Neutrino Flavor Conversions in Dense Media: Matter Stimulation and Dispersion Relations

PhD Defense

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Supervisor: Huaiyu Duan

Outline

1. Neutrino Oscillations
   1.1 Neutrinos as Fundamental Particles
   1.2 Why Do Neutrinos Oscillate

2. Neutrino Oscillations in Matter
   2.1 Matter Interactions and MSW Effect
   2.2 Neutrino Oscillations in Matter and Rabi Oscillations
   2.3 Multiple Frequencies in Matter Potential
   2.4 Summary of Neutrino Oscillations in Vacuum and Matter

3. Collective Oscillations
   3.1 Neutrino Self-interactions
   3.2 Linear Stability Analysis
   3.3 Dispersion Relations
   3.4 Summary of Collective Oscillations

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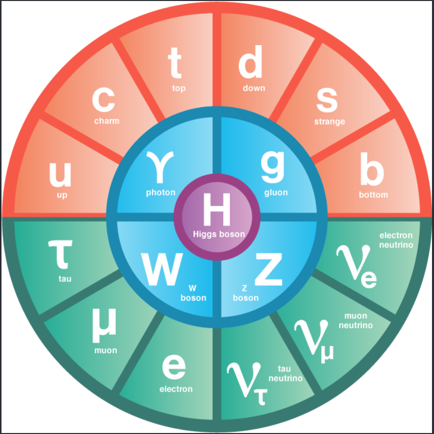
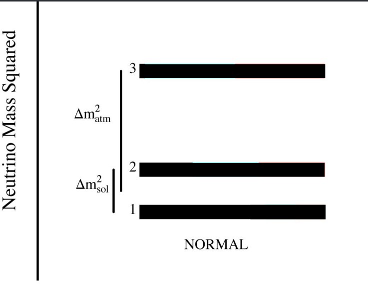
What are Neutrinos?



Elementary particles.
Source: symmetrymagazine.org

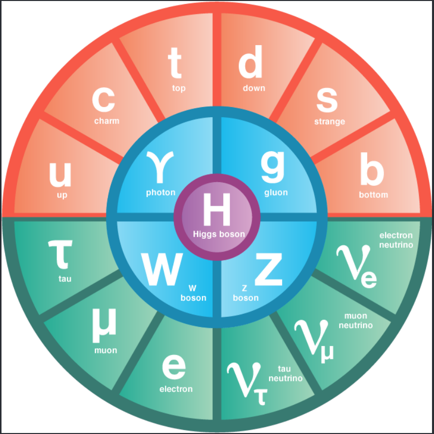
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- non-vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

What are Neutrinos?



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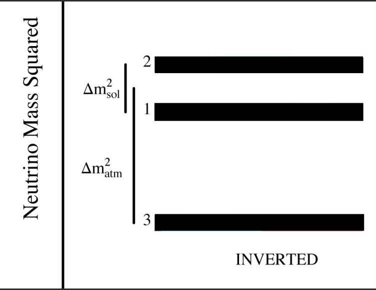

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Why Do Neutrinos Oscillate?

Two flavor senario

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

θ_v: vacuum mixing angle

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_v}{2} \left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \right)$$

- Mixing angle θ_v
- Oscillation frequency:

$$\omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

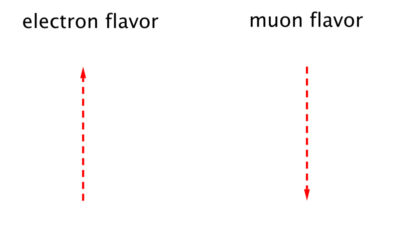
Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability:

$$P = \frac{1}{2} + s_3$$

Equation of motion:

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



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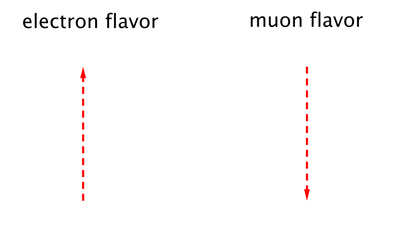
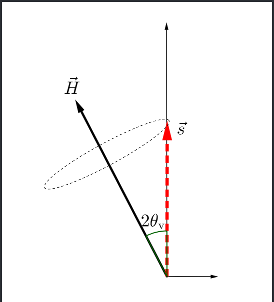
Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \ )$$

$$\to \cos 2\theta_v \begin{pmatrix} 0 \\ 0 \\ \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v \\ 0 \\ 0 \end{pmatrix}$$





Outline for Section 2

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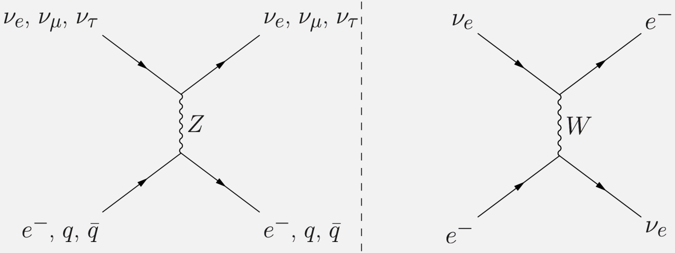
Matter Interaction



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-.

Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_\mathrm{v} = \delta m^2 / 2E$):

$$\mathbf{H} = \boxed{\frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\sigma_3 + \sin 2\theta_\mathrm{v}\sigma_1\right)} \boxed{+\frac{\lambda(x)}{2}\sigma_3}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2} G_\mathrm{F} n_\mathrm{e}(x)$

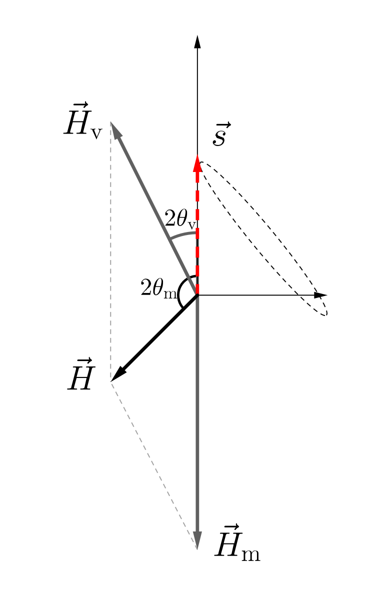
Matter Interaction

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

$$\rightarrow \omega_{\mathrm{v}}\begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

$$= \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)$$

Matter Interaction



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

$$\omega_\mathrm{v} = |\vec{H}_\mathrm{v}|$$

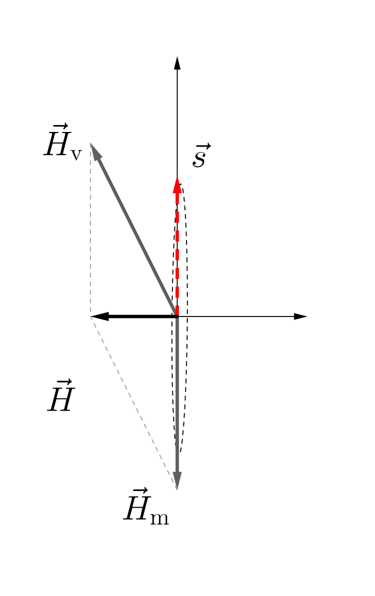
Oscillation frequency in **matter**:

$$\omega_\mathrm{m} = |\vec{H}|$$

Flavor states and mass states in matter

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{H} \end{pmatrix}$$

MSW Resonance



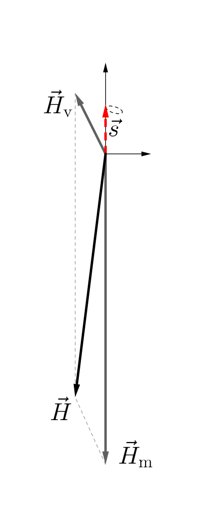
- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2}G_\mathrm{F} n_\mathrm{e} \equiv \omega_\mathrm{v} \cos 2\theta_\mathrm{v}$$

MSW Effect

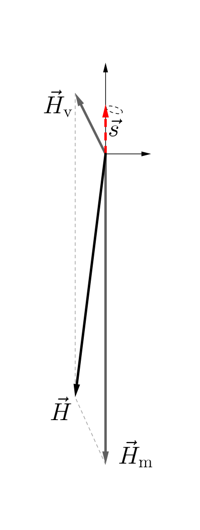
Adiabatic matter density change

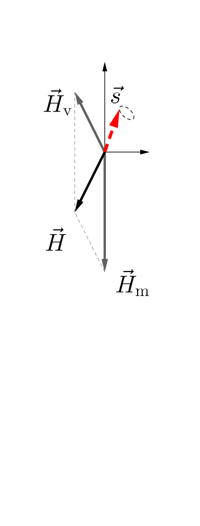
Large density



MSW Effect

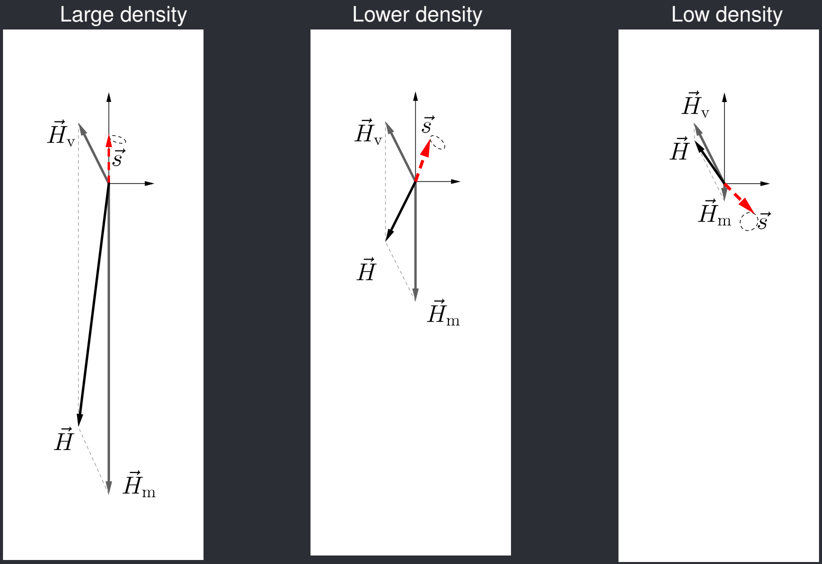
Adiabatic matter density change





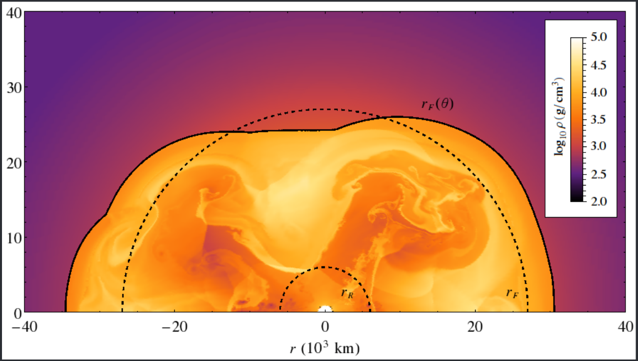
MSW Effect

Adiabatic matter density change



Supernova Matter Density Profile

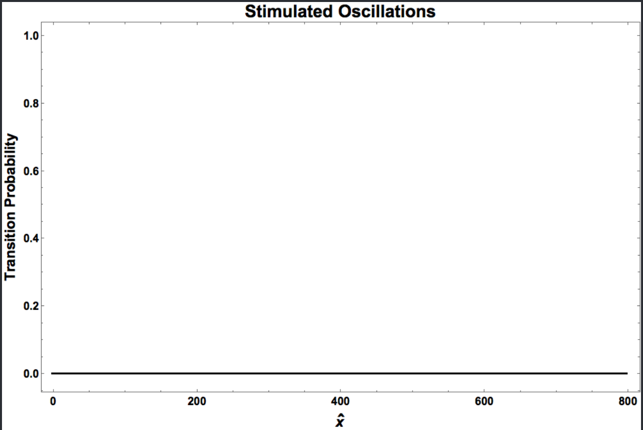
Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Neutrino Flavor Conversions in Matter

$$\lambda(x) = \lambda_0$$

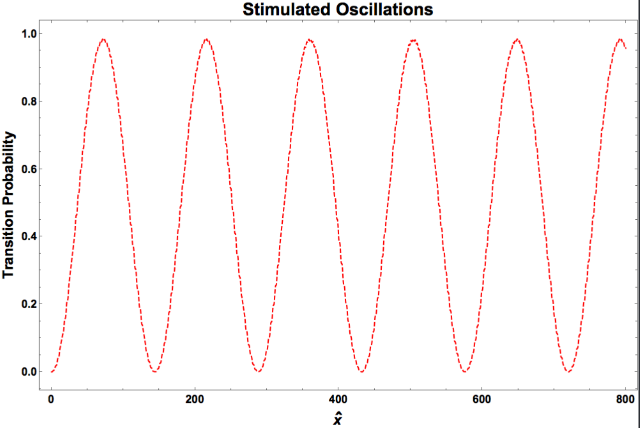


Transition probabilities between mass states in matter.

Neutrino Flavor Conversions in Matter

$A = 0.1\omega_m$
$k = 0.995\omega_m$
$\theta_m = \pi/6$

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



P. Krastev and A. Smirnov (1989); A. Friedland et al (2006); J. Kneller et al (2013); K. Patton et al (2014);

Rabi Oscillations

Rabi Oscillations

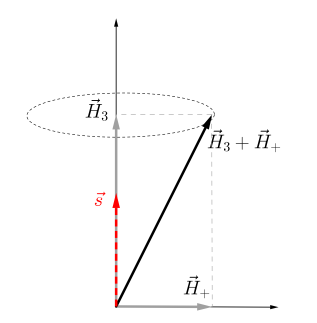
Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi Oscillations

Static Frame

$$\vec{H}_3 = \omega_\mathrm{m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Rabi Oscillations

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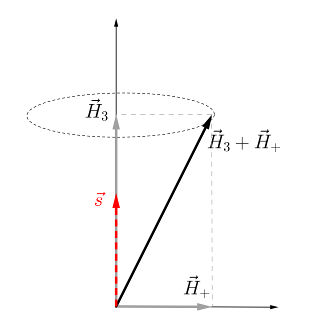
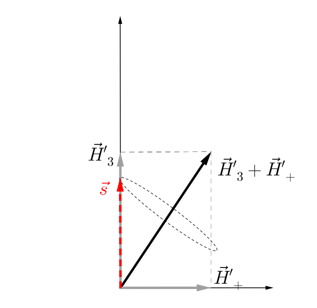


Corotating Frame

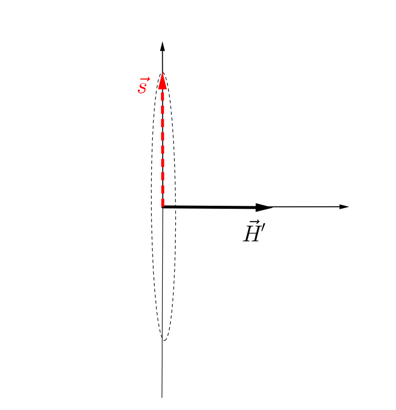
$$\vec{H}'_3 = (\omega_\mathrm{m} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_\mathrm{m} - k)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \Rightarrow k = \omega_\mathrm{m}$$



Rabi Oscillations

Rabi Oscillations

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = |\alpha|\sqrt{1+D^2}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0$$

Hamiltonian

matter basis:

$$\mathbf{H} = \frac{1}{2}(-\omega_{\mathrm{m}})\,\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_H\end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A \cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A \cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A \cos(kx)\sigma_1$$

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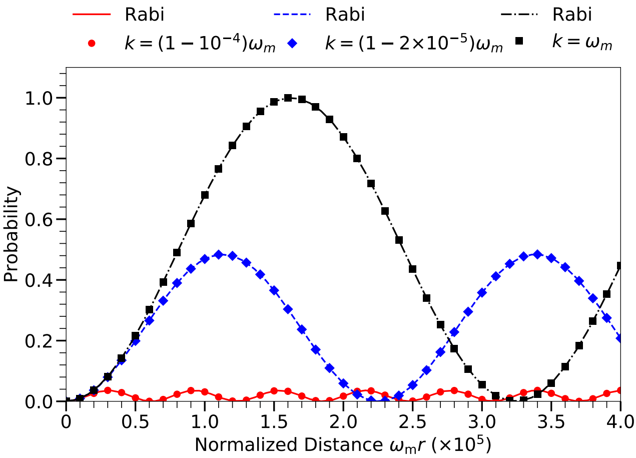
$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Rabi Formula Works



Transition between two mass states in background matter potential λ_0;
$A_1 = -10^{-4}\omega_\mathrm{m}$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{\mathrm{L}} \\ \tilde{\psi}_{\mathrm{H}} \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty}\begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_\mathrm{m} J_n(A\cos 2\theta_\mathrm{m}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty}\begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

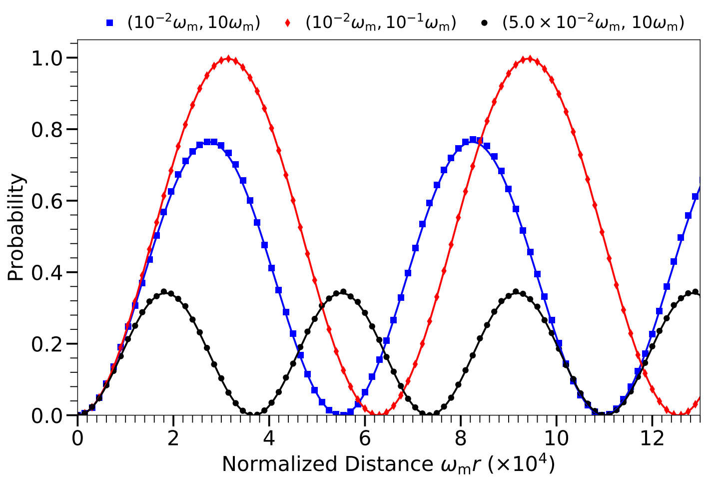
Resonance condition for each mode: $nk = \omega_{\mathrm{m}}$

Rabi Oscillations With Multiple Driving Frequencies

Relative detuning for two driving potentials, α_1, k_1 and α_2, k_2

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies



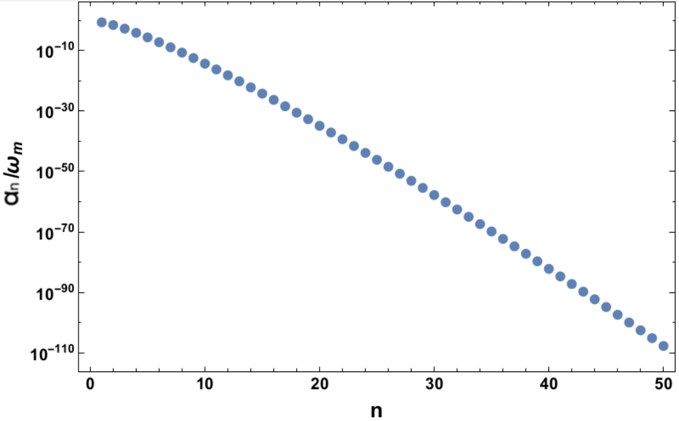
$A_1 = 10^{-4}\omega_m$, $k_1 = \omega_m$; Legend shows (A_2, k_2); Grid lines: amplitude predicted using $1/(1 + D'^2)$

Rabi Oscillations With Multiple Driving Frequencies

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
For $k_1 = \omega_\mathrm{m}$, survival of resonance requires

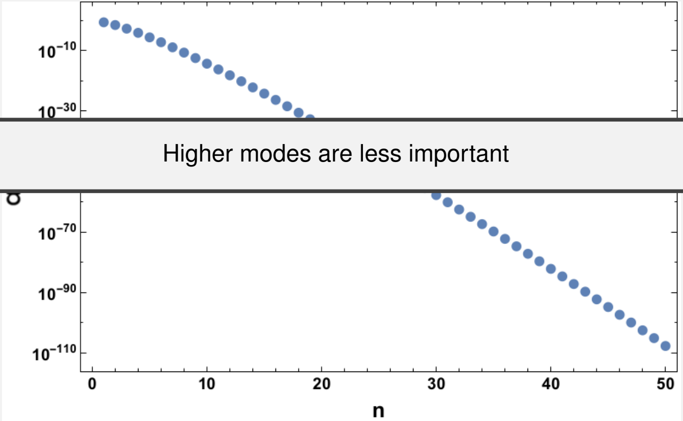
$$|\alpha_2| \ll \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential

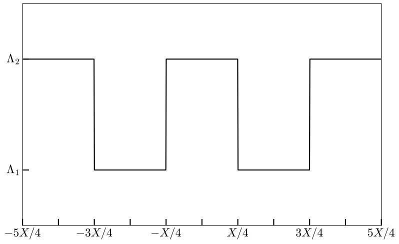


Higher modes are less important

Width of different modes given value of matter potential frequency k

Castle Wall Matter Potential

Rabi oscillation picture also works for matter potential with multiple frequencies.



Castle wall matter profile:
$\Lambda_2 = 0.7\omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}$
$\Lambda_1 = 0.3\omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}$

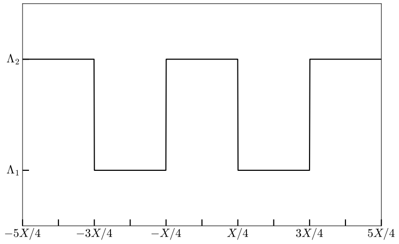
$$\lambda(x) = \lambda_0 + \sum_1^{\infty} \lambda_n \cos(k_n x)$$

where

$$\begin{aligned}\lambda_0 &= (\Lambda_1 + \Lambda_2)/2 \\ \lambda_n &= 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi) \\ k_n &= 2\pi(2n-1)/X\end{aligned}$$

Castle Wall Matter Potential

Rabi oscillation picture also works for matter potential with multiple frequencies.



Castle wall matter profile:
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$\Lambda_1 = 0.3\omega_v \cos 2\theta_v$

$$\lambda(x) = \lambda_0 + \sum_1^{\infty} \lambda_n \cos(k_n x)$$

where

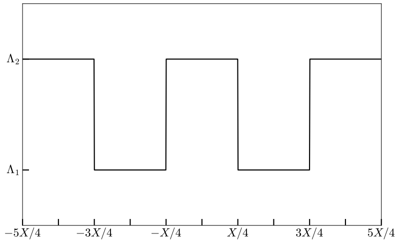
$$\lambda_0 = (\Lambda_1 + \Lambda_2)/2$$
$$\lambda_n = 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi)$$
$$k_n = 2\pi(2n-1)/X$$

Choose period $X = 2\pi/\omega_m$ so that

$$k_1 = \omega_m$$

Castle Wall Matter Potential

Rabi oscillation picture also works for matter potential with multiple frequencies.



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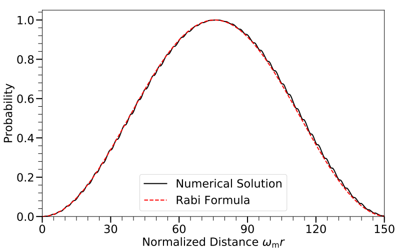
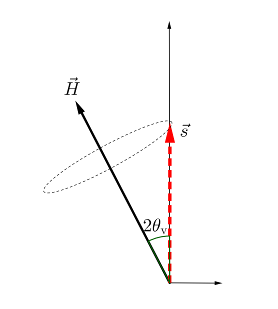


Transition probability is a Rabi resonance with small variations due to higher orders.

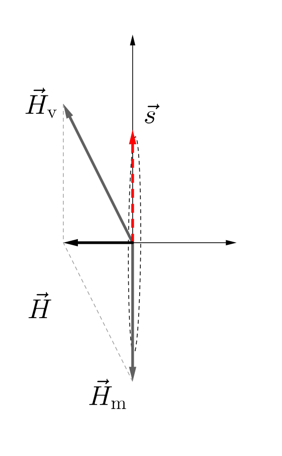
Summary of Neutrino Oscillations in Matter

1. Vacuum oscillations: flavor states are not mass states.



Summary of Neutrino Oscillations in Matter

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.



Summary of Neutrino Oscillations in Matter

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Neutrino oscillations in matter: variation in matter potential can cause resonances.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

Summary of Neutrino Oscillations in Matter

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Neutrino oscillations in matter: variation in matter potential can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

Outline for Section 3

1. Neutrino Oscillations
    1.1 Neutrinos as Fundamental Particles
    1.2 Why Do Neutrinos Oscillate

2. Neutrino Oscillations in Matter
    2.1 Matter Interactions and MSW Effect
    2.2 Neutrino Oscillations in Matter and Rabi Oscillations
    2.3 Multiple Frequencies in Matter Potential
    2.4 Summary of Neutrino Oscillations in Vacuum and Matter

3. Collective Oscillations
    3.1 Neutrino Self-interactions
    3.2 Linear Stability Analysis
    3.3 Dispersion Relations
    3.4 Summary of Collective Oscillations

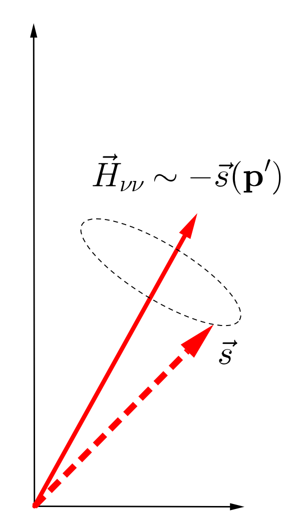
Neutrino Self-interactions

Interaction Hamiltonian $\mathbf{H}_{\nu\nu}$

$$\sqrt{2}G_F n(v')(1-\hat{v}\cdot\hat{v}')\rho(v')$$

In Flavor Isospin space

$$-2\sqrt{2}G_F n(v')(1-\hat{v}\cdot\hat{v}')\vec{s}(v')$$



Neutrino Self-interactions

Characteristic Energy Scales

- $\omega_{\rm v} = \delta m^2 / 2E$
- $\mu \sim G_{\rm F}(1 - \hat{v}_1 \cdot \hat{v}_2) n_\nu$

Vacuum oscillation frequencies

$$\omega_{\rm v} = \frac{\Delta m^2}{2E} \sim \frac{2\pi}{1\mathrm{km}} \left(\frac{\Delta m^2_{32}}{2.5 \times 10^{-3} \mathrm{eV}^2} \right) \left(\frac{1 MeV}{E} \right)$$

$$\sim \frac{2\pi}{33\mathrm{km}} \left(\frac{\Delta m^2_{12}}{7.5 \times 10^{-5} \mathrm{eV}^2} \right) \left(\frac{1\mathrm{MeV}}{E} \right)$$

Neutrino Self-interactions

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- $\omega_\mathrm{v} = \delta m^2 / 2E$
- $\mu \sim G_\mathrm{F}(1 - \hat{v}_1 \cdot \hat{v}_2) n_\nu$

Suppose we have neutrino flux $10^{50}\mathrm{ergs} \cdot \mathrm{s}^{-1}$. We estimate the potential at radius R to be

$$\mu \sim \frac{1}{0.01km} \left(\frac{100\mathrm{km}}{R} \right)^2 \left(\frac{1\mathrm{MeV}}{E} \right)$$

Two-Beam Model

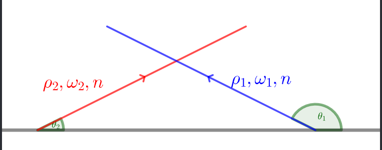
$$H_{\mathrm{v},1} = -\frac{1}{2}\omega_{\mathrm{v}}\sigma_3$$
$$H_{\mathrm{v},2} = \frac{1}{2}\omega_{\mathrm{v}}\sigma_3$$
$$H_{\nu\nu} = \frac{1}{2}(\mu\rho_1 - \mu\rho_2)$$

where

$$\mu = \sqrt{2}G_{\mathrm{F}}\xi n$$

Geometric factor

$$\xi = (1 - \cos(\theta_1 - \theta_2))$$
$$= 3/2$$

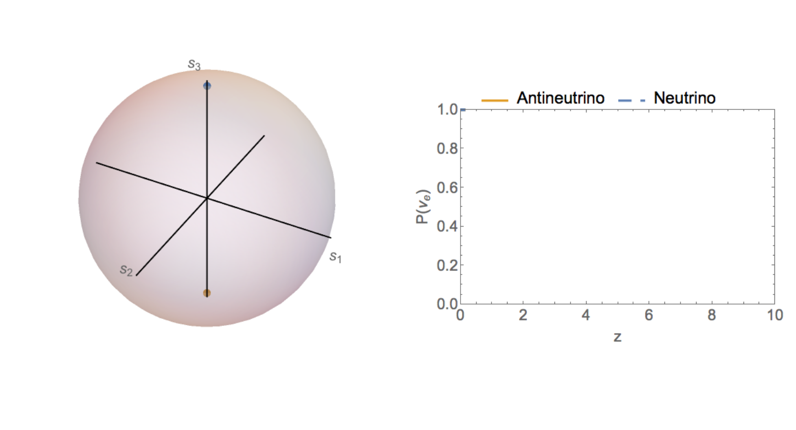


ρ_1: neutrinos; ρ_2: antineutrinos $\theta_1 = 5\pi/6$; $\theta_2 = \pi/6$

$$\theta_{\mathrm{v}} = 0$$

Neutrino Self-interactions

Inverted hierarchy;
$\mu = 5|\omega_{\mathrm{v}}|$



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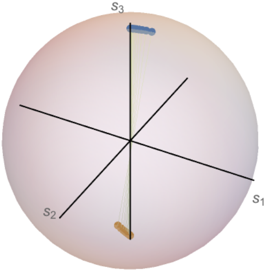

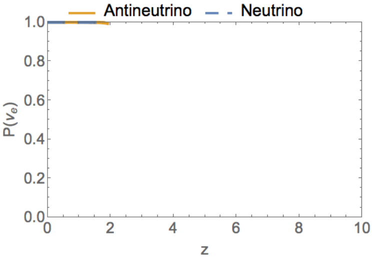

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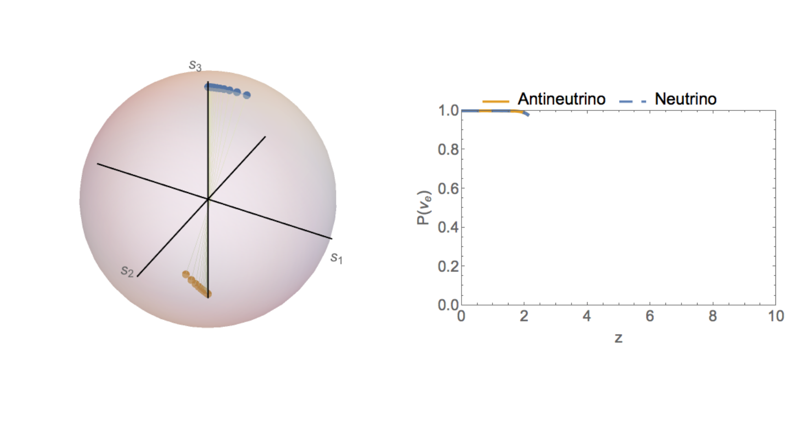

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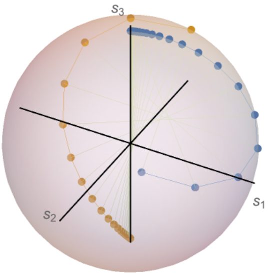

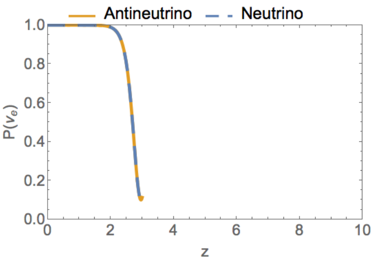

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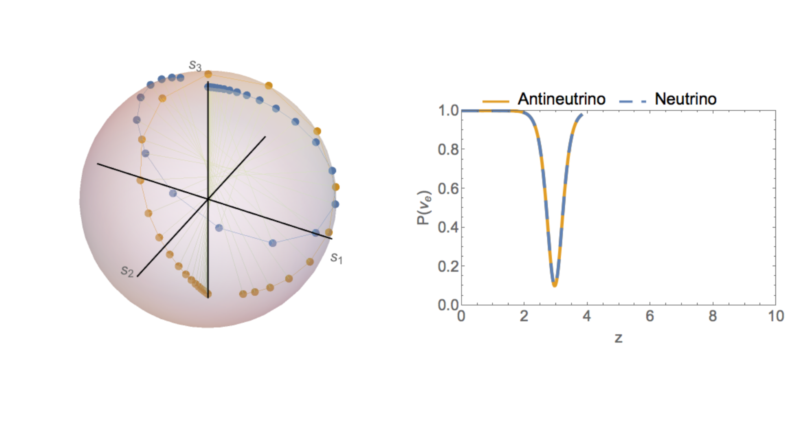

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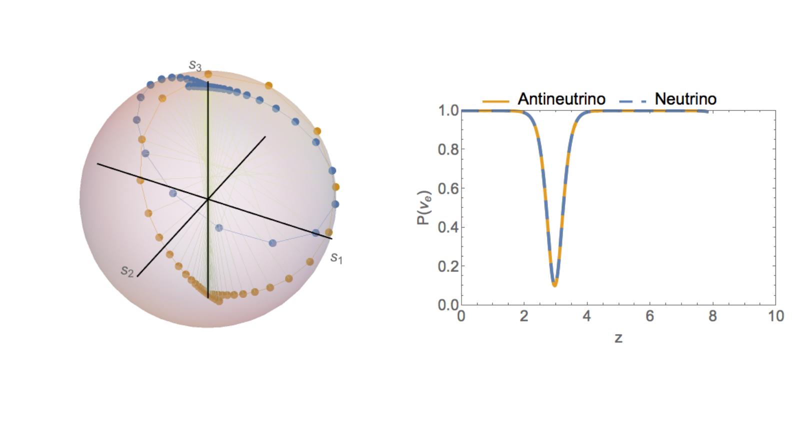

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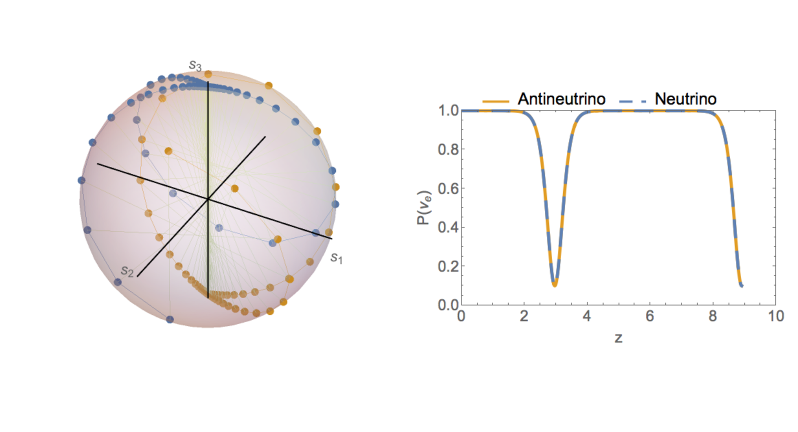

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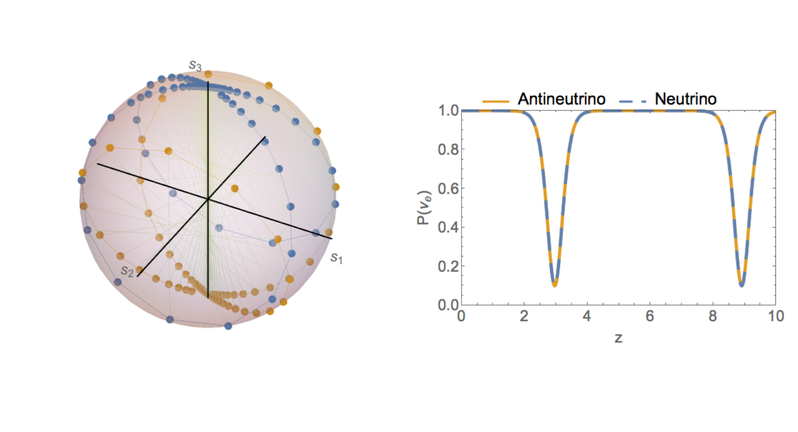
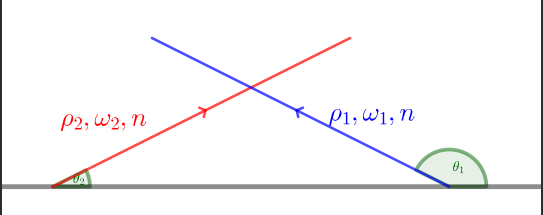


Linear Stability Analysis



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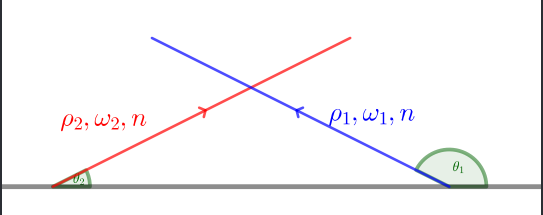

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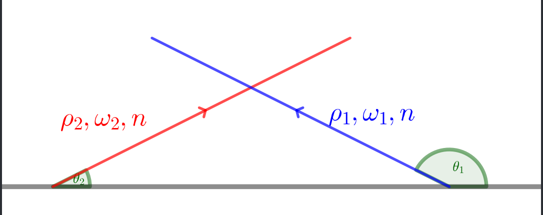

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$$i\partial_z \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} \mu/2 + \omega_v & -\mu/2 \\ \mu/2 & -\omega_v - \mu/2 \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Linear Stability Analysis

Collective mode

$$\begin{pmatrix} \epsilon_1(z) \\ \epsilon_2(z) \end{pmatrix} = \begin{pmatrix} \epsilon_1(0) \\ \epsilon_2(0) \end{pmatrix} e^{iK_z z}$$

Eigenvalues or collective oscillation frequencies

$$K_z = \pm\sqrt{\omega_{\mathrm{v}}(\omega_{\mathrm{v}} + \mu)}$$

Identify the condition for complex eigenvalues

$$\omega_{\mathrm{v}}(\omega_{\mathrm{v}} + \mu) < 0$$

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Similar analysis can be done for all four dimensions t, x, y, z,

$$(\Omega, K_x, K_y, K_z)$$

Dispersion Relation

Izaguirre, I., Raffelt, G., & Tamborra, I. (2017). *Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach*. Physical Review Letters, 118(2), 021101.

- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.

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- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.
- Instabilities and dispersion relation gaps are related.

Dispersion Relations

Equation of motion for off-diagonal element of density matrix (Izaguirre et al, 2017)

$$i(\partial_t + v \cdot \nabla_r)\epsilon(v) = v^{\mu}\left(\Lambda + \Phi\right)_{\mu} - \int d\Gamma' v^{\mu} v'_{\mu} G(v')\epsilon(v')$$

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- $\boxed{v^\mu}$: four-velocity of neutrinos $(1, \mathbf{v})$
- $\boxed{\Lambda}$: matter contribution $(\sqrt{2}G_\mathrm{F}n_\mathrm{e}, \sqrt{2}G_F n_e \mathbf{v}_\mathrm{e})$
- $\boxed{\Phi}$: neutrino flux $(\sqrt{2}G_\mathrm{F}n_\nu, \sqrt{2}G_F n_\nu \mathbf{v})$
- $\boxed{G(\mathbf{v}')}$: electron lepton number $\sqrt{2}G_\mathrm{F}\int_0^\infty \frac{E^2dE}{2\pi^2}\left(n_{\nu_\mathrm{e}} - n_{\bar{\nu}_\mathrm{e}}\right)$

Dispersion Relations

Collective mode of off-diagonal element

$$\epsilon \longrightarrow \tilde{\epsilon} e^{-i(\Omega t - \boldsymbol{K}\cdot\boldsymbol{r})}$$

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$$v^{\mu}(K_{\mu} - (\Lambda + \Phi)_{\mu})\tilde{\epsilon}(\boldsymbol{v}) = -\int d\Gamma' v^{\mu} v'_{\mu} G(\boldsymbol{v}')\tilde{\epsilon}(\boldsymbol{v}')$$

with $K_{\mu} \longrightarrow (\Omega, \boldsymbol{K})$

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Without neutrino self-interaction: $v^\mu k_\mu = 0$

Dispersion Relations

Rewrite

$$
\begin{aligned}
& -\int d\Gamma' \, v^{\mu} v'_{\mu} G(v')\tilde{\epsilon}(v') \\
= \; & v^{\mu} \left(-\int d\Gamma' \, v'_{\mu} G(v')\tilde{\epsilon}(v') \right) \\
\equiv \; & v^{\mu} a_{\mu}
\end{aligned}
$$

Dispersion Relations

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EoM

$$
v^\mu k_\mu \tilde{\epsilon}(v) = v^\mu a_\mu
$$

Dispersion Relations

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EoM

$$v^\mu k_\mu \tilde{\epsilon}(v) = v^\mu a_\mu$$

$$\Longrightarrow$$

$$\tilde{\epsilon}(v) = v^\mu a_\mu / v^\mu k_\mu$$

Collect all terms of a_μ

$$v^\mu \left(\delta^\nu_\mu + \int d\Gamma' \frac{G(v')v'_\mu v^\nu}{v^\alpha k_\alpha}\right) a_\nu = 0$$

Dispersion Relations

Axially Symmetric: $v^\alpha k_\alpha = \omega(1 - n\cos\theta)$ where $n = |\boldsymbol{k}|/\omega$

Condition for nontrivial solutions of a_μ,

$$v^\mu \left(\omega \delta^\nu{}_\mu + N^\nu{}_\mu\right) a_\nu = 0$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

$$N^\mu{}_\nu \to$$

$$\begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 & -\frac{1}{2}I_1 \\ 0 & -\frac{1}{4}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(I_0 - I_2) & 0 \\ \frac{1}{2}I_1 & 0 & 0 & -\frac{1}{2}I_2 \end{pmatrix}$$

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$$\Rightarrow$$

$$\left(\omega\delta^\nu{}_\mu + N^\nu{}_\mu\right) a_\nu = 0$$

$$\Rightarrow$$

$$\mathrm{Det}\,(\omega I + N) = 0,$$

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Dispersion Relations

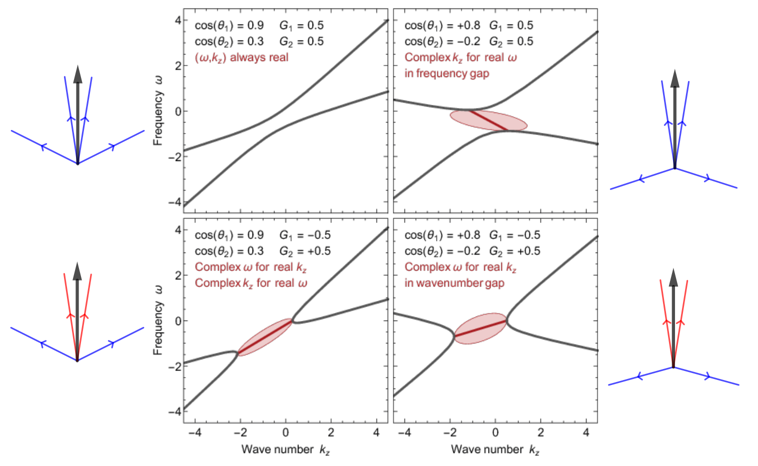
$$a_\mu = -\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')$$

$$\omega = \boxed{\frac{1}{4}(I_0 - I_2)}, \qquad \boxed{-\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)}$$

- Axial Symmetry Breaking solution
- Axially Symmetric solution

Dispersion Relations and Instabilities

Modified version of Fig. 1 in Izaguirre et al, 2017;



Two zenith angles

Dispersion Relations and Instabilities

Izaguirre, I., Raffelt, G., & Tamborra, I. (2017). *Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach*. Physical Review Letters, 118(2), 021101.

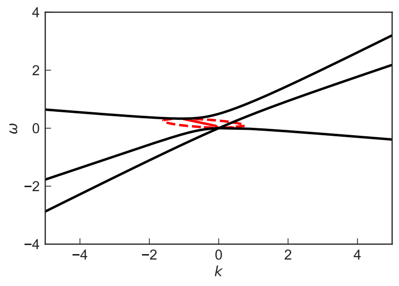
The left panels use forward modes ($0 < \cos\theta_{1,2} < 1$) as in traditional bulb emission. If ν_e dominates in both modes (upper left), both ω and k_z are real: no fast flavor conversion occurs. If one mode has a $\bar{\nu}_e$ excess ($G_1 < 0$), the DR has a gap, providing complex ω for real k_z and the other way around, as indicated by the red blob. Disturbances with k_z in the gap grow exponentially in time. A real ω imposed at the boundary causes exponential spatial growth. These conclusions carry over to more general $G(\theta)$ where one needs a crossing from positive to negative ELN intensities to obtain a dispersion gap, which, in turn, enables fast flavor conversion, similar to spectral crossings for slow modes [40–42].

The DR alone only indicates which solutions are consistent with the EOM, but not which ones will actually occur. We would be sure that the system was always stable if the DR did not have any gaps, which, however, seem to be generic. Except for quantum fluctuations or hypothetical flavor-violating interactions [46–48], M^2 is the only source of seed perturbations. However, which spectrum of flavor disturbances is produced, and where, remains to be better understood.

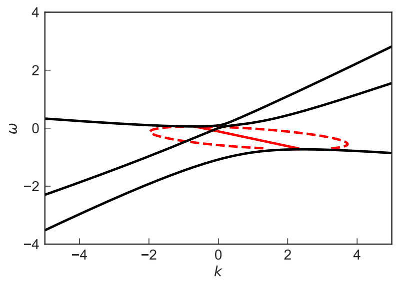
1. Gaps lead to Instabilities.
2. Instabilities do not occur without gap.

Dispersion Relations and Instabilities

Three zenith angles



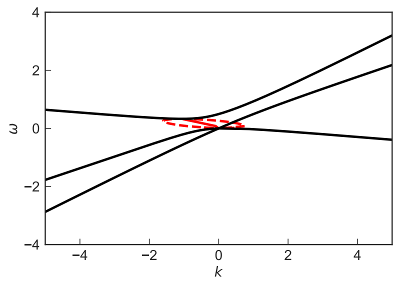
Axial Symmetry Breaking solutions



Axially Symmetric solution

Dispersion Relations and Instabilities

Three zenith angles



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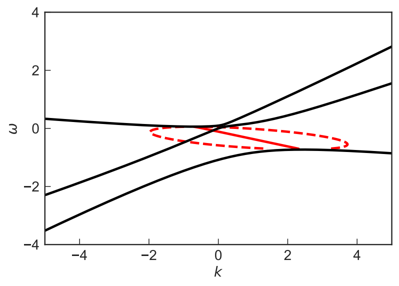

Axially Symmetric solution

Cubic equation of $k = |\boldsymbol{k}| \Rightarrow$ 3 solutions of k for given ω

Dispersion Relations and Instabilities

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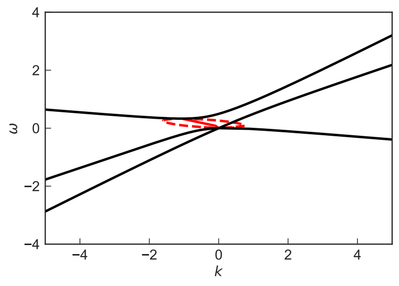

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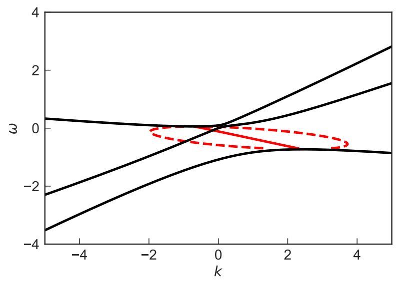

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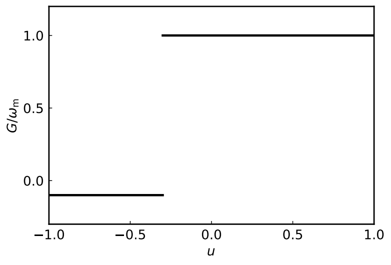
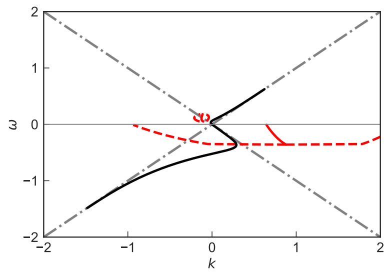
Instabilities occur without gaps.

Dispersion Relations and Instabilities

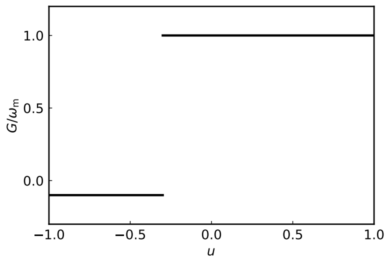


Box spectrum: -0.1 for $u \in [-1, -0.3)$; 1 for $u \in [-0.3, 1]$



Axially Symmetric solution: no gaps yet unstable in some regions

Dispersion Relations and Instabilities



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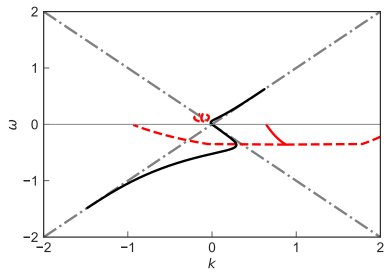

Axially Symmetric solution: no gaps yet unstable in some regions

Instabilities occur without gaps.

Summary of Dispersion Relations

- Neutrino oscillation instabilities might occur in DR gaps.
- Neutrino oscillation instabilities might occur even if DR has no gaps.

Acknowledgement

I am very thankful to my advisor Professor Huaiyu Duan, as well as my colleagues Dr. Sajad Abbar, Dr. Shashank Shalgar, and Joshua Martin, for all the help in both research and life.

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Backup Slides

Parameters

Vacuum oscillations: $\sin^2\theta_v = 0.093$

Bipolar model animation:

- $\theta_v = 0$
- $\alpha = 1$
- $\mu = 5$

Initial condition

- $$\vec{s} = \begin{pmatrix} 10^{-3} \\ 0 \\ 1 \end{pmatrix}$$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}}$$

Corotating frame of the second potential

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^{N} A_a \sin(k_a x)$$

Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\begin{pmatrix} 0 & B_{\{n_a\}} e^{i\sum_a n_a k_a x} \\ B^*_{\{n_a\}} e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a\right)\left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right)$$

Axial Symmetries

$$\tilde{\epsilon} = \frac{v^{\mu} a_{\mu}}{v^{\alpha} k_{\alpha}}$$

Dispersion Relations

Solve k for Axial Symmetry Breaking solutions

$$1 = \frac{1}{4k} \int \mathrm{d}u G(u) \frac{1-u^2}{\omega/k - u}.$$

around $\omega \to 0$.

Dispersion Relations

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Apply Stokhotski-Plemelj theorem

$$\mathrm{Re}(k) = \frac{1}{4} \left(\mathscr{P} \int \mathrm{d}u G(u) \frac{1-u^2}{-u} \right)$$
$$\mathrm{Im}(k) = \frac{\pi}{4} G(0)\, \mathrm{Sign}(\omega)\, \mathrm{Sign}(\mathrm{Im}(k)).$$

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- $G(0)\,\mathrm{Sign}(\omega) > 0$: $|\mathrm{Im}(k)| = \frac{\pi}{4}|G(0)|$
- $G(0)\,\mathrm{Sign}(\omega) < 0$: $|\mathrm{Im}(k)| = 0$

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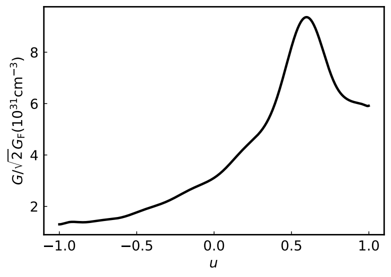
Gap between dispersion relation and $\omega = 0$

Dispersion Relations and Instabilities

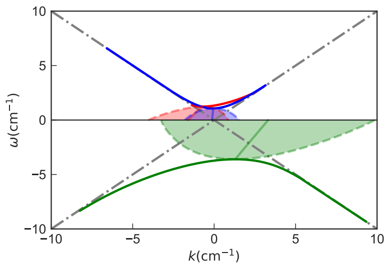
Remake of Fig.3 of Izaguirre et al, 2017

Define $u = \cos\theta$

Garching spectrum:



Garching spectrum $G(u)$



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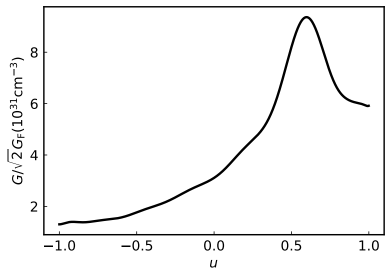

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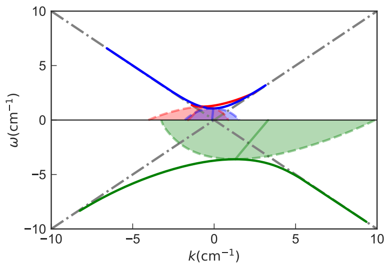

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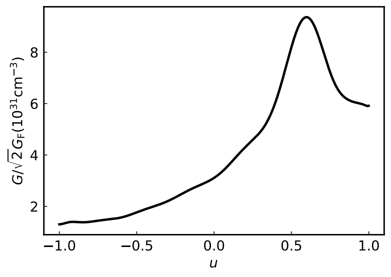

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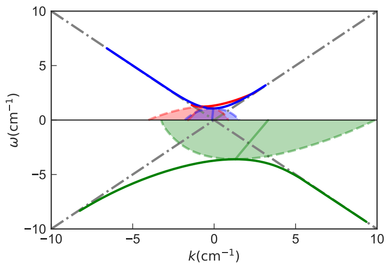

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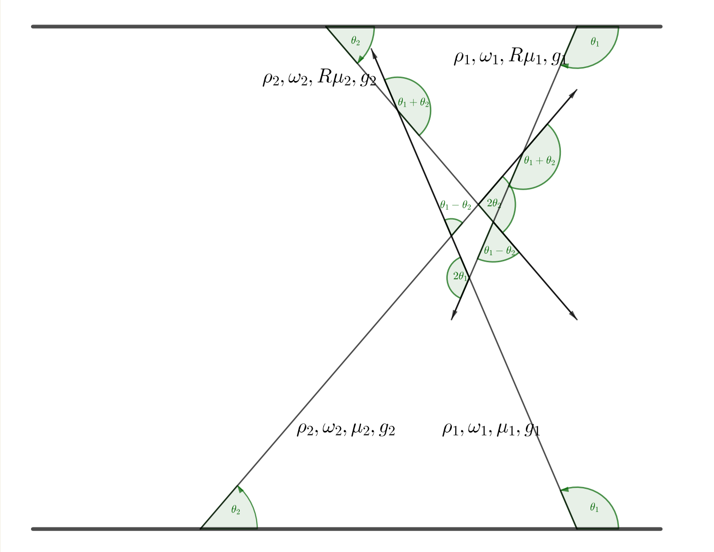
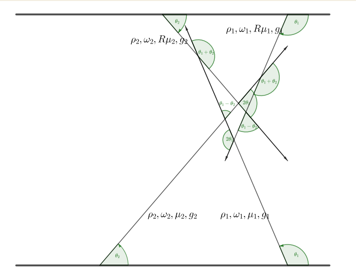
Neutrino Halo



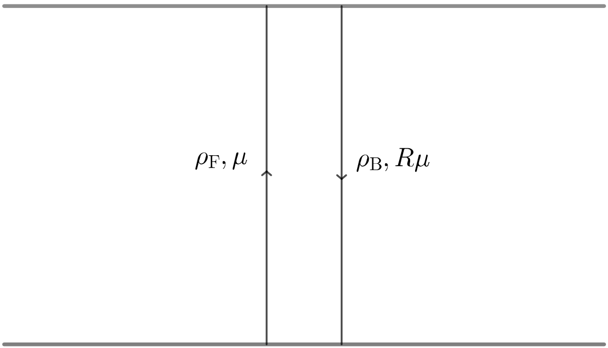
Neutrino Halo

Assumptions

- Neutrinos are translational symmetric on the emission line.
- Reflection obays Snell's law.
- Neutrinos are reflected on a fixed surface $z = L$.
- Neutrino reflections are translational symmetric.



Flavor Isospin

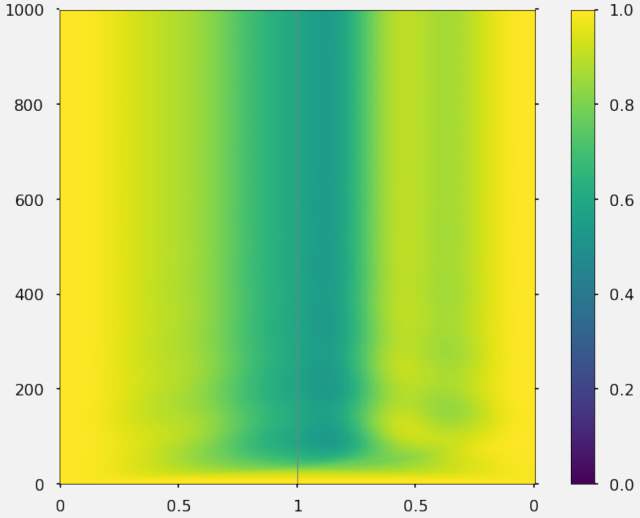


Relaxation Scheme

Algorithm

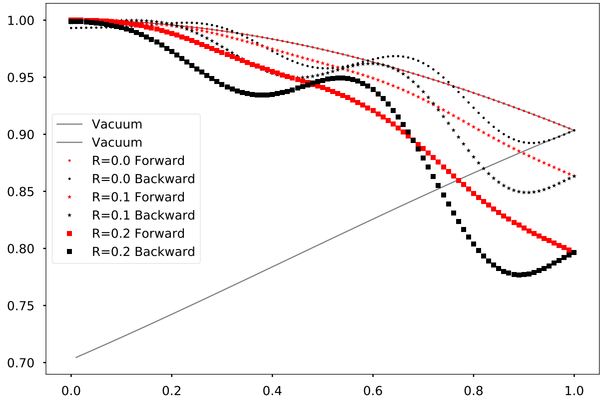
1. Calculate forward beam using null backward beam;
2. Calculate backward beam using forward beam calculated in step 1;
3. Calculate forward beam using backward beam calculated in step 2;
4. Repeat 2 and 3 until the beams reach equilibrium.

Numerical Method



Horizontal axis is the location of neutrinos; Vertical axis is the number of iteration steps; Color indicates the electron flavor probability.

Numerical Method



Linear Stability Analysis

EoM

$$
\begin{aligned}
i\partial_t \vec{s}_F &= \mathrm{s}_F \times (\vec{H}_v + R\mu\vec{s}_B) \\
i\partial_t \vec{s}_B &= \vec{s}_B \times (-\vec{H}_v - \mu\vec{s}_F).
\end{aligned}
$$

Compare with bipolar

$$
\begin{aligned}
i\partial_t \vec{s} &= \mathrm{s} \times (\eta\vec{H}_v + \alpha\mu\bar{\vec{s}}) \\
i\partial_t \bar{\vec{s}} &= \bar{\vec{s}} \times (\eta\vec{H}_v + \mu\vec{s})
\end{aligned}
$$

Linear Stability Analysis