Neutrino Flavor Conversions in Dense Medium: Matter Stimulation, Dispersion Relation, and Neutrino Halo

PhD Defense

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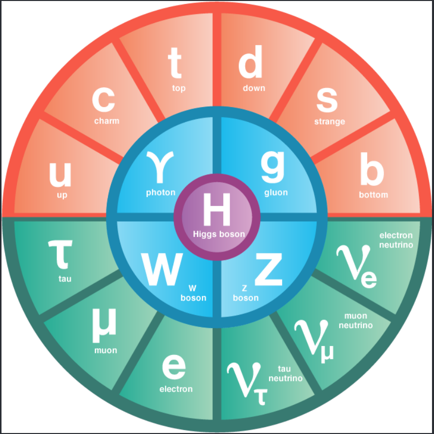
Outline for Section 1

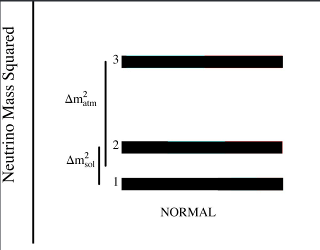
What are Neutrinos?



Elementary particles.
Source: symmetrymagazine.org

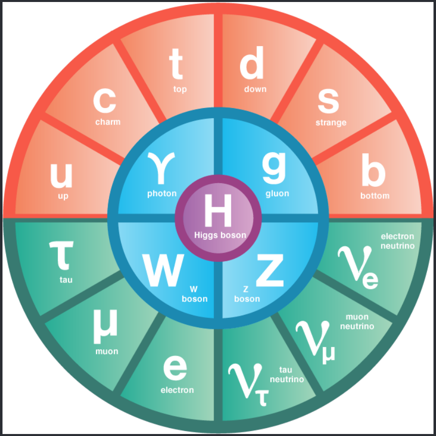
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- none vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

What are Neutrinos?



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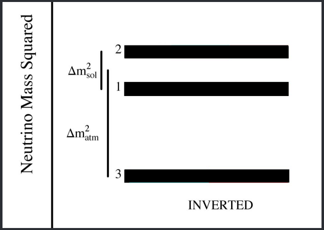

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Why Do Neutrinos Oscillate?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_v}{2}\left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1\right)$$

- Oscillation frequency:

$$\omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_v

Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

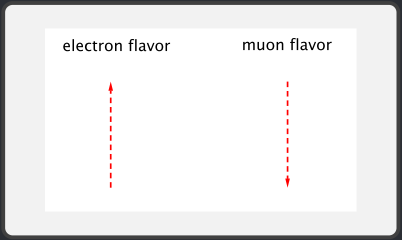
Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



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Equation of motion

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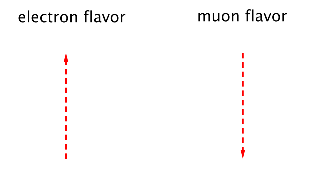
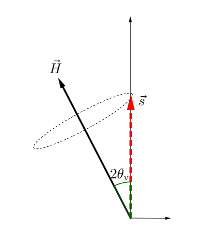
Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2}(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1)$$

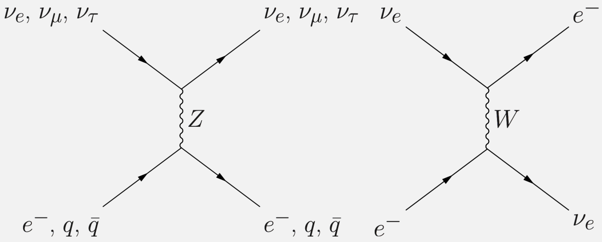
$$\rightarrow \cos 2\theta_v \begin{pmatrix} 0 \\ 0 \\ \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v \\ 0 \\ 0 \end{pmatrix}$$





Outline for Section 2

Matter Interaction



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, etc.

Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_\mathrm{v} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma_3} + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma_1}\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_\mathrm{F}n_\mathrm{e}(x)$

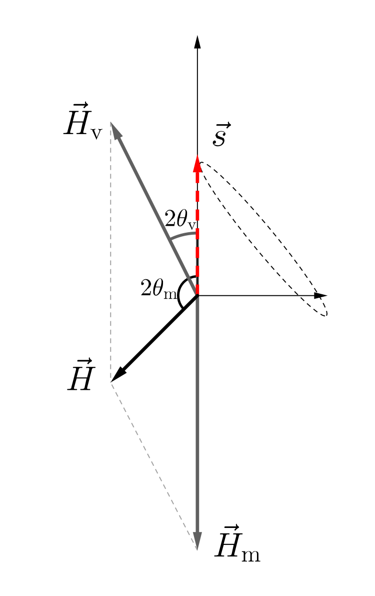
MSW Effect

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

$$\to \omega_{\mathrm{v}} \begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

$$= \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)$$

MSW Effect



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

$$\omega_{\mathrm{v}} = |\vec{H}_{\mathrm{v}}|$$

Oscillation frequency in **matter**:

$$\omega_{\mathrm{m}} = |\vec{H}|$$

Flavor states and mass states in matter

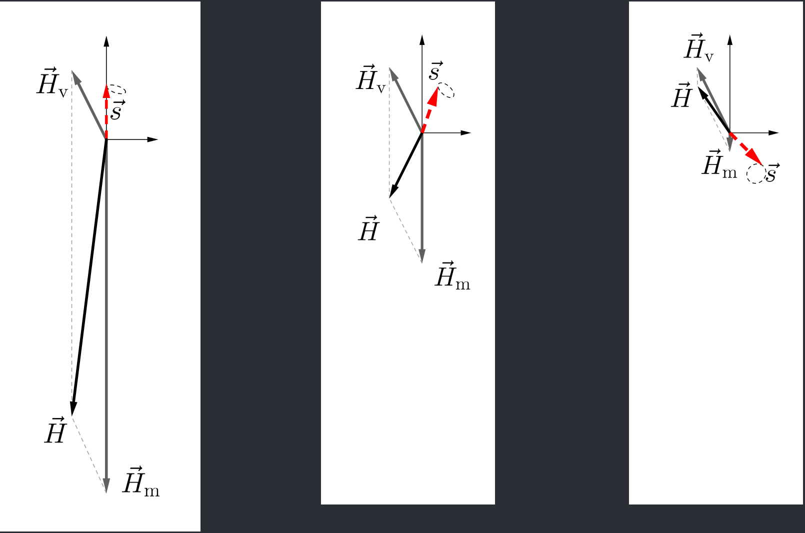
$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & \sin\theta_{\mathrm{m}} \\ -\sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix}$$

MSW Effect

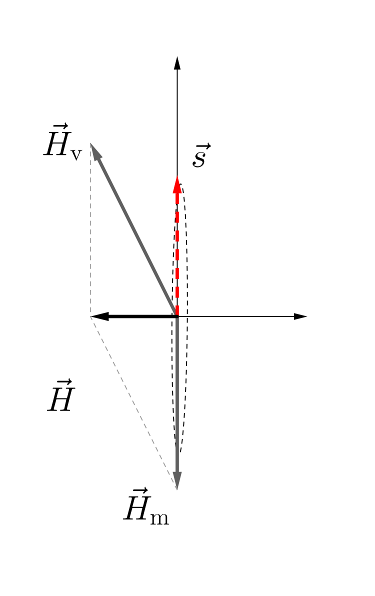
Adiabatic matter density change

Large density

Low density



MSW Effect

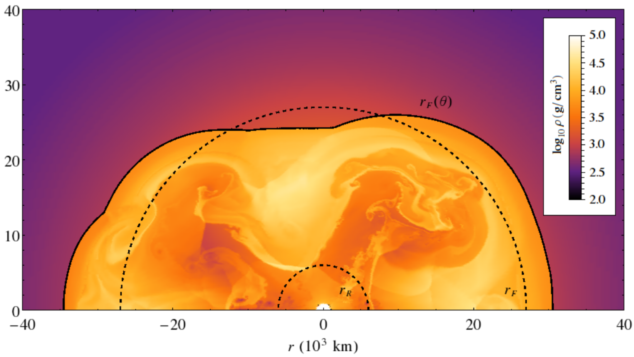


- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2} G_{\mathrm{F}} n_{\mathrm{e}} \equiv \omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}$$

Supernova Matter Density Profile

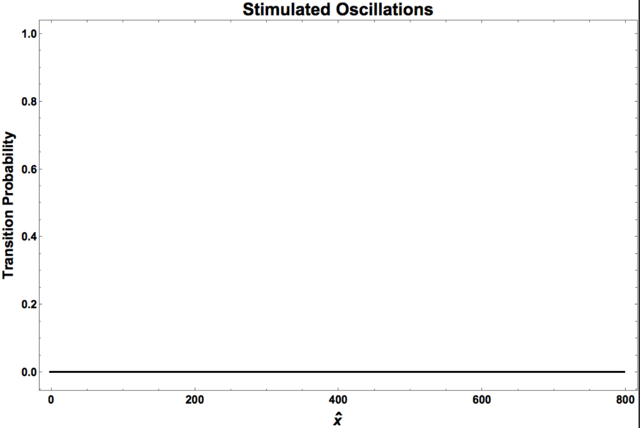
Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

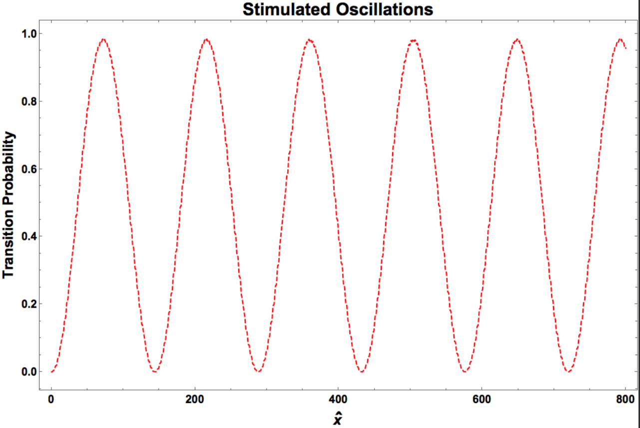
$$\lambda(x) = \lambda_0$$



Transition probabilities between mass states in matter.

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



P. Krastev and A. Smirnov (1989); A. Friedland et al (2006); J. Kneller et al (2013); K. Patton et al (2014);

Rabi Oscillations

Rabi Oscillation

Hamiltonian

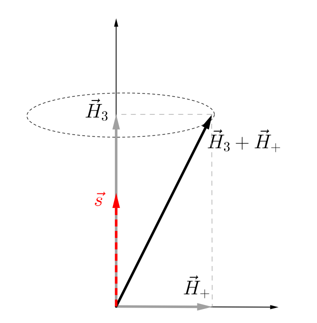
$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3-\frac{\alpha}{2}\begin{pmatrix}0 & e^{ikt}\\ e^{-ikt} & 0\end{pmatrix}$$

Scheme

Rabi Oscillations

Static Frame

$$\vec{H}_3 = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Rabi Oscillations

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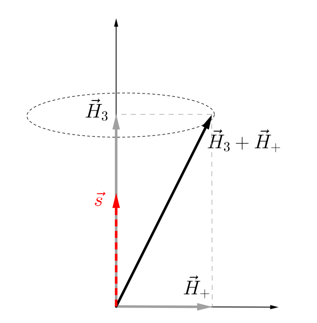
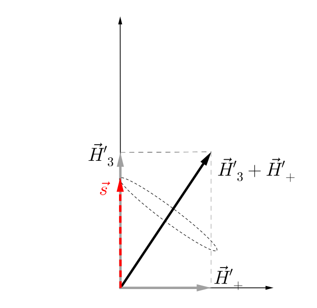


Corotating Frame

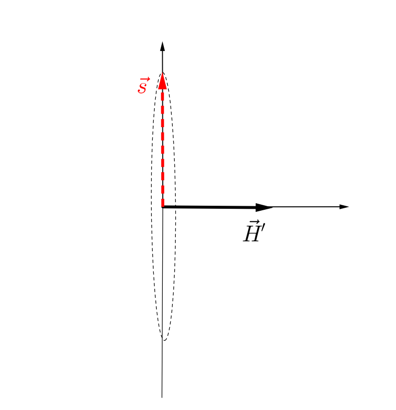
$$\vec{H}'_3 = (\omega_\mathrm{m} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_{\mathrm{m}} - k)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow k = \omega_{\mathrm{m}}$$



Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_\mathrm{m}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Scheme

Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_\mathrm{R}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_\mathrm{m} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_\mathrm{R} = |\alpha|\sqrt{1+D^2}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}}\right)\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_H\end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_m$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_m + \cancel{\cos 2\theta_m A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_m}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_m}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

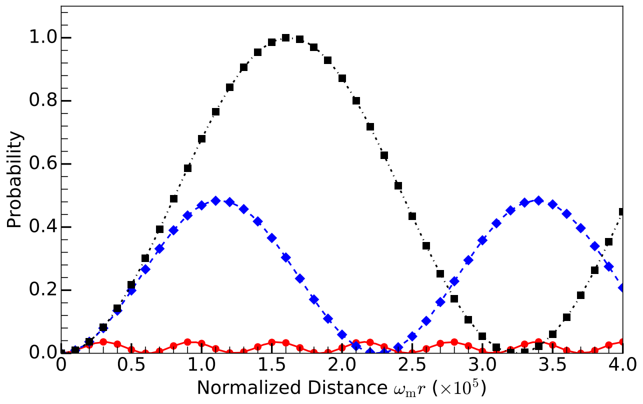
$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \cancel{\alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Rabi Formula Works



Transition between two mass states in background matter potential λ_0
Lines: Rabi formula
Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_{\mathrm{m}}$, $k = (1 - 2 \times 10^{-5})\omega_{\mathrm{m}}$, and $k = (1 - 10^{-4})\omega_{\mathrm{m}}$ respectively.

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \cancel{\alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}}$$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}(-\omega_\mathrm{m} + A\cos(kx)\cos 2\theta_\mathrm{m})\boldsymbol{\sigma}_3 - \frac{A\cos(kx)}{2}\sin\theta_\mathrm{m}\boldsymbol{\sigma}_1.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{H} \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_\mathrm{L} \\ \tilde{\psi}_\mathrm{H} \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_\mathrm{m}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

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where $\alpha_n = -(-i)^n nk \tan 2\theta_\mathrm{m} J_n(A\cos 2\theta_\mathrm{m}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Resonance condition for each mode: $nk = \omega_\mathrm{m}$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

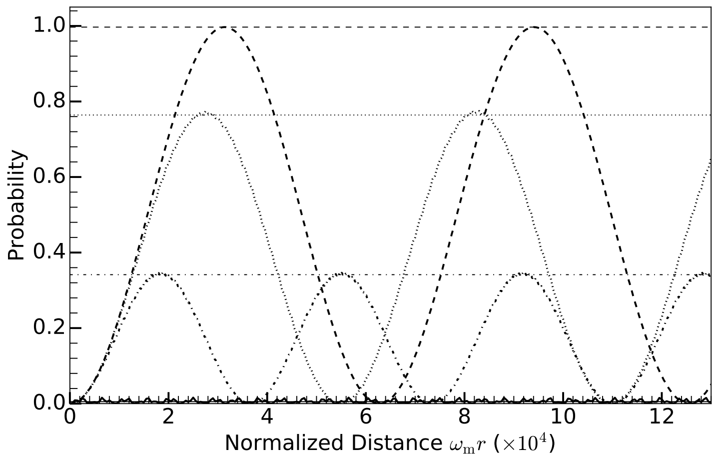
Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_\mathrm{m} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_\mathrm{m} - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$



Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_m, 10\omega_m$	$10^{-2}\omega_m, 10^{-1}\omega_m$	$5.0 \times 10^{-2}\omega_m, 10\omega_m$	$5 \times 10^{-2}\omega_m, 10^{-1}\omega_m$

Rabi Oscillations With Multiple Driving Frequencies

Consider $k_1 = \omega_\mathrm{m}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_\mathrm{m} - k_2)} \right|$$

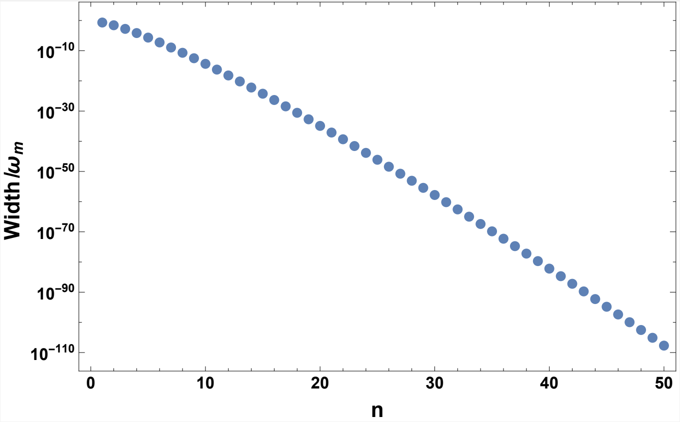
Amplitude reduces from 1 to 1/2 if

$$D' = 1 \Rightarrow \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
For $k_1 = \omega_\mathrm{m}$, survival of resonance requires

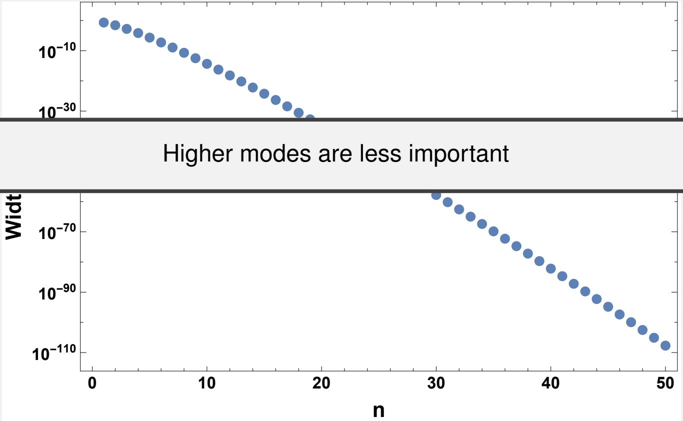
$$|\alpha_2| \ll \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential



Higher modes are less important

Width of different modes given value of matter potential frequency *k*

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^{N} A_a \sin(k_a x)$$

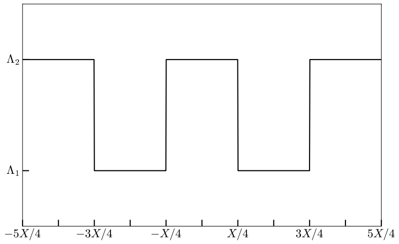
Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \begin{pmatrix} 0 & B_{\{n_a\}} e^{i\sum_a n_a k_a x} \\ B^*_{\{n_a\}} e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a\right) \left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right)$$

Castle Wall Matter Potential



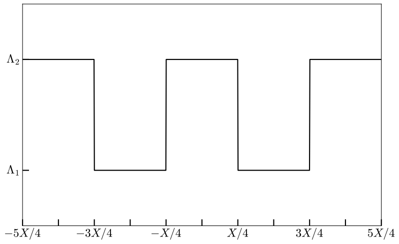
Castle wall matter profile:
$\Lambda_2 = 0.35\omega_{\rm v}\cos 2\theta_{\rm v}$,
$\Lambda_1 = 0.15\omega_{\rm v}\cos 2\theta_{\rm v}$ and period
$X = 2\pi/\omega_{\rm m}$

$$\lambda(x) = \lambda_0 + \sum_1^\infty \lambda_n \cos(k_n x)$$

where

$$\lambda_0 = (\Lambda_1 + \Lambda_2)/2$$
$$\lambda_n = 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi)$$
$$k_n = 2\pi(2n-1)/X$$

Castle Wall Matter Potential



Castle wall matter profile:
$\Lambda_2 = 0.35\omega_\mathrm{v} \cos 2\theta_\mathrm{v}$,
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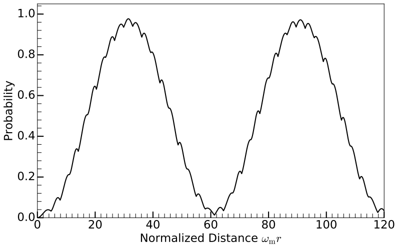
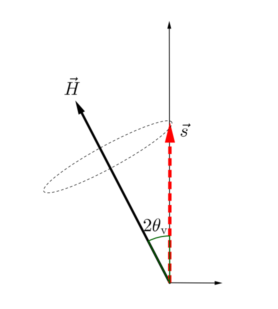


Transition probability is a Rabi resonance with small variations due to higher orders.

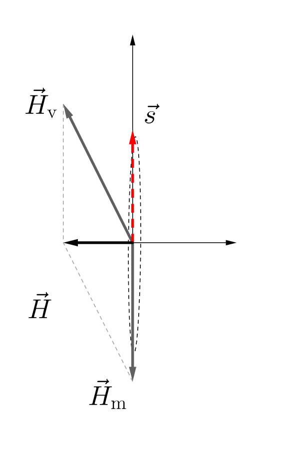
Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

Outline for Section 3

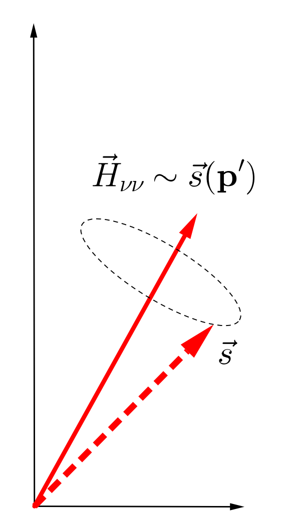
Neutrino Self-interactions

Interaction Hamiltonian $\mathbf{H}_{\nu\nu}$

$$\sqrt{2}G_F(1-\hat{p}\cdot\hat{p}')\rho(\mathbf{p}')$$

In Flavor Isospin space

$$-2\sqrt{2}G_F(1-\hat{p}\cdot\hat{p}')n(\mathbf{p}')\vec{s}(\mathbf{p}')$$



Neutrino Self-interactions

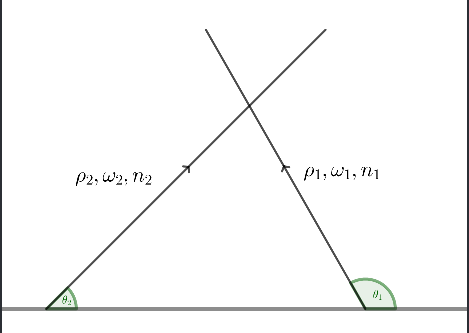
- $H_\nu = -\eta \frac{1}{2} \omega \sigma_3$
- $H_m = \frac{1}{2} \lambda \sigma_3$
- $H_{\nu\nu,2} = \frac{1}{2} \mu_1 \rho_1 \xi$
- $H_{\nu\nu,1} = \frac{1}{2} \mu_2 \rho_2 \xi$

where

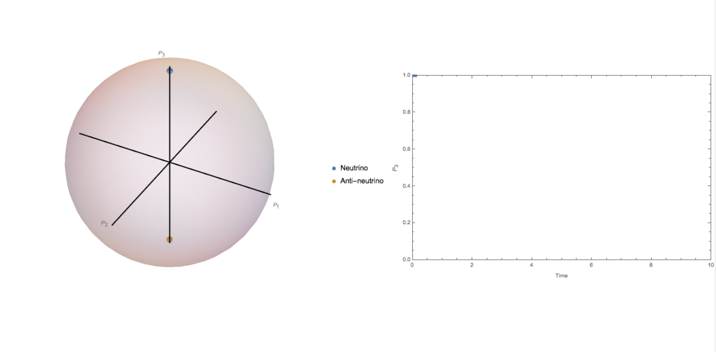
$$\mu_i = \sqrt{2} G_\mathrm{F} \xi n_i$$

Geometric factor

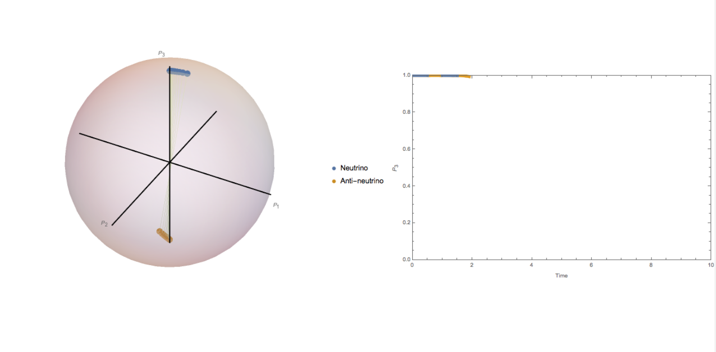
$$\xi = (1 - \cos(\theta_1 - \theta_2))$$



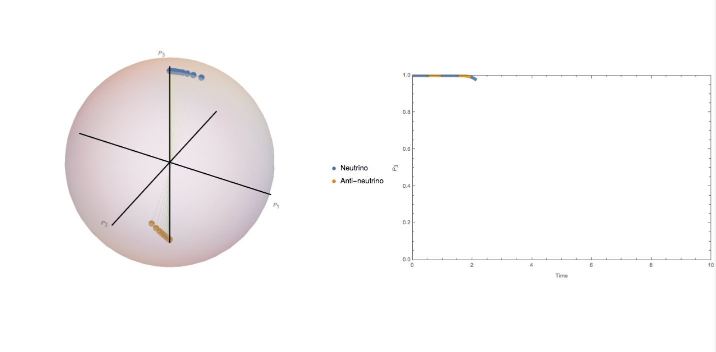
Neutrino Self-interactions



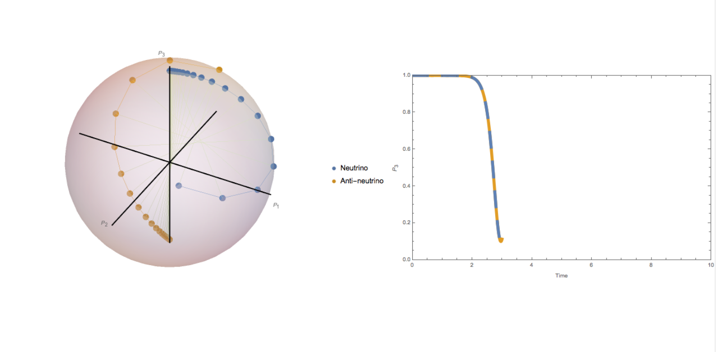
Neutrino Self-interactions



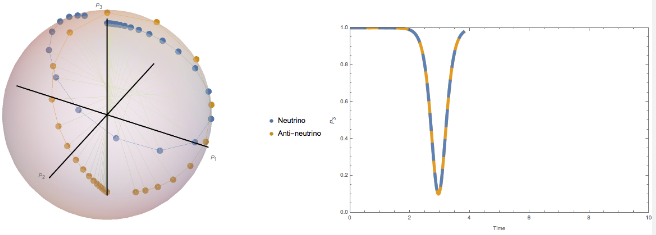
Neutrino Self-interactions



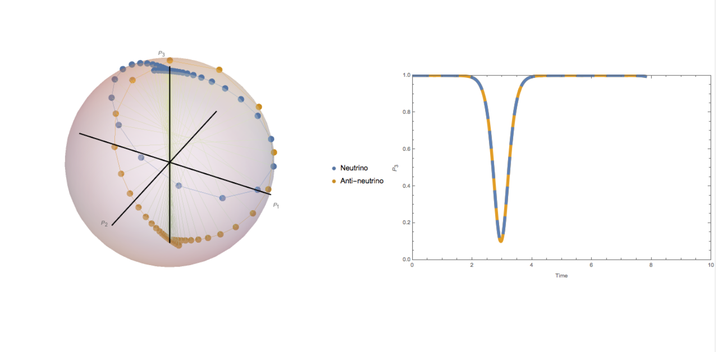
Neutrino Self-interactions



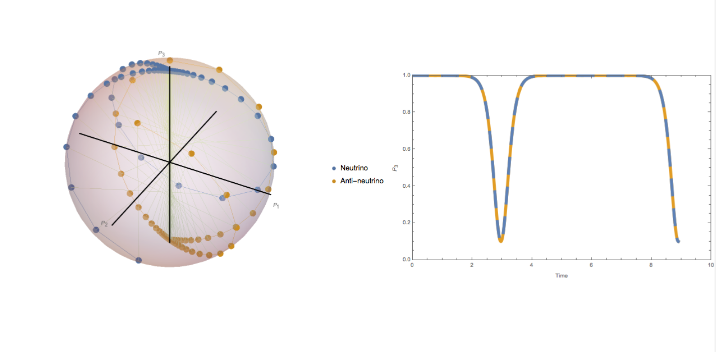
Neutrino Self-interactions



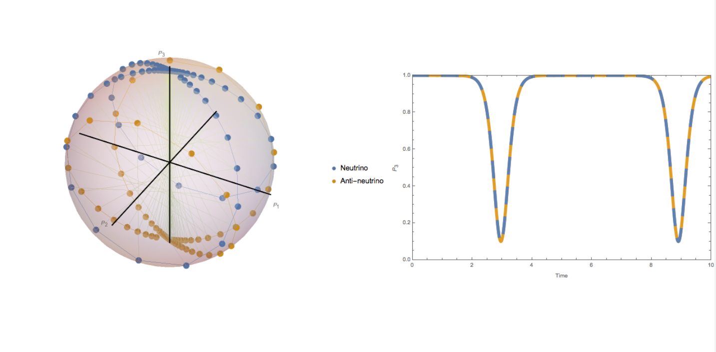
Neutrino Self-interactions



Neutrino Self-interactions



Neutrino Self-interactions



Neutrino Self-interactions

Characteristic Length Scales

- $\omega_{\mathrm{v}} = \delta m^2/2E$
- $\lambda \sim G_{\mathrm{F}} n_e$
- $\mu \sim G_{\mathrm{F}}(1 - \mathbf{p}_1 \cdot \mathbf{p}_2) n_\nu$

Vacuum oscillation oscillation frequencies

$$\omega_{\mathrm{v}} = \frac{\Delta m^2}{2E} \sim \frac{2\pi}{1\mathrm{km}} \left(\frac{\Delta m^2_{32}}{2.5 \times 10^{-3}\mathrm{eV}^2} \right) \left(\frac{1MeV}{E} \right)$$

$$\sim \frac{2\pi}{33\mathrm{km}} \left(\frac{\delta m^2_{12}}{7.5 \times 10^{-5}\mathrm{eV}^2} \right) \left(\frac{1\mathrm{MeV}}{E} \right)$$

Neutrino Self-interactions

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$$\sim \frac{2\pi}{33\text{km}} \left(\frac{\delta m^2_{12}}{7.5 \times 10^{-5}\text{eV}^2} \right) \left(\frac{1\text{MeV}}{E} \right)$$

Neutrino self-interactions might lead to faster oscillations, since $\mu \gg \omega_{\text{v}}$.

Neutrino Self-interactions

Characteristic Length Scales

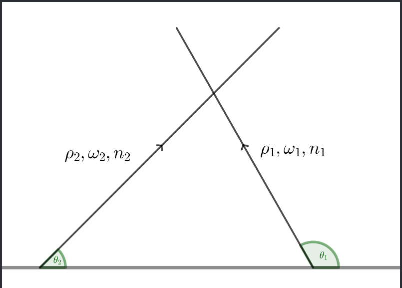
- $\omega_{\mathrm{v}} = \delta m^2/2E$
- $\lambda \sim G_{\mathrm{F}} n_e$
- $\mu \sim G_{\mathrm{F}}(1 - \mathbf{p}_1 \cdot \mathbf{p}_2) n_\nu$

Suppose we have neutrino flux $10^{50}\mathrm{ergs} \cdot \mathrm{s}^{-1}$. We estimate the potential at radius R to be

$$\mu \sim \frac{1}{0.01km} \left(\frac{100\mathrm{km}}{R} \right)^2 \left(\frac{1\mathrm{MeV}}{E} \right)$$

Linear Stability Analysis

$$H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1\xi, \qquad H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2\xi$$

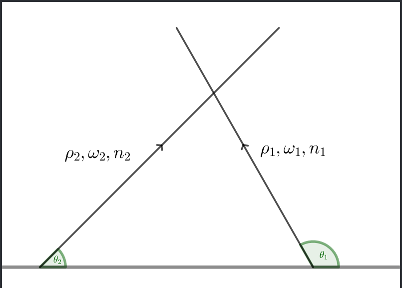


ρ_1: neutrinos; ρ_2: antineutrinos

$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = 2\pi/3, \theta_2 = \pi/6$$

Linear Stability Analysis



$$H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1\xi, \qquad H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2\xi$$

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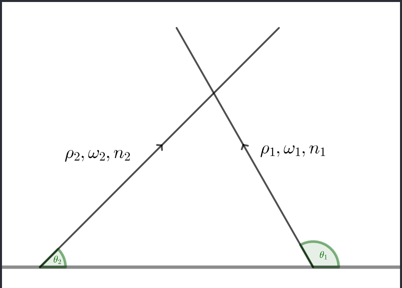
$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = 2\pi/3, \theta_2 = \pi/6$$

$$\rho_i = \frac{1}{2}\begin{pmatrix} 1 & \epsilon_i \\ \epsilon_i^* & -1 \end{pmatrix}$$

Linear Stability Analysis



$$H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1\xi, \qquad H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2\xi$$

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$$\rho_i = \frac{1}{2}\begin{pmatrix} 1 & \epsilon_i \\ \epsilon_i^* & -1 \end{pmatrix}$$

$$i\partial_z\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} \omega_v - \mu\xi & \mu\xi \\ -\mu\xi & -\omega_v + \mu\xi \end{pmatrix}$$

Linear Stability Analysis

Eigenvalues

$$\pm\sqrt{\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi)}$$

Identify the condition for complex eigenvalues

$$\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi) < 0$$

Linear Stability Analysis

Eigenvalues

$$\pm\sqrt{\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi)}$$

Identify the condition for complex eigenvalues

$$\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi) < 0$$

- Normal hierarchy: $\omega_v > 0$, requires $\mu > \omega_{\mathrm{v}}/2\xi$;
- Inverted hierarchy: $\omega_v < 0$, requires $\mu < \omega_{\mathrm{v}}/2\xi$.

Linear Stability Analysis

Eigenvalues

$$\pm\sqrt{\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi)}$$

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- Normal hierarchy: $\omega_v > 0$, requires $\mu > \omega_{\mathrm{v}}/2\xi$;
- Inverted hierarchy: $\omega_v < 0$, requires $\mu < \omega_{\mathrm{v}}/2\xi$.

Continuous emission angles?

Equation of Motion with Self-interactions

$$i(\partial_t + \hat{v})$$

Dispersion Relation

$$\mathrm{Det}(\omega + N^{\nu}{}_{\nu}) = 0,$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

$$N^{\mu}{}_{\nu} = \omega P^{\mu}{}_{\nu} \longrightarrow \begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 & -\frac{1}{2}I_1 \\ 0 & -\frac{1}{4}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(I_0 - I_2) & 0 \\ \frac{1}{2}I_1 & 0 & 0 & -\frac{1}{2}I_2 \end{pmatrix}$$

Dispersion Relation

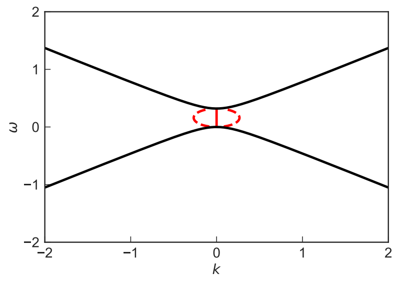
$$\mathrm{Det}(\omega + N^{\nu}{}_{\nu}) = 0,$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

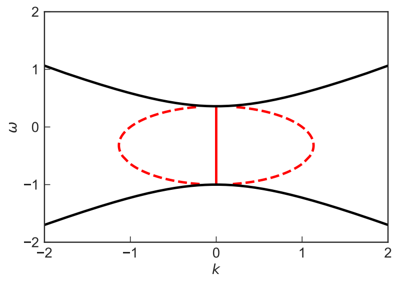
Solutions

$$\omega = \frac{1}{4}(I_0 - I_2), \quad -\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$$

Dispersion Relation



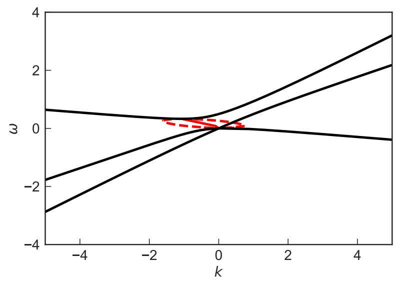
MAA solutions



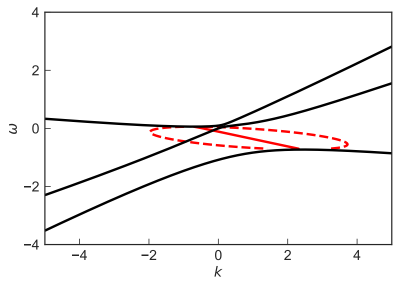
MZA solutions

Two beams

Dispersion Relation



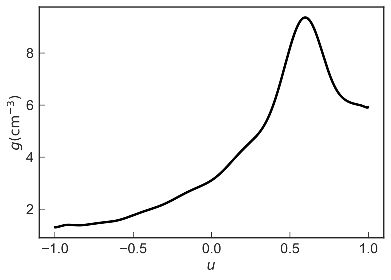
MAA solutions

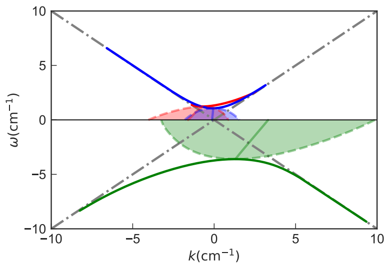


MZA solutions

Three beams

Dispersion Relations





Dispersion relation and linear stability analysis (right panel) for a spectrum constructed from Garching 1D simulation data (left panel). Solid red line is dispersion relation for MAA solution while blue and green lines are for MZA solutions. Light red (green and green) blob is instability for MAA (MZA) solution.

Dispersion Relations

$$k = \frac{1}{4}\int \mathrm{d}u G(u)\frac{1-u^2}{\omega/k-u}.$$

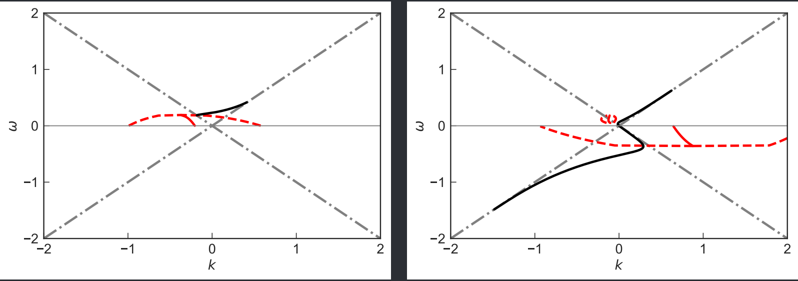
$$\mathrm{Re}(k) = \frac{1}{4}\left(\mathscr{P}\int \mathrm{d}u G(u)\frac{1-u^2}{-u}\right)$$

$$\mathrm{Im}(k) = \frac{\pi}{4}G(0)\,\mathrm{Sign}\,(\omega)\,\mathrm{Sign}\,(\mathrm{Im}(k)).$$

$$|\mathrm{Im}(k)| = \frac{\pi}{4}|G(0)|.$$

$$\left(4\,\mathrm{Re}(k) - \mathscr{P}\int \frac{G(u)}{u}\mathrm{d}u + U_1\right)^2 - (\mathrm{Sign}(\omega\,\mathrm{Im}(k))\pi G(0) + 4\,\mathrm{Im}(k))^2$$
$$= -\left(\mathscr{P}\int \frac{G(u)}{u}\mathrm{d}u + U_1\right)\pi\,\mathrm{Sign}(\omega\,\mathrm{Im}(k))G(0),$$

Dispersion Relations



Dispersion relation and linear stability analysis for box spectrum. The box spectrum is defined to be -0.1 within range $u \in [-1, -0.3)$ and 1 within range $u \in [-0.3, 1]$. Left panel shows the dispersion relation and the complex k for real ω for MAA solution. Right panel is the corresponding result for MZA solution. Dash-dotted gray lines are $\omega = \pm k$ which sets the boundaries of the forbidden region for dispersion relation.

Outline for Section 4

Summary

- The fact that neutrino flavor sates are not mass states causes vacuum oscillations.
- MSW resonance happens when matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
- Even matter profile doesn't match MSW requirement, variation in matter profile can cause resonances.
- Single frequency perturbations in matter profile is a combination of many Rabi oscillations.

- How to understand and calculate systems with multi-frequency matter profile (turbulence).
- Combine periodic or even turbulent matter profile with neutrino self-interaction.

Acknowledgement

I am very thankful to my advisor Professor Huaiyu Duan, Dr. Sajad Abbar, and Dr. Shashank Shalgar, and Joshua Martin, for all the help in both research and life.

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Citations

TeX, LaTeX, and Beamer

TeX is a programming language for the typesetting of documents. It was created by Donald Erwin Knuth in the late 1970s and it is documented in *The TeXbook* [1].

In the early 1980s, Leslie Lamport created the initial version of LaTeX, a high-level language on top of TeX, which is documented in *LaTeX: A Document Preparation System* [2]. There exists a healthy ecosystem of packages that extend the base functionality of LaTeX; *The LaTeX Companion* [3] acts as a guide through the ecosystem.

In 2003, Till Tantau created the initial version of Beamer, a LaTeX package for the creation of presentations. Beamer is documented in the *User's Guide to the Beamer Class* [4].

[1] Donald E. Knuth. *The TEXbook*. Addison-Wesley, 1984.

[2] Leslie Lamport. *LATEX: A Document Preparation System*. Addison-Wesley, 1986.

[3] M. Goossens, F. Mittelbach, and A. Samarin. *The LATEX Companion*. Addison-Wesley, 1994.

[4] Till Tantau. *User's Guide to the Beamer Class Version 3.01*. Available at `http://latex-beamer.sourceforge.net`.

[5] A. Mertz and W. Slough. Edited by B. Beeton and K. Berry. *Beamer by example* In TUGboat, Vol. 26, No. 1., pp. 68-73.

Hamiltonian, and Basis, and Rabi Oscillations

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1.$$

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0,

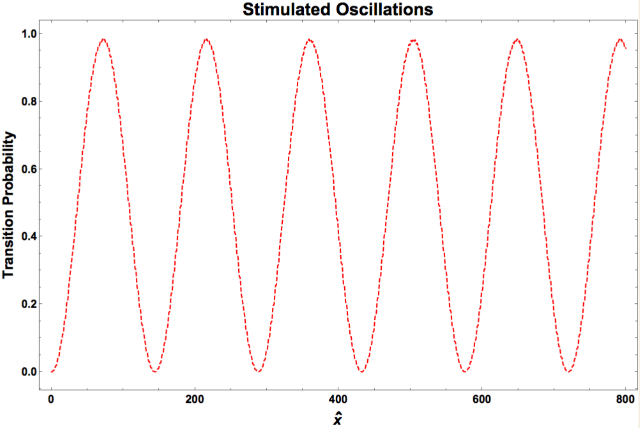
$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

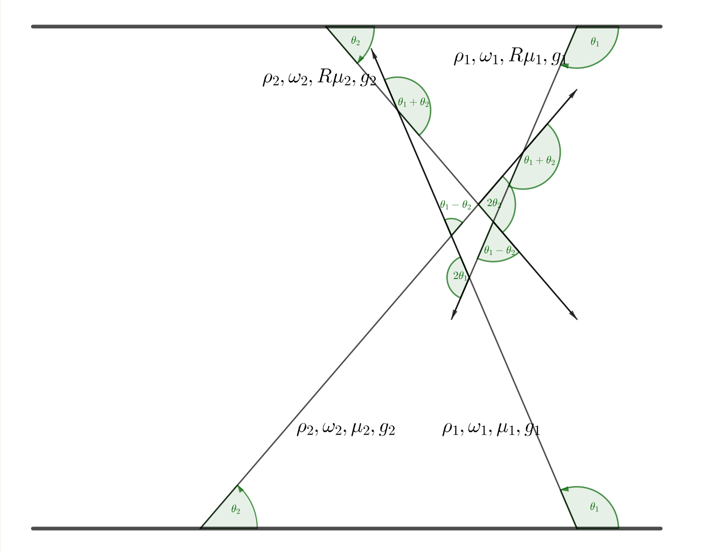
Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\sin(kx)$ with $\hat{x} = \omega_m x$, $A = 0.1\omega_m$, $k = 0.995\omega_m$, $\theta_m = \pi/6$

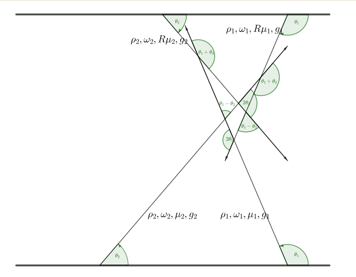
Neutrino Halo



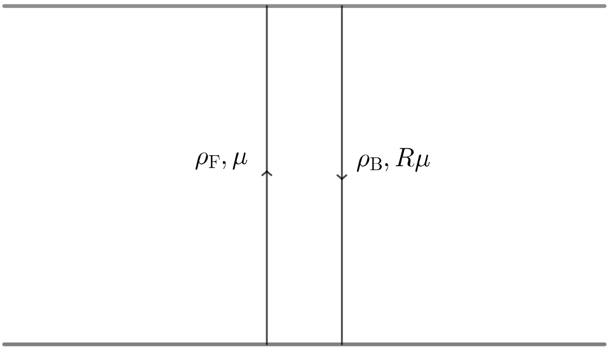
Neutrino Halo

Assumptions

- Neutrinos are translational symmetric on the emission line.
- Reflection obays Snell's law.
- Neutrinos are reflected on a fixed surface $z = L$.
- Neutrino reflections are translational symmetric.



Flavor Isospin

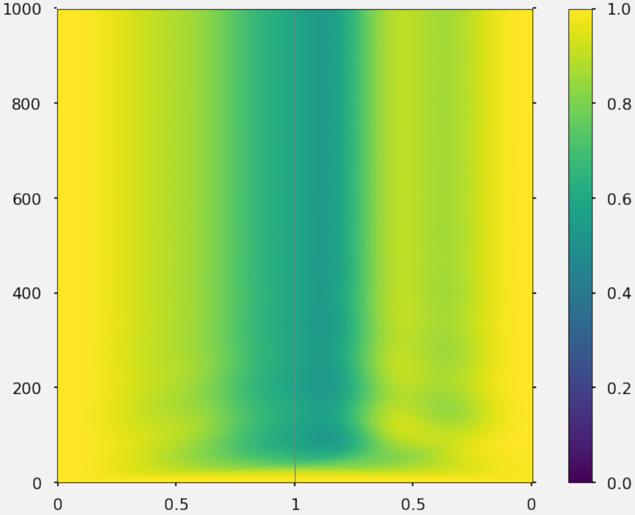


Relaxation Scheme

Algorithm

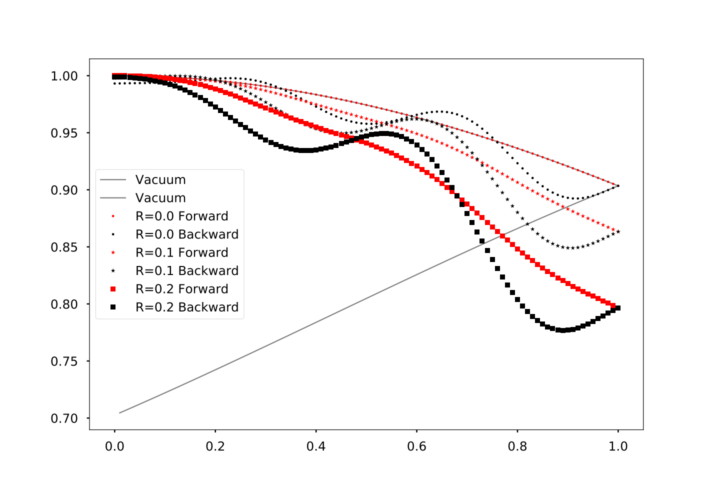
1. Calculate forward beam using null backward beam;
2. Calculate backward beam using forward beam calculated in step 1;
3. Calculate forward beam using backward beam calculated in step 2;
4. Repeat 2 and 3 until the beams reach equilibrium.

Numerical Method



Horizontal axis is the location of neutrinos; Vertical axis is the number of iteration steps; Color indicates the electron flavor probability.

Numerical Method



Linear Stability Analysis

EoM

$$i\partial_t \vec{s}_F = \mathrm{s}_F \times (\vec{H}_v + R\mu\vec{s}_B)$$
$$i\partial_t \vec{s}_B = \vec{s}_B \times (-\vec{H}_v - \mu\vec{s}_F).$$

Compare with bipolar

$$i\partial_t \vec{s} = \mathrm{s} \times (\eta\vec{H}_v + \alpha\mu\vec{\bar{s}})$$
$$i\partial_t \vec{\bar{s}} = \vec{\bar{s}} \times (\eta\vec{H}_v + \mu\vec{s})$$

Linear Stability Analysis