



Neutrino Flavor Conversions in Dense Medium: Matter Stimulation, Dispersion Relation, and Neutrino Halo

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PhD Defense

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Supervisor: Huaiyu Duan

Outline for Section 1

1. Neutrino Oscillations

1.1 Neutrinos as Fundamental Particles

1.2 Why Do Neutrinos Oscillate

2. Matter Stimulated Oscillations

2.1 Matter Interactions, MSW Effect, and Solar Neutrino Problem

2.2 Stimulated Neutrino Oscillations and Rabi Oscillations

2.3 Basis and Formalism

2.4 Multiple Frequencies in Matter Potential

3. Neutrino Oscillations and Dispersion Relation

3.1 Neutrino Self-interactions

3.2 Linear Stability Analysis

3.3 Dispersion Relation

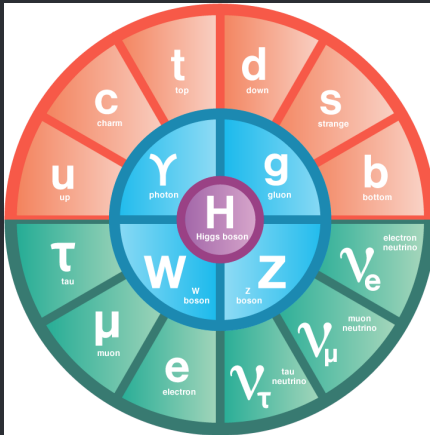
4. Neutrino Halo Problem

4.1 Flavor Isospin Picture

4.2 Numerical Method

5. Summary

What are Neutrinos?

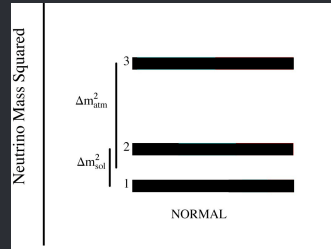


Elementary particles.

Source: symmetrymagazine.org

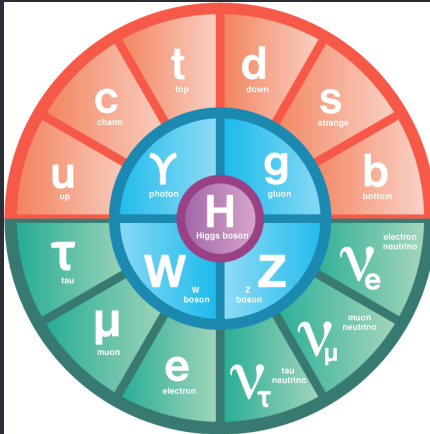
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- none vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

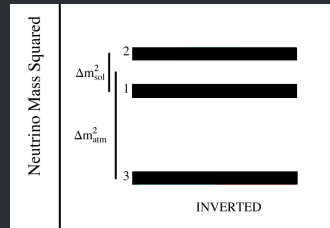
What are Neutrinos?



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Why Do Neutrinos Oscillate?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_\nu}{2} (-\cos 2\theta_\nu \sigma_3 + \sin 2\theta_\nu \sigma_1)$$

- Oscillation frequency:

$$\omega_\nu = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_ν

Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

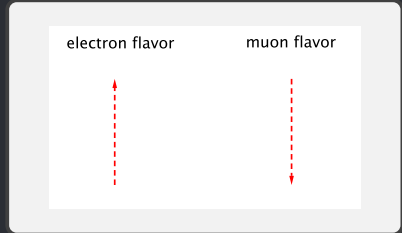
Flavor isospin: $\vec{S} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Equation of motion

$$\dot{\vec{S}} = \vec{S} \times \vec{H}$$



Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{S} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Electron flavor survival probability

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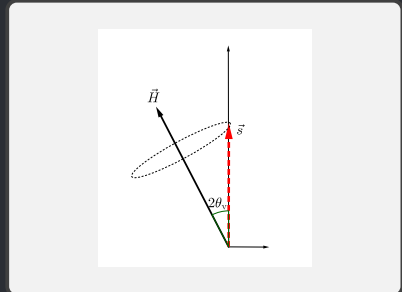
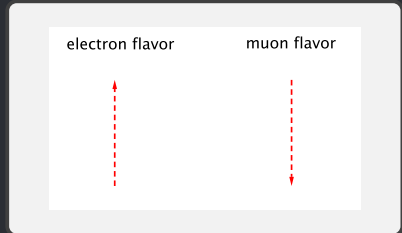
Equation of motion

$$\dot{\vec{S}} = \vec{S} \times \vec{H}$$

Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1)$$

$$\rightarrow \cos 2\theta_v \begin{pmatrix} 0 & \\ 0 & \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v & \\ 0 & 0 \end{pmatrix}$$



Outline for Section 2

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- 2.1 Matter Interactions, MSW Effect, and Solar Neutrino Problem
- 2.2 Stimulated Neutrino Oscillations and Rabi Oscillations
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- 2.4 Multiple Frequencies in Matter Potential

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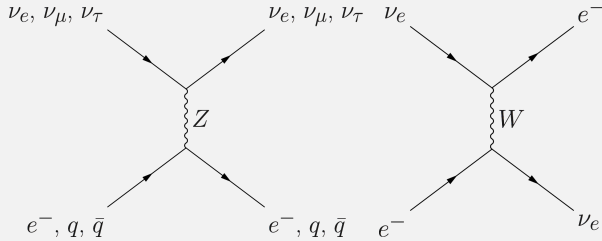
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- 4.1 Flavor Isospin Picture
- 4.2 Numerical Method

5. Summary

Matter Interaction



Neutral current interaction between ν_e, ν_μ, ν_τ , and e^-, q, \bar{q} , etc.

Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_v = \delta m^2/2E$):

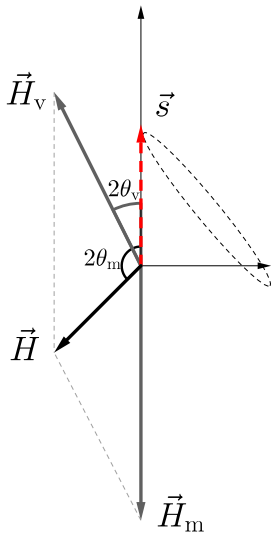
$$\mathbf{H} = \frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1) + \frac{\lambda(x)}{2} \sigma_3$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_F n_e(x)$

MSW Effect

$$\begin{aligned} \mathbf{H} &= \frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1) + \frac{\lambda(x)}{2} \sigma_3 \\ &\rightarrow \omega_v \begin{pmatrix} -\sin 2\theta_v & \\ & 0 \\ & & \cos 2\theta_v \end{pmatrix} + \begin{pmatrix} 0 & \\ & 0 \\ & & -\lambda(x) \end{pmatrix} \\ &= \tilde{H}_v + \tilde{H}_m(x) \end{aligned}$$

MSW Effect



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

$$\omega_v = |\vec{H}_v|$$

Oscillation frequency in **matter**:

$$\omega_m = |\vec{H}|$$

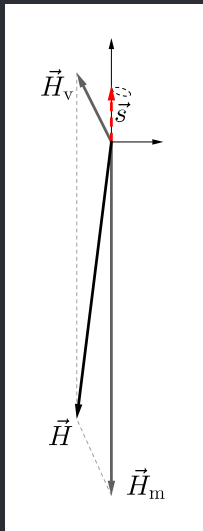
Flavor states and mass states in matter

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

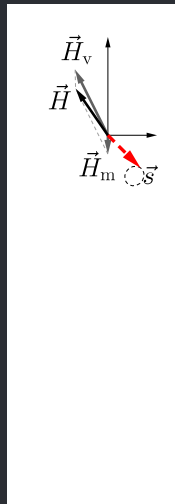
MSW Effect

Adiabatic matter density change

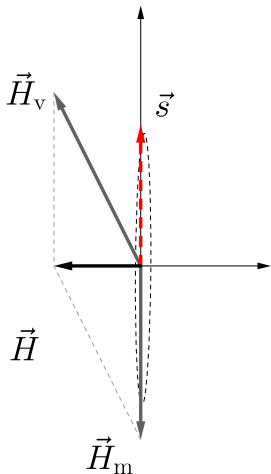
Large density



Low density



MSW Effect

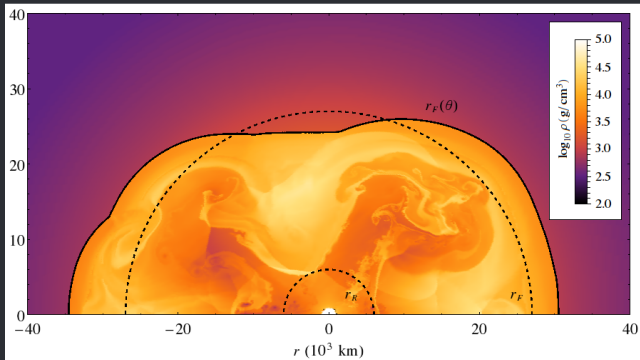


- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2}G_F n_e \equiv \omega_v \cos 2\theta_v$$

Supernova Matter Density Profile

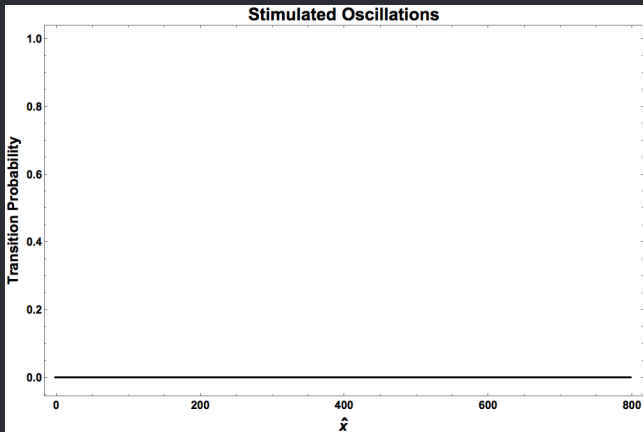
Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

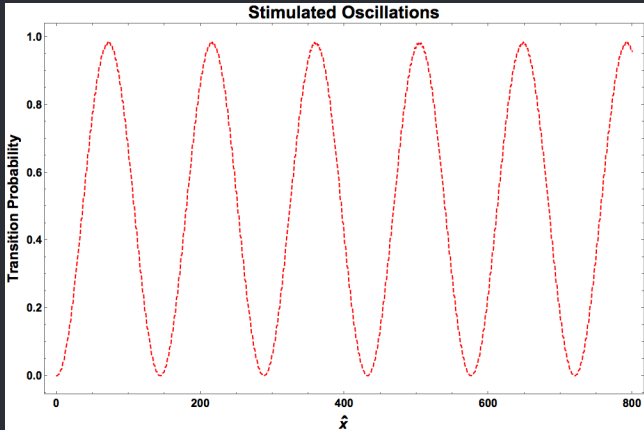
$$\lambda(x) = \lambda_0$$



Transition probabilities between mass states in matter.

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A \cos(kx)$$



P. Krastev and A. Smirnov (1989); A. Friedland et al (2006); J. Kneller et al (2013); K. Patton et al (2014);

Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2} \begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

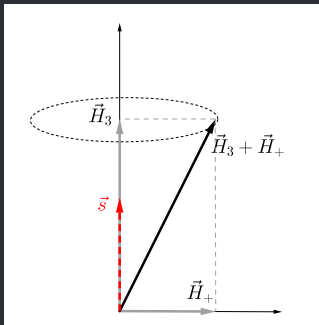
Scheme



Rabi Oscillations

Static Frame

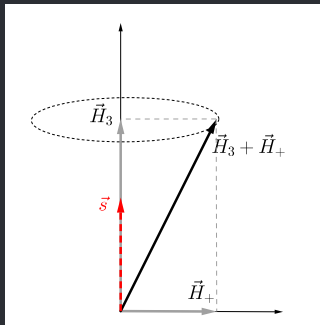
$$\vec{H}_3 = \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Rabi Oscillations

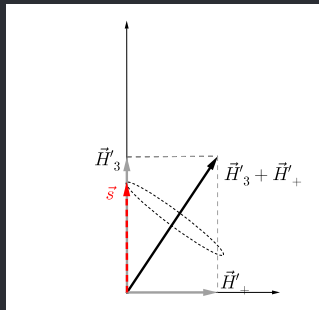
Static Frame

$$\vec{H}_3 = \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Corotating Frame

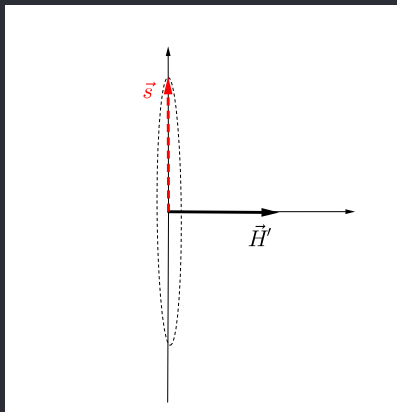
$$\vec{H}'_3 = (\omega_m - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_m - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow k = \omega_m$$



Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2} \begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi formula

$$P_{1 \rightarrow 2} = \frac{1}{1 + D^2} \sin^2 \left(\frac{\Omega_R}{2} t \right).$$

Relative detuning

$$D = \left| \frac{\omega_m - k}{\alpha} \right|.$$

Rabi frequency

$$\Omega_R = |\alpha| \sqrt{1 + D^2}$$

Scheme



Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

matter basis:

$$\mathbf{H} = \frac{1}{2} (-\omega_m) \sigma_3$$

Hamiltonian in Matter Basis

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A \cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2} (-\omega_m + A \cos(kx) \cos 2\theta_m) \sigma_3 - \frac{A \cos(kx)}{2} \sin 2\theta_m \sigma_1$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_m$$

$$\mathbf{H} = \frac{1}{2} \left(-\omega_m + \cancel{\cos 2\theta_m} A \cos(kx) \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1$$

$$\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_m}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

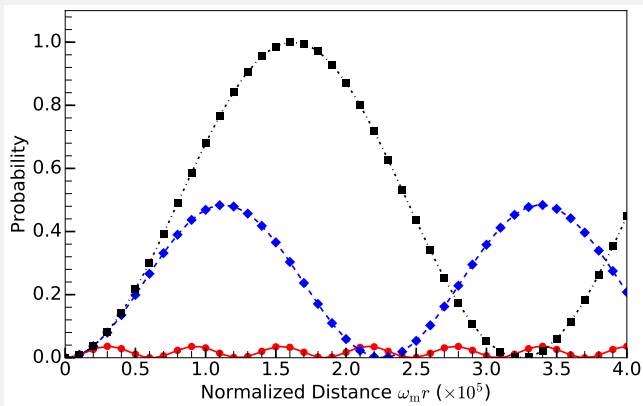
$$k \sim \omega_m$$

$$\mathbf{H} = \frac{1}{2} \left(-\omega_m + \cancel{\cos 2\theta_m A \cos(kx)} \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1$$

$$\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

$$\alpha = \frac{\sin 2\theta_m}{2} A$$

Rabi Formula Works



Transition between two mass states in background matter potential λ_0

Lines: Rabi formula

Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_m$, $k = (1 - 2 \times 10^{-5})\omega_m$, and $k = (1 - 10^{-4})\omega_m$ respectively.

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2} \left(-\omega_m + \cancel{\cos 2\theta_m A \cos(kx)} \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1$$

$$\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2} (-\omega_m + A \cos(kx) \cos 2\theta_m) \sigma_3 - \frac{A \cos(kx)}{2} \sin \theta_m \sigma_1.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_L \\ \tilde{\psi}_H \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_m}{2} \int_0^x A \cos(k\tau) d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A \cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\tilde{H} = -\frac{\omega_m}{2} \sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2} \alpha_n e^{i(nk)x} \\ \frac{1}{2} \alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_m J_n(A \cos 2\theta_m/k)$.

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A \cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

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where $\alpha_n = -(-i)^n nk \tan 2\theta_m J_n(A \cos 2\theta_m/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Resonance condition for each mode: $nk = \omega_m$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\hat{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}$$

Corotating frame of the second potential,

$$\tilde{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

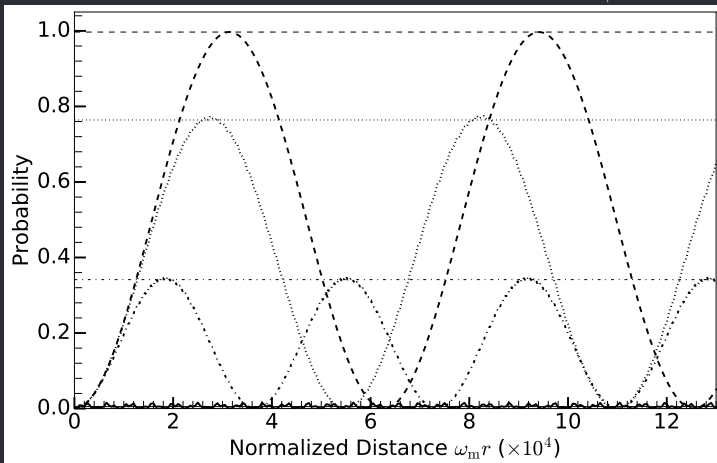
Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$



Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_m, 10\omega_m$	$10^{-2}\omega_m, 10^{-1}\omega_m$	$5.0 \times 10^{-2}\omega_m, 10\omega_m$	$5 \times 10^{-2}\omega_m, 10^{-1}\omega_m$

Rabi Oscillations With Multiple Driving Frequencies

Consider $k_1 = \omega_m$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$

Amplitude reduces from 1 to 1/2 if

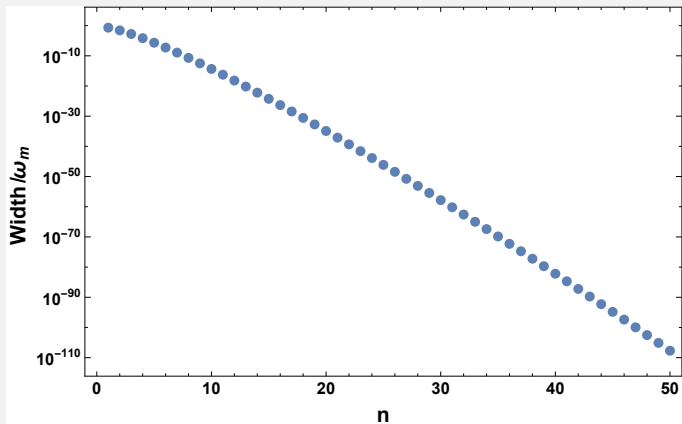
$$D' = 1 \Rightarrow \alpha_{2,C} \equiv \sqrt{2|\alpha_1(k_2 - \omega_m)|}.$$

Two driving frequencies k_1 , and k_2 , with amplitude α_1 , and α_2

For $k_1 = \omega_m$, survival of resonance requires

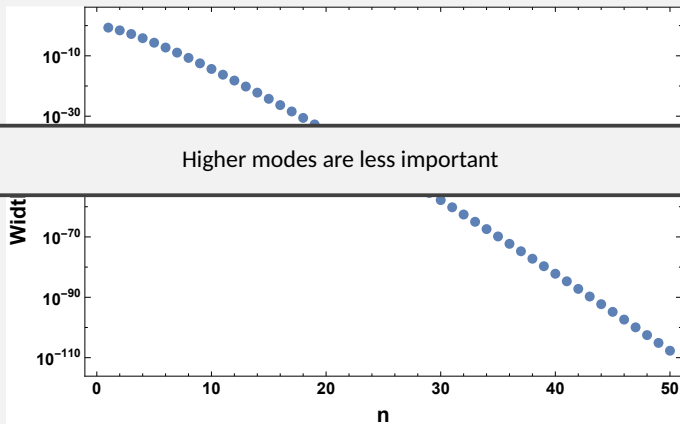
$$|\alpha_2| \ll \alpha_{2,C} \equiv \sqrt{2|\alpha_1(k_2 - \omega_m)|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^N A_a \sin(k_a x)$$

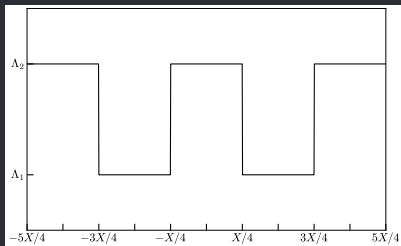
Hamiltonian in Rabi Basis

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2} \sigma_3 + \frac{1}{2} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \begin{pmatrix} 0 & B_{\{n_a\}} e^{i \sum_a n_a k_a x} \\ B_{\{n_a\}}^* e^{-i \sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a \right) \left(\prod_a J_{n_a} \left(\frac{A_a}{k_a} \cos 2\theta_m \right) \right)$$

Castle Wall Matter Potential



Castle wall matter profile:

$\Lambda_2 = 0.35\omega_v \cos 2\theta_v$, $\Lambda_1 = 0.15\omega_v \cos 2\theta_v$
and period $X = 2\pi/\omega_m$

$$\lambda(x) = \lambda_0 + \sum_1^{\infty} \lambda_n \cos(k_n x)$$

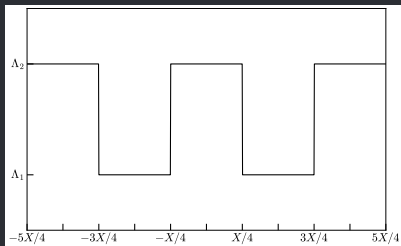
where

$$\lambda_0 = (\Lambda_1 + \Lambda_2)/2$$

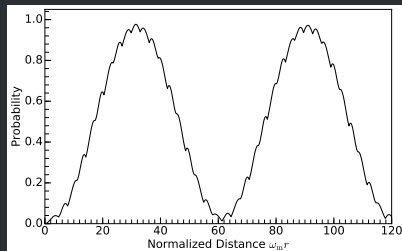
$$\lambda_n = 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi)$$

$$k_n = 2\pi(2n - 1)/X$$

Castle Wall Matter Potential



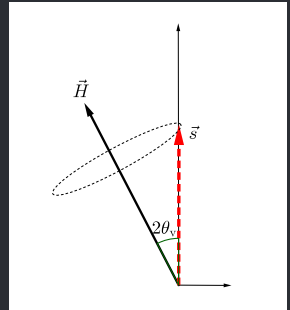
Castle wall matter profile:
 $\Lambda_2 = 0.35\omega_v \cos 2\theta_v$, $\Lambda_1 = 0.15\omega_v \cos 2\theta_v$
and period $X = 2\pi/\omega_m$



Transition probability is a Rabi resonance with small variations due to higher orders.

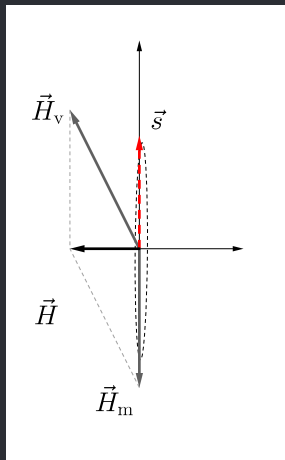
Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor states are not mass states.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.



Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.

For matter potential

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

Resonance condition

$$nk = \omega_m$$

Summary of Stimulated Oscillations

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter potential can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

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 - 4.2 Numerical Method
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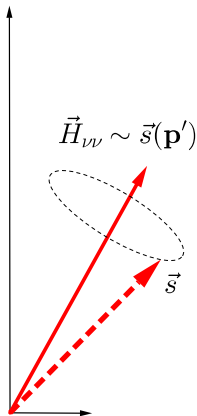
Neutrino Self-interactions

Interaction Hamiltonian $\mathbf{H}_{\nu\nu}$

$$\sqrt{2}G_F(1 - \hat{p} \cdot \hat{p}')\rho(\mathbf{p}')$$

In Flavor Isospin space

$$-2\sqrt{2}G_F(1 - \hat{p} \cdot \hat{p}')n(\mathbf{p}')\vec{s}(\mathbf{p}')$$



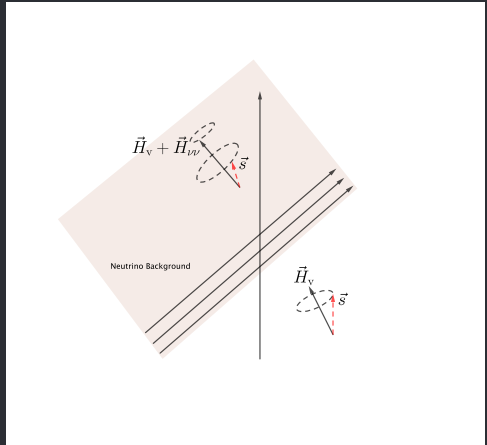
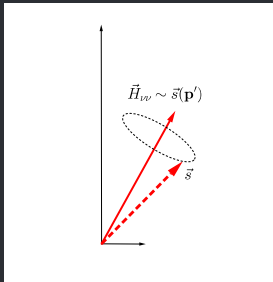
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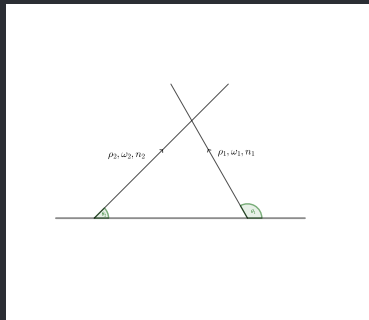


Neutrino Self-interactions

- $H_v = -\eta \frac{1}{2} \omega \sigma_3$
- $H_m = -\frac{1}{2} \lambda \sigma_3$
- $H_{\nu\nu,2} = -\frac{1}{2} \mu_1 \rho_1 \xi$
- $H_{\nu\nu,1} = -\frac{1}{2} \mu_2 \rho_2 \xi$

$$\mu_i = \sqrt{2} G_F (1 - \cos(\theta_1 - \theta_2)) n_i$$

$$\xi = (1 - \cos(\theta_1 - \theta_2))$$



Fast Mode

Characteristic Length Scales

- $\omega_v = \delta m^2 / 2E$
- $\lambda \sim G_F n_e$
- $\mu \sim G_F (1 - \mathbf{p}_1 \cdot \mathbf{p}_2) n_\nu$

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Vacuum oscillation frequencies

$$\begin{aligned}\omega_v &= \frac{\Delta m^2}{2E} \sim 6.3 \times 10^{-3} \text{m}^{-1} \frac{\Delta m_{32}^2}{2.5 \times 10^{-3} \text{eV}^2} \frac{1 \text{MeV}}{E} \\ &\sim 1.90 \times 10^{-4} \text{m}^{-1} \frac{\delta m_{12}^2}{7.5 \times 10^{-5} \text{eV}^2} \frac{1 \text{MeV}}{E}\end{aligned}$$

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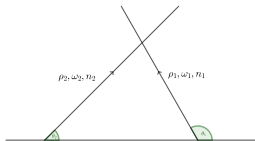
Neutrino self-interactions might lead to Faster oscillations.

Linear Stability Analysis

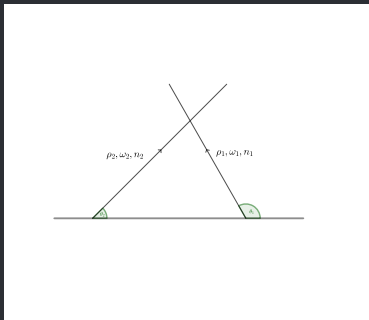
$$H_{vv,2} = \frac{1}{2}\mu_1\rho_1\xi, \quad H_{vv,1} = \frac{1}{2}\mu_2\rho_2\xi$$

$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = \theta_2 \equiv \theta$$



Linear Stability Analysis



$$H_{vv,2} = \frac{1}{2}\mu_1\rho_1\xi, \quad H_{vv,1} = \frac{1}{2}\mu_2\rho_2\xi$$

$$i\partial_z\rho_i = [H_i, \rho_i]$$

$$\theta_1 = \theta_2 \equiv \theta$$

$$\rho_i = \frac{1}{2} \begin{pmatrix} 1 & \epsilon_i \\ \epsilon_i^* & -1 \end{pmatrix}$$

Linear Stability Analysis

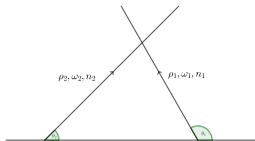
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$$\rho_i = \frac{1}{2} \begin{pmatrix} 1 & \epsilon_i \\ \epsilon_i^* & -1 \end{pmatrix}$$

$$i\partial_z \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \frac{1}{2\sin\theta} \begin{pmatrix} \omega'_i + \mu f_2 \xi & -\mu f_2 \xi \\ -\mu f_1 \xi & \omega'_2 + \mu f_1 \xi \end{pmatrix}$$



Linear Stability Analysis

Identify the condition for complex eigenvalues

$$-2\sqrt{-f_1 f_2 \xi^2} < \eta(-\omega_1 + \omega_2)/\mu + (f_2 - f_1)\xi < 2\sqrt{-f_1 f_2 \xi^2}$$

Dispersion Relation

$$\text{Det}(\omega + N_{\nu}^{\nu}) = 0,$$

where

$$N_{\nu}^{\mu} = \omega P_{\nu}^{\mu} \rightarrow \begin{pmatrix} \frac{1}{2}l_0 & 0 & 0 & -\frac{1}{2}l_1 \\ 0 & -\frac{1}{4}(l_0 - l_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(l_0 - l_2) & 0 \\ \frac{1}{2}l_1 & 0 & 0 & -\frac{1}{2}l_2 \end{pmatrix}$$

$$I_n(\theta) = \int_{\cos \theta_2}^{\cos \theta_1} d \cos \theta G(\theta) \frac{\cos^n \theta}{1 - n \cos \theta}$$

Dispersion Relation

$$\text{Det}(\omega + N_{\nu}^{\nu}) = 0,$$

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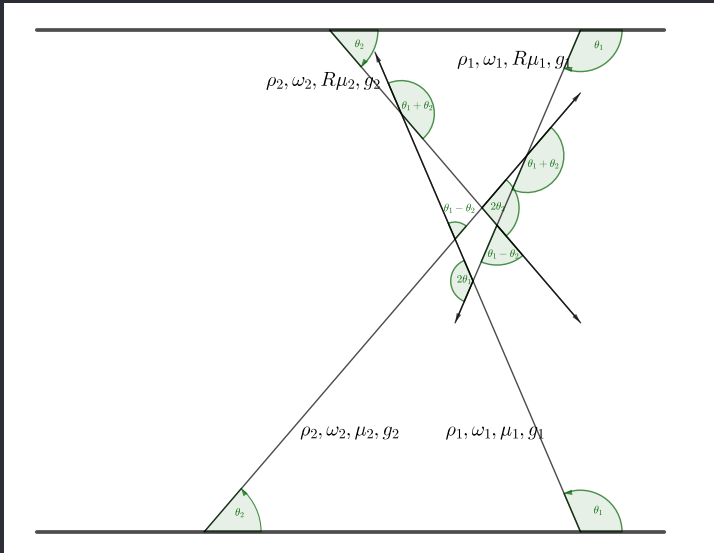
Solutions

$$\omega = \frac{1}{4}(l_0 - l_2), \quad -\frac{1}{4} \left(l_0 - l_2 \pm \sqrt{(l_0 - 2l_1 + l_2)(l_0 + 2l_1 + l_2)} \right)$$

Outline for Section 4

1. Neutrino Oscillations
 - 1.1 Neutrinos as Fundamental Particles
 - 1.2 Why Do Neutrinos Oscillate
2. Matter Stimulated Oscillations
 - 2.1 Matter Interactions, MSW Effect, and Solar Neutrino Problem
 - 2.2 Stimulated Neutrino Oscillations and Rabi Oscillations
 - 2.3 Basis and Formalism
 - 2.4 Multiple Frequencies in Matter Potential
3. Neutrino Oscillations and Dispersion Relation
 - 3.1 Neutrino Self-interactions
 - 3.2 Linear Stability Analysis
 - 3.3 Dispersion Relation
4. Neutrino Halo Problem
 - 4.1 Flavor Isospin Picture
 - 4.2 Numerical Method
5. Summary

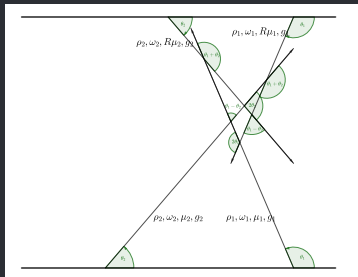
Neutrino Halo



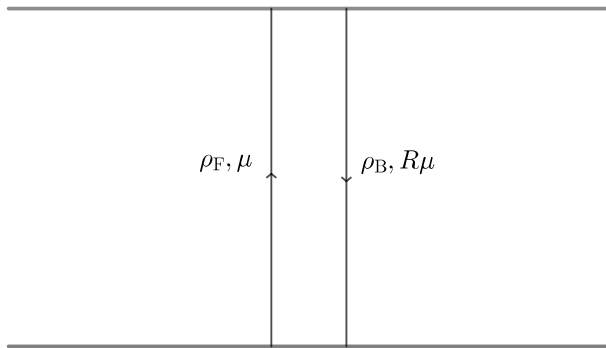
Neutrino Halo

Assumptions

- Neutrinos are translational symmetric on the emission line.
- Reflection obeys Snell's law.
- Neutrinos are reflected on a fixed surface $z = L$.
- Neutrino reflections are translational symmetric.



Flavor Isospin

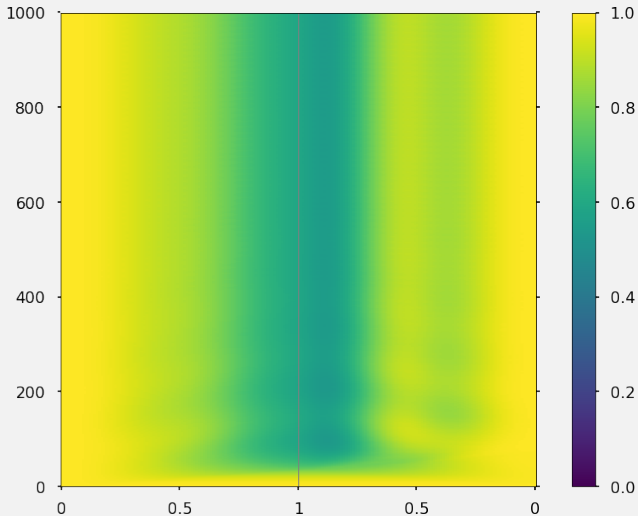


Relaxation Scheme

Algorithm

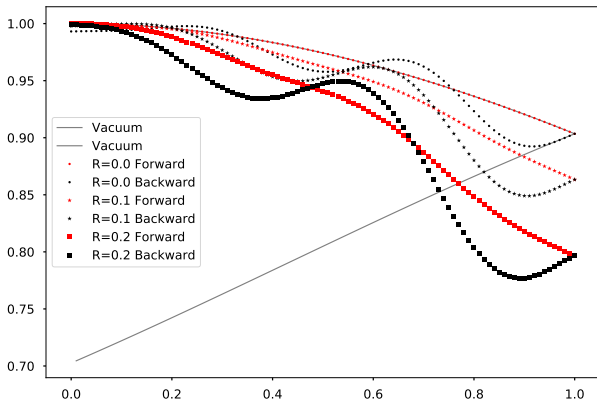
1. Calculate forward beam using null backward beam;
2. Calculate backward beam using forward beam calculated in step 1;
3. Calculate forward beam using backward beam calculated in step 2;
4. Repeat 2 and 3 until the beams reach equilibrium.

Numerical Method



Horizontal axis is the location of neutrinos; Vertical axis is the number of iteration steps; Color indicates the electron flavor probability.

Numerical Method



Linear Stability Analysis

EoM

$$i\partial_t \vec{\mathfrak{s}}_F = \mathbf{s}_F \times (\vec{H}_\nu + R\mu \vec{\mathfrak{s}}_B)$$

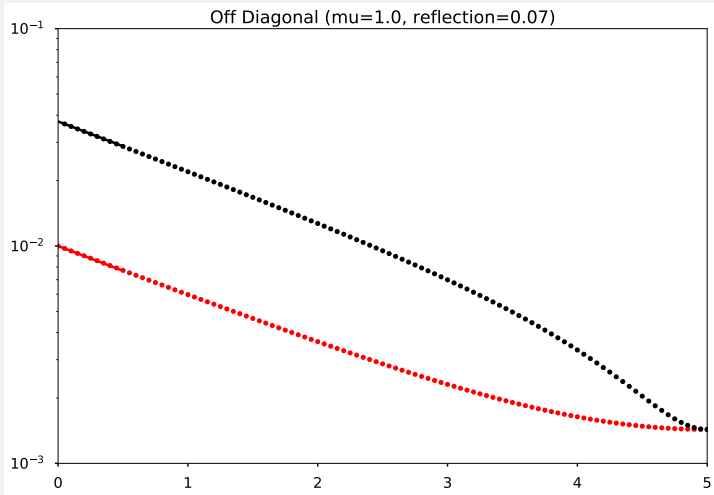
$$i\partial_t \vec{\mathfrak{s}}_B = \vec{\mathfrak{s}}_B \times (-\vec{H}_\nu - \mu \vec{\mathfrak{s}}_F).$$

Compare with bipolar

$$i\partial_t \vec{\mathfrak{s}} = \mathbf{s} \times (\eta \vec{H}_\nu + \alpha \mu \vec{\bar{\mathfrak{s}}})$$

$$i\partial_t \vec{\bar{\mathfrak{s}}} = \vec{\bar{\mathfrak{s}}} \times (\eta \vec{H}_\nu + \mu \vec{\mathfrak{s}})$$

Linear Stability Analysis



Outline for Section 5

1. Neutrino Oscillations
 - 1.1 Neutrinos as Fundamental Particles
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5. Summary

Summary

- The fact that neutrino flavor states are not mass states causes vacuum oscillations.
- MSW resonance happens when matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
- Even matter profile doesn't match MSW requirement, variation in matter profile can cause resonances.
- Single frequency perturbations in matter profile is a combination of many Rabi oscillations.
- How to understand and calculate systems with multi-frequency matter profile (turbulence).
- Combine periodic or even turbulent matter profile with neutrino self-interaction.

Acknowledgement

I am very thankful to my advisor Professor Huaiyu Duan, Dr. Sajad Abbar, and Dr. Shashank Shalgar, and Joshua Martin, for all the help in both research and life. Supported by DOE EPSCoR grant #DE-SC0008142 at UNM.

Citations

\TeX , \LaTeX , and Beamer

\TeX is a programming language for the typesetting of documents. It was created by Donald Erwin Knuth in the late 1970s and it is documented in *The \TeX book* [1].

In the early 1980s, Leslie Lamport created the initial version of \LaTeX , a high-level language on top of \TeX , which is documented in *\LaTeX : A Document Preparation System* [2]. There exists a healthy ecosystem of packages that extend the base functionality of \LaTeX ; *The \LaTeX Companion* [3] acts as a guide through the ecosystem.

In 2003, Till Tantau created the initial version of Beamer, a \LaTeX package for the creation of presentations. Beamer is documented in the *User's Guide to the Beamer Class* [4].

Bibliography

\TeX , \LaTeX , and Beamer

- [1] Donald E. Knuth. *The \TeX book*. Addison-Wesley, 1984.
- [2] Leslie Lamport. *\LaTeX : A Document Preparation System*. Addison-Wesley, 1986.
- [3] M. Goossens, F. Mittelbach, and A. Samarin. *The \LaTeX Companion*. Addison-Wesley, 1994.
- [4] Till Tantau. *User's Guide to the Beamer Class Version 3.01*. Available at <http://latex-beamer.sourceforge.net>.
- [5] A. Mertz and W. Slough. Edited by B. Beeton and K. Berry. *Beamer by example*. In TUGboat, Vol. 26, No. 1., pp. 68-73.

Hamiltonian, and Basis, and Rabi Oscillations

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2} (-\omega_m + \delta\lambda(x) \cos 2\theta_m) \sigma_3 - \frac{\delta\lambda(x)}{2} \sin 2\theta_m \sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$$\mathbf{H} = \frac{1}{2} (-\omega_m + \cos 2\theta_m A \cos(kx)) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1.$$

Stimulated Neutrino Oscillations

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis: Hamiltonian is diagonalized with only background matter profile λ_0 ,

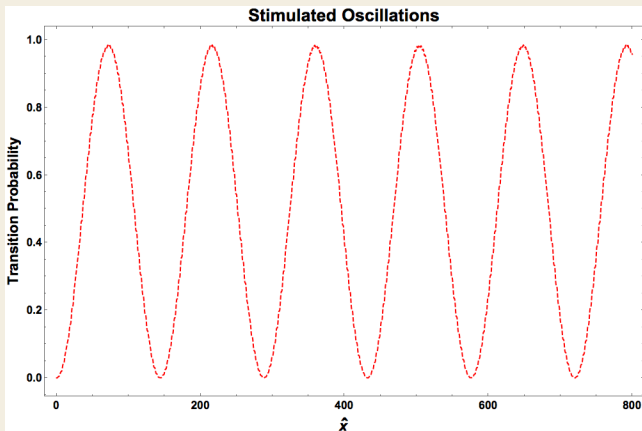
$$H_{\text{background}} = -\frac{\omega_m}{2}\sigma_3.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}(-\omega_m + \delta\lambda(x)\cos 2\theta_m)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin\theta_m\sigma_1.$$

Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A \sin(kx)$ with $\hat{x} = \omega_m x$, $A = 0.1\omega_m$, $k = 0.995\omega_m$, $\theta_m = \pi/6$