Neutrino Flavor Conversions in Dense Medium: Matter Stimulation, Dispersion Relation, and Neutrino Halo

PhD Defense

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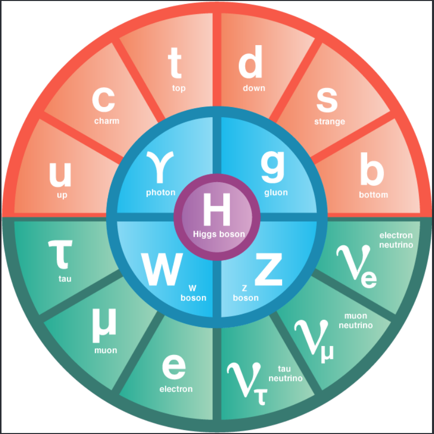
Outline for Section 1

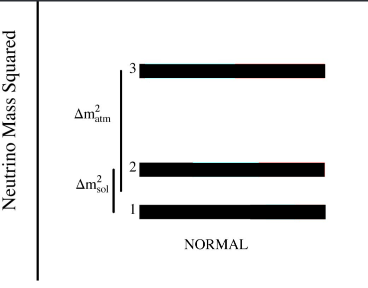
What are Neutrinos?



Elementary particles.
Source: symmetrymagazine.org

Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- none vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

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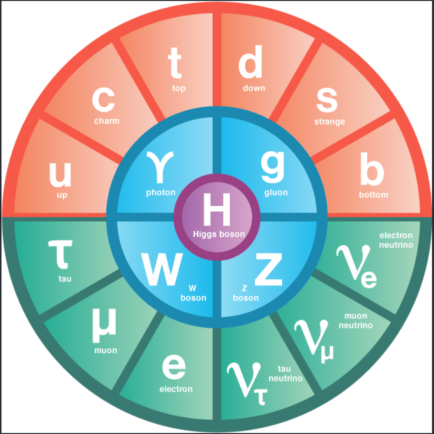

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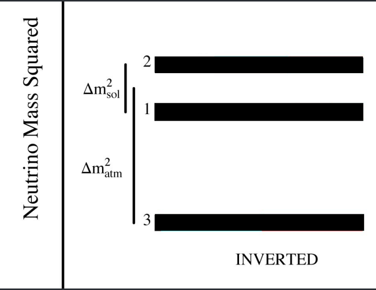

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Why Do Neutrinos Oscillate?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\text{v} & \sin\theta_\text{v} \\ -\sin\theta_\text{v} & \cos\theta_\text{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\sigma_3 + \sin 2\theta_{\mathrm{v}}\sigma_1\right)$$

- Mixing angle θ_{v}
- Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

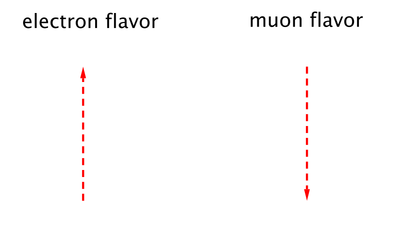
Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability:

$$P = \frac{1}{2} + s_3$$

Equation of motion:

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



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Equation of motion:

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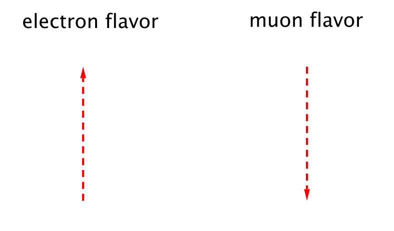
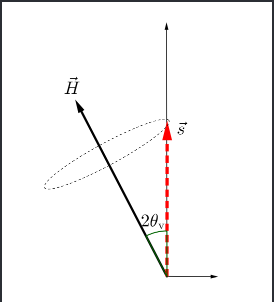
Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2} \left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \right)$$

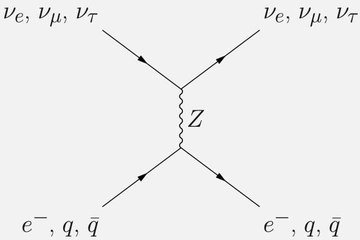
$$\to \cos 2\theta_v \begin{pmatrix} 0 \\ 0 \\ \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v \\ 0 \\ 0 \end{pmatrix}$$



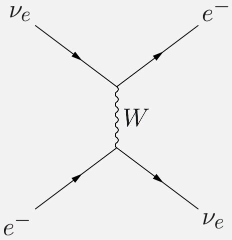


Outline for Section 2

Matter Interaction



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-.



Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2 / 2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2} G_{\mathrm{F}} n_{\mathrm{e}}(x)$

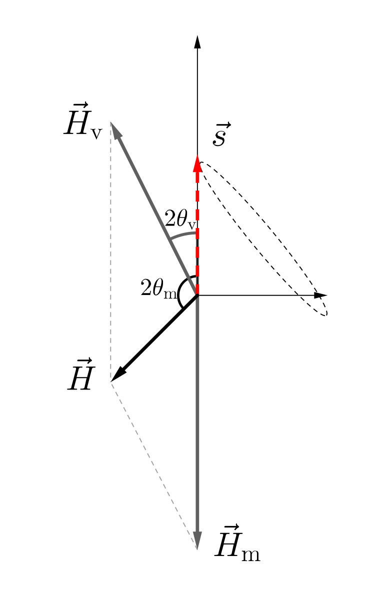
MSW Effect

$$\mathbf{H} = \boxed{\frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma}_3 + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma}_1\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma}_3}$$

$$\to \boxed{\omega_\mathrm{v}\begin{pmatrix}-\sin 2\theta_\mathrm{v}\\ 0 \\ \cos 2\theta_\mathrm{v}\end{pmatrix}} \boxed{+\begin{pmatrix}0\\ 0\\ -\lambda(x)\end{pmatrix}}$$

$$= \boxed{\vec{H}_\mathrm{v}} \boxed{+\vec{H}_\mathrm{m}(x)}$$

MSW Effect



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

$$\omega_\mathrm{v} = |\vec{H}_\mathrm{v}|$$

Oscillation frequency in **matter**:

$$\omega_\mathrm{m} = |\vec{H}|$$

Flavor states and mass states in matter

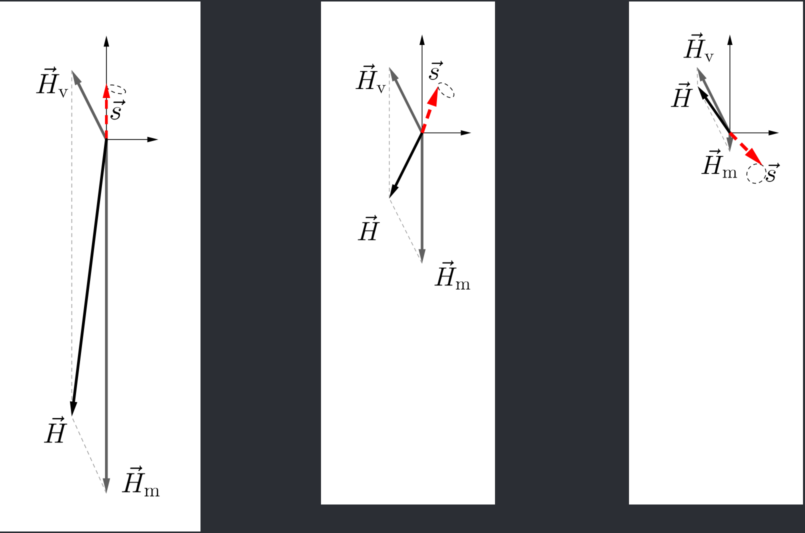
$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{H} \end{pmatrix}$$

MSW Effect

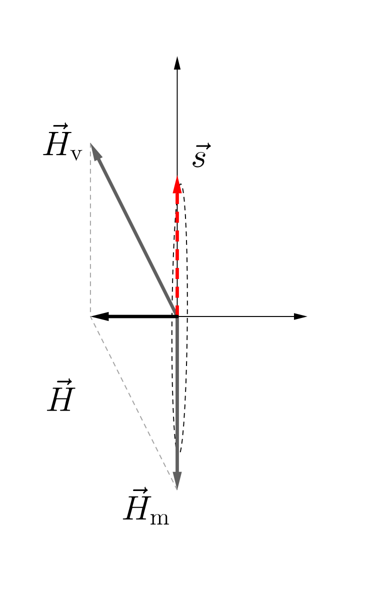
Adiabatic matter density change

Large density

Low density



MSW Effect

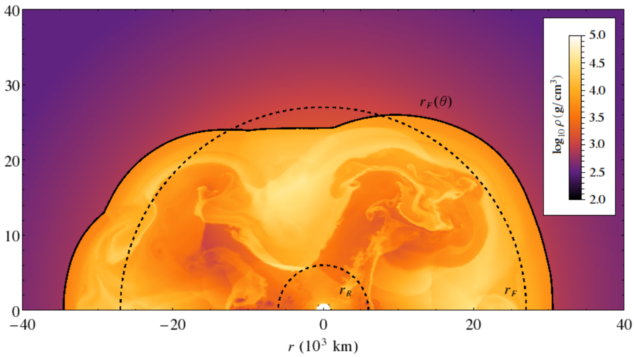


- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2}G_\mathrm{F}n_\mathrm{e} \equiv \omega_\mathrm{v}\cos 2\theta_\mathrm{v}$$

Supernova Matter Density Profile

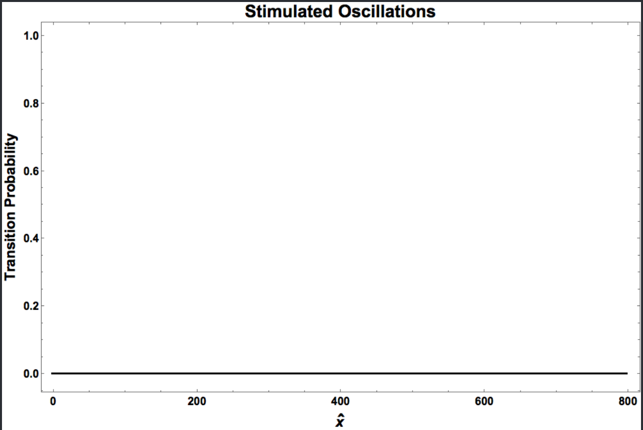
Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

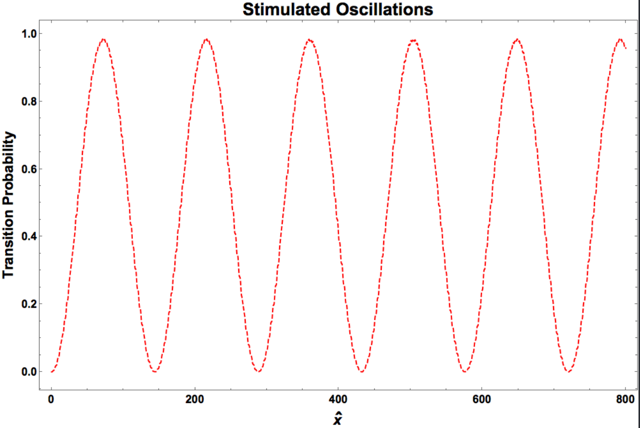
$$\lambda(x) = \lambda_0$$



Transition probabilities between mass states in matter.

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



P. Krastev and A. Smirnov (1989); A. Friedland et al (2006); J. Kneller et al (2013); K. Patton et al (2014);

Rabi Oscillations

Scheme

Rabi Oscillation

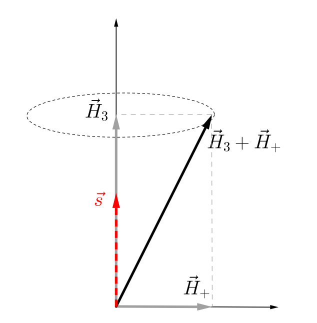
Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi Oscillations

Static Frame

$$\vec{H}_3 = \omega_\mathrm{m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Rabi Oscillations

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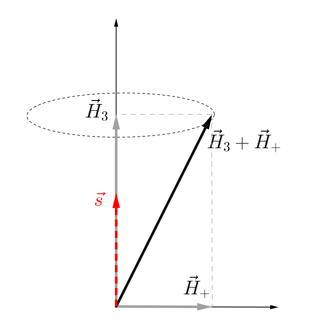
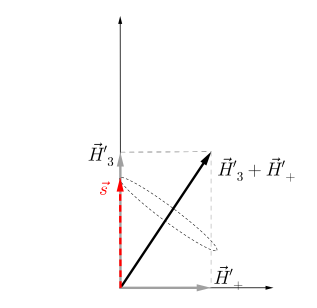


Corotating Frame

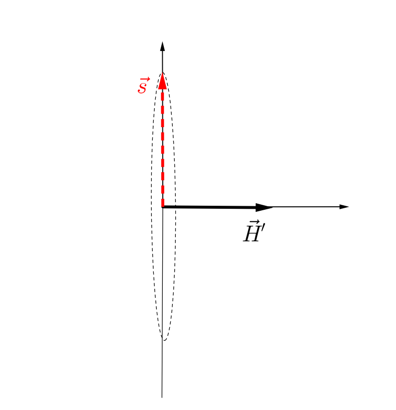
$$\vec{H}'_3 = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_\mathrm{m} - k)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \Rightarrow k = \omega_\mathrm{m}$$



Rabi Oscillations

Scheme

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = |\alpha|\sqrt{1+D^2}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

matter basis:

$$\mathbf{H} = \frac{1}{2}(-\omega_{\mathrm{m}})\,\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

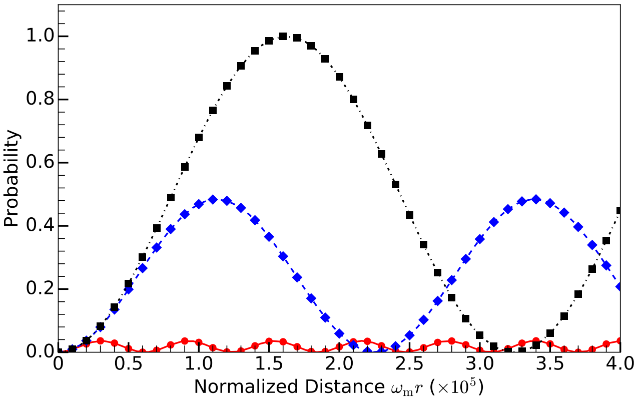
$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \cancel{\alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Rabi Formula Works



Transition between two mass states in background matter potential λ_0
Lines: Rabi formula
Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_{\rm m}$, $k = (1 - 2 \times 10^{-5})\omega_{\rm m}$, and $k = (1 - 10^{-4})\omega_{\rm m}$ respectively.

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_\mathrm{m} + \cancel{\cos 2\theta_\mathrm{m} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_\mathrm{m}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_\mathrm{m}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\-\sin(kx)\\0\end{pmatrix} + \cancel{\alpha\begin{pmatrix}\cos(-kx)\\-\sin(-kx)\\0\end{pmatrix}}$$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{\mathrm{L}} \\ \tilde{\psi}_{\mathrm{H}} \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

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where $\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Resonance condition for each mode: $nk = \omega_{\mathrm{m}}$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}}$$

Corotating frame of the second potential

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

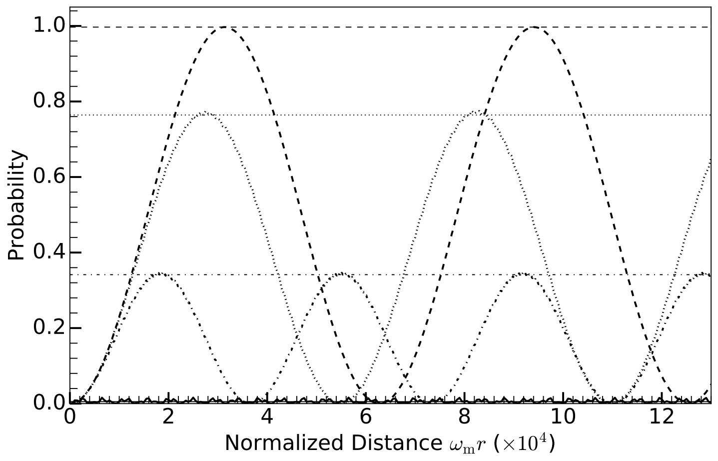
Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$



$A_1 = 10^{-4}\omega_m$, $k_1 = \omega_m$; Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_2 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_m, 10\omega_m$	$10^{-2}\omega_m, 10^{-1}\omega_m$	$5.0 \times 10^{-2}\omega_m, 10\omega_m$	$5 \times 10^{-2}\omega_m, 10^{-1}\omega_m$

Rabi Oscillations With Multiple Driving Frequencies

Consider $k_1 = \omega_\mathrm{m}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_\mathrm{m} - k_2)} \right|$$

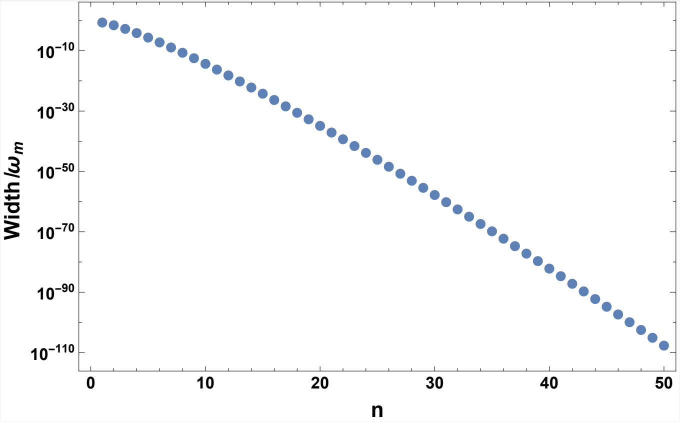
Amplitude reduces from 1 to 1/2 if

$$D' = 1 \Rightarrow \alpha_{2,C} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
For $k_1 = \omega_\mathrm{m}$, survival of resonance requires

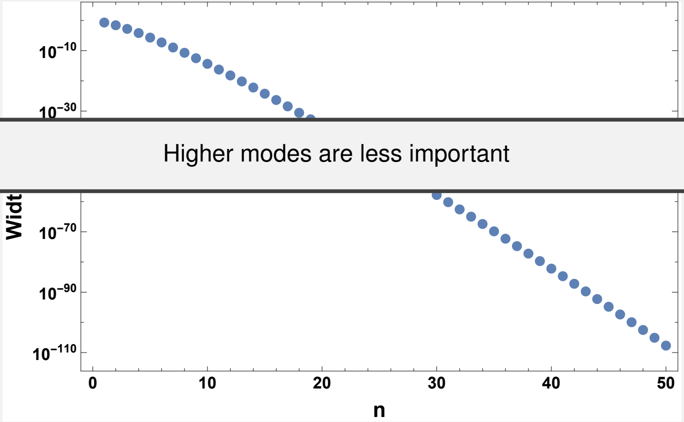
$$|\alpha_2| \ll \alpha_{2,C} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential



Higher modes are less important

Width of different modes given value of matter potential frequency k

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^{N} A_a \sin(k_a x)$$

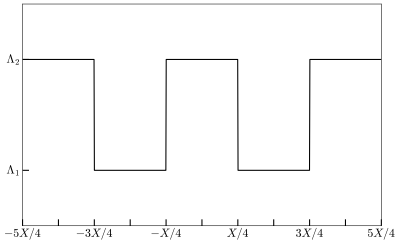
Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\begin{pmatrix} 0 & B_{\{n_a\}} e^{i\sum_a n_a k_a x} \\ B^*_{\{n_a\}} e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a}\tan 2\theta_m \left(\sum_a n_a k_a\right)\left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right)$$

Castle Wall Matter Potential



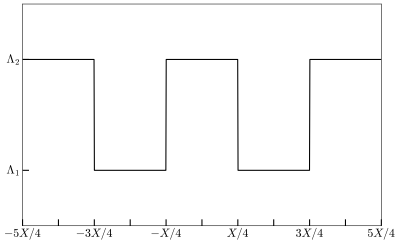
Castle wall matter profile:
$\Lambda_2 = 0.35\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$,
$\Lambda_1 = 0.15\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$ and period
$X = 2\pi/\omega_\mathrm{m}$

$$\lambda(x) = \lambda_0 + \sum_1^\infty \lambda_n \cos(k_n x)$$

where

$$\begin{aligned}
\lambda_0 &= (\Lambda_1 + \Lambda_2)/2 \\
\lambda_n &= 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi) \\
k_n &= 2\pi(2n-1)/X
\end{aligned}$$

Castle Wall Matter Potential



Castle wall matter profile:
$\Lambda_2 = 0.35\omega_v \cos 2\theta_v$,
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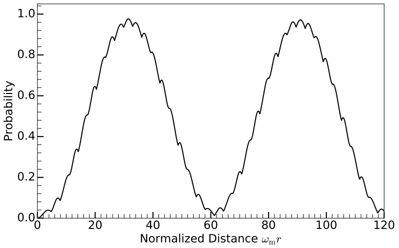
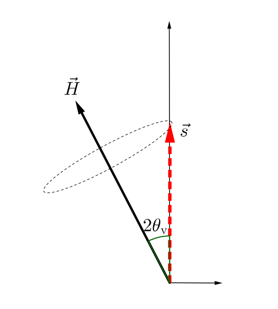


Transition probability is a Rabi resonance with small variations due to higher orders.

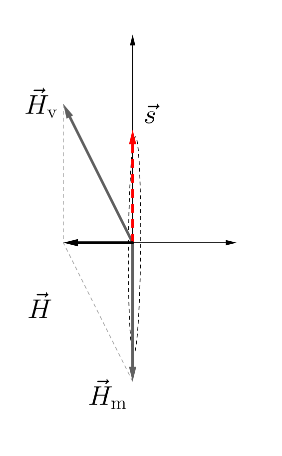
Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

Outline for Section 3

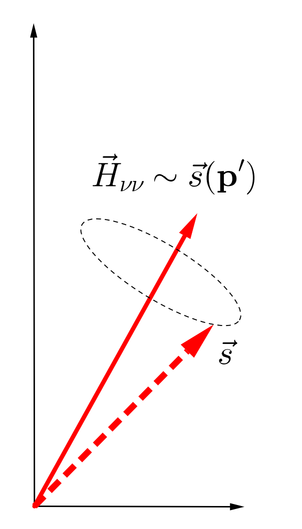
Neutrino Self-interactions

Interaction Hamiltonian $\mathbf{H}_{\nu\nu}$

$$\sqrt{2}G_F n(p')(1 - \hat{p}\cdot\hat{p}')\rho(p')$$

In Flavor Isospin space

$$-2\sqrt{2}G_F n(p')(1 - \hat{p}\cdot\hat{p}')\vec{s}(p')$$



Neutrino Self-interactions

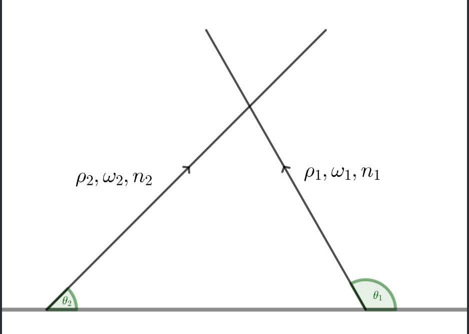
- $H_\nu = -\frac{1}{2}\omega\sigma_3$
- $H_m = \frac{1}{2}\lambda\sigma_3$
- $H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1$
- $H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2$

where

$$\mu_i = \sqrt{2}G_\mathrm{F}\xi n_i$$

Geometric factor

$$\xi = (1-\cos(\theta_1-\theta_2))$$



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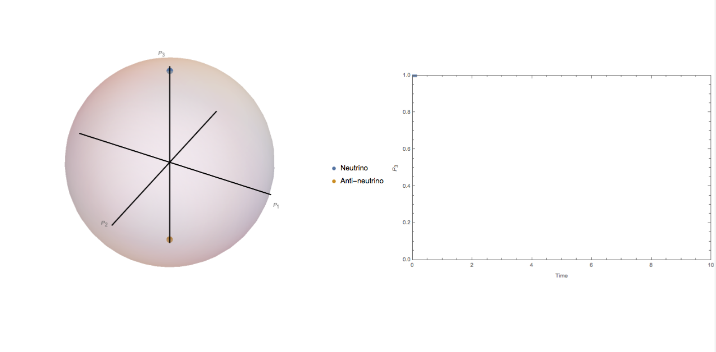

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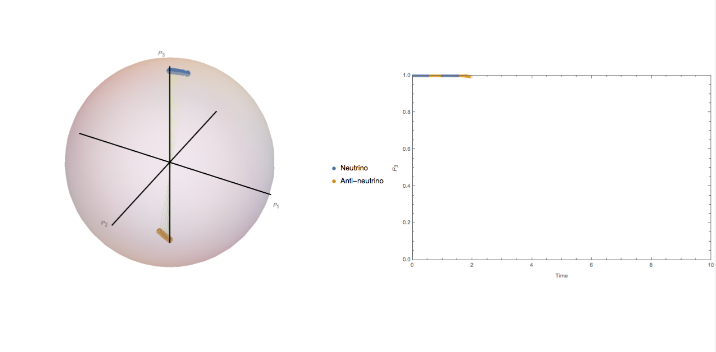

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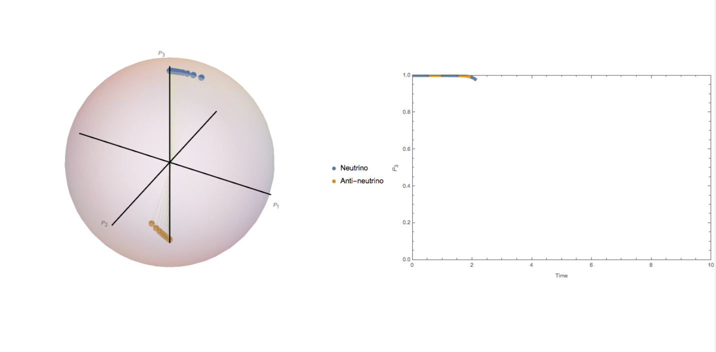

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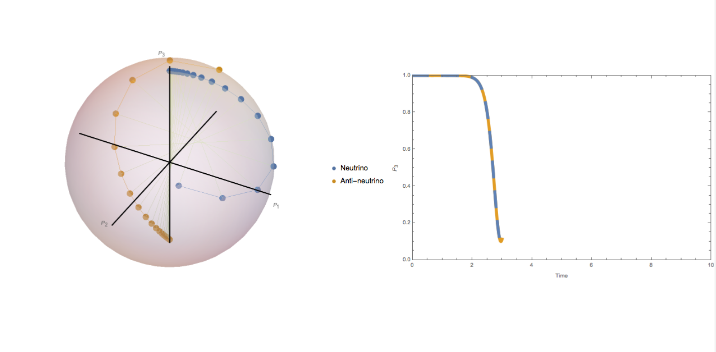

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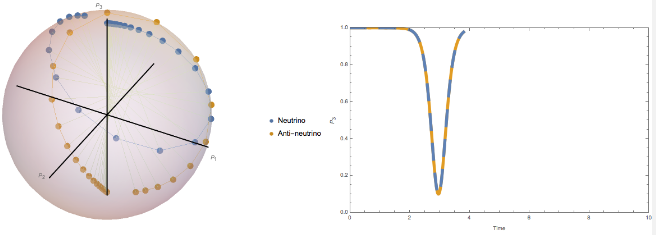

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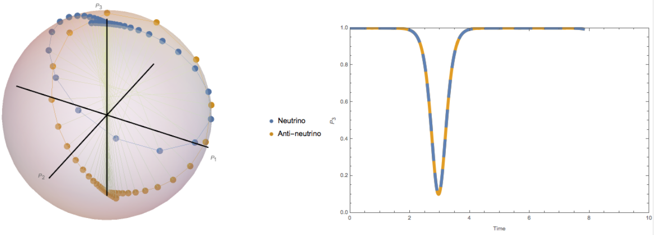

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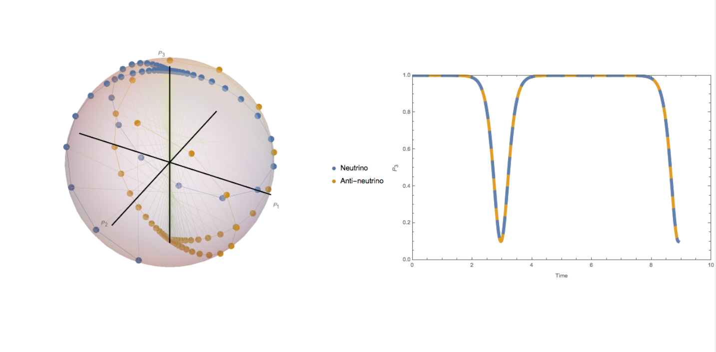

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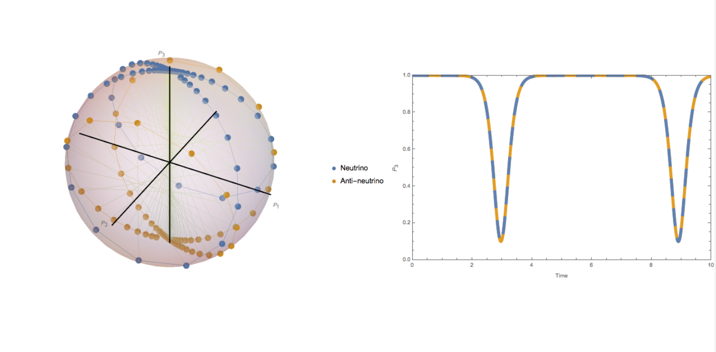

Neutrino Self-interactions

Characteristic Energy Scales

- $\omega_\mathrm{v} = \delta m^2/2E$
- $\lambda \sim G_\mathrm{F} n_\mathrm{e}$
- $\mu \sim G_\mathrm{F}(1 - \hat{v}_1 \cdot \hat{v}_2) n_\nu$

Vacuum oscillation oscillation frequencies

$$\omega_\mathrm{v} = \frac{\Delta m^2}{2E} \sim \frac{2\pi}{1\mathrm{km}} \left(\frac{\Delta m^2_{32}}{2.5 \times 10^{-3}\mathrm{eV}^2} \right) \left(\frac{1MeV}{E} \right)$$

$$\sim \frac{2\pi}{33\mathrm{km}} \left(\frac{\Delta m^2_{12}}{7.5 \times 10^{-5}\mathrm{eV}^2} \right) \left(\frac{1\mathrm{MeV}}{E} \right)$$

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Neutrino self-interactions might lead to faster oscillations, since $\mu \gg \omega_\mathrm{v}$.

Neutrino Self-interactions

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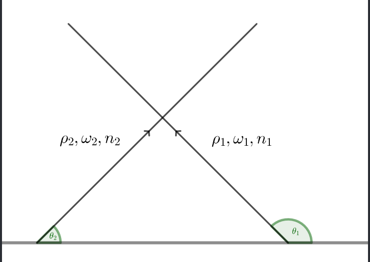
Suppose we have neutrino flux $10^{50}\mathrm{ergs} \cdot \mathrm{s}^{-1}$. We estimate the potential at radius R to be

$$\mu \sim \frac{1}{0.01 km} \left(\frac{100\mathrm{km}}{R} \right)^2 \left(\frac{1\mathrm{MeV}}{E} \right)$$

Linear Stability Analysis

$$H_{\nu\nu,2} = \frac{1}{2}\mu_1\rho_1\xi, \qquad H_{\nu\nu,1} = \frac{1}{2}\mu_2\rho_2\xi$$



ρ_1: neutrinos;

ρ_2: antineutrinos

$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = 2\pi/3, \theta_2 = \pi/6$$

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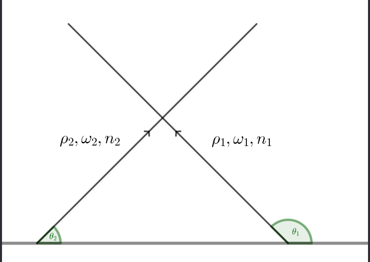

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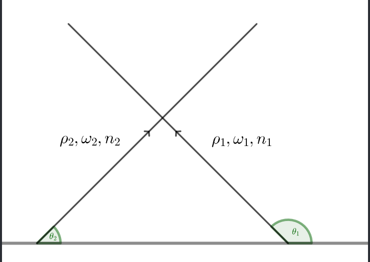

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$$i\partial_z \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \begin{pmatrix} \omega_\mathrm{v} - \mu\xi & \mu\xi \\ -\mu\xi & -\omega_\mathrm{v} + \mu\xi \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

Linear Stability Analysis

Solution of the form

$$\begin{pmatrix} \epsilon_1(z) \\ \epsilon_2(z) \end{pmatrix} = \begin{pmatrix} \epsilon_1(0) \\ \epsilon_2(0) \end{pmatrix} e^{iK_z z}$$

Eigenvalues or collective oscillation frequencies

$$K_z = \pm\sqrt{\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi)}$$

Identify the condition for complex eigenvalues

$$\omega_{\mathrm{v}}(\omega_{\mathrm{v}} - 2\mu\xi) < 0$$

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K_z is instability in z direction for our model.

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Similar analysis can be done for all four dimensions t, x, y, z,

$$\left(\Omega, K_x, K_y, K_z\right)$$

Dispersion Relation

Izaguirre, I., Raffelt, G., & Tamborra, I. (2017). *Fast Pairwise Conversion of Supernova Neutrinos: A Dispersion Relation Approach*. Physical Review Letters, 118(2), 021101.

- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.

Dispersion Relation

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- Linear stability analysis → dispersion relation for Ω and $\mathbf{K}$.
- Instabilities occur in dispersion relation gaps.

Dispersion Relation

Equation of motion for off-diagonal element of density matrix (Izaguirre et al, 2017)

$$i(\partial_t + v \cdot \nabla_r)\epsilon(v) = v^\mu (\Lambda + \Phi)_\mu - \int d\Gamma' v^\mu v'_\mu G(v')\epsilon(v')$$

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- v^μ: four-velocity of neutrinos $(1, v)$
- Λ: matter contribution $(\sqrt{2}G_F n_e, \sqrt{2}G_F n_e v_e)$
- Φ: neutrino flux $(\sqrt{2}G_F n_\nu, \sqrt{2}G_F n_\nu v)$
- $G(\mathrm{v}')$: electron lepton number of neutrinos

  $$\sqrt{2}G_F \int_0^\infty \frac{E^2 dE}{2\pi^2} \left(n_{\nu_e} - n_{\bar{\nu}_e}\right)$$

Dispersion Relation

Collective mode of off-diagonal element

$$\epsilon \longrightarrow \tilde{\epsilon} e^{-i(\Omega t - \boldsymbol{K}\cdot\boldsymbol{r})}$$

Dispersion Relation

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Replacement:

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$$v^{\mu}(K_{\mu} - (\Lambda + \Phi)_{\mu})\tilde{\epsilon}(v) = -\int d\Gamma' v^{\mu} v'_{\mu} G(v')\tilde{\epsilon}(v')$$

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with $\boxed{K_\mu \longrightarrow (\Omega, \boldsymbol{K})}$

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with $\boxed{k_\mu \longrightarrow (\omega, \boldsymbol{k})}$

Without neutrino self-interaction: $v^\mu k_\mu = 0$

Dispersion Relation

Rewrite

$$
\begin{aligned}
& -\int d\Gamma' \, v^\mu v'_\mu G(v')\tilde{\epsilon}(v') \\
= \, & v^\mu \left(-\int d\Gamma' \, v'_\mu G(v')\tilde{\epsilon}(v') \right) \\
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EoM

$$
v^\mu k_\mu \tilde{\epsilon}(v) = v^\mu a_\mu
$$

Dispersion Relation

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Rewrite

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$$= \boxed{v^\mu} \boxed{\left(-\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')\right)}$$

$$\equiv v^\mu \boxed{a_\mu}$$

EoM

$$v^\mu k_\mu \tilde{\epsilon}(v) = v^\mu a_\mu$$

$$\Longrightarrow$$

$$\tilde{\epsilon}(v) = v^\mu a_\mu / v^\mu k_\mu$$

Collect all terms of a_μ

$$v^\mu \left(\delta^\nu_\mu + \int d\Gamma' \frac{G(v')v'_\mu v^\nu}{v^\alpha k_\alpha}\right) a_\nu = 0$$

Dispersion Relation

Axial symmetryc: $v^\alpha k_\alpha = \omega(1 - n\cos\theta)$ where $n = |\boldsymbol{k}|/\omega$

Nontrivial solutions to EoM requires

$$v^\mu \left(\omega\delta^\nu{}_\mu + N^\nu{}_\mu\right) a_\nu = 0$$

$$I_n(\theta) = \int_{\cos\theta_2}^{\cos\theta_1} d\cos\theta G(\theta) \frac{\cos^n\theta}{1 - n\cos\theta}$$

$$N^\mu{}_\nu \longrightarrow$$

$$\begin{pmatrix} \frac{1}{2}I_0 & 0 & 0 & -\frac{1}{2}I_1 \\ 0 & -\frac{1}{4}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(I_0 - I_2) & 0 \\ \frac{1}{2}I_1 & 0 & 0 & -\frac{1}{2}I_2 \end{pmatrix}$$

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$$\left(\omega\delta^{\nu}{}_{\mu} + N^{\nu}{}_{\mu}\right)a_{\nu} = 0$$

$$\Rightarrow$$

$$\mathrm{Det}\left(\omega I + N\right) = 0,$$

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$$\omega = \frac{1}{4}(I_0 - I_2), \quad -\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$$

Dispersion Relation

$$a_\mu = -\int d\Gamma' v'_\mu G(v')\tilde{\epsilon}(v')$$

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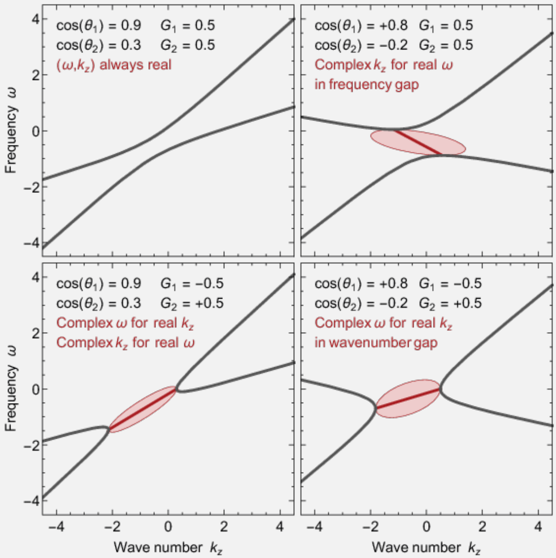
- $\frac{1}{4}(I_0 - I_2)$: MAA solution
- $-\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 - 2I_1 + I_2)(I_0 + 2I_1 + I_2)}\right)$: MZA solution

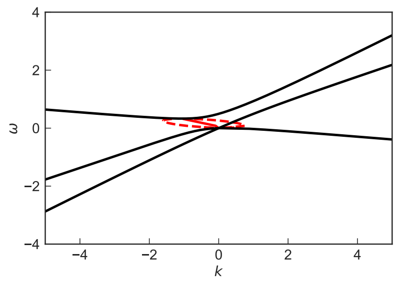
Dispersion Relation

Izaguirre et al, 2017

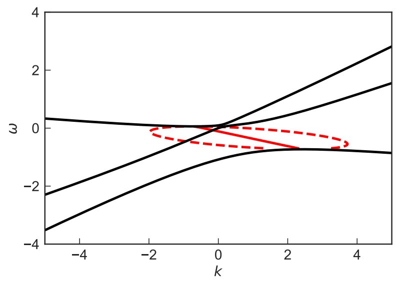


Dispersion Relation

Three beams



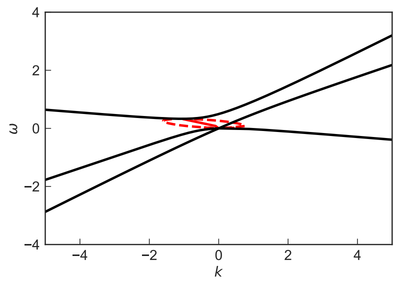
MAA solutions



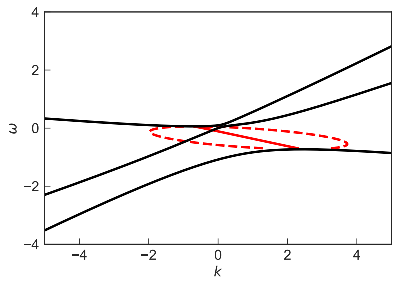
MZA solutions

Dispersion Relation

Three beams



MAA solutions



MZA solutions

Gap and instability correspondance does NOT hold in three beams case.

Dispersion Relations

Solve k for MAA solutions

$$1 = \frac{1}{4k} \int \mathrm{d}u G(u) \frac{1-u^2}{\omega/k - u}.$$

around $\omega \to 0$.

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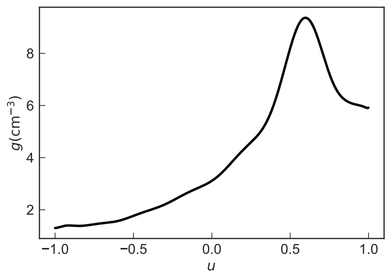
Gap between dispersion relation and $\omega = 0$

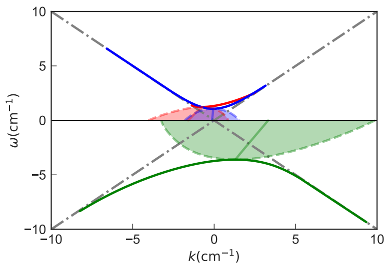
Dispersion Relations

Remake of Fig.3 of Izaguirre et al, 2017

Define $u = \cos\theta$
Garching spectrum:



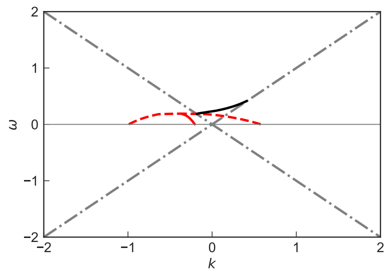
Garching spectrum $G(u)$



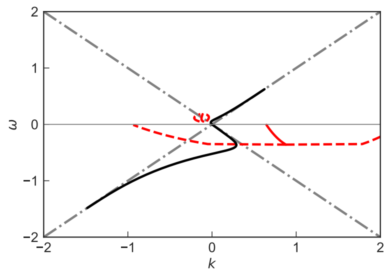
MAA: red; MZA: blue and green

Dispersion Relations

Box spectrum: -0.1 within range $u \in [-1, -0.3)$ and 1 within range $u \in [-0.3, 1]$



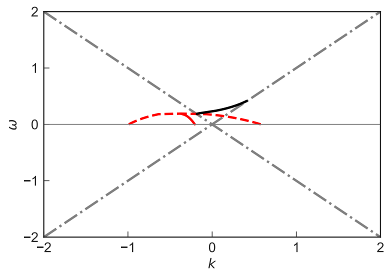
MAA solution



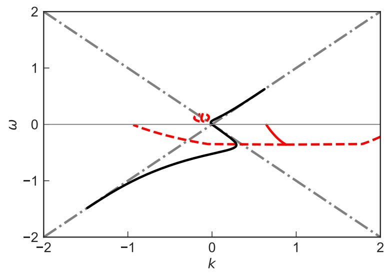
MZA solution

Dispersion Relations

Box spectrum: -0.1 within range $u \in [-1, -0.3)$ and 1 within range $u \in [-0.3, 1]$



MAA solution



MZA solution

Instabilities do not only show up as gaps.

Summary of Dispersion Relation

- Neutrino oscillation instability corresponds to gaps of dispersion relations for two beams model;
- It can NOT be generalized to multiple emission beams;
- Gaps should be defined as the gap between dispersion relation and $\Omega = 0$ instead of the gaps between dispersion relations.

Acknowledgement

I am very thankful to my advisor Professor Huaiyu Duan, Dr. Sajad Abbar, and Dr. Shashank Shalgar, and Joshua Martin, for all the help in both research and life.
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Backup Slides

Hamiltonian, and Basis, and Rabi Oscillations

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1.$$

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0,

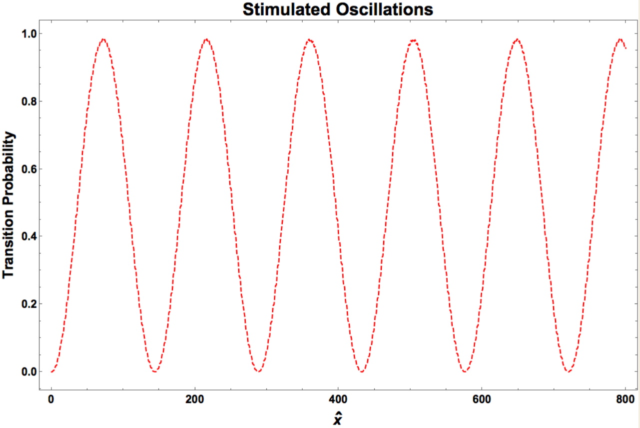
$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

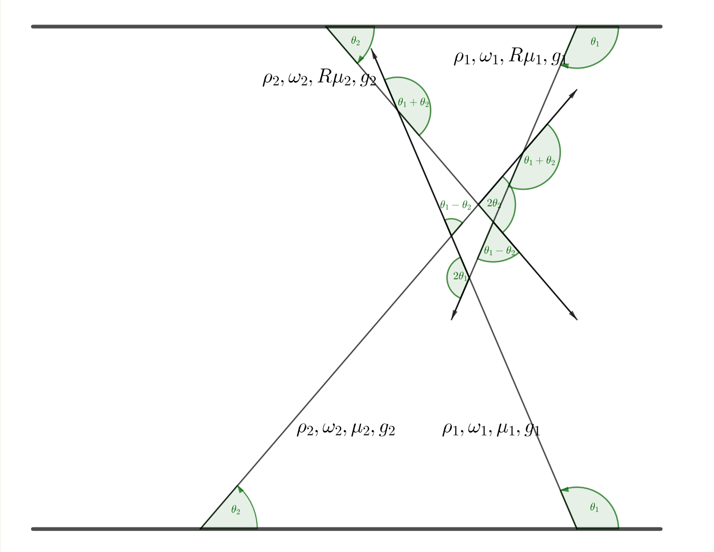
Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\sin(kx)$ with $\hat{x} = \omega_m x$, $A = 0.1\omega_m$, $k = 0.995\omega_m$, $\theta_m = \pi/6$

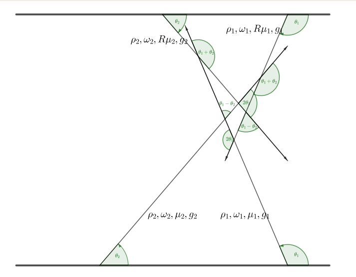
Neutrino Halo



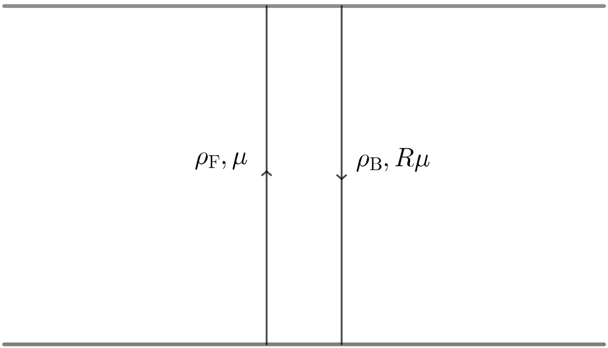
Neutrino Halo

Assumptions

- Neutrinos are translational symmetric on the emission line.
- Reflection obays Snell's law.
- Neutrinos are reflected on a fixed surface $z = L$.
- Neutrino reflections are translational symmetric.



Flavor Isospin

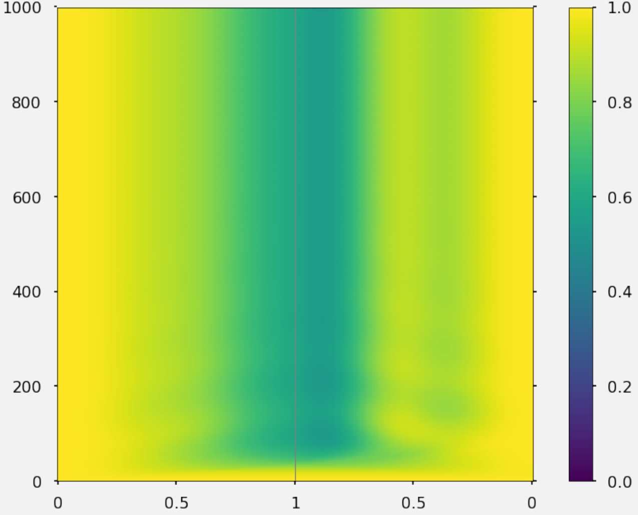


Relaxation Scheme

Algorithm

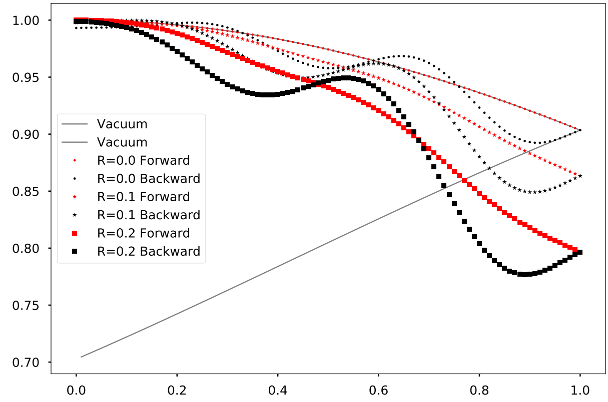
1. Calculate forward beam using null backward beam;
2. Calculate backward beam using forward beam calculated in step 1;
3. Calculate forward beam using backward beam calculated in step 2;
4. Repeat 2 and 3 until the beams reach equilibrium.

Numerical Method



Horizontal axis is the location of neutrinos; Vertical axis is the number of iteration steps; Color indicates the electron flavor probability.

Numerical Method



Linear Stability Analysis

EoM

$$
\begin{aligned}
i\partial_t \vec{s}_F &= \mathrm{s}_F \times (\vec{H}_v + R\mu\vec{s}_B) \\
i\partial_t \vec{s}_B &= \vec{s}_B \times (-\vec{H}_v - \mu\vec{s}_F).
\end{aligned}
$$

Compare with bipolar

$$
\begin{aligned}
i\partial_t \vec{s} &= \mathrm{s} \times (\eta\vec{H}_v + \alpha\mu\bar{\vec{s}}) \\
i\partial_t \bar{\vec{s}} &= \bar{\vec{s}} \times (\eta\vec{H}_v + \mu\vec{s})
\end{aligned}
$$

Linear Stability Analysis