Cross Products

Most of the material presented here is taken from [1].

Given a finite dimensional vector space $\mathcal{V}$ over a field $\mathbb{F}$ with characteristic zero and with an inner product $g : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ define a bilinear map $E : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with the following properties:

I. Bilinear property: $E(a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{z}) + bE(\boldsymbol{y}, \boldsymbol{z})$, $E(\boldsymbol{x}, a\boldsymbol{y} + b\boldsymbol{z}) = aE(\boldsymbol{x}, \boldsymbol{y}) + bE(\boldsymbol{x}, \boldsymbol{z})$ where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{V}$ and $a, b \in \mathbb{F}$.

II. Antisymmetric property:

$$g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$. This is an expression of the antisymmetric property since

$$\begin{aligned} g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{y}, \boldsymbol{x})) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{y}) & g(\boldsymbol{x}, \boldsymbol{x}) \\ g(\boldsymbol{y}, \boldsymbol{y}) & g(\boldsymbol{y}, \boldsymbol{x}) \end{vmatrix} = -\begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, \boldsymbol{y}) \\ g(\boldsymbol{y}, \boldsymbol{x}) & g(\boldsymbol{y}, \boldsymbol{y}) \end{vmatrix} \\ &= -g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) = g(E(\boldsymbol{x}, \boldsymbol{y}), -E(\boldsymbol{x}, \boldsymbol{y})) \end{aligned}$$

and so we must have $E(\boldsymbol{x}, \boldsymbol{y}) = -E(\boldsymbol{y}, \boldsymbol{x})$.

III. Orthogonal property: $g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) = g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y}) = 0$.

We can use the bilinear map E to induce a linear map $L_x : \mathcal{V} \to \mathcal{V}$ as follows:

$$L_x(\boldsymbol{y}) = E(\boldsymbol{x}, \boldsymbol{y}), \text{ for all } \boldsymbol{y} \in \mathcal{V}.$$

This follows from property I. We note that

$$L_x^2(\boldsymbol{y}) = L_x E(\boldsymbol{x}, \boldsymbol{y}) = E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})).$$

From property II we have

$$\begin{aligned} g(E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})), E(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y}))) &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & g(\boldsymbol{x}, E(\boldsymbol{x}, \boldsymbol{y})) \\ g(E(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{x}) & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= \begin{vmatrix} g(\boldsymbol{x}, \boldsymbol{x}) & 0 \\ 0 & g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \end{vmatrix} \\ &= g(\boldsymbol{x}, \boldsymbol{x}) g(E(\boldsymbol{x}, \boldsymbol{y}), E(\boldsymbol{x}, \boldsymbol{y})) \\ &= g(\boldsymbol{x}, \boldsymbol{x}) \big(g(\boldsymbol{x}, \boldsymbol{x}) g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 \big) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) - g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &= g(\boldsymbol{x}, \boldsymbol{x})^2 g(\boldsymbol{y}, \boldsymbol{y}) + g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \\ &\quad - 2g(\boldsymbol{x}, \boldsymbol{y})^2 g(\boldsymbol{x}, \boldsymbol{x}) \end{aligned}$$

$$= g(g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}, g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}).$$

Therefore

$$E(\boldsymbol{x}, E(\boldsymbol{x},\boldsymbol{y})) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}$$

and

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}.$$

The tensor product $\boldsymbol{x} \otimes \boldsymbol{y}$ induces a linear map $T_{x\otimes y} : \mathcal{V} \to \mathcal{V}$ defined as

$$T_{x\otimes y}(\boldsymbol{z}) = g(\boldsymbol{y},\boldsymbol{z})\boldsymbol{x}, \text{ for all } \boldsymbol{z} \in \mathcal{V}.$$

From these definitions we can also generate the composite maps

$$T_{x\otimes y} L_z(\boldsymbol{u}) = T_{x\otimes y}(E(\boldsymbol{z},\boldsymbol{u})) = g(\boldsymbol{y}, E(\boldsymbol{z},\boldsymbol{u}))\boldsymbol{x}$$

and

$$L_x T_{y\otimes z}(\boldsymbol{u}) = L_x(g(\boldsymbol{z},\boldsymbol{u})\boldsymbol{y}) = g(\boldsymbol{z},\boldsymbol{u})E(\boldsymbol{x},\boldsymbol{y}) = T_{E(x,y)\otimes z}\boldsymbol{u}. \tag{1}$$

Note that since

$$g(\boldsymbol{x}+\boldsymbol{y}, E(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})) = 0$$

(using property III) we have

$$\begin{aligned} g(\boldsymbol{x}+\boldsymbol{y}, E(\boldsymbol{x}+\boldsymbol{y},\boldsymbol{z})) &= g(\boldsymbol{x}+\boldsymbol{y}, E(\boldsymbol{x},\boldsymbol{z}) + E(\boldsymbol{y},\boldsymbol{z})) \\ &= g(\boldsymbol{x}, E(\boldsymbol{x},\boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{x},\boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y},\boldsymbol{z})) + g(\boldsymbol{y}, E(\boldsymbol{y},\boldsymbol{z})) \\ &= g(\boldsymbol{y}, E(\boldsymbol{x},\boldsymbol{z})) + g(\boldsymbol{x}, E(\boldsymbol{y},\boldsymbol{z})) = 0. \end{aligned}$$

So we obtain

$$g(\boldsymbol{y}, E(\boldsymbol{z},\boldsymbol{x})) = g(\boldsymbol{x}, E(\boldsymbol{y},\boldsymbol{z}))$$

using the antisymmetric property of E. In the same way we can prove that this property is cyclic, i.e.

$$g(\boldsymbol{x}, E(\boldsymbol{y},\boldsymbol{z})) = g(\boldsymbol{z}, E(\boldsymbol{x},\boldsymbol{y})) = g(\boldsymbol{y}, E(\boldsymbol{z},\boldsymbol{x})). \tag{2}$$

Using this property $g(\boldsymbol{y}, E(\boldsymbol{z},\boldsymbol{u})) = g(\boldsymbol{u}, E(\boldsymbol{y},\boldsymbol{z}))$ and so

$$T_{x\otimes y} L_z(\boldsymbol{u}) = g(\boldsymbol{u}, E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{x} = T_{x\otimes E(y,z)}(\boldsymbol{u}). \tag{3}$$

Note that

$$L_x^2(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} = T_{x\otimes x}(\boldsymbol{y}) - g(\boldsymbol{x},\boldsymbol{x})I(\boldsymbol{y})$$

where $I : \mathcal{V} \to \mathcal{V}$ is the identity map. So

$$L_x^2 = T_{x\otimes x} - g(\boldsymbol{x},\boldsymbol{x})I. \tag{4}$$

We can obtain the linearization of L_x^2 by writing

$$\begin{aligned} L_{x+h}^2(\boldsymbol{y}) &= g(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{y})(\boldsymbol{x}+\boldsymbol{h}) - g(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{x}+\boldsymbol{h})\boldsymbol{y} \\ L_{x+h}^2 &= L_{x+h}L_{x+h} = L_x^2 + L_xL_h + L_hL_x + L_h^2 \end{aligned}$$

The r.h.s. of the first equation expands to

$$\begin{aligned} L_{x+h}^2(\boldsymbol{y}) = {}& g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \\ &+ g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{h},\boldsymbol{h})\boldsymbol{y} \\ = {}& L_x^2(\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + L_h^2(\boldsymbol{y}). \end{aligned}$$

We obtain

$$L_xL_h(\boldsymbol{y}) + L_hL_x(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} + g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

If

$$L_xL_h(\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \tag{5}$$

then

$$L_hL_x(\boldsymbol{y}) = g(\boldsymbol{h},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{h},\boldsymbol{x})\boldsymbol{y}.$$

Note that this linearization agrees with the property

$$\begin{aligned} L_xL_h(\boldsymbol{y}) &= g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h} - g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} \\ &= -(g(\boldsymbol{x},\boldsymbol{h})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{h}) \\ &= -L_xL_y(\boldsymbol{h}). \end{aligned}$$

We can use the linearization of L_x^2 to complete the following derivation:

$$\begin{aligned} \left(L_{E(x,y)} + L_xL_y\right)(\boldsymbol{z}) &= E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{z}) + L_xL_y(\boldsymbol{z}) \\ &= -E(\boldsymbol{z},E(\boldsymbol{x},\boldsymbol{y})) + L_xL_y(\boldsymbol{z}) \\ &= -L_zL_x(\boldsymbol{y}) + L_xL_y(\boldsymbol{z}) \\ &= -(g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{z},\boldsymbol{x})\boldsymbol{y}) + g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z} \\ &= 2g(\boldsymbol{x},\boldsymbol{z})\boldsymbol{y} - g(\boldsymbol{z},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{z}. \end{aligned}$$

This result can be used to write

$$L_{E(x,y)} + L_xL_y = 2T_{y\otimes x} - T_{x\otimes y} - g(\boldsymbol{x},\boldsymbol{y})I. \tag{6}$$

From this equation we get

$$\begin{aligned} L_{E(x,y)}L_x + L_xL_yL_x &= 2T_{y\otimes x}L_x - T_{x\otimes y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x + 2T_{y\otimes x}L_x - T_{x\otimes y}L_x - g(\boldsymbol{x},\boldsymbol{y})L_x \\ L_xL_yL_x &= -L_{E(x,y)}L_x - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \end{aligned} \tag{7}$$

since using (3)

$$T_{y\otimes x}L_x(\boldsymbol{z}) = T_{y\otimes E(x,x)} = 0.$$

($E(x,x)=0$ follows from property II). From (6)

$$\begin{aligned} L_{E(x,y)}L_x &= -L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x} - g(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{y})I \\ &= -L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x}; \end{aligned}$$

this produces a second equation for $L_xL_yL_x$:

$$\begin{aligned} L_xL_yL_x &= -\Big(-L_{E(E(x,y),x)} + 2T_{x\otimes E(x,y)} - T_{E(x,y)\otimes x}\Big) - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - T_{x\otimes E(y,x)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - 2T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} + T_{x\otimes E(x,y)} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= L_{E(E(x,y),x)} - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - g(\boldsymbol{x},\boldsymbol{y})L_x. \end{aligned} \tag{8}$$

Next

$$E(E(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{x}) = -E(\boldsymbol{x},E(\boldsymbol{x},\boldsymbol{y})) = -(g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x} - g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y}) = g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x}.$$

Therefore

$$\begin{aligned} L_{E(E(x,y),x)}(\boldsymbol{z}) &= E(g(\boldsymbol{x},\boldsymbol{x})\boldsymbol{y} - g(\boldsymbol{x},\boldsymbol{y})\boldsymbol{x},\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{x})E(\boldsymbol{y},\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})E(\boldsymbol{x},\boldsymbol{z}) \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y(\boldsymbol{z}) - g(\boldsymbol{x},\boldsymbol{y})L_x(\boldsymbol{z}). \end{aligned}$$

The third equation for $L_xL_yL_x$ is:

$$\begin{aligned} L_xL_yL_x &= g(\boldsymbol{x},\boldsymbol{x})L_y - g(\boldsymbol{x},\boldsymbol{y})L_x - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x} - g(\boldsymbol{x},\boldsymbol{y})L_x \\ &= g(\boldsymbol{x},\boldsymbol{x})L_y - 2g(\boldsymbol{x},\boldsymbol{y})L_x - T_{x\otimes E(x,y)} + T_{E(x,y)\otimes x}. \end{aligned} \tag{9}$$

We choose an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^d$ where $\dim\mathcal{V} = d$. Define the following linear map $S : \mathrm{end}(\mathcal{V}) \to \mathrm{end}(\mathcal{V})$

$$f \mapsto \sum_{i=1}^{d} L_{e_i}\circ f\circ L_{e_i}.$$

If $f = I$ then using (4)

$$S(I) = \sum_{i=1}^{d} L_{e_i}^2 = \sum_{i=1}^{d} T_{e_i\otimes e_i} - \sum_{i=1}^{d} I = \sum_{i=1}^{d} T_{e_i\otimes e_i} - dI.$$

Since

$$\sum_{i=1}^{d} T_{e_i\otimes e_i}(\boldsymbol{x}) = \sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{x})\boldsymbol{e}_i = \boldsymbol{x},$$

$$S(I) = (1-d)I. \tag{10}$$

If $f = T_{x\otimes y}$ then

$$\begin{aligned} S(T_{x\otimes y}) &= \sum_{i=1}^{d} L_{e_i} \circ T_{x\otimes y} \circ L_{e_i} \\ &= \sum_{i=1}^{d} L_{e_i} \circ T_{x\otimes E(y,e_i)} && \text{using (3)} \\ &= \sum_{i=1}^{d} T_{E(e_i,x)\otimes E(y,e_i)} && \text{using (1).} \end{aligned}$$

Using (2)

$$\begin{aligned} T_{E(e_i,x)\otimes E(y,e_i)}(\boldsymbol{z}) &= g(E(\boldsymbol{y},\boldsymbol{e}_i),\boldsymbol{z})E(\boldsymbol{e}_i,\boldsymbol{x}) \\ &= g(\boldsymbol{z},E(\boldsymbol{y},\boldsymbol{e}_i))E(\boldsymbol{e}_i,\boldsymbol{x}) \\ &= g(\boldsymbol{e}_i,E(\boldsymbol{z},\boldsymbol{y}))E(\boldsymbol{e}_i,\boldsymbol{x}) \\ &= g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))E(\boldsymbol{x},\boldsymbol{e}_i) \\ &= E(\boldsymbol{x},g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i) \end{aligned}$$

which substituted back in the summation gives

$$\begin{aligned} S(T_{x\otimes y}) &= \sum_{i=1}^{d} E(\boldsymbol{x},g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i) \\ &= E\left(\boldsymbol{x},\sum_{i=1}^{d} g(\boldsymbol{e}_i,E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i\right) \\ &= E(\boldsymbol{x},E(\boldsymbol{y},\boldsymbol{z})) = L_x L_y(\boldsymbol{z}) \end{aligned}$$

and so

$$S(T_{x\otimes y}) = L_x L_y. \tag{11}$$

If $f = L_y$ then using (9)

$$\begin{aligned} S(L_y) &= \sum_{i=1}^{d} L_{e_i} \circ L_y \circ L_{e_i} \\ &= \sum_{i=1}^{d} \Big(g(\boldsymbol{e}_i,\boldsymbol{e}_i)L_y - 2g(\boldsymbol{e}_i,\boldsymbol{y})L_{e_i} - T_{e_i\otimes E(e_i,y)} + T_{E(e_i,y)\otimes e_i}\Big). \end{aligned}$$

Since $g(\boldsymbol{e}_i,\boldsymbol{e}_i) = 1$ the first term is simply dL_y; for the second term write

$$\begin{aligned} \sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{y})L_{e_i}(\boldsymbol{z}) = \sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{y})E(\boldsymbol{e}_i,\boldsymbol{z}) &= \sum_{i=1}^{d} E(g(\boldsymbol{e}_i,\boldsymbol{y})\boldsymbol{e}_i,\boldsymbol{z}) \\ = E\left(\sum_{i=1}^{d} g(\boldsymbol{e}_i,\boldsymbol{y})\boldsymbol{e}_i,\boldsymbol{z}\right) &= E(\boldsymbol{y},\boldsymbol{z}) = L_y(\boldsymbol{z}). \end{aligned}$$

For the third term

$$\sum_{i=1}^{d} T_{e_i\otimes E(e_i,y)}(\boldsymbol{z}) = \sum_{i=1}^{d} g(\boldsymbol{z}, E(\boldsymbol{e}_i,\boldsymbol{y}))\boldsymbol{e}_i = \sum_{i=1}^{d} g(\boldsymbol{e}_i, E(\boldsymbol{y},\boldsymbol{z}))\boldsymbol{e}_i$$
$$= E(\boldsymbol{y},\boldsymbol{z}) = L_y(\boldsymbol{z}).$$

In a similar way the fourth term is

$$\sum_{i=1}^{d} T_{E(e_i,y)\otimes e_i}(\boldsymbol{z}) = \sum_{i=1}^{d} E(\boldsymbol{e}_i,\boldsymbol{y})g(\boldsymbol{z},\boldsymbol{e}_i)$$
$$= E\left(\sum_{i=1}^{d} g(\boldsymbol{z},\boldsymbol{e}_i)\boldsymbol{e}_i, \boldsymbol{y}\right) = E(\boldsymbol{z},\boldsymbol{y}) = -E(\boldsymbol{y},\boldsymbol{z}) = -L_y(\boldsymbol{z}).$$

Hence we conclude that

$$S(L_y) = (d-2-1-1)L_y = (d-4)L_y. \tag{12}$$

If $f = L_xL_y$ then

$$\begin{aligned}
S(L_xL_y) &= -S(L_{E(x,y)}) + 2S(T_{y\otimes x}) - S(T_{x\otimes y}) - g(\boldsymbol{x},\boldsymbol{y})S(I) && \text{using (6)}\\
&= -(d-4)L_{E(x,y)} + 2S(T_{y\otimes x}) - S(T_{x\otimes y}) - g(\boldsymbol{x},\boldsymbol{y})S(I) && \text{using (12)}\\
&= -(d-4)L_{E(x,y)} + 2L_yL_x - L_xL_y - g(\boldsymbol{x},\boldsymbol{y})S(I) && \text{using (11)}\\
&= -(d-4)L_{E(x,y)} + 2L_yL_x - L_xL_y - (1-d)g(\boldsymbol{x},\boldsymbol{y})I && \text{using (10).}
\end{aligned} \tag{13}$$

Using these derivations we can compute the following transformation:

$$g = \sum_{i=1}^{d}\sum_{j=1}^{d} L_{e_i}\circ L_x \circ L_{e_j} \circ L_{e_i} \circ L_{e_j}.$$

First using $S(L_{e_i}) = \sum_{j=1}^{d} L_{e_j}\circ L_{e_i}\circ L_{e_j}$ and (12) we have

$$g = \sum_{i=1}^{d} L_{e_i}\circ L_x \circ (d-4)L_{e_i} = (d-4)\sum_{i=1}^{d} L_{e_i}\circ L_x\circ L_{e_i} = (d-4)S(L_x) = (d-4)^2L_x.$$

Another way of computing the same transformation is to write

$$\begin{aligned}
g &= \sum_{j=1}^{d} S(L_xL_{e_j})\circ L_{e_j}\\
&= \sum_{j=1}^{d}\Big(-(d-4)L_{E(x,e_j)} + 2L_{e_j}L_x - L_xL_{e_j} - (1-d)g(\boldsymbol{x},\boldsymbol{e}_j)I\Big)\circ L_{e_j}.
\end{aligned}$$

using (13). Starting from the last term we have

$$g(\boldsymbol{x},\boldsymbol{e}_j)I\circ L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{e}_j)L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{x},\boldsymbol{e}_j)E(\boldsymbol{e}_j,\boldsymbol{z})$$

which after applying the summation w.r.t. j becomes

$$\sum_{j=1}^{d} g(\boldsymbol{x},\boldsymbol{e}_j) I \circ L_{e_j}(\boldsymbol{z}) = E\left(\sum_{j=1}^{d} g(\boldsymbol{x},\boldsymbol{e}_j)\boldsymbol{e}_j, \boldsymbol{z}\right) = E(\boldsymbol{x},\boldsymbol{z}) = L_x(\boldsymbol{z}).$$

For $L_{e_j} L_x$ using (12) we have

$$2\sum_{j=1}^{d} L_{e_j} \circ L_x \circ L_{e_j} = 2S(L_x) = 2(d-4)L_x$$

while for $L_x L_{e_j}$ we have using (10)

$$L_x \sum_{j=1}^{d} L_{e_j} \circ L_{e_j} = L_x \sum_{j=1}^{d} L_{e_j} \circ I \circ L_{e_j} = L_x S(I) = (1-d)L_x.$$

For $L_{E(x,e_j)}$ write using (7)

$$L_{E(x,e_j)} L_{e_j} = -L_{E(e_j,x)} L_{e_j} = -\Big(-L_{e_j} L_x L_{e_j} - L_{e_j \otimes E(x,e_j)} - g(\boldsymbol{e}_j,\boldsymbol{x}) L_{e_j}\Big)$$

For the second term

$$L_{e_j \otimes E(x,e_j)}(\boldsymbol{z}) = g(\boldsymbol{z}, E(\boldsymbol{x},\boldsymbol{e}_j))\boldsymbol{e}_j = g(\boldsymbol{e}_j, E(\boldsymbol{z},\boldsymbol{x}))\boldsymbol{e}_j$$

which after summation w.r.t. j is equal to $E(\boldsymbol{z},\boldsymbol{x})$. For the third term

$$g(\boldsymbol{e}_j,\boldsymbol{x}) L_{e_j}(\boldsymbol{z}) = g(\boldsymbol{e}_j,\boldsymbol{x}) E(\boldsymbol{e}_j,\boldsymbol{z}) = E(g(\boldsymbol{e}_j,\boldsymbol{x})\boldsymbol{e}_j, \boldsymbol{z})$$

which after summation w.r.t. j is equal to $E(\boldsymbol{x},\boldsymbol{z})$. Therefore

$$\sum_{j=1}^{d} L_{E(x,e_j)} L_{e_j} = \sum_{j=1}^{d} L_{e_j} L_x L_{e_j} + E(\boldsymbol{z},\boldsymbol{x}) + E(\boldsymbol{x},\boldsymbol{z}) = S(L_x) = (d-4)L_x$$

and so computing the transformation using this method we have

$$\begin{aligned}
g &= \sum_{j=1}^{d} \Big(-(d-4)L_{E(x,e_j)} + 2L_{e_j} L_x - L_x L_{e_j} - (1-d) g(\boldsymbol{x},\boldsymbol{e}_j) I\Big) \circ L_{e_j} \\
&= -(d-4) \underbrace{\sum_{j=1}^{d} L_{E(x,e_j)} L_{e_j}}_{(d-4)L_x} + 2 \underbrace{\sum_{j=1}^{d} L_{e_j} L_x L_{e_j}}_{(d-4)L_x} - \underbrace{L_x \sum_{j=1}^{d} L_{e_j} L_{e_j}}_{(1-d)L_x} - (1-d) \underbrace{\sum_{j=1}^{d} g(\boldsymbol{x},\boldsymbol{e}_j) I L_{e_j}}_{L_x} \\
&= \Big(-(d-4)^2 + 2(d-4) - (1-d) - (1-d)\Big) L_x \\
&= \Big(-(d-4)^2 + 2(d-4) - 2(1-d)\Big) L_x
\end{aligned}$$

Since both methods compute the same transformation we must have

$$-(d-4)^2 + 2(d-4) - 2(1-d) = (d-4)^2$$

or

$$(d-4)^2-(d-4)-(d-1)=d^2-10d+21=(d-3)(d-7)=0.$$

For a vector space $\mathcal{V}$ with $\dim\mathcal{V}=3$, given a linear map $\phi:\mathcal{V}\to\mathcal{V}$, define a linear map $\bar{\phi}\in\mathrm{end}(\mathcal{V})$

$$\bar{\phi}E(\boldsymbol{x},\boldsymbol{y})=E(\phi\boldsymbol{x},\phi\boldsymbol{y}).$$

In this vector space the *volume form* is a (0,3) antisymmetric tensor

$$\omega(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)=\epsilon_\sigma\omega(\boldsymbol{x}_{\sigma(1)},\boldsymbol{x}_{\sigma(2)},\boldsymbol{x}_{\sigma(3)})$$

where $\sigma\in S_3$ the symmetry group of permutations of $\{1,2,3\}$ and $\epsilon_\sigma=(-1)^{N(\sigma)}$ where $N(\sigma)$ is the number of single transpositions that generate σ. Define the volume form:

$$\omega(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)=g(\boldsymbol{x}_1,E(\boldsymbol{x}_2,\boldsymbol{x}_3)); \tag{14}$$

We can use (2) to show that the definition is antisymmetric. For the symmetry group S_3, a cyclic transposition is even and this agrees with (2). We are left with $f_1=\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$, $f_2=\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)$ and $f_3=\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$. These are all odd permutations and,

$$\begin{aligned}
\omega(\boldsymbol{x}_1,\boldsymbol{x}_3,\boldsymbol{x}_2)&=g(\boldsymbol{x}_1,E(\boldsymbol{x}_3,\boldsymbol{x}_2))=-g(\boldsymbol{x}_1,E(\boldsymbol{x}_2,\boldsymbol{x}_3))\\
&=-\omega(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3),\\
\omega(\boldsymbol{x}_3,\boldsymbol{x}_2,\boldsymbol{x}_1)&=g(\boldsymbol{x}_3,E(\boldsymbol{x}_2,\boldsymbol{x}_1))=-g(\boldsymbol{x}_3,E(\boldsymbol{x}_1,\boldsymbol{x}_2))\\
&=-g(\boldsymbol{x}_1,E(\boldsymbol{x}_2,\boldsymbol{x}_3))=-\omega(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3),\\
\omega(\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_3)&=g(\boldsymbol{x}_2,E(\boldsymbol{x}_1,\boldsymbol{x}_3))=-g(\boldsymbol{x}_2,E(\boldsymbol{x}_3,\boldsymbol{x}_1))\\
&=-g(\boldsymbol{x}_1,E(\boldsymbol{x}_2,\boldsymbol{x}_3))=-\omega(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3).
\end{aligned}$$

Given a volume form the definition of the determinant is

$$\det(\phi)=\frac{\omega(\phi\boldsymbol{e}_1,\phi\boldsymbol{e}_2,\phi\boldsymbol{e}_2)}{\omega(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)}$$

where $\{\boldsymbol{e}_i\}_{i=1}^3$ is a basis of $\mathcal{V}$. This is unique since the vector space of volume forms has dimension 1. It is also independent of the choice of basis. To prove this choose a basis $\tilde{\boldsymbol{e}}_i=\Lambda^j{}_i\boldsymbol{e}_j$. Write

$$\det(\phi)=\frac{\omega(\phi\tilde{\boldsymbol{e}}_1,\phi\tilde{\boldsymbol{e}}_2,\phi\tilde{\boldsymbol{e}}_2)}{\omega(\tilde{\boldsymbol{e}}_1,\tilde{\boldsymbol{e}}_2,\tilde{\boldsymbol{e}}_3)}=\frac{\omega(\phi\Lambda^i{}_1\boldsymbol{e}_i,\phi\Lambda^j{}_2\boldsymbol{e}_j,\phi\Lambda^k{}_3\boldsymbol{e}_k)}{\omega(\Lambda^i{}_1\boldsymbol{e}_i,\Lambda^j{}_2\boldsymbol{e}_j,\Lambda^k{}_3\boldsymbol{e}_k)}.$$

Since the volume form is trilinear

$$\det(\phi)=\frac{\Lambda^i{}_1\Lambda^j{}_2\Lambda^k{}_3\omega(\phi\boldsymbol{e}_i,\phi\boldsymbol{e}_j,\phi\boldsymbol{e}_k)}{\Lambda^i{}_1\Lambda^j{}_2\Lambda^k{}_3\omega(\boldsymbol{e}_i,\boldsymbol{e}_j,\boldsymbol{e}_k)}.$$

The only combinations of indices i,j,k that do not set $\omega(\phi\boldsymbol{e}_i,\phi\boldsymbol{e}_j,\phi\boldsymbol{e}_k)$, $\omega(\boldsymbol{e}_i,\boldsymbol{e}_j,\boldsymbol{e}_k)$ to zero are those in the symmetry group S_3. So

$$\Lambda^i{}_1\Lambda^j{}_2\Lambda^k{}_3\omega(\phi\boldsymbol{e}_i,\phi\boldsymbol{e}_j,\phi\boldsymbol{e}_k)=\sum_\sigma\Lambda^{\sigma(1)}{}_1\Lambda^{\sigma(2)}{}_2\Lambda^{\sigma(3)}{}_3\omega(\phi\boldsymbol{e}_{\sigma(1)},\phi\boldsymbol{e}_{\sigma(2)},\phi\boldsymbol{e}_{\sigma(3)})$$

$$= \sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_3 \epsilon_\sigma \omega(\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_3)$$
$$= K_\Lambda \omega(\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_3).$$

Note that since Λ is a basis transformation tensor, $\Lambda_1, \Lambda_2, \Lambda_3$ are independent vectors and so $K_\Lambda \neq 0$. If, for example, $\Lambda_3 = a^1 \Lambda_1 + a^2 \Lambda_2$ then

$$\begin{aligned}\sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_3 \epsilon_\sigma &= \sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \left(a^1 \Lambda^{\sigma(3)}{}_1 + a^2 \Lambda^{\sigma(3)}{}_2 \right) \\ &= a^1 \sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_1 \epsilon_\sigma + a^2 \sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_2 \epsilon_\sigma.\end{aligned}$$

In the first sum, interchanging the first and third elements in each product does not change the total. Another way of saying this is that

$$\sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_1 \epsilon_\sigma = \sum_\sigma \Lambda^{\tau\sigma(1)}{}_1 \Lambda^{\tau\sigma(2)}{}_2 \Lambda^{\tau\sigma(3)}{}_1 \epsilon_\sigma \tag{15}$$

where τ denotes this single transposition. However, if we apply this single transposition to all elements of S_3 we would still obtain the same set of permutations. Therefore

$$\sum_\sigma \Lambda^{\sigma(1)}{}_1 \Lambda^{\sigma(2)}{}_2 \Lambda^{\sigma(3)}{}_1 \epsilon_\sigma = \sum_\sigma \Lambda^{\tau\sigma(1)}{}_1 \Lambda^{\tau\sigma(2)}{}_2 \Lambda^{\tau\sigma(3)}{}_1 \epsilon_{\tau\sigma}. \tag{16}$$

Since $\epsilon_{\tau\sigma} = \epsilon_\tau \epsilon_\sigma = -\epsilon_\sigma$ combine (15) and (16) to obtain

$$\sum_\sigma \Lambda^{\tau\sigma(1)}{}_1 \Lambda^{\tau\sigma(2)}{}_2 \Lambda^{\tau\sigma(3)}{}_1 \epsilon_\sigma = -\sum_\sigma \Lambda^{\tau\sigma(1)}{}_1 \Lambda^{\tau\sigma(2)}{}_2 \Lambda^{\tau\sigma(3)}{}_1 \epsilon_\sigma.$$

Therefore the first sum is zero and the same procedure can be used to show that the second sum is also zero. Note that K_Λ is the standard definition of the determinant of Λ found in introductory textbooks. In the same way,

$$\Lambda^i{}_1 \Lambda^j{}_2 \Lambda^k{}_3 \omega(\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) = K_\Lambda \omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3).$$

Therefore,

$$\det(\phi) = \frac{\omega(\phi \tilde{\boldsymbol{e}}_1, \phi \tilde{\boldsymbol{e}}_2, \phi \tilde{\boldsymbol{e}}_2)}{\omega(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2, \tilde{\boldsymbol{e}}_3)} = \frac{K_\Lambda \omega(\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_2)}{K_\Lambda \omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)} = \frac{\omega(\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_2)}{\omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)}.$$

This definition conforms with all the properties of a determinant. If ϕ is singular then $\{\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_2\}$ are not independent vectors and so $\det(\phi) = 0$ since $\omega(\phi \boldsymbol{e}_1, \phi \boldsymbol{e}_2, \phi \boldsymbol{e}_2) = 0$. For two non-singular endomorphisms ϕ, ψ

$$\begin{aligned}\det(\phi\psi) &= \frac{\omega(\phi\psi \boldsymbol{e}_1, \phi\psi \boldsymbol{e}_2, \phi \boldsymbol{e}_2)}{\omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)} \\ &= \frac{\omega(\phi\psi \boldsymbol{e}_1, \phi\psi \boldsymbol{e}_2, \phi \boldsymbol{e}_2)}{\omega(\psi \boldsymbol{e}_1, \psi \boldsymbol{e}_2, \psi \boldsymbol{e}_3)} \frac{\omega(\psi \boldsymbol{e}_1, \psi \boldsymbol{e}_2, \psi \boldsymbol{e}_3)}{\omega(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)} = \det(\phi)\det(\psi)\end{aligned}$$

since $\{\psi \boldsymbol{e}_i\}_{i=1}^3$ is a basis of $\mathcal{V}$. Using the volume form definition from (14)

$$\det(\phi) = \frac{g(\phi \boldsymbol{e}_1, E(\phi \boldsymbol{e}_2, \phi \boldsymbol{e}_3))}{g(\boldsymbol{e}_1, E(\boldsymbol{e}_2, \boldsymbol{e}_3))}.$$

Wlog, to simplify the derivations assume that $\{\boldsymbol{e}_i\}_{i=1}^3$ is the standard orthonormal basis with $E(\boldsymbol{e}_1, \boldsymbol{e}_2) = \boldsymbol{e}_3$, $E(\boldsymbol{e}_2, \boldsymbol{e}_3) = \boldsymbol{e}_1$ and $E(\boldsymbol{e}_3, \boldsymbol{e}_1) = \boldsymbol{e}_2$. So

$$g(\boldsymbol{e}_1, E(\boldsymbol{e}_2, \boldsymbol{e}_3)) = g(\boldsymbol{e}_1, \boldsymbol{e}_1) = 1.$$

From the definition of $\bar{\phi}$

$$\det(\phi) = g(\phi \boldsymbol{e}_1, \bar{\phi} E(\boldsymbol{e}_2, \boldsymbol{e}_3)) = g(\phi \boldsymbol{e}_1, \bar{\phi} \boldsymbol{e}_1).$$

The adjoint of ϕ, ϕ^*, is defined as the endomorphism with the property

$$g(\phi \boldsymbol{e}_1, \bar{\phi} \boldsymbol{e}_1) = g(\boldsymbol{e}_1, \phi^* \bar{\phi} \boldsymbol{e}_1).$$

So we must have

$$\det(\phi) = g(\boldsymbol{e}_1, \phi^* \bar{\phi} \boldsymbol{e}_1)$$

which holds only if

$$\phi^* \bar{\phi} = \det(\phi) I.$$

This can be rewritten as

$$\bar{\phi} = \det(\phi)(\phi^*)^{-1}.$$

For $\mathbb{R}^3$ we note that $\phi^* \equiv \phi^\mathsf{T}$ and so $\bar{\phi} = \det(\phi)\phi^{-\mathsf{T}}$.

References

[1] Alberto Elduque. Vector cross products. *Electronic copy found at:* http://www.unizar.es/matematicas/algebra/elduque/Talks/crossproducts.pdf, 2004.