Neutrino Flavor Conversions in Dense Medium: Matter Stimulation, Dispersion Relation, and Neutrino Halo

PhD Defense

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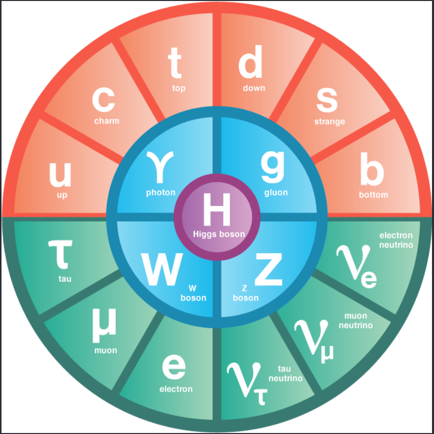
Outline for Section 1

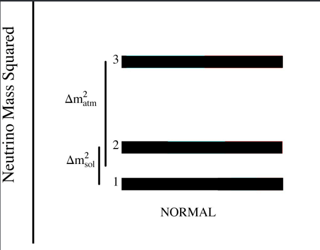
What are Neutrinos?



Elementary particles.
Source: symmetrymagazine.org

Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- none vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

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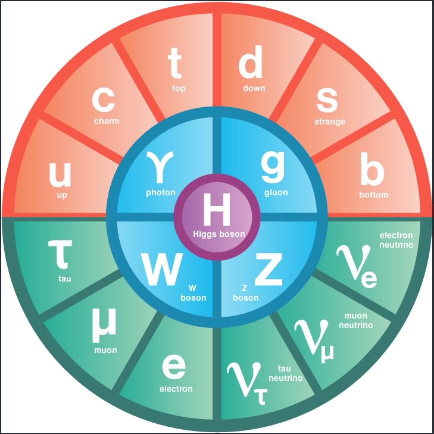

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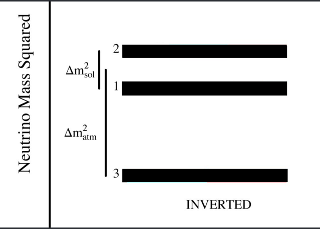


Adapted from Olga Mena & Stephen Parke (2004)

Why Do Neutrinos Oscillate?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\sigma_3 + \sin 2\theta_{\mathrm{v}}\sigma_1\right)$$

- Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_{v}

Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

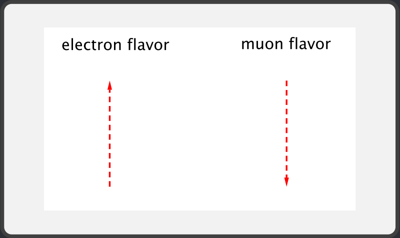
Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



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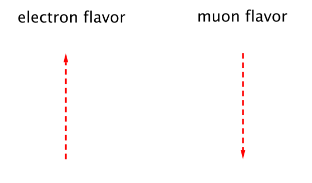
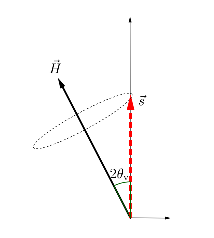
Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2} \left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \right)$$

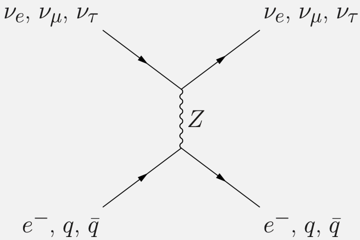
$$\rightarrow \cos 2\theta_v \begin{pmatrix} 0 \\ 0 \\ \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v \\ 0 \\ 0 \end{pmatrix}$$



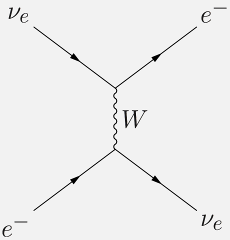


Outline for Section 2

Matter Interaction



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, etc.



Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_v = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_v}{2}(-\cos 2\theta_v \boldsymbol{\sigma_3} + \sin 2\theta_v \boldsymbol{\sigma_1})} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2} G_F n_e(x)$

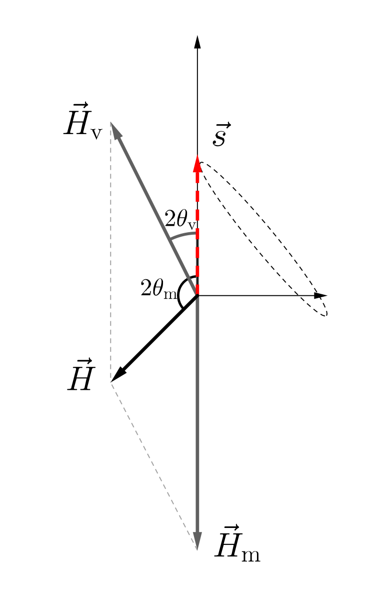
MSW Effect

$$\mathbf{H} = \frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma_3} + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

$$\to \omega_\mathrm{v}\begin{pmatrix} -\sin 2\theta_\mathrm{v} \\ 0 \\ \cos 2\theta_\mathrm{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

$$= \vec{H}_\mathrm{v} + \vec{H}_\mathrm{m}(x)$$

MSW Effect



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

$$\omega_{\mathrm{v}} = |\vec{H}_{\mathrm{v}}|$$

Oscillation frequency in **matter**:

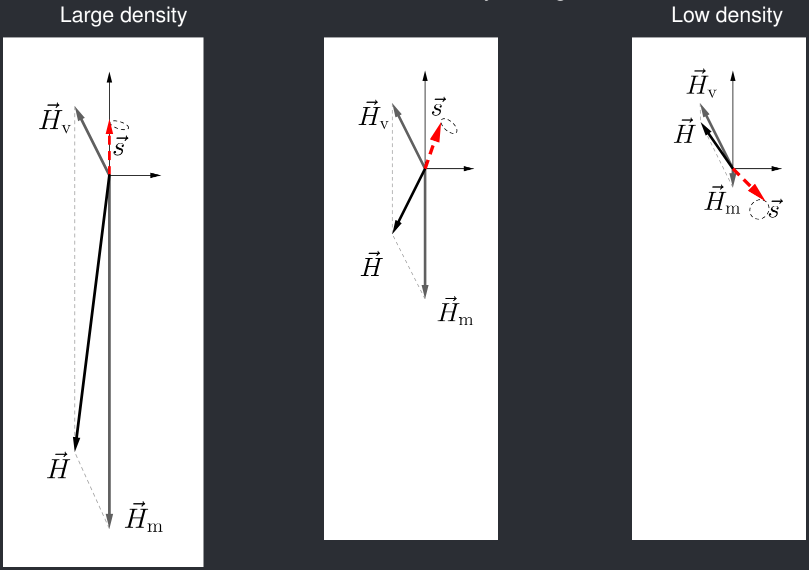
$$\omega_{\mathrm{m}} = |\vec{H}|$$

Flavor states and mass states in matter

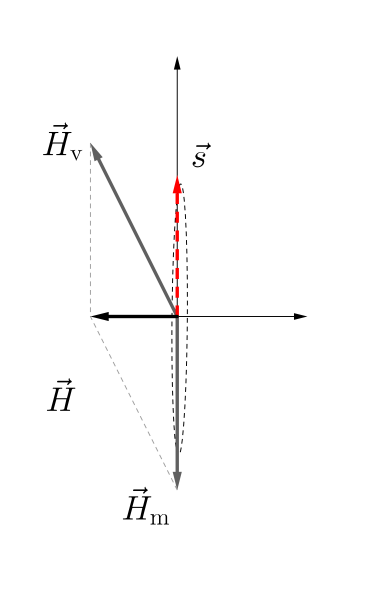
$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & \sin\theta_{\mathrm{m}} \\ -\sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix}$$

MSW Effect

Adiabatic matter density change



MSW Effect

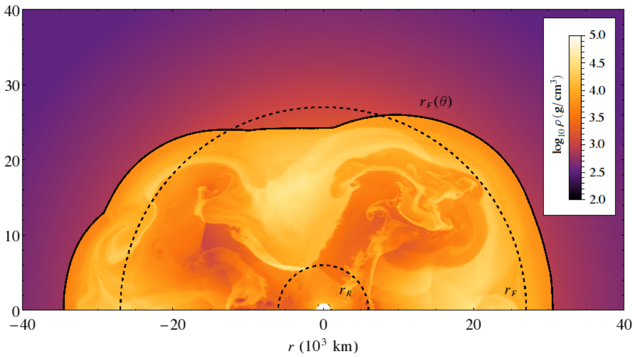


- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}} \equiv \omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}$$

Supernova Matter Density Profile

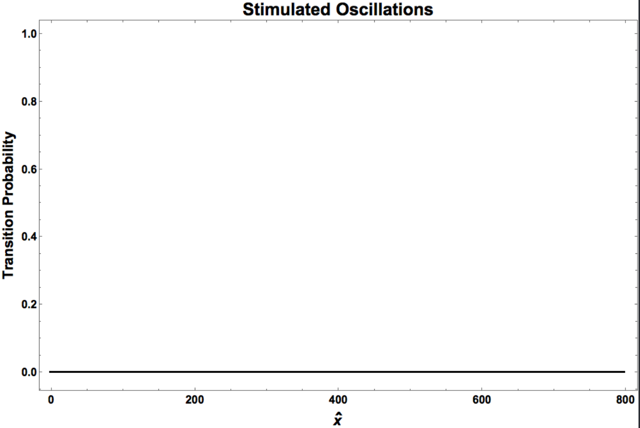
Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

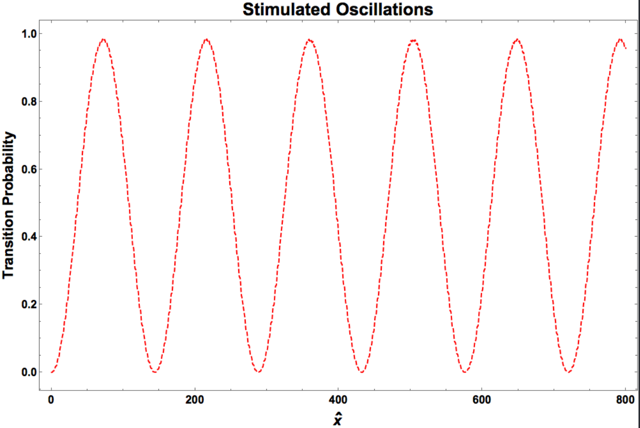
$$\lambda(x) = \lambda_0$$



Transition probabilities between mass states in matter.

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



P. Krastev and A. Smirnov (1989); A. Friedland et al (2006); J. Kneller et al (2013); K. Patton et al (2014);

Rabi Oscillations

Rabi Oscillation

Hamiltonian

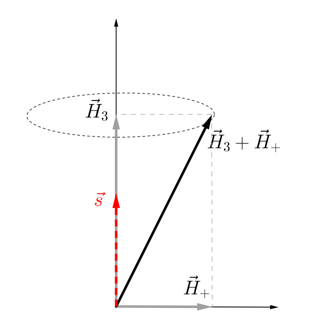
$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Scheme

Rabi Oscillations

Static Frame

$$\vec{H}_3 = \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Rabi Oscillations

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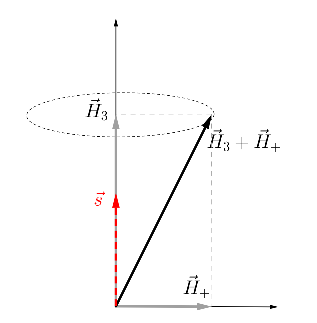
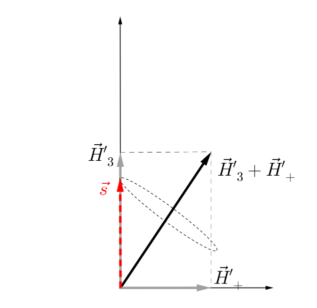


Corotating Frame

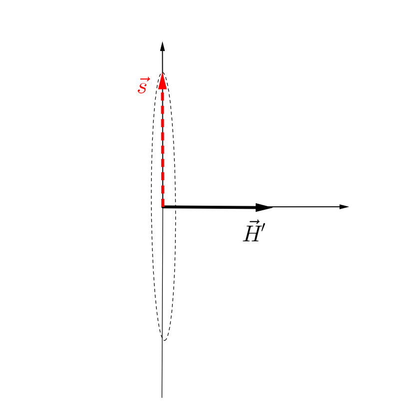
$$\vec{H}'_3 = (\omega_\mathrm{m} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_\mathrm{m} - k)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \Rightarrow k = \omega_\mathrm{m}$$



Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3-\frac{\alpha}{2}\begin{pmatrix}0 & e^{ikt}\\ e^{-ikt} & 0\end{pmatrix}$$

Scheme

Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}}-k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = |\alpha|\sqrt{1+D^2}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_H\end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}}\right)\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_H\end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_m$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_m + \cancel{\cos 2\theta_m A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_m}{2} A\cos(kx)\sigma_1$$

$$\to \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_m}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

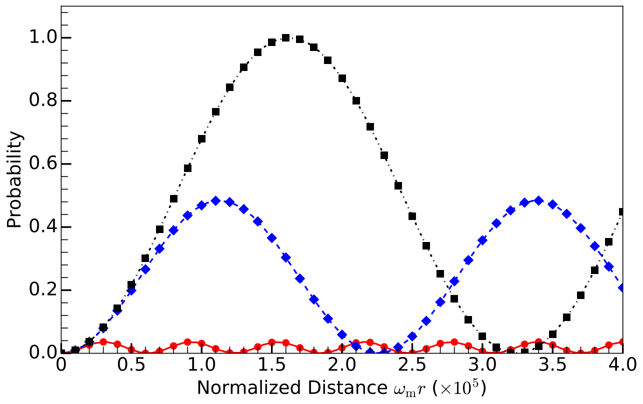
$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\to \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \alpha\,\cancel{\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Rabi Formula Works



Transition between two mass states in background matter potential λ_0
Lines: Rabi formula
Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_{\mathrm{m}}$, $k = (1 - 2 \times 10^{-5})\omega_{\mathrm{m}}$, and $k = (1 - 10^{-4})\omega_{\mathrm{m}}$ respectively.

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \cancel{\alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}}$$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma}_3 - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma}_1.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{\mathrm{L}} \\ \tilde{\psi}_{\mathrm{H}} \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty}\begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk\tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Single Frequency Matter Potential

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The Hamiltonian

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where $\alpha_n = -(-i)^n nk \tan 2\theta_\mathrm{m} J_n(A\cos 2\theta_\mathrm{m}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Resonance condition for each mode: $nk = \omega_\mathrm{m}$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

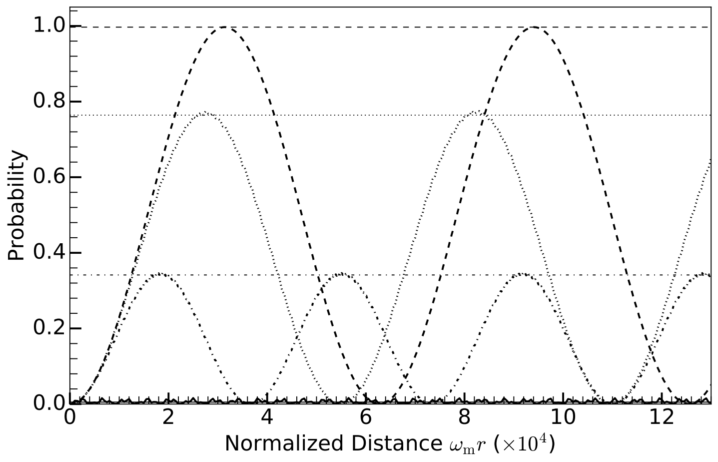
Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$



Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$	$5.0 \times 10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$5 \times 10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$

Rabi Oscillations With Multiple Driving Frequencies

Consider $k_1 = \omega_\mathrm{m}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_\mathrm{m} - k_2)} \right|$$

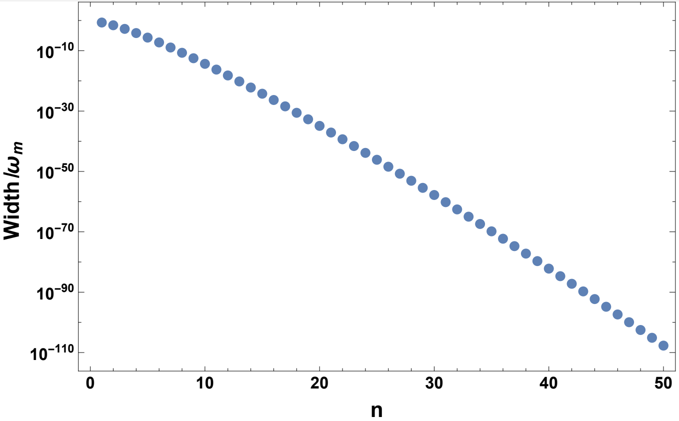
Amplitude reduces from 1 to 1/2 if

$$D' = 1 \Rightarrow \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
For $k_1 = \omega_\mathrm{m}$, survival of resonance requires

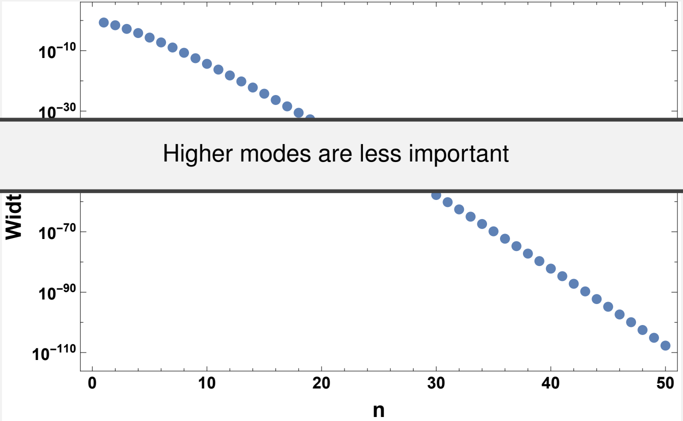
$$|\alpha_2| \ll \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential



Higher modes are less important

Width of different modes given value of matter potential frequency k

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^{N} A_a \sin(k_a x)$$

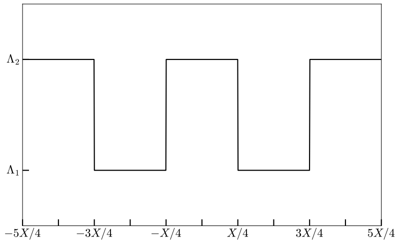
Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\begin{pmatrix} 0 & B_{\{n_a\}}e^{i\sum_a n_a k_a x} \\ B^*_{\{n_a\}}e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a}\tan 2\theta_m\left(\sum_a n_a k_a\right)\left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right)$$

Castle Wall Matter Potential



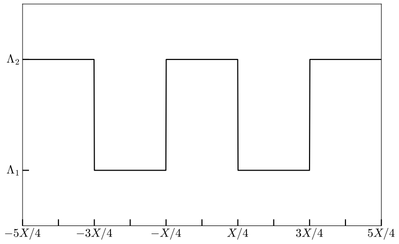
Castle wall matter profile:
$\Lambda_2 = 0.35\omega_v \cos 2\theta_v$,
$\Lambda_1 = 0.15\omega_v \cos 2\theta_v$ and period
$X = 2\pi/\omega_m$

$$\lambda(x) = \lambda_0 + \sum_1^\infty \lambda_n \cos(k_n x)$$

where

$$\begin{aligned} \lambda_0 &= (\Lambda_1 + \Lambda_2)/2 \\ \lambda_n &= 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi) \\ k_n &= 2\pi(2n-1)/X \end{aligned}$$

Castle Wall Matter Potential



Castle wall matter profile:
$\Lambda_2 = 0.35\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$,
$\Lambda_1 = 0.15\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$ and period
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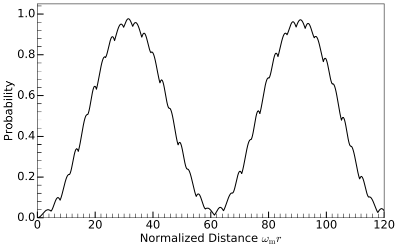
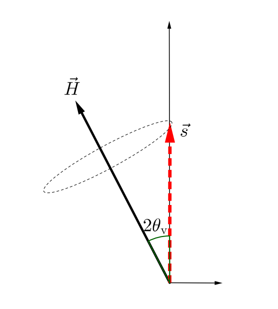


Transition probability is a Rabi resonance with small variations due to higher orders.

Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.



Summary of Stimulated Oscillations

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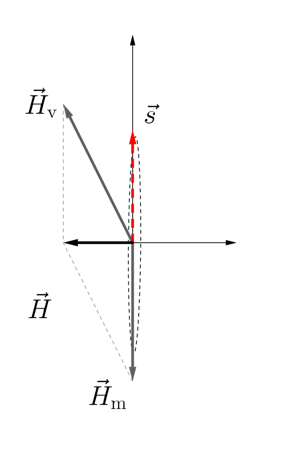
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.



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1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

Outline for Section 3

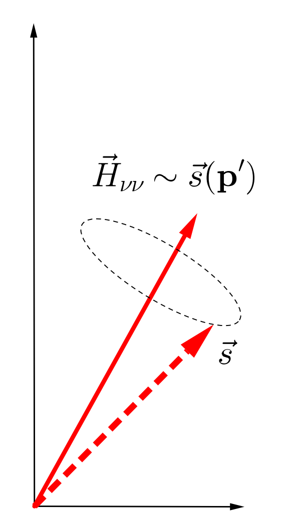
Neutrino Self-interactions

Interaction Hamiltonian $\mathbf{H}_{\nu\nu}$

$$\sqrt{2}G_F(1-\hat{p}\cdot\hat{p}')\rho(\mathbf{p}')$$

In Flavor Isospin space

$$-2\sqrt{2}G_F(1-\hat{p}\cdot\hat{p}')n(\mathbf{p}')\vec{s}(\mathbf{p}')$$



Neutrino Self-interactions

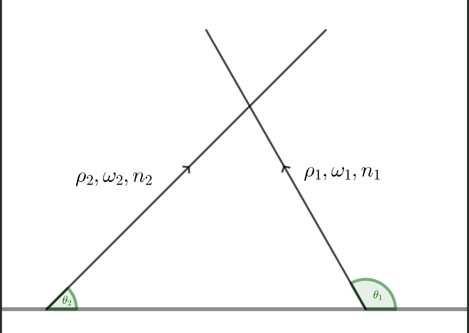
- $H_\nu = -\eta \frac{1}{2} \omega \sigma_3$
- $H_m = \frac{1}{2} \lambda \sigma_3$
- $H_{\nu\nu,2} = \frac{1}{2} \mu_1 \rho_1 \xi$
- $H_{\nu\nu,1} = \frac{1}{2} \mu_2 \rho_2 \xi$

where

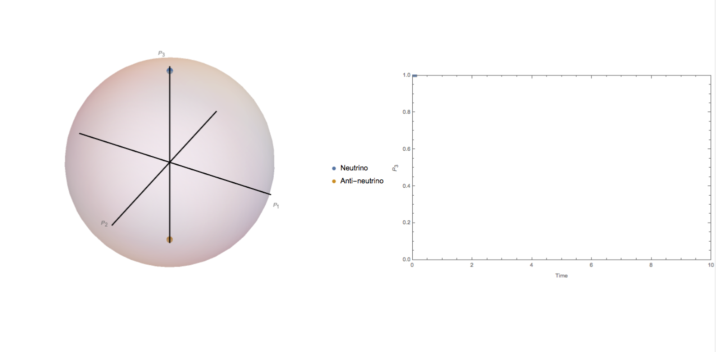
$$\mu_i = \sqrt{2} G_F \xi n_i$$

Geometric factor

$$\xi = (1 - \cos(\theta_1 - \theta_2))$$



Neutrino Self-interactions



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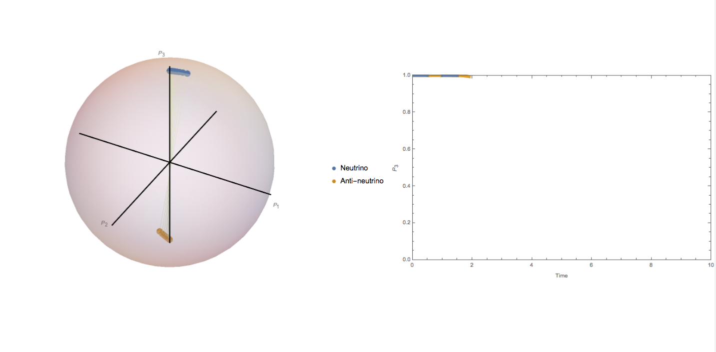

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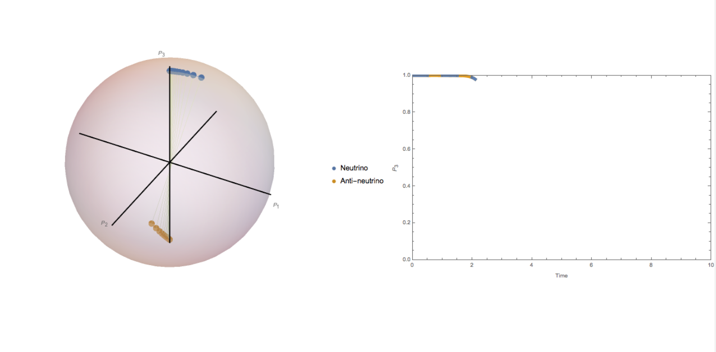
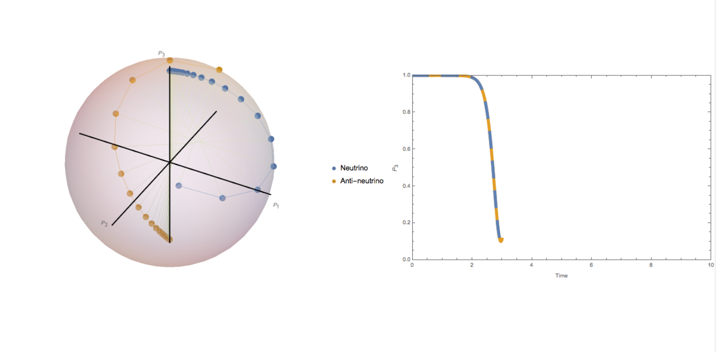


Neutrino Self-interactions



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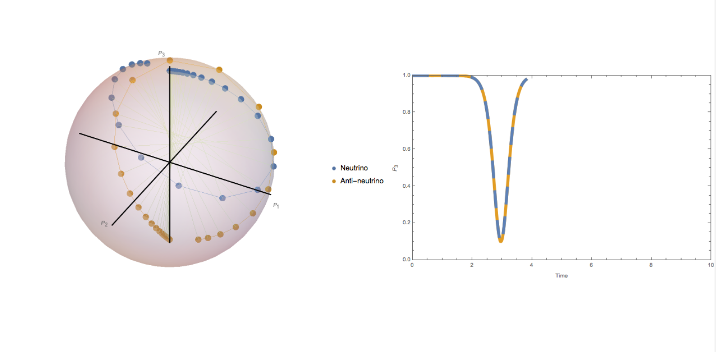

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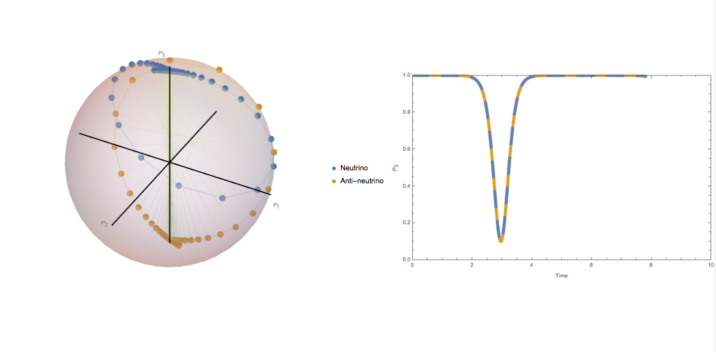

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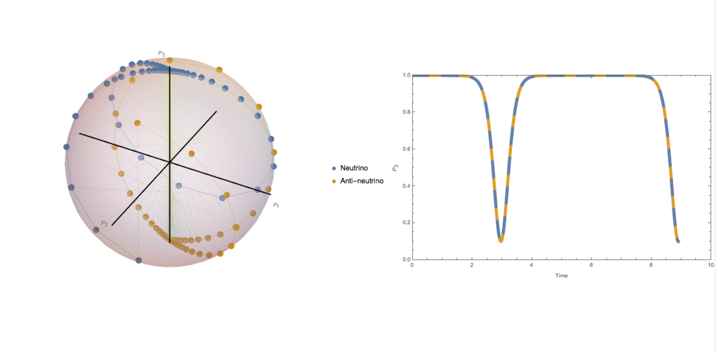

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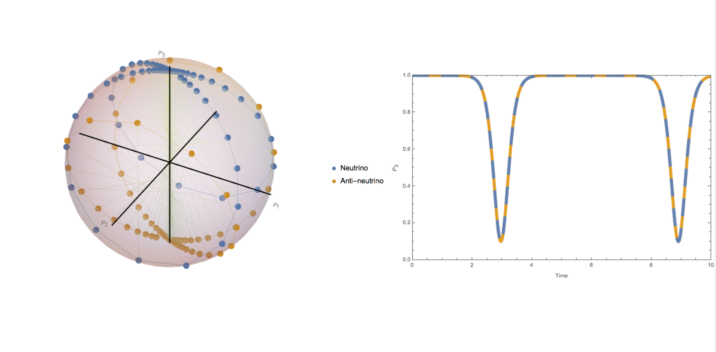

Neutrino Self-interactions

Characteristic Length Scales

- $\omega_{\rm v} = \delta m^2/2E$
- $\lambda \sim G_{\rm F} n_e$
- $\mu \sim G_{\rm F}(1 - \mathbf{p}_1 \cdot \mathbf{p}_2) n_\nu$

Vacuum oscillation oscillation frequencies

$$\omega_{\rm v} = \frac{\Delta m^2}{2E} \sim \frac{2\pi}{1\text{km}} \left(\frac{\Delta m^2_{32}}{2.5 \times 10^{-3}\text{eV}^2} \right) \left(\frac{1MeV}{E} \right)$$
$$\sim \frac{2\pi}{33\text{km}} \left(\frac{\delta m^2_{12}}{7.5 \times 10^{-5}\text{eV}^2} \right) \left(\frac{1\text{MeV}}{E} \right)$$

Neutrino Self-interactions

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$$\sim \frac{2\pi}{33\text{km}} \left(\frac{\delta m^2_{12}}{7.5 \times 10^{-5}\text{eV}^2} \right) \left(\frac{1\text{MeV}}{E} \right)$$

Neutrino self-interactions might lead to faster oscillations, since $\mu \gg \omega_{\text{v}}$.

Neutrino Self-interactions

Characteristic Length Scales

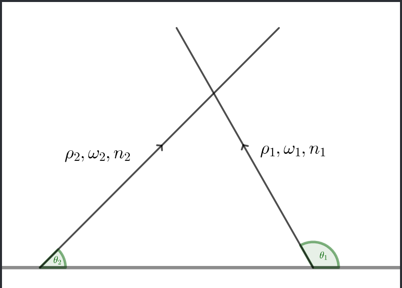
- $\omega_{\mathrm{v}} = \delta m^2/2E$
- $\lambda \sim G_{\mathrm{F}} n_e$
- $\mu \sim G_{\mathrm{F}}(1 - \mathbf{p}_1 \cdot \mathbf{p}_2) n_\nu$

Suppose we have neutrino flux $10^{50}\mathrm{ergs} \cdot \mathrm{s}^{-1}$. We estimate the potential at radius R to be

$$\mu \sim \frac{1}{0.01km} \left(\frac{100\mathrm{km}}{R} \right)^2 \left(\frac{1\mathrm{MeV}}{E} \right)$$

Linear Stability Analysis

$$H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1\xi, \qquad H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2\xi$$



ρ_1: neutrinos; ρ_2: antineutrinos

$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = 2\pi/3, \theta_2 = \pi/6$$

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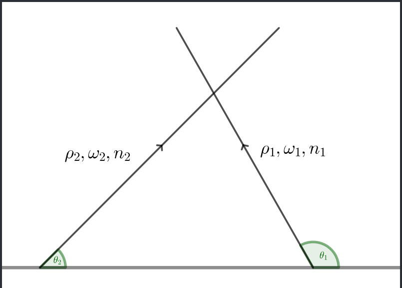

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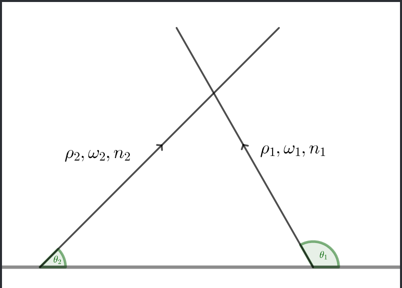
Linear Stability Analysis



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$$i\partial_z\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} \omega_v - \mu\xi & \mu\xi \\ -\mu\xi & -\omega_v + \mu\xi \end{pmatrix}$$

Linear Stability Analysis

Eigenvalues or collective oscillation frequencies

$$K_z = \pm\sqrt{\omega_\mathrm{v}(\omega_\mathrm{v} - 2\mu\xi)}$$

Identify the condition for complex eigenvalues

$$\omega_\mathrm{v}(\omega_\mathrm{v} - 2\mu\xi) < 0$$

Linear Stability Analysis

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K_z is instability in z direction for our model.

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K_z is instability in z direction for our model.

Similar analysis can be done for all four dimensions t, x, y, z,

$$(\Omega, K_x, K_y, K_z)$$

Dispersion Relation

Izaguirre, I., Raffelt, G., & Tamborra, I. (2017). *Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach*. Physical Review Letters, 118(2), 021101.

- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.

Dispersion Relation

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- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.
- Instabilities occur in dispersion relation gaps.

Dispersion Relation

Equation of motion for off-diagonal element of density matrix (Izaguirre et al, 2017)

$$i(\partial_t + v \cdot \nabla_r)\epsilon(v) = v^\mu (\Lambda + \Phi)_\mu - \int d\Gamma' v^\mu v'_\mu G(v')\epsilon(v')$$

Dispersion Relation

Equation of motion for off-diagonal element of density matrix (Izaguirre et al, 2017)

$$i(\partial_t + \boldsymbol{v} \cdot \nabla_r)\epsilon(\boldsymbol{v}) = \boxed{v^\mu}(\boxed{\Lambda} + \boxed{\Phi})_\mu - \int d\Gamma' v^\mu v'_\mu \boxed{G(\boldsymbol{v}')}\,\epsilon(\boldsymbol{v}')$$

- $\boxed{v^\mu}$: four-velocity of neutrinos $(1, \boldsymbol{v})$
- $\boxed{\Lambda}$: matter contribution $(\sqrt{2}G_\mathrm{F} n_\mathrm{e}, \sqrt{2}G_F n_e \boldsymbol{v}_\mathrm{e})$
- $\boxed{\Phi}$: neutrino
- $\boxed{G(\mathbf{v}')}$: electron lepton number of neutrinos

  $$\sqrt{2}G_\mathrm{F}\int_0^\infty \frac{E^2\,dE}{2\pi^2}\left(n_{\nu_\mathrm{e}} - n_{\bar{\nu}_\mathrm{e}}\right)$$

Dispersion Relation

Collective mode of off-diagonal element

$$\epsilon \longrightarrow \tilde{\epsilon} e^{-i(\Omega t - \boldsymbol{K}\cdot\boldsymbol{r})}$$

Dispersion Relation

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Replacement:

- $\epsilon \longrightarrow \tilde{\epsilon}$
- $\partial_t \longrightarrow -i\Omega$, $\nabla_{\boldsymbol{r}} \longrightarrow i\boldsymbol{K}$

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Collective mode

$$v^\mu (K_\mu - (\Lambda + \Phi)_\mu)\tilde{\epsilon}(\boldsymbol{v}) = -\int d\Gamma' v^\mu v'_\mu G(\boldsymbol{v}')\tilde{\epsilon}(\boldsymbol{v}')$$

with $K_\mu \longrightarrow (\Omega, \boldsymbol{K})$

Without neutrino self-interaction: $v^\mu K_\mu = 0$

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Dispersion Relation

Rewrite

$$
\begin{aligned}
&-\int d\Gamma' \, \boxed{v^\mu} \, v'_\mu G(v')\tilde{\epsilon}(v') \\
&= \boxed{v^\mu} \, \boxed{\left(-\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')\right)} \\
&\equiv v^\mu \, \boxed{a_\mu}
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$$

Dispersion Relation

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$$

EoM

$$v^\mu K_\mu \tilde{\epsilon}(v) = v^\mu a_\mu$$

Dispersion Relation

Rewrite

$$v^\mu K_\mu \tilde{\epsilon}(v) = -\int d\Gamma' v^\mu v'_\mu G(v')\tilde{\epsilon}(v')$$

$$-\int d\Gamma' \, \boxed{v^\mu} \, v'_\mu G(v')\tilde{\epsilon}(v')$$

$$= \boxed{v^\mu} \, \boxed{\left(-\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')\right)}$$

$$\equiv v^\mu \, \boxed{a_\mu}$$

EoM

$$v^\mu K_\mu \tilde{\epsilon}(v) = v^\mu a_\mu$$

$$\Longrightarrow$$

$$\tilde{\epsilon}(v) = v^\mu a_\mu / v^\mu K_\mu$$

Collect all terms of a_μ

$$v^\mu \left(\delta^\nu_\mu + \int d\Gamma' \frac{G(v')v'_\mu v^\nu}{v^\alpha K_\alpha}\right) a_\nu = 0$$

Dispersion Relation

$v^\alpha K_\alpha = \Omega(1 - n\cos\theta)$ where $n = |\boldsymbol{K}|/\Omega$

Nontrivial solutions to EoM requires

$$v^\mu \left(\Omega I + N^\nu{}_\mu\right) a_\nu = 0$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

$$N^\mu{}_\nu =$$

$$\begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 & -\frac{1}{2}I_1 \\ 0 & -\frac{1}{4}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(I_0 - I_2) & 0 \\ \frac{1}{2}I_1 & 0 & 0 & -\frac{1}{2}I_2 \end{pmatrix}$$

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$$\omega = \frac{1}{4}(I_0 - I_2), \quad -\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$$

Dispersion Relation

$$v^\alpha K_\alpha = \Omega(1 - n\cos\theta) \text{ where } n = |\boldsymbol{K}|/\Omega$$

Nontrivial solutions to EoM requires

$$v^\mu \boxed{\left(\Omega I + N^\nu{}_\mu\right) a_\nu} = 0$$

$$\Rightarrow$$

$$\left(\Omega I + N^\nu_\mu\right) a_\nu = 0$$

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$$v^\mu \boxed{\left(\Omega I + N^\nu{}_\mu\right) a_\nu} = 0$$

$$\Rightarrow$$

$$\left(\Omega I + N^\nu{}_\mu\right) a_\nu = 0$$

$$\Rightarrow$$

$$\mathrm{Det}\left(\omega I + N^\nu{}_\mu\right) = 0,$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

$$N^\mu{}_\nu =$$

$$\begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 & -\frac{1}{2}I_1 \\ 0 & -\frac{1}{4}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(I_0 - I_2) & 0 \\ \frac{1}{2}I_1 & 0 & 0 & -\frac{1}{2}I_2 \end{pmatrix}$$

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Dispersion Relation

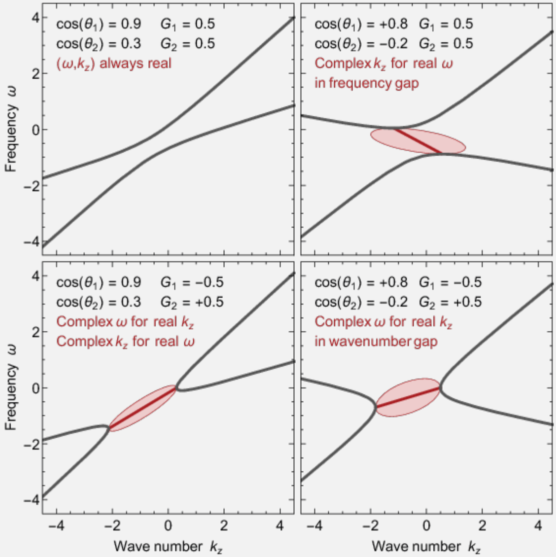
$$a_\mu = -\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')$$

$$\omega = \frac{1}{4}(I_0 - I_2)\,, \qquad -\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$$

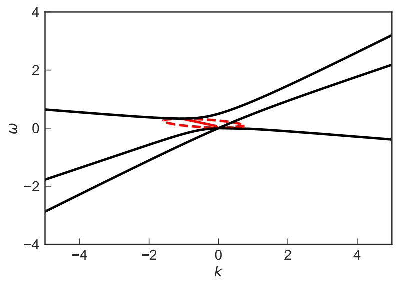
- $\frac{1}{4}(I_0 - I_2)$: MAA solution
- $-\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$: MZA solution

Dispersion Relation

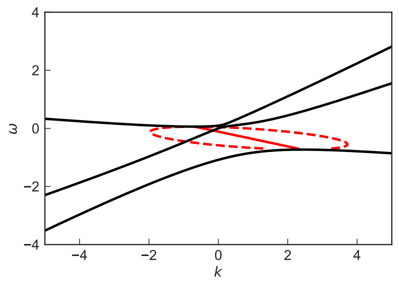
Izaguirre et al, 2017



Dispersion Relation



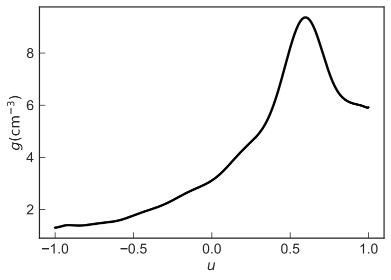
MAA solutions



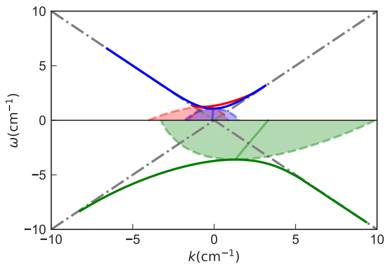
MZA solutions

Three beams

Dispersion Relations

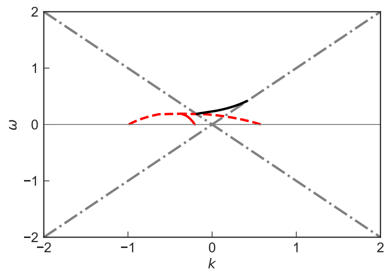


Spectrum $G(u)$, where $u = \cos\theta$

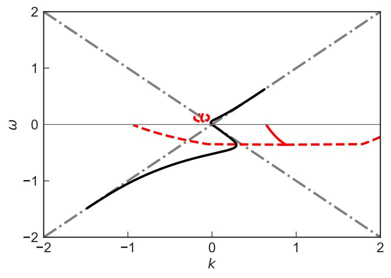


MAA: red; MZA: blue and green

Dispersion Relations



MAA solution



MZA solution

Outline for Section 4

Summary

- The fact that neutrino flavor sates are not mass states causes vacuum oscillations.
- MSW resonance happens when matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
- Even matter profile doesn't match MSW requirement, variation in matter profile can cause resonances.
- Single frequency perturbations in matter profile is a combination of many Rabi oscillations.

- How to understand and calculate systems with multi-frequency matter profile (turbulence).
- Combine periodic or even turbulent matter profile with neutrino self-interaction.

Acknowledgement

I am very thankful to my advisor Professor Huaiyu Duan, Dr. Sajad Abbar, and Dr. Shashank Shalgar, and Joshua Martin, for all the help in both research and life.
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Citations

TEX, LATEX, and Beamer

Bibliography

TEX, LATEX, and Beamer

[1] Izaguirre, I., Raffelt, G., and Tamborra, I. (2017). *Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach.* Physical Review Letters, 118(2), 021101.

[2] Donald E. Knuth. *The TEXbook*. Addison-Wesley, 1984.

[3] A. Mertz and W. Slough. Edited by B. Beeton and K. Berry. *Beamer by example* In TUGboat, Vol. 26, No. 1., pp. 68-73.

Hamiltonian, and Basis, and Rabi Oscillations

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1.$$

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0,

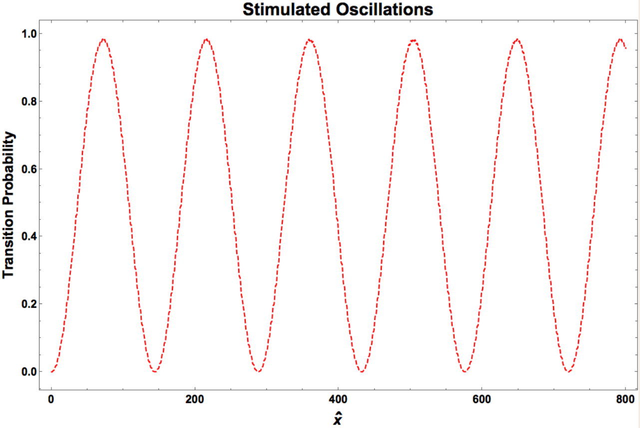
$$\mathrm{H}_{\text{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

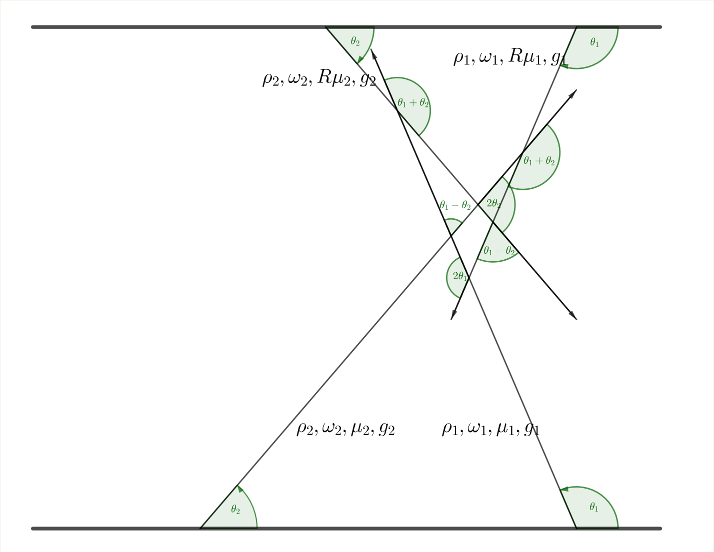
Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\sin(kx)$ with $\hat{x} = \omega_m x$, $A = 0.1\omega_m$, $k = 0.995\omega_m$, $\theta_m = \pi/6$

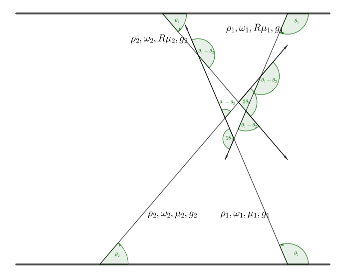
Neutrino Halo



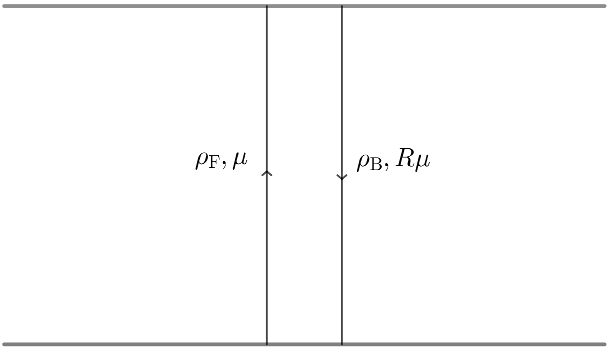
Neutrino Halo

Assumptions

- Neutrinos are translational symmetric on the emission line.
- Reflection obays Snell's law.
- Neutrinos are reflected on a fixed surface $z = L$.
- Neutrino reflections are translational symmetric.



Flavor Isospin

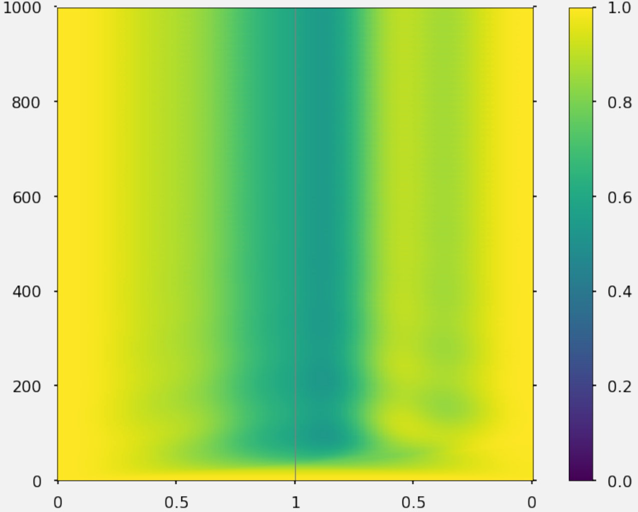


Relaxation Scheme

Algorithm

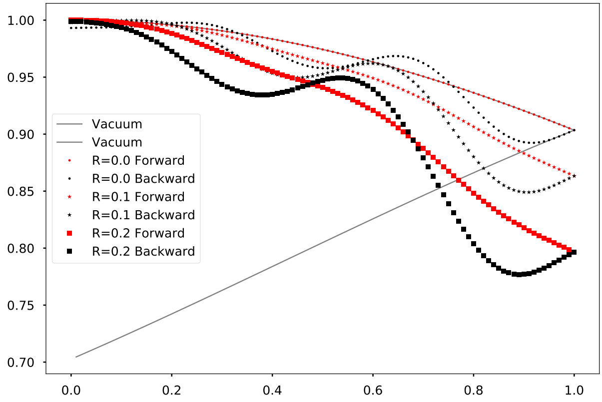
1. Calculate forward beam using null backward beam;
2. Calculate backward beam using forward beam calculated in step 1;
3. Calculate forward beam using backward beam calculated in step 2;
4. Repeat 2 and 3 until the beams reach equilibrium.

Numerical Method



Horizontal axis is the location of neutrinos; Vertical axis is the number of iteration steps; Color indicates the electron flavor probability.

Numerical Method



Linear Stability Analysis

EoM

$$i\partial_t \vec{s}_F = \mathbf{s}_F \times (\vec{H}_v + R\mu\vec{s}_B)$$
$$i\partial_t \vec{s}_B = \vec{s}_B \times (-\vec{H}_v - \mu\vec{s}_F).$$

Compare with bipolar

$$i\partial_t \vec{s} = \mathbf{s} \times (\eta\vec{H}_v + \alpha\mu\bar{\vec{s}})$$
$$i\partial_t \bar{\vec{s}} = \bar{\vec{s}} \times (\eta\vec{H}_v + \mu\vec{s})$$

Linear Stability Analysis