Proof of formula 6.145, Chapter 6, page 222 mml-book: https://github.com/mml-book/mml-book.github.io/blob/master/book/chapter06.pdf.

X is a random variable of pdf, $f_X(x)$, and $U : range(X) \to I \subseteq \mathbb{R}$ an invertible C^1-function, with its inverse, U^{-1}, also of class C^1. Then the pdf of the random variable $Y = U(X)$ is related to the pdf of X by:

$$f_Y(y) = f_X(U^{-1}(y))\,|(U^{-1}(y))'|,$$

where $|(U^{-1}(y))'|$ is the modulus of the derivative of the function U^{-1}.

An invertible function between two real intervals is either strictly increasing or strictly decreasing.

Case 1. If U is increasing, then its inverse U^{-1} is also increasing, and of positive derivative (You considered ONLY this case, without pointing it out), and the cdf of the rv Y can be expressed as:

$$F_Y(y) = P(Y \leq y) = P(U(X) \leq y) = P(U^{-1}(U(X)) \leq U^{-1}(y)) = P(X \leq U^{-1}(y) = F_X(U^{-1}(y))$$

Taking into account the relationship cdf-pdf, and the last relation deduced above, we get:

$$f_Y(y) = F_Y'(y) = F_X'(U^{-1}(y))(U^{-1}(y))' = f_X(U^{-1}(y))(U^{-1}(y))'$$

Case 2. If U is decreasing, its inverse has the same property, i.e. $U^{-1}(x_1 \geq U^{-1}(x_2)$ iff $x_1 \leq x_2$, and moreover its derivative is negative. In this case the cdf of the rv Y is:

$$F_Y(y) = P(U(X) \leq y) \overset{U^{-1}\text{descreasing}}{=} P(X \geq U^{-1}(y)) = 1 - P(X < U^{-1}(y)) = 1 - F_X(U^{-1}(y))$$

Computing the derivative of the first and last term, we get:

$$f_Y(y) = -f_X(U^{-1}(y))\underbrace{(U^{-1}(y))'}_{<0} = f_X(U^{-1}(y))|(U^{-1}(y))'|$$

The last equality is based on the modulus definition:

$$|(U^{-1}(y))'| = \begin{cases} -(U^{-1}(y))' & \text{if } (U^{-1}(y))' < 0 \\ (U^{-1}(y))' & \text{otherwise} \end{cases}$$

Hence we can conclude that the pdfs of the two random variables are related by:

$$f_Y(y) = f_X(U^{-1}(y))\,|(U^{-1}(y))'|$$