Exercises 1

Prove by mathematical induction the following (1-9) identities:

1. $1+2+3+...+n=\dfrac{n(n+1)}{2}$, *for* $n\in N$.
2. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$, *for* $n\in N$.
3. $1^3+2^3+3^3+\cdots+n^3=\left[\dfrac{n(n+1)}{2}\right]^2$, *for* $n\in N$.
4. $1^3+2^3+3^3+\cdots+n^3=(1+2+3+...+n)^2$, *for* $n\in N$.
5. $1+3+5+...+(2n-1)=n^2$, *for* $n\in N$.
6. $\dfrac{1}{1\cdot 3}+\dfrac{1}{3\cdot 5}+\dfrac{1}{5\cdot 7}+\cdots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$, *for* $n\in N$.
7. $\dfrac{1}{a(a+1)}+\dfrac{1}{(a+1)(a+2)}+\cdots+\dfrac{1}{(a+n-1)(a+n)}=\dfrac{n}{a(a+n)}$, *for* $a,n\in N$.
8. $\dfrac{0}{1!}+\dfrac{1}{2!}+...+\dfrac{n-1}{n!}=1-\dfrac{1}{n!}$, *for* $n\in N$.

Find the formulas for the following sums (9-12) by examining the values of these expressions for small values of n. Use mathematical induction to prove your results.

9. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\ldots+\dfrac{1}{n(n+1)}$, $n\in N$;
10. $\dfrac{1}{2\cdot 7}+\dfrac{1}{7\cdot 12}+\ldots+\dfrac{1}{(5n-3)(5n+2)}$, $n\in N$;
11. $\dfrac{1}{1\cdot 3}+\dfrac{7}{3\cdot 5}+\ldots+\dfrac{2n^2-1}{(2n-1)(2n+1)}$, $n\in N$;
12. $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)$, $n\in N$;

Use mathematical induction to prove the following inequalities (problems 13-17):

13. $2^n\geq 2n+1$, *if* $n\in N$ *and* $\mathrm{n}\geq ?$.
14. $2^n>n^2$, *if* $n\in N$ *and* $\mathrm{n}\geq ?$.
15. $3^n<n!$, *if* $n\in N$ *and* $\mathrm{n}\geq ?$.
16. $n!<n^n$, *if* $n\in N$ *and* $\mathrm{n}\geq 2$.
17. $1+\dfrac{1}{4}+\dfrac{1}{9}+\ldots+\dfrac{1}{n^2}<2-\dfrac{1}{n}$, $n\in N$ *and* $n\geq 2$.

Prove (problems 18-29), that for arbitrary positive integer n

18. $n^3+(n+1)^3+(n+2)^3$ is divisible by 9;
19. n^3+5n is divisible by 6;
20. n^3+11n is divisible by 6;
21. 7^n+3n-1 is divisible by 9;
22. $4^n+15n-1$ is divisible by 9;
23. $5^n+2\cdot 3^n+5$ is divisible by 8;
24. 5^n-3^n+2n is divisible by 4;
25. $5^{n+3}+11^{3n+1}$ is divisible by 17;
26. $5\cdot 2^{3n-2}+3^{3n-1}$ is divisible by 19;
27. $6^{2n}+3^{n+2}+3^n$ is divisible by 11;
28. $7^{n+2}+8^{2n+1}$ is divisible by 57.
29. 4^n-3^n-7 is divisible by 84 if $n\in N$ and n is even number.

Out of class exercises*:*

30. Prove, that any number composed of 3^n units, is divisible by 3^n**.**
31. *Arithmetic progression.* The sequence a_n is defined by the following recurrence relations: $a_n=a_{n-1}+d,\ n=2,3,...;\ a_1=a;$ d- is a *"difference"*, a - is known. Find the formulas for the a_n and the sum $\sum_{n=1}^{m} a_n$. Use mathematical induction to prove your results.
32. *Geometric progression.* The sequence b_n is defined by the following recurrence relations: $b_n=q\cdot b_{n-1},\ n=2,3,...;\ b_1=b;$ q- is a *"ratio"*, b- is known. Find the formulas for the b_n and the sum $\sum_{n=1}^{m} b_n$. Use mathematical induction to prove your results.
33. Suppose, that: $y_1=\frac{3}{8},\ y_n=\frac{3}{8}+\frac{y_{n-1}^2}{2},\ n=2,3,4...,$

    and prove the following inequalities $y_{n-1}<y_n<\frac{1}{2},\ n=2,3,4...$

34. Suppose, that the sequence u_n is defined by the following recurrence relations: $u_1=1,\ u_n=u_{n-1}+n,\ n=2,3,...$ Prove, that $u_n+u_{n+1}=(n+1)^2,\ for\ n\in N.$