Neutrino Oscillations in Matter

PhD Candidacy Exam

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OUTLINE

1. Introduction
   - History of Neutrinos
   - What are Neutrinos
   - Neutrino Oscillations
   - Why Oscillations
2. Matter Effect
   - Matter Interaction
   - MSW Effect
   - Solar Neutrino Problem
   - Stimulated Neutrino Oscillations
3. Understanding Stimulated Oscillations
   - Hamiltonian, and Basis
   - Single Frequency Matter Profile
4. Summary & Future Work

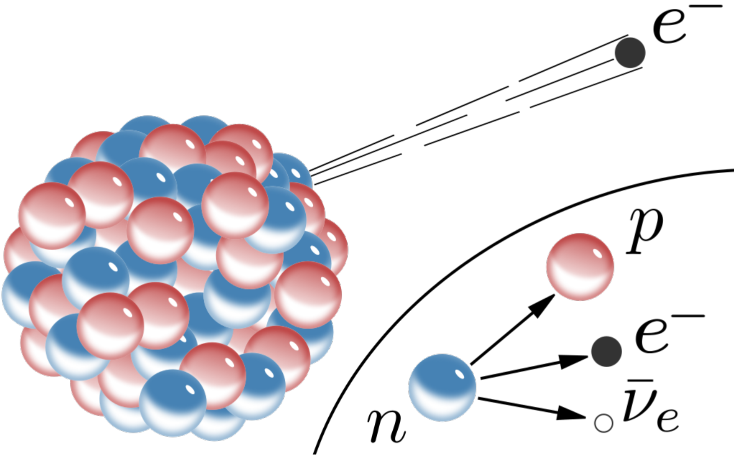
Overview

NEUTRINO TIMELINE

History of Neutrino (Partial)

- 1930 — Pauli, letter to “Radioactive Ladies and Gentlemen"
- 1933 — Fermi, the name “neutrino"
- 1956 — Reines & Cowan, reactor neutrinos
- 1968 — Homestake, solar neutrinos
- 1978 & 1985 — Mikheyev–Smirnov–Wolfenstein (MSW) effect
- 1987 — SN1987A neutrinos $\sim$ 20 events
- 1998 & 2001 — Super–Kamiokande & SNO, neutrino oscillations

What are Neutrinos?



Beta decay and antineutrino production. Source: Beta_Decay@Wikipedia

What are Neutrinos?

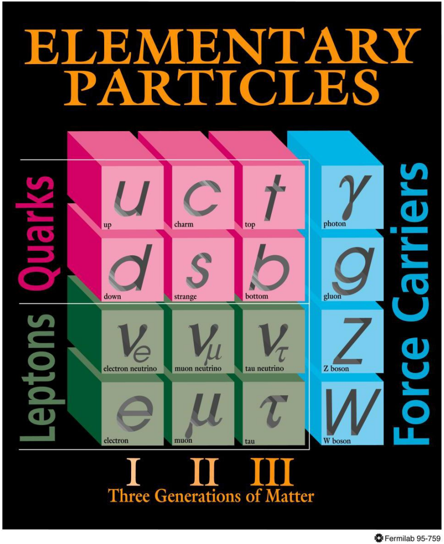
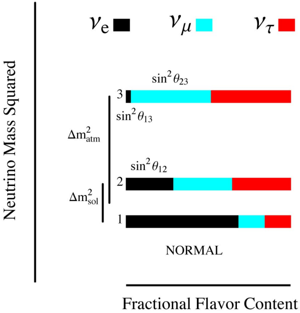


Table of elementary particles. Source: Fermilab

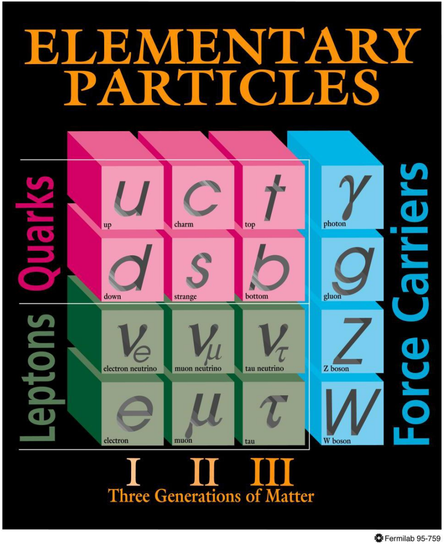
Neutrinos are

- Fermions,
- electrically neutral,
- light.



Adapted from Olga Mena & Stephen Parke (2004)

WHAT ARE NEUTRINOS?



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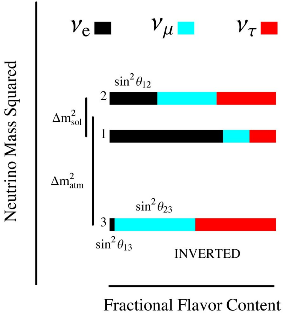

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What is Neutrino Oscillation?

Neutrino Oscillation

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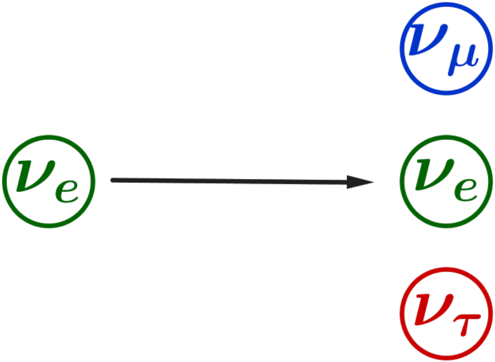
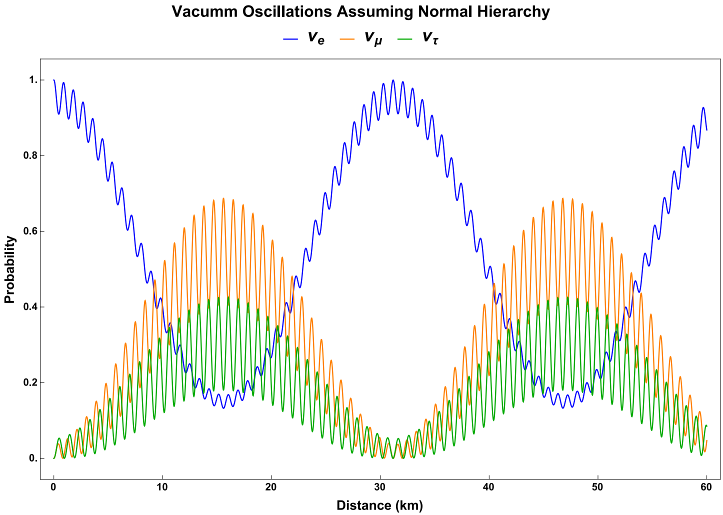
Neutrino Flavor Conversion



Neutrino Oscillations

What is Neutrino Oscillation?



Probabilities of finding neutrinos to be in each flavors.

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x |\Psi\rangle = \hat{\mathbf{H}} |\Psi\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{|\nu_1\rangle, |\nu_2\rangle\}$.
-
$$\mathbf{H} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$
- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp(i\omega_v x/2) \\ \langle\nu_2|\Psi(0)\rangle \exp(-i\omega_v x/2) \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_\mathrm{e}\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

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θ_v: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\omega_\mathrm{v}}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} = \frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma_3} + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma_1}\right)$$

NATURE OF NEUTRINO OSCILLATION

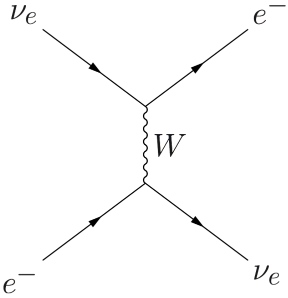
Transition Probability

$$P(|\nu_\mathrm{e}\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_\mathrm{v})\sin^2(\omega_\mathrm{v}x/2)$$

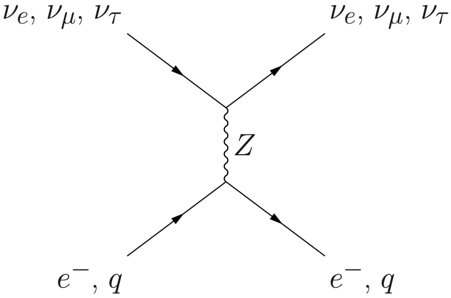
- $\omega_\mathrm{v} = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength
- Mixing angle determines flavor oscillation amplitude

OVERVIEW

MATTER INTERACTION



Charged current interaction between ν_e and e^-



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, quarks.

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\begin{pmatrix} -\cos 2\theta_{\mathrm{v}} & \sin 2\theta_{\mathrm{v}} \\ \sin 2\theta_{\mathrm{v}} & \cos 2\theta_{\mathrm{v}} \end{pmatrix}} + \boxed{\sqrt{2}G_{\mathrm{F}} n_{\mathrm{e}}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

- Vacuum Hamiltonian
- Matter interaction

MATTER INTERACTION

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)} + \boxed{\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)$

Matter Interaction

Hamiltonian with Matter Potential

$$\mathbf{H} = \frac{\lambda(x) - \omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_3} + \frac{\omega_{\mathrm{v}} \sin 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_1}$$

Hamiltonian in Vacuum

$$\frac{\omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_3} + \frac{\omega_{\mathrm{v}} \sin 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_1}$$

MSW Effect

Hamiltonian with Matter Potential

$$\mathbf{H} = \frac{\omega_\mathrm{m}(x)\cos 2\theta_\mathrm{m}(x)}{2}\boldsymbol{\sigma}_\mathbf{3} + \frac{\omega_\mathrm{m}(x)\sin 2\theta_\mathrm{m}(x)}{2}\boldsymbol{\sigma}_\mathbf{1},$$

where

$$\omega_\mathrm{m}(x) = \omega_\mathrm{v}\sqrt{\left(\frac{\lambda(x)}{\omega_\mathrm{v}} - \cos 2\theta_\mathrm{v}\right)^2 + \sin^2 2\theta_\mathrm{v}},$$

$$\tan 2\theta_\mathrm{m}(x) = \frac{\sin 2\theta_\mathrm{v}}{\cos 2\theta_\mathrm{v} - \lambda(x)/\omega_\mathrm{v}}.$$

Significance of θ_m

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix}$$

Constant Matter Density

Transition Probability for Constant Matter Density

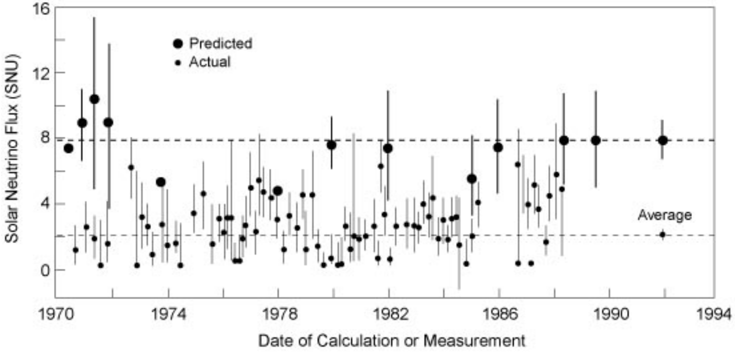
$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_m)\sin^2(\omega_m x)$$

where

$$\omega_m(x) = \omega_v\sqrt{\left(\frac{\lambda(x)}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v},$$

$$\tan 2\theta_m(x) = \frac{\sin 2\theta_v}{\cos 2\theta_v - \lambda(x)/\omega_v}.$$

Solar Neutrino Problem

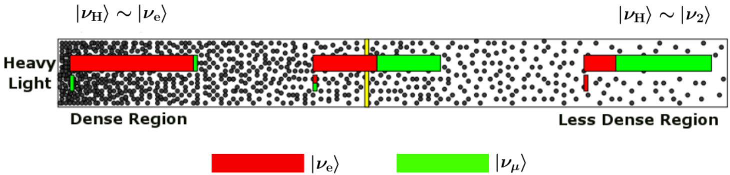


Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW Effect and Solar Neutrinos

$$\mathbf{H} = \frac{\lambda(x) - \omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_3} + \frac{\omega_{\mathrm{v}} \sin 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & \sin\theta_{\mathrm{m}} \\ -\sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix}$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW Effect

Suppose $\omega_{\mathrm{v}} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H}' = \boxed{-\frac{\omega_{\mathrm{v}}}{2}\begin{pmatrix} -\cos 2\theta_{\mathrm{v}} & \sin 2\theta_{\mathrm{v}} \\ \sin 2\theta_{\mathrm{v}} & \cos 2\theta_{\mathrm{v}} \end{pmatrix}} + \boxed{\sqrt{2}G_{\mathrm{F}} n_{\mathrm{e}}(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\downarrow$$

$$\mathbf{H}' = \left(\frac{-\omega_{\mathrm{v}}}{2}\cos 2\theta_{\mathrm{v}} + \frac{\lambda(x)}{2}\right)\boldsymbol{\sigma}_{\mathbf{3}} - \frac{\omega_{\mathrm{v}}}{2}\sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma}_{\mathbf{1}}$$

Length Scales

Characteristic Scales

- Vacuum problem: only one length scale

$$l_{\mathrm{v}} \sim \frac{1}{\omega_{\mathrm{v}}}$$

- Constant matter profile λ_0:

$$l_{\mathrm{v}}, \quad l_{\mathrm{m}} \sim \frac{1}{\omega_{\mathrm{m}}}$$

- Varying matter profile $\lambda(x) = \lambda_0 + A\sin(kx)$,

$$l_{\mathrm{v}}, \quad l_{\mathrm{m}}, \quad l_k \sim \frac{1}{k}$$

MSW Resonance

$$l_{\mathrm{v}} \sim l_{\mathrm{m}} \sin 2\theta_{\mathrm{v}}$$

LENGTH SCALES

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$$l_{\mathrm{v}}, \quad l_{\mathrm{m}}, \quad l_k \sim \frac{1}{k}$$

Other Resonance?

l_{m}, and l_k?

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0,

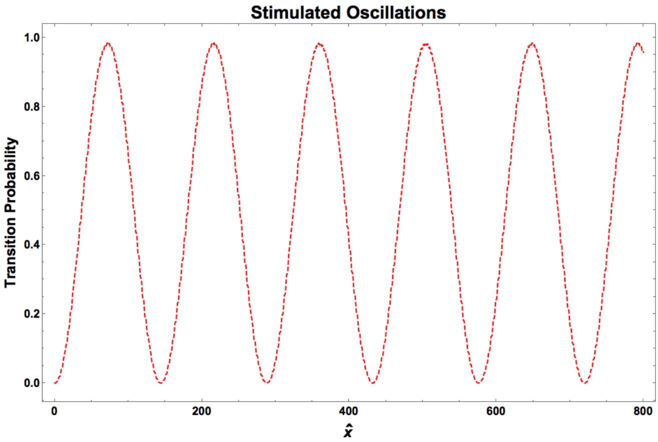
$$-\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

Stimulated Neutrino Oscillations

P. Krastev (1989); F. Loreti (1994); A. Balantekin, et al (1996);
E. Akhmedov (2000);
J. Kneller, et al (2013); K. Patton, et al (2014); $\cdots$



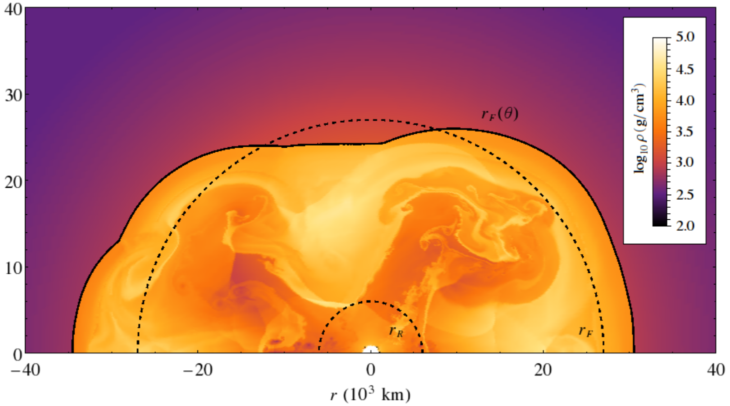
Stimulated oscillations. $\delta\lambda(x) = A\sin(\hat{k}\hat{x})$ with $\hat{A} = 0.1$, $\hat{k} = 0.995$, $\theta_m = \pi/6$

OVERVIEW

Supernova Matter Density Profile

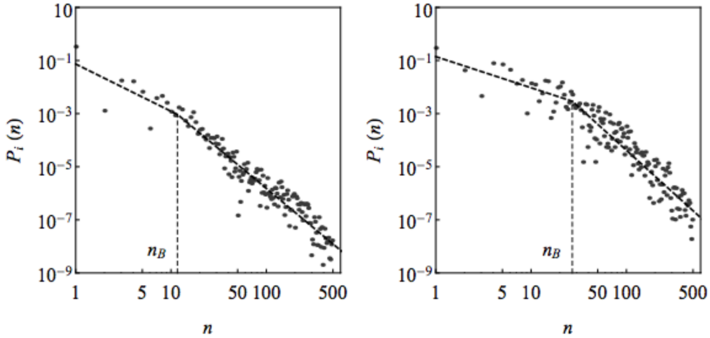
Why Do We Care

Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

SUPERNOVA MATTER DENSITY PROFILE



Supernova matter density power spectra from three simulations. E. Borriello, et al (2014)

$$P = 4|c_n|^2$$

$$\Delta\rho(r) = \sum_n c_n e^{ik_n r}$$

UNDERSTANDING STIMULATED OSCILLATIONS

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Remove diagonal elements

New basis $\{|\tilde{\nu}_{\mathrm{L}}\rangle, |\tilde{\nu}_{\mathrm{H}}\rangle\}$ is related to background matter basis $\{|\nu_{\mathrm{L}}\rangle, |\nu_{\mathrm{H}}\rangle\}$ through

$$\begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_{\mathrm{L}}\rangle \\ |\tilde{\nu}_{\mathrm{H}}\rangle \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = -\frac{\omega_{\mathrm{m}}}{2}x + \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x \delta\lambda(\tau)d\tau.$$

Single Frequency Matter Profile

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}} \begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix}$$

For matter profile

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in New Basis

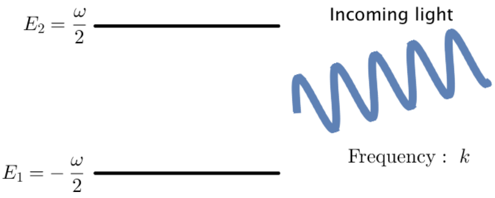
$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2} e^{2i\eta(x)} \\ &= \frac{i}{4}\left[\exp\left(i(k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right. \\ &\qquad \left. - \exp\left(i(-k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right] \end{aligned}$$

Rabi Oscillation

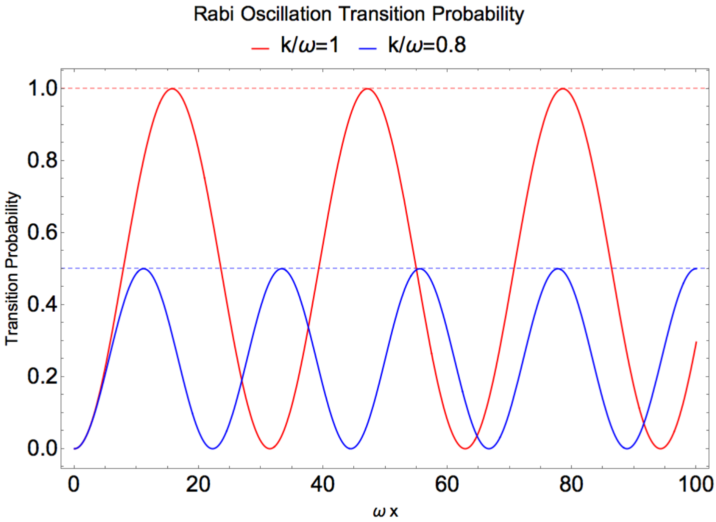
Rabi Oscillation

Hamiltonian

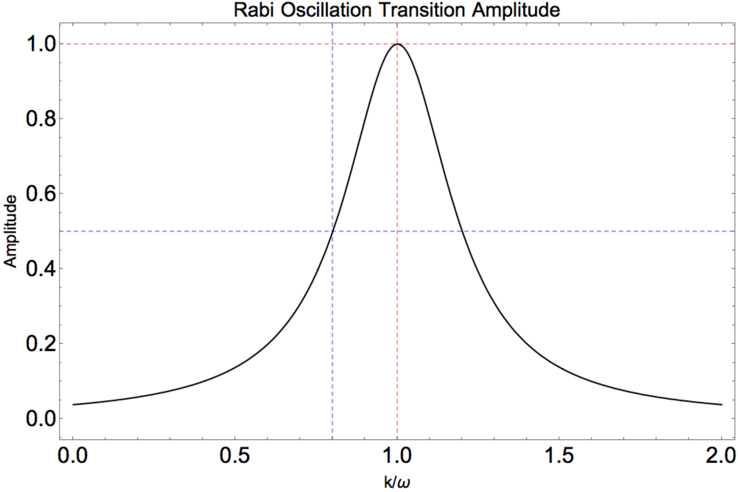
$$\begin{pmatrix} -\omega/2 & \alpha\omega e^{ikx} \\ \alpha\omega e^{-ikx} & \omega/2 \end{pmatrix},$$



RABI OSCILLATIONS



RABI OSCILLATIONS



Resonance

Single Frequency Matter Profile

Off-diagonal Term in Our System

$$h \propto \Big[\exp\left(i(k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right) \\ - \exp\left(i(-k+\omega_{\mathrm{m}})x + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\Big]$$

Jacobi-Anger expansion

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ is Bessel's function of the first kind.

Single Frequency Matter Profile

Scaled Quantities

Characteristic scale: ω_{m}

- $\hat{A} = A/\omega_{\mathrm{m}}$
- $\hat{k} = k/\omega_{\mathrm{m}}$
- $\hat{x} = \omega_{\mathrm{m}} x$
- $\hat{h} = h/\omega_{\mathrm{m}}$

Single Frequency Matter Profile

Rotation Wave Approximation

The off-diagonal element of Hamiltonian

$$\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2} \hat{B}_n e^{i(n\hat{k}-1)\hat{x}},$$

where $\hat{B}_n = -(-i)^n n\hat{k} \tan 2\theta_{\mathrm{m}} J_n(\hat{A}\cos 2\theta_{\mathrm{m}}/\hat{k})$.

Near Resonance

small $(n\hat{k} - 1) \rightarrow$ important term

Find integer $n_0 = \mathrm{Round}\left[1/\hat{k}\right]$ that minimizes $n\hat{k} - 1$.

Single Frequency Matter Profile

Transition Probability

$$P_{1\to 2} = \frac{|\Gamma/2|^2}{q^2} \sin^2\left(\frac{q}{2}x\right),$$

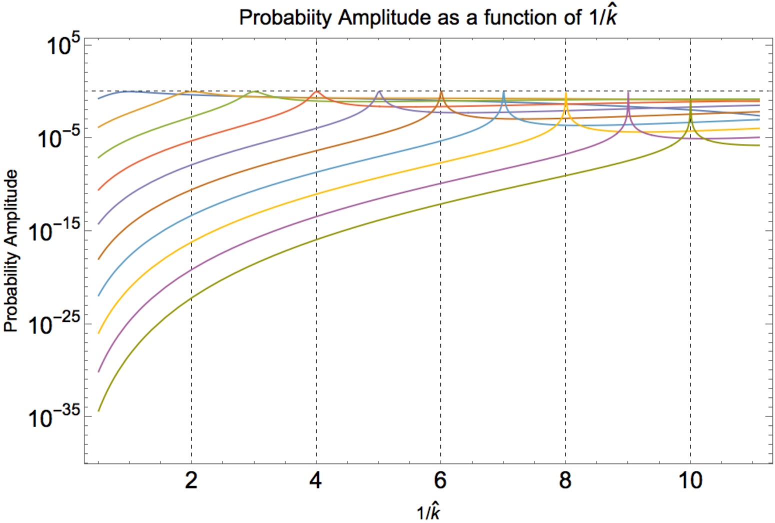
where

$$q^2 = |\Gamma/2|^2 + (n_0\hat{k} - 1)^2, \quad \text{frequency of oscillations}$$

$$\Gamma = \left|\hat{B}_{n_0}\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

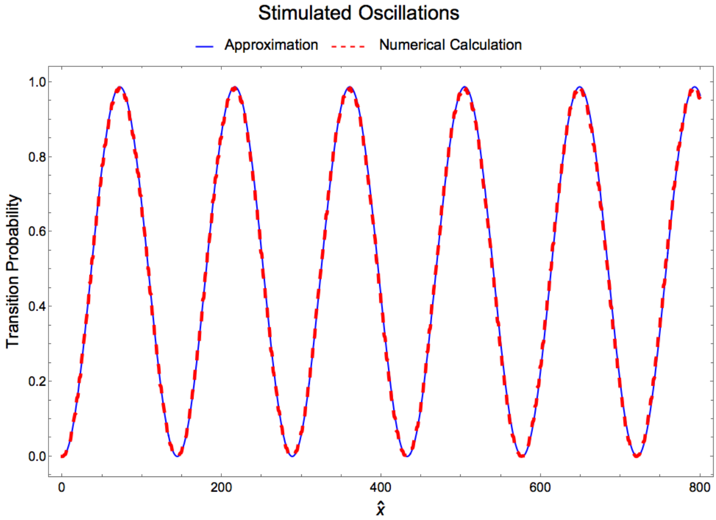
$$n_0 = \mathrm{Round}\left[1/\hat{k}\right]$$

Single Frequency Matter Profile



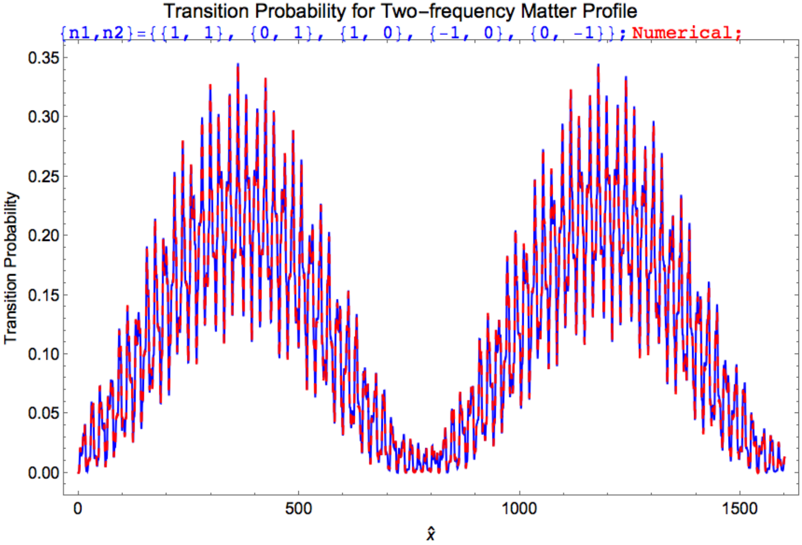
Transition probability amplitude as a function of $1/\hat{k}$ (the phase is $(n - 1/\hat{k})$). The colors represent different orders of n_0.

SINGLE FREQUENCY MATTER PROFILE



RWA works. $\hat{A} = 0.1$, $\hat{k} = 0.995$, $\theta_{\mathrm{m}} = \pi/6$

Two-frequency Matter Profile



Comparison of approximation and numerical solution for two-frequency matter profile

OVERVIEW

Summary & Future Work

- Mass squared differences + mixing angles $\rightarrow$ Vacuum oscillations
- Matter oscillations depend on matter profile
- Simple periodic matter profile $\delta\lambda(x) \sim \sin(kx) \rightarrow$ resonance and width

- Multi-frequency matter profile
- Combine matter effect with neutrino self-interactions

ACKNOWLEDGEMENT

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Supported by DOE EPSCoR grant #desc0008142 at UNM.

BACKUP SLIDES

BACKUP SLIDES

Single Frequency Matter Profile

Why Does RWA Work?

$$J_n(n \operatorname{sech} \alpha) \sim \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}, \quad \text{for large } n$$

$\Rightarrow$

$$\Gamma \propto \hat{B}_n \propto \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}$$

Small perturbation $\Rightarrow$ Small $\hat{A}$ $\Rightarrow$ Large α $\Rightarrow$ Drops fast at large n.

Two-frequency Matter Profile

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

TWO-FREQUENCY MATTER PROFILE $\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}}$,

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2) e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}},$$

where

$$\begin{aligned}&\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)\\ =& -(-i)^{n_1+n_2}(n_1\hat{k}_1+n_2\hat{k}_2) J_{n_1}\left(\frac{\hat{A}_1\cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right) J_{n_2}\left(\frac{\hat{A}_2\cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)\end{aligned}$$

Which terms are important?

Two-frequency Matter Profile

Resonance Lines

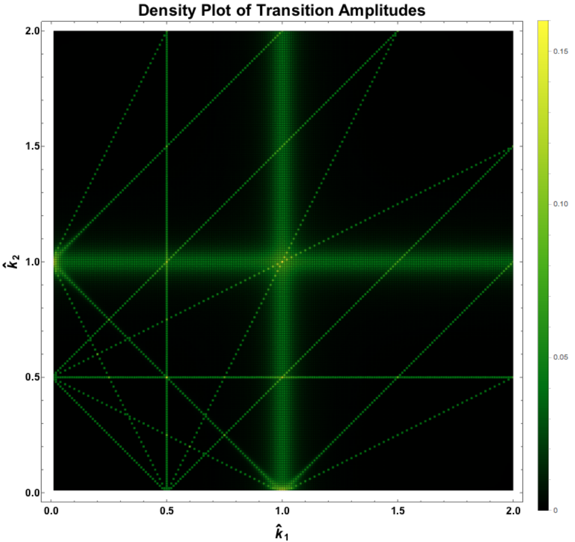
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.

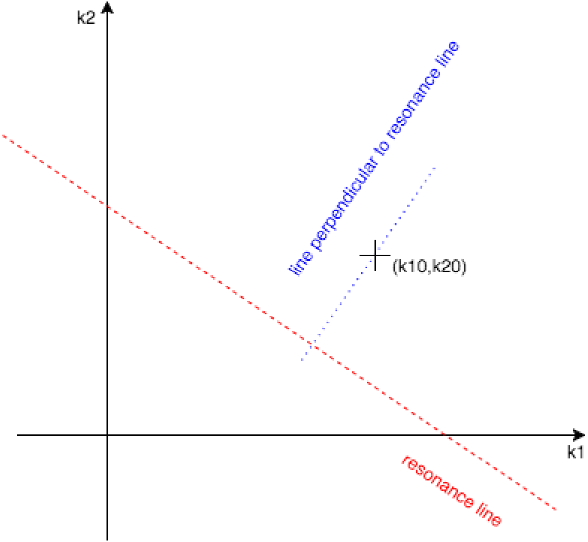
Two-frequency Matter Profile

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

Two-frequency Matter Profile



Resonance line, distance to resonance, and width

Two-frequency Matter Profile

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\text{intercept}}, \hat{k}_{2,\text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

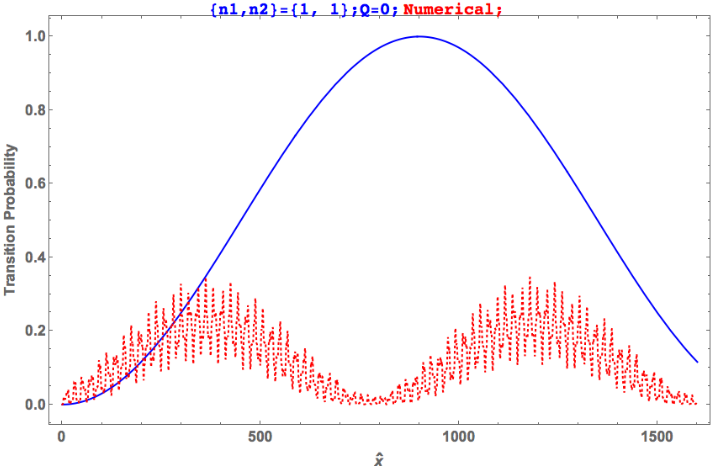
Distance to Resonance Line

$$d = \frac{|n_1\hat{k}_{10} + n_2\hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

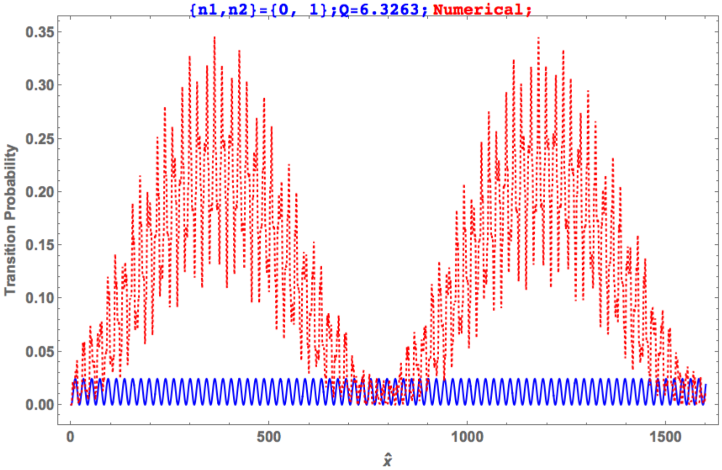
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



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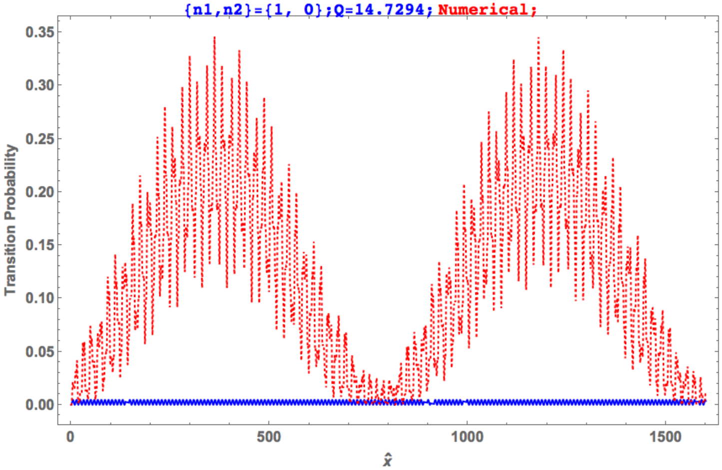
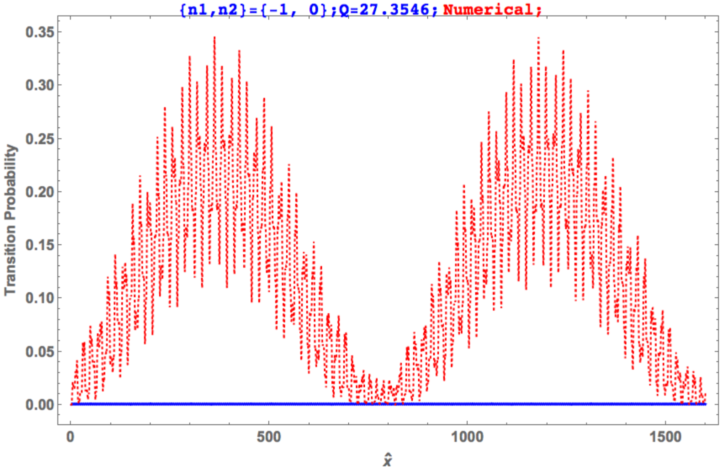
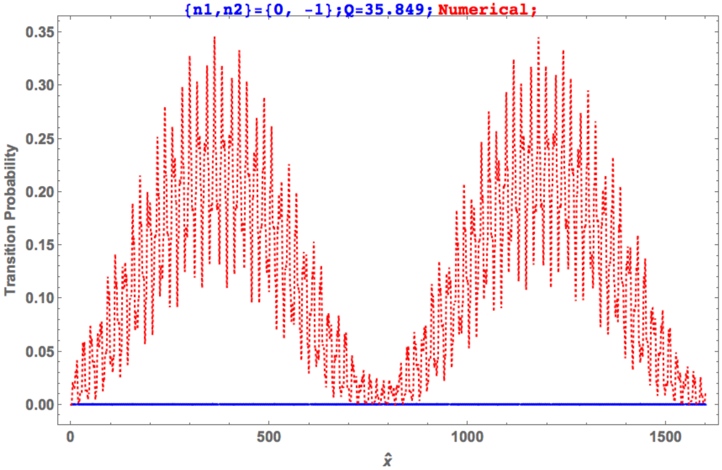


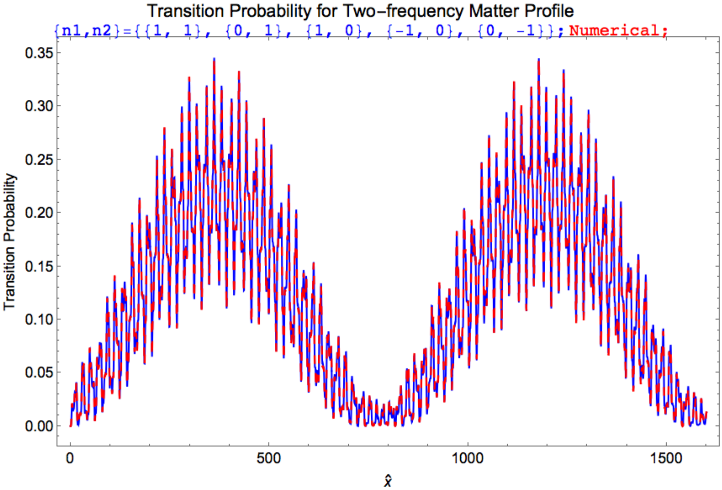
Two-frequency Matter Profile



TWO-FREQUENCY MATTER PROFILE



Two-frequency Matter Profile



Bessel's Function

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I