Neutrino Oscillations in Matter

PhD Candidacy Exam

Lei Ma
Supervisor: Huaiyu Duan

Department of Physics
UNM

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Outline

1. Introduction
   - What are Neutrinos
   - Neutrino Oscillations
   - Why Do Neutrinos Oscillate
2. Matter Effect
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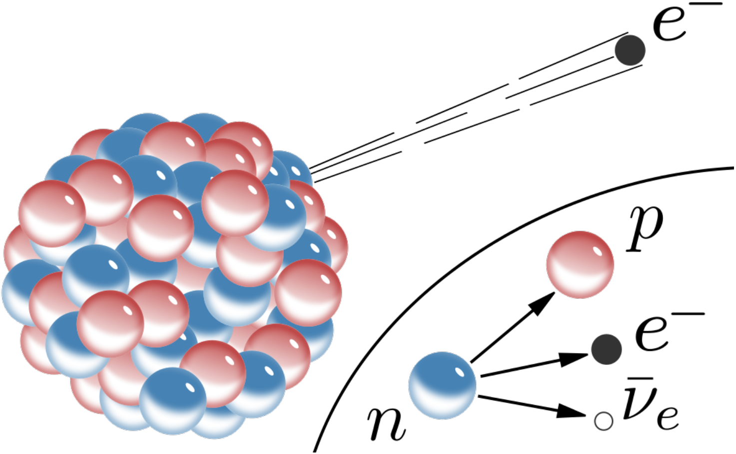
   - MSW Effect
   - Solar Neutrino Problem
   - Stimulated Neutrino Oscillations
3. Understanding Stimulated Oscillations
   - Hamiltonian, and Basis
   - Single Frequency Matter Profile
   - Two-frequency Matter Profile
4. Summary & Future Work

OVERVIEW

What are Neutrinos?



Beta decay and antineutrino production. Source: Beta_Decay@Wikipedia

WHAT ARE NEUTRINOS?

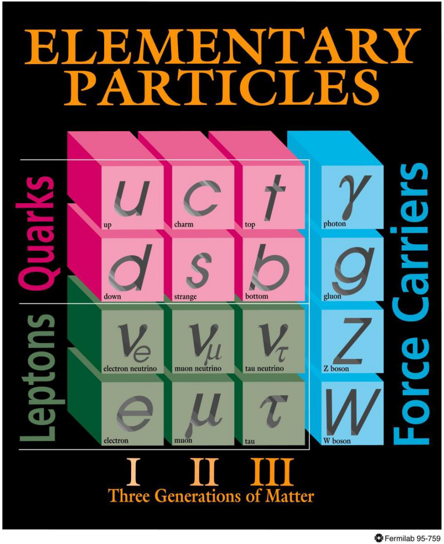
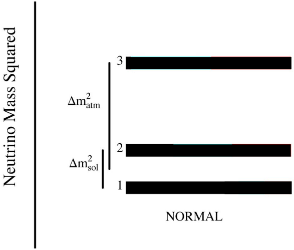


Table of elementary particles. Source: Fermilab

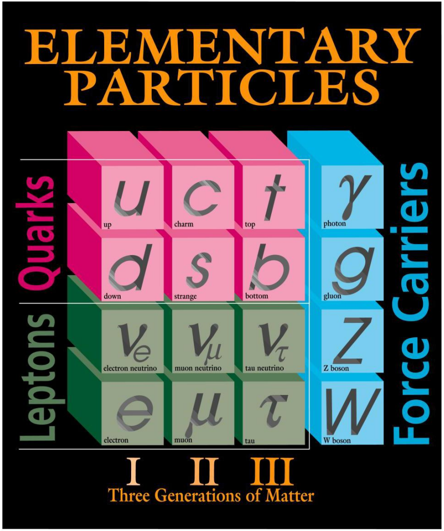
Neutrinos are

- fermions,
- electrically neutral,
- light.



Adapted from Olga Mena & Stephen Parke (2004)

What are Neutrinos?



Fermilab 95-759

Table of elementary particles. Source: Fermilab

Neutrinos are

- fermions,
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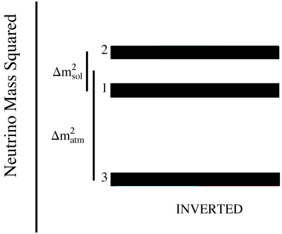


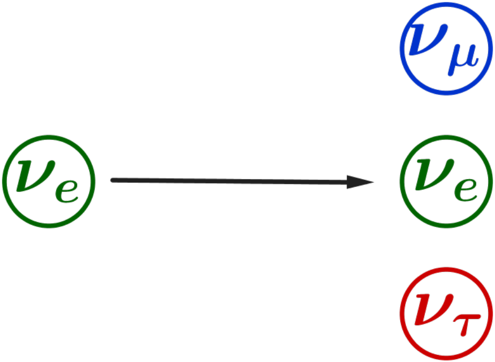
Adapted from Olga Mena & Stephen Parke (2004)

WHAT IS NEUTRINO OSCILLATION?

Neutrino Oscillation

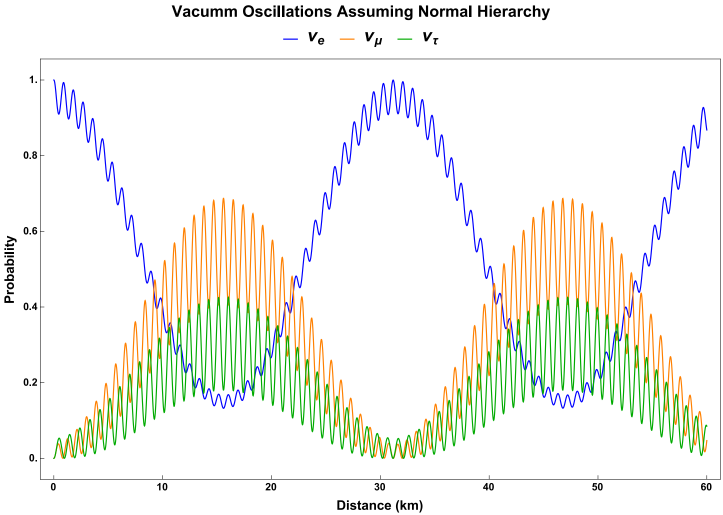
||

Neutrino Flavor Conversion



Neutrino Oscillations

What is Neutrino Oscillation?



Probabilities of finding neutrinos to be in each flavor.

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x |\Psi\rangle = \hat{\mathbf{H}} |\Psi\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{|\nu_1\rangle, |\nu_2\rangle\}$.
- $$\mathbf{H} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$
- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp(i\omega_v x/2) \\ \langle\nu_2|\Psi(0)\rangle \exp(-i\omega_v x/2) \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_\mathrm{e}\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\omega_\mathrm{v}}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} = \frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma_3} + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma_1}\right)$$

Nature of Neutrino Oscillation

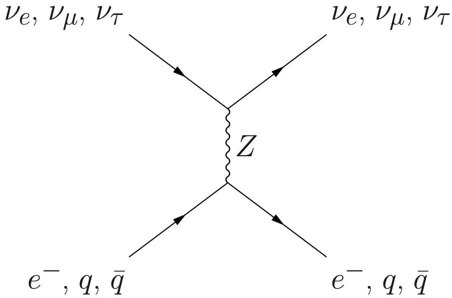
Transition Probability

$$P(|\nu_\mathrm{e}\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_\mathrm{v}) \sin^2(\omega_\mathrm{v} x/2)$$

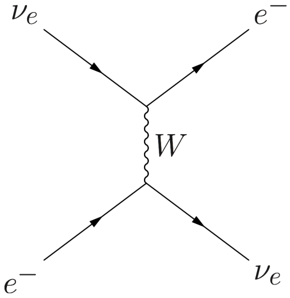
- $\omega_\mathrm{v} = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- Mixing angle θ_v determines flavor oscillation amplitude.

OVERVIEW

Matter Interaction



Neutral current interaction between
ν_e, ν_μ, ν_τ,
and e^-, quarks and antiquarks.



Charged current interaction between
ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_\mathrm{v} = \delta m^2/2E$):

$$\mathbf{H} = \frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} \pm \sqrt{2} G_\mathrm{F} n_\mathrm{e}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- Vacuum Hamiltonian
- Matter interaction

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)$

MSW EFFECT

Hamiltonian in Vacuum

$$\mathbf{H}_{\text{vacuum}} = \frac{\omega_{\text{v}} \cos 2\theta_{\text{v}}}{2}\boldsymbol{\sigma_3} + \frac{\omega_{\text{v}} \sin 2\theta_{\text{v}}}{2}\boldsymbol{\sigma_1}$$

$$\begin{aligned}\mathbf{H} &= \frac{\lambda(x) - \omega_{\text{v}} \cos 2\theta_{\text{v}}}{2}\boldsymbol{\sigma_3} + \frac{\omega_{\text{v}} \sin 2\theta_{\text{v}}}{2}\boldsymbol{\sigma_1} \\ &= \frac{\omega_{\text{m}}(x) \cos 2\theta_{\text{m}}(x)}{2}\boldsymbol{\sigma_3} + \frac{\omega_{\text{m}}(x) \sin 2\theta_{\text{m}}(x)}{2}\boldsymbol{\sigma_1},\end{aligned}$$

where

$$\begin{aligned}\omega_{\text{m}}(x) &= \sqrt{(\lambda(x) - \omega_{\text{v}} \cos 2\theta_{\text{v}})^2 + \omega_{\text{v}}^2 \sin^2 2\theta_{\text{v}}}, \\ \tan 2\theta_{\text{m}}(x) &= \frac{\omega_{\text{v}} \sin 2\theta_{\text{v}}}{\omega_{\text{v}} \cos 2\theta_{\text{v}} - \lambda(x)}.\end{aligned}$$

MSW EFFECT

Constant matter profile λ_0 as an example,

Significance of θ_{m}

Define matter basis $\{|\nu_{\mathrm{L}}\rangle, |\nu_{\mathrm{H}}\rangle\}$

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & \sin\theta_{\mathrm{m}} \\ -\sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}$$

MSW Resonance

Hamiltonian with Matter Potential

$$\begin{aligned}\mathbf{H} &= \frac{\lambda(x) - \omega_\mathrm{v}\cos 2\theta_\mathrm{v}}{2}\boldsymbol{\sigma_3} + \frac{\omega_\mathrm{v}\sin 2\theta_\mathrm{v}}{2}\boldsymbol{\sigma_1} \\ &= \frac{\omega_\mathrm{m}(x)\cos 2\theta_\mathrm{m}(x)}{2}\boldsymbol{\sigma_3} + \frac{\omega_\mathrm{m}(x)\sin 2\theta_\mathrm{m}(x)}{2}\boldsymbol{\sigma_1}\end{aligned}$$

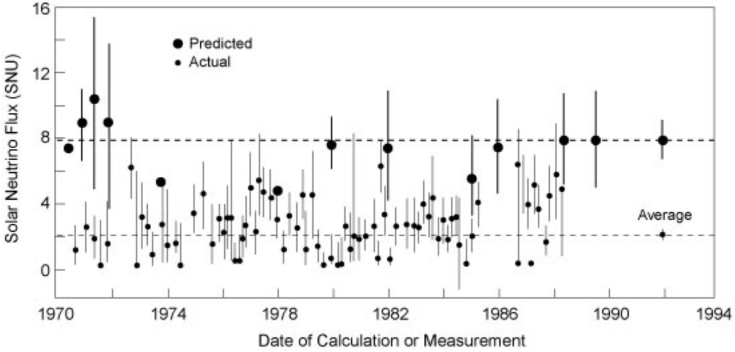
$$\tan 2\theta_\mathrm{m}(x) = \frac{\omega_\mathrm{v}\sin 2\theta_\mathrm{v}}{\omega_\mathrm{v}\cos 2\theta_\mathrm{v} - \lambda(x)}.$$

$$\begin{pmatrix}|\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle\end{pmatrix} = \begin{pmatrix}\cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m}\end{pmatrix}\begin{pmatrix}|\nu_\mathrm{L}\rangle \\ |\nu_\mathrm{H}\rangle\end{pmatrix}$$

Transition Probability

$$P(|\nu_\mathrm{e}\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_\mathrm{m})\sin^2\left(\omega_\mathrm{m}x\right)$$

Solar Neutrino Problem



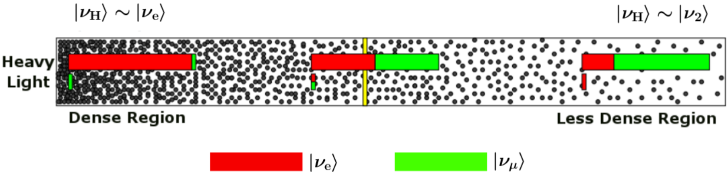
Chlorine detector (Homestake experiment) results and theory predictions. SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW Effect and Solar Neutrinos

$$\mathbf{H} = \frac{\lambda(x) - \omega_{\rm v}\cos 2\theta_{\rm v}}{2}\boldsymbol{\sigma_3} + \frac{\omega_{\rm v}\sin 2\theta_{\rm v}}{2}\boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_{\rm L}\rangle \\ |\nu_{\rm H}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\rm m} & -\sin\theta_{\rm m} \\ \sin\theta_{\rm m} & \cos\theta_{\rm m} \end{pmatrix} \begin{pmatrix} |\nu_{\rm e}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\rm m}}{2}\boldsymbol{\sigma_3}$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW EFFECT

Suppose $\omega_\mathrm{v} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = -\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} + \sqrt{2} G_\mathrm{F} n_\mathrm{e}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

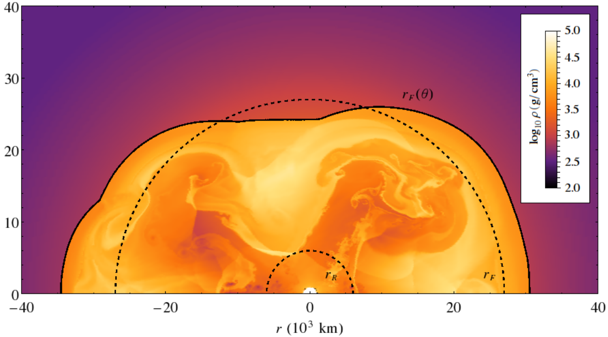
$$\downarrow$$

$$\mathbf{H} = \left(\frac{-\omega_\mathrm{v}}{2} \cos 2\theta_\mathrm{v} + \frac{\lambda(x)}{2} \right) \boldsymbol{\sigma_3} - \frac{\omega_\mathrm{v}}{2} \sin 2\theta_\mathrm{v} \boldsymbol{\sigma_1}$$

SUPERNOVA MATTER DENSITY PROFILE

Why Do We Care

Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

$$\Delta n_e(r) = \sum_n c_n \sin(k_n r + \phi_n)$$

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0,

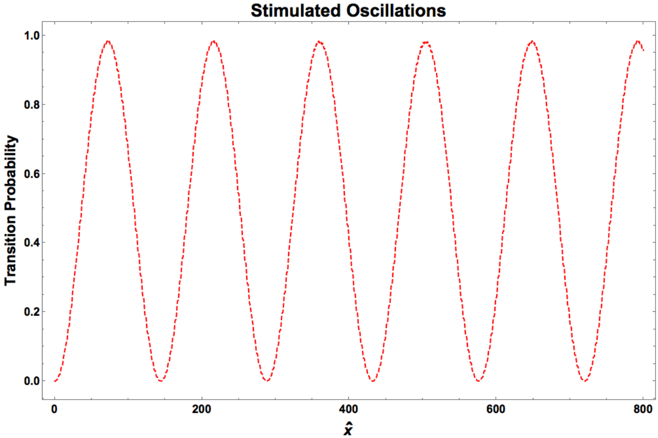
$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013); K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\sin(kx)$ with $\hat{x} = \omega_{\rm m} x$, $A = 0.1\omega_{\rm m}$, $k = 0.995\omega_{\rm m}$, $\theta_{\rm m} = \pi/6$

Overview

UNDERSTANDING STIMULATED OSCILLATIONS

Matter profile

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define new basis $\{|\tilde{\nu}_{\mathrm{L}}\rangle, |\tilde{\nu}_{\mathrm{H}}\rangle\}$ is related to background matter basis $\{|\nu_{\mathrm{L}}\rangle, |\nu_{\mathrm{H}}\rangle\}$ through

$$\begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_{\mathrm{L}}\rangle \\ |\tilde{\nu}_{\mathrm{H}}\rangle \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = -\frac{\omega_{\mathrm{m}}}{2}x + \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x \delta\lambda(\tau)d\tau.$$

Single Frequency Matter Profile

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}} \begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

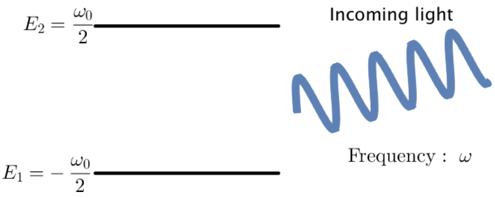
$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2} e^{2i\eta(x)} \\ &= \frac{i}{4}\left[\exp\left(i(k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right. \\ &\quad \left. -\exp\left(i(-k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right] \end{aligned}$$

Rabi Oscillation

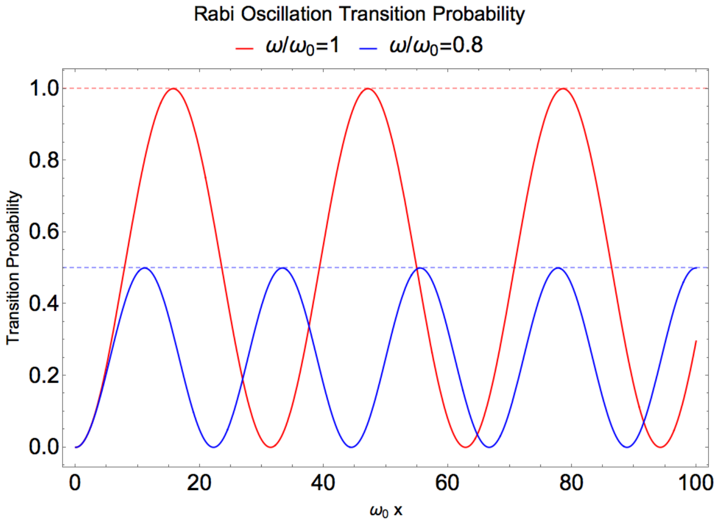
Rabi Oscillation

Hamiltonian

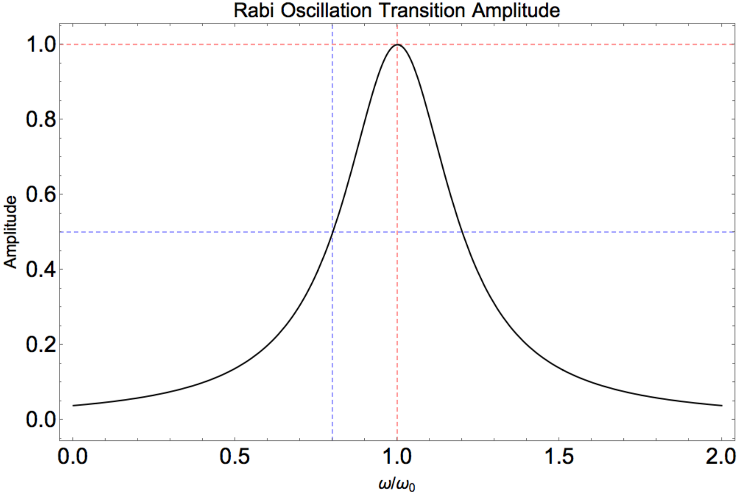
$$\begin{pmatrix} -\omega_0/2 & \alpha\omega_0 e^{i\omega x} \\ \alpha\omega_0 e^{-ikx} & \omega_0/2 \end{pmatrix},$$



Rabi Oscillations



Rabi Oscillations



Resonance

Single Frequency Matter Profile

Off-diagonal Term in Our System

$$\widetilde{\mathbf{H}} = \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$h \propto \Big[\exp\left(i(k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right) \\ - \exp\left(i(-k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\Big]$$

Jacobi-Anger expansion

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

Single Frequency Matter Profile

Scaled Quantities

Characteristic scale: ω_{m}

- $\hat{A} = A/\omega_{\mathrm{m}}$
- $\hat{k} = k/\omega_{\mathrm{m}}$
- $\hat{x} = \omega_{\mathrm{m}} x$
- $\hat{h} = h/\omega_{\mathrm{m}}$

Single Frequency Matter Profile

Rotation Wave Approximation

The off-diagonal element of Hamiltonian

$$\widetilde{\mathbf{H}} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}} \\ \frac{1}{2}\hat{B}_n^* e^{-i(n\hat{k}-1)\hat{x}} & 0 \end{pmatrix}$$

where $\hat{B}_n = -(-i)^n n\hat{k} \tan 2\theta_{\mathrm{m}} J_n(\hat{A}\cos 2\theta_{\mathrm{m}}/\hat{k})$.

Single Frequency Matter Profile

Transition Probability

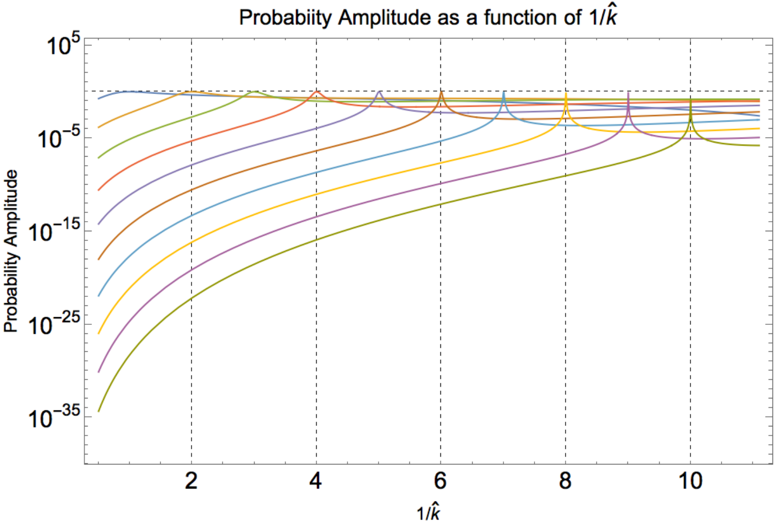
$$P^{(n)}_{\mathrm{L}\to\mathrm{H}} = \frac{\left|\left|\hat{B}_n\right|/2\right|^2}{\left|\left|\hat{B}_n\right|/2\right|^2 + (n\hat{k}-1)^2} \sin^2\left(\frac{q^{(n)}}{2}x\right),$$

where

$$q^{(n)} = \sqrt{\left|\Gamma^{(n)}/2\right|^2 + (n\hat{k}-1)^2}, \quad \text{frequency of oscillations}$$

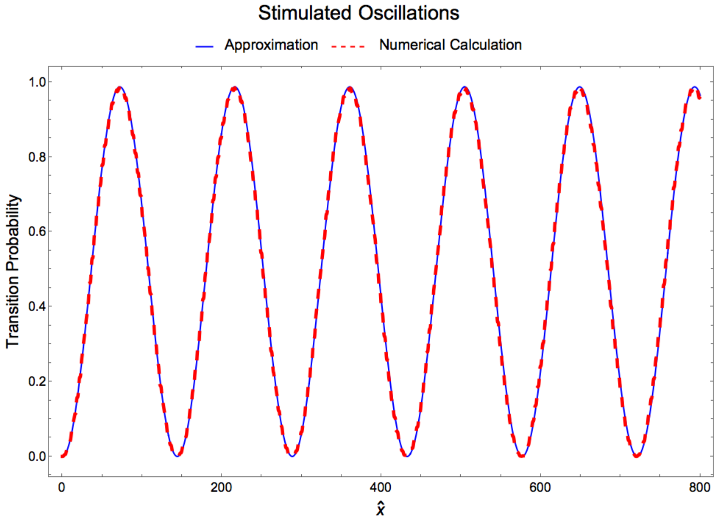
$$\Gamma^{(n)} = \left|\hat{B}_n\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

SINGLE FREQUENCY MATTER PROFILE



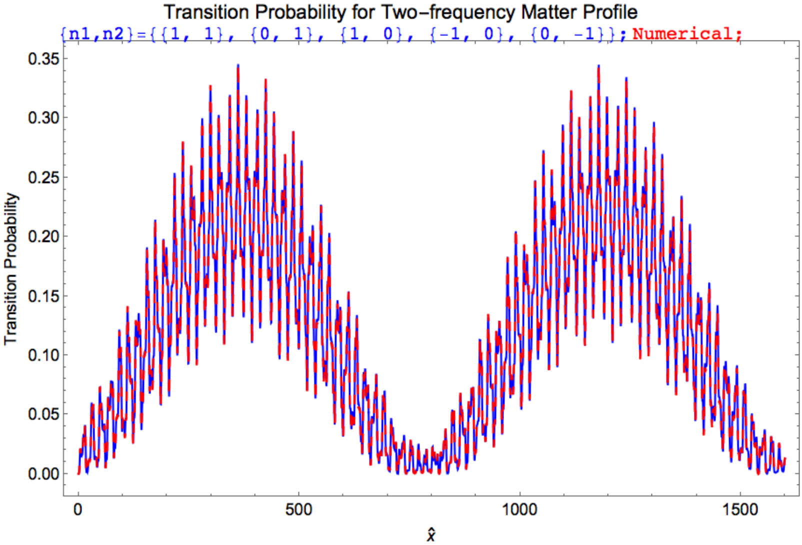
Resonances of different $n = 1/\hat{k}$.

Single Frequency Matter Profile



$$\hat{A} = 0.1,\ \hat{k} = 0.995,\ \theta_{\mathrm{m}} = \pi/6$$

Two-frequency Matter Profile



$\lambda(x) = \lambda_0 + A_1 \sin(k_1 x) + A_2 \sin(k_2 x)$. $\hat{k}_1 = 0.3$, $\hat{k}_2 = 0.7$, $A_1 = A_2 = 0.1$, $\theta_m = \pi/5$.

Overview

SUMMARY & FUTURE WORK

- The fact that neutrino flavor sates are not mass states causes vacuum oscillations.
- MSW resonance happens when matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
- Even matter profile doesn't match MSW requirement, variation in matter profile can cause resonances.
- Single frequency perturbations in matter profile is a combination of many Rabi oscillations.

- How to understand and calculate systems with multi-frequency matter profile (turbulence).
- Combine periodic or even turbulent matter profile with neutrino self-interaction.

ACKNOWLEDGEMENT

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BACKUP SLIDES

BACKUP SLIDES

Parameters Used for Vacuum Oscillations

$\theta_{12} = 33.36/180\pi$; $\theta_{13} = 8.66/180\pi$; $\theta_{23} = 40/180 * \pi$; $\delta_{cp} = 0$;
$m_1^2 = 0.01$; $m_2^2 = m_1^2 + 0.000079$; $E = 1\text{MeV}$

Single Frequency Matter Profile

Why Does It Work?

$$J_n(n \operatorname{sech} \alpha) \sim \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}, \quad \text{for large } n$$

$\Rightarrow$

$$\Gamma \propto \hat{B}_n \propto \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}$$

Small perturbation $\Rightarrow$ Small $\hat{A}$ $\Rightarrow$ Large α $\Rightarrow$ Drops fast at large n.

Two-frequency Matter Profile

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

Two-Frequency Matter Profile $\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}}$,

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1\hat{k}_1 + n_2\hat{k}_2 - 1)\hat{x}},$$

where

$$\begin{aligned}&\hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2)\\ =& -(-i)^{n_1+n_2}(n_1\hat{k}_1 + n_2\hat{k}_2) J_{n_1}\left(\frac{\hat{A}_1 \cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right) J_{n_2}\left(\frac{\hat{A}_2 \cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)\end{aligned}$$

Which terms are important?

TWO-FREQUENCY MATTER PROFILE

Resonance Lines

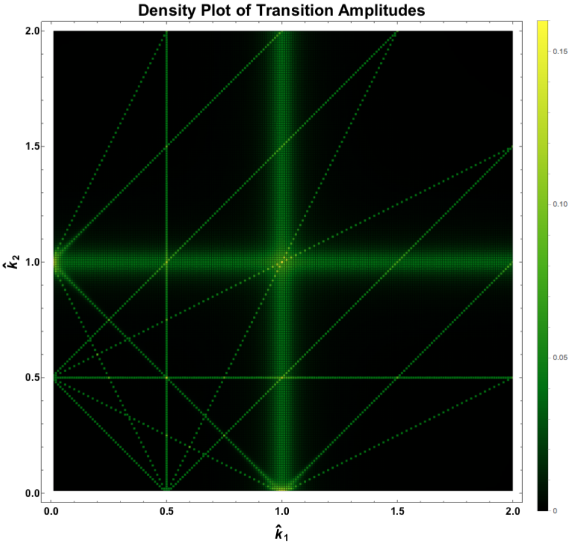
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.

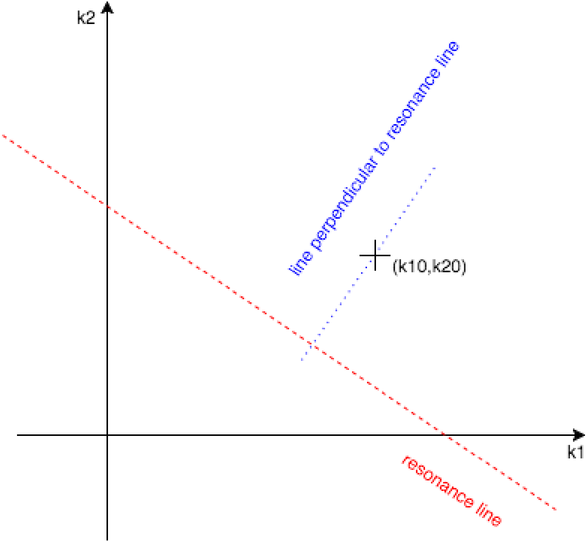
Two-frequency Matter Profile

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

Two-frequency Matter Profile



Resonance line, distance to resonance, and width

Two-frequency Matter Profile

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\text{intercept}}, \hat{k}_{2,\text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

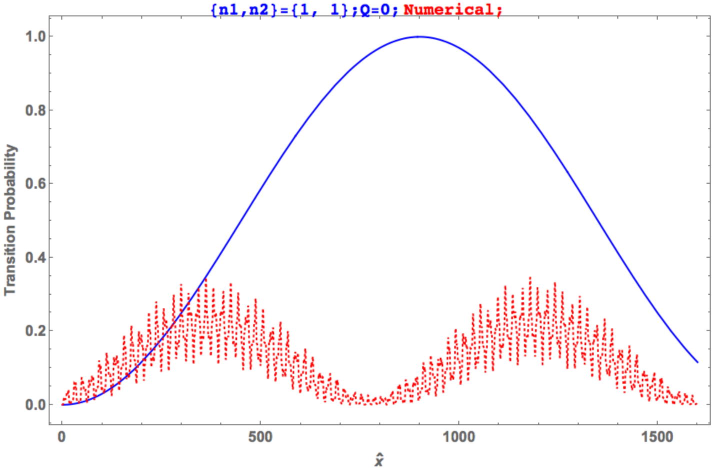
Distance to Resonance Line

$$d = \frac{|n_1\hat{k}_{10} + n_2\hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

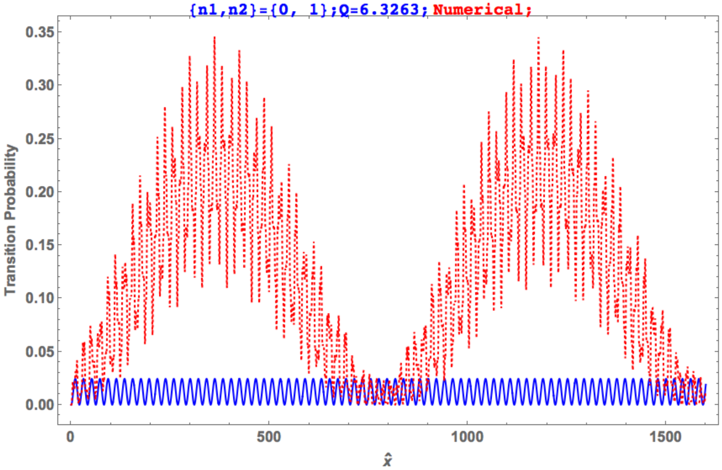
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

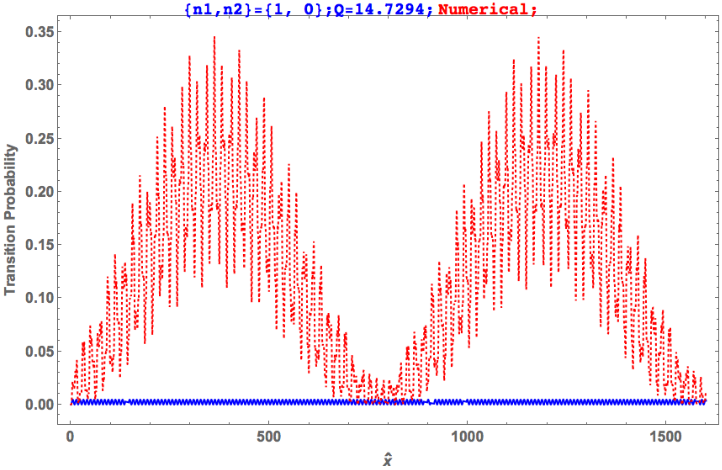
Two-frequency Matter Profile



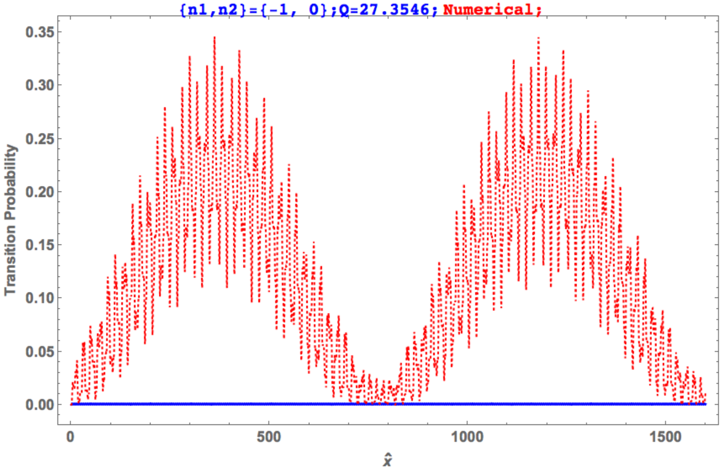
Two-frequency Matter Profile



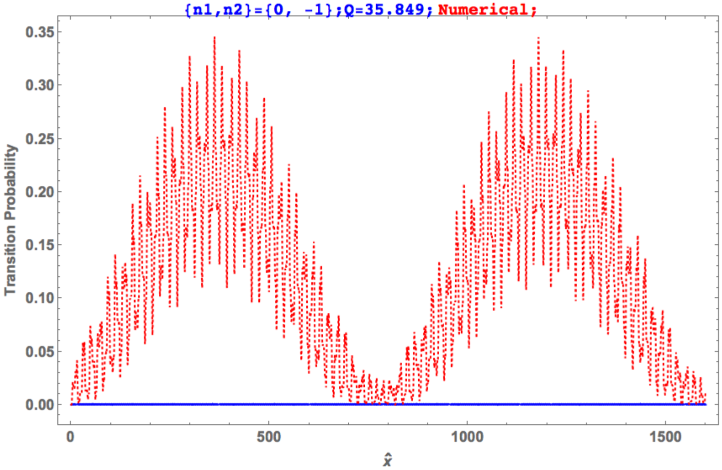
Two-frequency Matter Profile



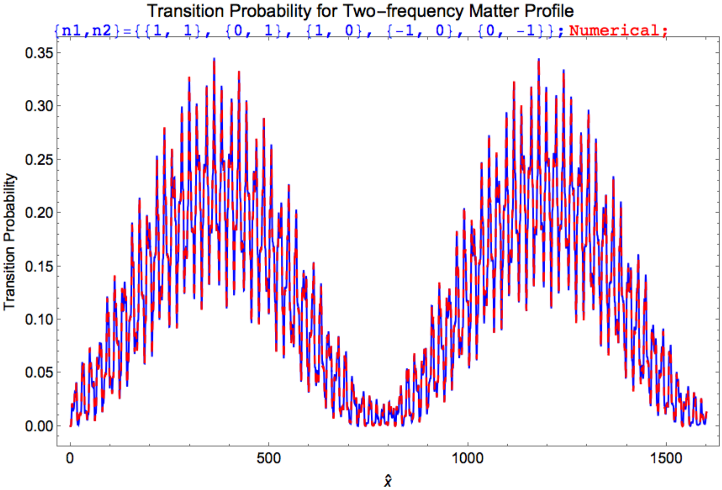
TWO-FREQUENCY MATTER PROFILE



Two-frequency Matter Profile



Two-frequency Matter Profile



BESSEL'S FUNCTION

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I