4) - 2(1-d) \right) L_x \end{aligned}$$

Since both methods compute the same transformation we must have

$$-(d-4)^2 + 2(d-4) - 2(1-d) = (d-4)^2$$

or

$$(d-4)^2 - (d-4) - (d-1) = d^2 - 10d + 21 = (d-3)(d-7) = 0.$$

For a vector space  $\mathcal{V}$  with  $\dim \mathcal{V} = 3$ , given a linear map  $\phi : \mathcal{V} \rightarrow \mathcal{V}$ , define a linear map  $\bar{\phi} \in \text{end}(\mathcal{V})$

$$\bar{\phi}E(\mathbf{x}, \mathbf{y}) = E(\phi\mathbf{x}, \phi\mathbf{y}).$$

In this vector space the *volume form* is a  $(0,3)$  antisymmetric tensor

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \epsilon_\sigma \omega(\mathbf{x}_{\sigma(1)}, \mathbf{x}_{\sigma(2)}, \mathbf{x}_{\sigma(3)})$$

where  $\sigma \in S_3$  the symmetry group of permutations of  $\{1, 2, 3\}$  and  $\epsilon_\sigma = (-1)^{N(\sigma)}$  where  $N(\sigma)$  is the number of single transpositions that generate  $\sigma$ . Define the volume form:

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)); \quad (14)$$

We can use (2) to show that the definition is antisymmetric. For the symmetry group  $S_3$ , a cyclic transposition is even and this agrees with (2). We are left with  $f_1 = (\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix})$ ,  $f_2 = (\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix})$  and  $f_3 = (\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix})$ . These are all odd permutations and,

$$\begin{aligned} \omega(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2) &= g(\mathbf{x}_1, E(\mathbf{x}_3, \mathbf{x}_2)) = -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) \\ &= -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \\ \omega(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) &= g(\mathbf{x}_3, E(\mathbf{x}_2, \mathbf{x}_1)) = -g(\mathbf{x}_3, E(\mathbf{x}_1, \mathbf{x}_2)) \\ &= -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) = -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \\ \omega(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) &= g(\mathbf{x}_2, E(\mathbf{x}_1, \mathbf{x}_3)) = -g(\mathbf{x}_2, E(\mathbf{x}_3, \mathbf{x}_1)) \\ &= -g(\mathbf{x}_1, E(\mathbf{x}_2, \mathbf{x}_3)) = -\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \end{aligned}$$

Given a volume form the definition of the determinant is

$$\det(\phi) = \frac{\omega(\phi\mathbf{e}_1, \phi\mathbf{e}_2, \phi\mathbf{e}_3)}{\omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)}$$

where  $\{\mathbf{e}_i\}_{i=1}^3$  is a basis of  $\mathcal{V}$ . This is unique since the vector space of volume forms has dimension 1. It is also independent of the choice of basis. To prove this choose a basis  $\tilde{\mathbf{e}}_i = \Lambda^j{}_i \mathbf{e}_j$ . Write

$$\det(\phi) = \frac{\omega(\phi\tilde{\mathbf{e}}_1, \phi\tilde{\mathbf{e}}_2, \phi\tilde{\mathbf{e}}_3)}{\omega(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)} = \frac{\omega(\phi\Lambda^i{}_1 \mathbf{e}_i, \phi\Lambda^j{}_2 \mathbf{e}_j, \phi\Lambda^k{}_3 \mathbf{e}_k)}{\omega(\Lambda^i{}_1 \mathbf{e}_i, \Lambda^j{}_2 \mathbf{e}_j, \Lambda^k{}_3 \mathbf{e}_k)}.$$

Since the volume form is trilinear

$$\det(\phi) = \frac{\Lambda^i{}_1 \Lambda^j{}_2 \Lambda^k{}_3 \omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k)}{\Lambda^i{}_1 \Lambda^j{}_2 \Lambda^k{}_3 \omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)}.$$

The only combinations of indices  $i, j, k$  that do not set  $\omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k)$ ,  $\omega(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$  to zero are those in the symmetry group  $S_3$ . So

$$\Lambda^i{}_1 \Lambda^j{}_2 \Lambda^k{}_3 \omega(\phi\mathbf{e}_i, \phi\mathbf{e}_j, \phi\mathbf{e}_k) = \sum_{\sigma} \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_3 \omega(\phi\mathbf{e}_{\sigma(1)}, \phi\mathbf{e}_{\sigma(2)}, \phi\mathbf{e}_{\sigma(3)})$$

$$\begin{aligned}
&= \sum_{\sigma} \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_3 \epsilon_{\sigma} \omega(\phi e_1, \phi e_2, \phi e_3) \\
&= K_{\Lambda} \omega(\phi e_1, \phi e_2, \phi e_3).
\end{aligned}$$

In the same way,

$$\Lambda^i{}_1 \Lambda^j{}_2 \Lambda^k{}_3 \omega(e_i, e_j, e_k) = K_{\Lambda} \omega(e_1, e_2, e_3).$$

Therefore,

$$\det(\phi) = \frac{\omega(\phi \tilde{e}_1, \phi \tilde{e}_2, \phi \tilde{e}_3)}{\omega(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)} = \frac{K_{\Lambda} \omega(\phi e_1, \phi e_2, \phi e_3)}{K_{\Lambda} \omega(e_1, e_2, e_3)} = \frac{\omega(\phi e_1, \phi e_2, \phi e_3)}{\omega(e_1, e_2, e_3)}.$$

This definition conforms with all the properties of a determinant. If  $\phi$  is singular then  $\{\phi e_1, \phi e_2, \phi e_3\}$  are not independent vectors and so  $\det(\phi) = 0$  since  $\omega(\phi e_1, \phi e_2, \phi e_3) = 0$ . For two non-singular endomorphisms  $\phi, \psi$

$$\begin{aligned}
\det(\phi \psi) &= \frac{\omega(\phi \psi e_1, \phi \psi e_2, \phi \psi e_3)}{\omega(e_1, e_2, e_3)} \\
&= \frac{\omega(\phi \psi e_1, \phi \psi e_2, \phi e_3)}{\omega(\psi e_1, \psi e_2, \psi e_3)} \frac{\omega(\psi e_1, \psi e_2, \psi e_3)}{\omega(e_1, e_2, e_3)} = \det(\phi) \det(\psi)
\end{aligned}$$

since  $\{\psi e_i\}_{i=1}^3$  is a basis of  $\mathcal{V}$ . Using the volume form definition from (14)

$$\det(\phi) = \frac{g(\phi e_1, E(\phi e_2, \phi e_3))}{g(e_1, E(e_2, e_3))}.$$

Wlog, to simplify the derivations assume that  $\{e_i\}_{i=1}^3$  is the standard orthonormal basis with  $E(e_1, e_2) = e_3$ ,  $E(e_2, e_3) = e_1$  and  $E(e_3, e_1) = e_2$ . So

$$g(e_1, E(e_2, e_3)) = g(e_1, e_1) = 1.$$

From the definition of  $\bar{\phi}$

$$\det(\phi) = g(\phi e_1, \bar{\phi} E(e_2, e_3)) = g(\phi e_1, \bar{\phi} e_1).$$

The adjoint of  $\phi$ ,  $\phi^*$ , is defined as the endomorphism with the property

$$g(\phi e_1, \bar{\phi} e_1) = g(e_1, \phi^* \bar{\phi} e_1).$$

So we must have

$$\det(\phi) = g(e_1, \phi^* \bar{\phi} e_1)$$

which holds only if

$$\phi^* \bar{\phi} = \det(\phi) I.$$

This can be rewritten as

$$\bar{\phi} = \det(\phi) (\phi^*)^{-1}.$$

For a vector space like  $\mathbb{R}^3$  we note that  $\phi^* \equiv \phi^T$  and so this equation can be rewritten as  $\bar{\phi} = \det(\phi) \phi^{-T}$ .

## References

- [1] Alberto Elduque. Vector cross products. *Electronic copy found at:* <http://www.unizar.es/matematicas/algebra/elduque/Talks/crossproducts.pdf>, 2004.