Dispersion Relation Gaps and Neutrino Flavor Instabilities in Fast Modes*

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Abstract

ABSTRACT PLACEHOLDER

I. INTRODUCTION

Neutrino flavor conversions in vacuum are linear effects in Schrodinger equation, in other words, the vacuum Hamiltonian doesn't depend on the state of neutrino flavors. However, neutrinos propagate in dense neutrino media demonstrate highly nonlinear flavor transformations due to forward scattering interactions of neutrinos. Such interactions lead to flavor conversions instabilities. The technique used to investigate the nonlinear neutrino flavor conversion is linear stability analysis [1, 2]. Recent studies by I. Izaguirre, G. Raffelt, and I. Tamborra show that linear stability analysis indicates the dispersion relation [3]. They also showed that dispersion relation can be defined and calculated in linear regime of neutrino flavor conversions. Moreover, they conclued that neutrino flavor conversion instabilities occur within gaps in dispersion relation. We argue that neutrino flavor conversion instabilities is not exactly mapped to gaps in dispersion relation for discrete emission with more than two angles or continuous angular distributions of neutrino emissions. To begin with, we review reference 3 to explain the dispersion relation in Sec. II.

II. DISPERSION RELATION OF NEUTRINO FLAVOR CONVERSION

We consider two-flavor scenario of neutrino oscillations. As an initial condition, all neutrinos and antineutrinos are emitted approximately as electron flavors. For the purpose of linear stability analysis, the single particle density matrix for neutrinos is explicitly defined as

$$\rho = \frac{1}{2} \begin{pmatrix} 1 & \epsilon \\ \epsilon^* & -1 \end{pmatrix}. \tag{1}$$

To determine the flavor evolution, Liouville equation of neutrinos is used,

$$i(\partial_t + \mathbf{v} \cdot \nabla)\rho = [H, \rho], \tag{2}$$

where H is the Hamiltonian for neutrino oscillations. Density matrix and equation of motion for antineutrinos are defined in the same maner with the corresponding Hamiltonian for antineutrinos.

In principle, neutrino oscillations Hamiltonian depend on three different contributions, vacuum oscillations H_{v}, interactions with matter H_{m}, and interactions with neutrinos themselves $H_{\nu\nu}$. The concentration of this work is on fast neutrino oscillations, which would

occur even without neutrino mass differences. However, the vacuum term can always be combined with matter term by redefining new matter potential. Thus neglecting vacuum term doesn't change the formalism of linear stability analysis. For completeness, we write down the vacuum oscillations term

$$H_{\mathrm{v}} = -\frac{\omega_{\mathrm{v}}}{2}\sigma_3, \tag{3}$$

where $\omega = \frac{\delta m^2}{2}$ with δm^2 being the mass squared difference in this two flavor scenario. Interaction with matter is described by matter potential

$$H_{\mathrm{m}} = \frac{1}{2}\lambda\sigma_3. \tag{4}$$

where $\lambda = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}$. G_{F} is the Fermi constant and n_{e} is the number density of electrons in the background.

In order to calculate the neutrino forward scattering, the spectrum of neutrino (antineutrinos) distributions $f_{\nu_{\mathrm{e}}(\bar{\nu}_{\mathrm{e}})}(\hat{\mathbf{v}}, E)$ and $f_{\nu_{\mathrm{x}}(\bar{\nu}_{\mathrm{x}})}(\hat{\mathbf{v}}, E)$ is required, where $\hat{\mathbf{v}}$ is the velocity direction of neutrinos (antineutrinos). Neutrino forward scattering potential is

$$H_{\nu\nu} = \sqrt{2}G_{\mathrm{F}} \iint \frac{\mathrm{d}\cos\theta'\mathrm{d}\phi'}{4\pi} v^{\mu}v'_{\mu} \int \frac{E'^2\mathrm{d}E'}{2\pi^2}(\rho_n - \bar{\rho}_n), \tag{5}$$

where $v^{\mu} = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}$ is the four velocity of (anti)neutrinos in spherical coordinate system, ρ_n and $\bar{\rho}_n$ are the flavor density matrix of neutrino ensemble,

$$\rho_n(\hat{\mathbf{v}}, E) = \left(f_{\nu_{\mathrm{e}}}(\hat{\mathbf{v}}, E) - f_{\nu_{\mathrm{x}}}(\hat{\mathbf{v}}, E)\right)\rho \tag{6}$$

$$\bar{\rho}_n(\hat{\mathbf{v}}, E) = \left(f_{\bar{\nu}_{\mathrm{e}}}(\hat{\mathbf{v}}, E) - f_{\bar{\nu}_{\mathrm{x}}}(\hat{\mathbf{v}}, E)\right)\bar{\rho}. \tag{7}$$

In 3 the authors assumed that the distribution of ν_x and $\bar{\nu}_x$ are the same, namely $f_{\nu_{\mathrm{x}}}(\hat{\mathbf{v}}, E) - f_{\bar{\nu}_{\mathrm{x}}}(\hat{\mathbf{v}}, E) = 0$. We follow the same simplifications in this paper. In addition, we following the same definition of electron lepton number (ELN) of neutrinos travelling in direction $\hat{\mathbf{v}}$,

$$G(\hat{\mathbf{v}}) = \sqrt{2}G_{\mathrm{F}} \int \frac{E'^2\mathrm{d}E'}{2\pi^2}(f_{\nu_{\mathrm{e}}}(\cos\theta', \phi', E') - f_{\bar{\nu}_{\mathrm{e}}}(\cos\theta', \phi', E')). \tag{8}$$

The ELN is useful in linear stability analysis where we assumed that ϵ is small and evolve with the form $\epsilon_0 e^{-i(\Omega t - \mathbf{K}\cdot\mathbf{x})}$. In linear regime, constant matter background simply shifts the Fourier mode frequencies and doesn't generate or eliminate gaps in dispersion relation. Accordingly we neglect matter background.

Linear stability analysis of Eq. (2) for axial symmetric neutrino emission shows that

$$\det\left[\omega\mathrm{I} + \frac{1}{2}\begin{pmatrix} I_0 & 0 & 0 & -I_1 \\ 0 & -\frac{1}{2}(I_0 - I_2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(I_0 - I_2) & 0 \\ I_1 & 0 & 0 & -I_2 \end{pmatrix}\right] = 0, \tag{9}$$

where

$$I_m = \int_{-1}^{1} du\, G(u) \frac{u^m}{1 - (|k|/\omega)\, u}. \tag{10}$$

where we define $u = \cos\theta$. Two categories of solutions was found, namely the multi-azimuthal angle (MAA) solution and multi-zenith angle (MZA) solution [2]. The MAA solution is related to symmetry breaking in azimuthal angle only. Its formal solution is

$$\omega = \frac{1}{4}(I_0 - I_2). \tag{11}$$

The MZA solution is related to symmetry breaking in both azimuthal angle and zenith angle, which has formal solutions

$$\omega = -\frac{1}{4}\left(I_0 - I_2 \pm \sqrt{(I_0 + I_2 - 2I_1)(I_0 + I_2 + 2I_1)}\right). \tag{12}$$

Eq. (9) is equivalent to dispersion relation defined in 3. In other words, the solutions (11) and (12) are dispersion relations $D(\omega, \mathbf{k})$ for a chosen direction of $\hat{\mathbf{k}} = \hat{\mathbf{z}}$.

HAVE TO EXPLAIN THE IDEA OF GAP AND INSTABILITY HERE. Maybe Later?

III. DISCRETE ZENITH ANGLE MODEL

In 3, the authors relate gaps to instabilities of neutrino oscillations. The definition of gaps of dispersion relation becomes clear in examples. We repeat the two zenith ranges example discussed in 3 as a review.

We continue the discussion of axial symmetric neutrino emissions but with discretized zenith angles θ thus discretized u. In this case, the ELN is independent of azimuth angle ϕ. For neutrino emission with 2 zenith angles, the ELN spectrum can be written as

$$G(u) = \sum_{i=1}^{2} g_i \delta(u - u_i). \tag{13}$$

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The MAA solution becomes an equation of hypobola for ω and k, which has asymptotic lines $\omega = ku_i$ for $i = 1, 2$. Mathematically speaking, the hyperbola equation has two solutions of ω (k) for any given real k (ω). The solutions are either real which indicates stable solutions or complex which indicates exponential growth in linear regime. On the other hand, non-existence of real solution of ω (k) for given real k (ω) is equivalent to gap in dispersion relation. Thus the equivalence of gap and instabilities is guaranteed in neutrino emission with two-zenith-angle emission.
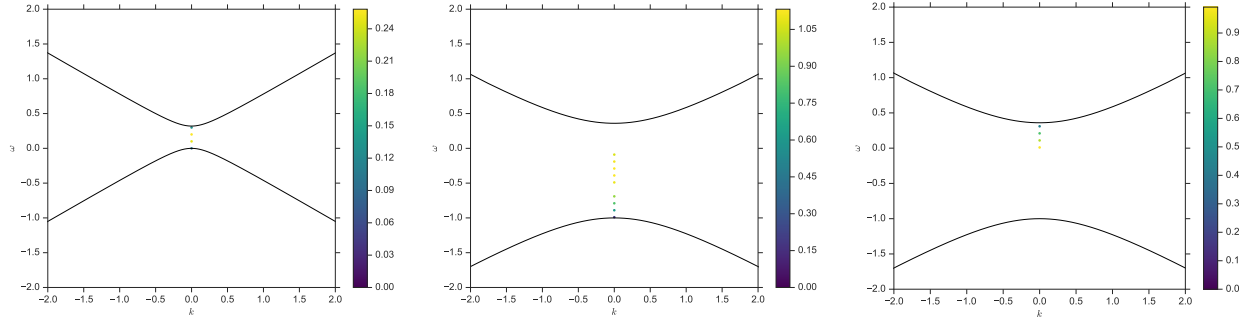

FIG. 1. Dispersion relation and instabilities of two zenith angles spectrum. **More about the figures**

However, this conclusion can not be generalized to arbitrary number of emission angles. As an example, we calculate the three zenith angles emission configuration. For three zenith angles, there are three solutions of k (ω) for given real ω (k). As long as real solutions disappear, complex solutions emerge, which leads to instabilities occur even without an actual gap.

Place the dispersion relation as well as LSA for three zenith angles here

Emphasize here that the number of solutions is the key.

The forbidden region is determined by the emission angles.

IV. CONTINUOUS SPECTRUM

More realistic models of supernova explosions involves continuous zenith angle ELN spectra. In this section we analyze dispersion relation and instabilities for box-shaped spectra and spectra from Garching group, both of which are assumed to be axial symmetric.

A. Box Spectra

We choose to calculate the dispersion relation and linear stability analysis for a box spectrum for simplicity.

A figure of box spectrum and the corresponding DR+LSA.

B. Instabilities at $\omega \to 0$

$$4 = \bar{I}_0 - \bar{I}_2 \tag{14}$$

$$4 = \frac{1}{k} \int G(u) \frac{1 - u^2}{\omega/k - u} \tag{15}$$

$$k_R = \frac{1}{4} \left(P \int G(u) \frac{1 - u^2}{-u} \right) \tag{16}$$

$$k_I = \frac{\pi}{4} G(0) \operatorname{Sign}(\omega) \operatorname{Sign}(\operatorname{Im}(k)) \tag{17}$$

$$|k_I| = \frac{\pi}{4} G(0) \operatorname{Sign}(\omega). \tag{18}$$

V. CONCLUSION

Appendix A: Plan of the paper

- ~~Review fast mode oscillations~~

- State what has been done in Raffelt's paper.

- The conclusion is not true.

- Two beams examples to prove that the number of solutions is the key.

- Show that the continuous case is not related to gap at all. Box spectrum?

- Continuous spectrum or Garching group, data to show that we can prove the location of the instability.

But I have a question. Is it really reliable? Should I use principal value integral for box spectrum DR?

We are still not crystal clear about the relation between gap and lsa.

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[1] A. Banerjee, A. Dighe, and G. Raffelt, Physical Review D **84**, 053013 (2011), arXiv:1107.2308 [hep-ph].

[2] G. Raffelt, S. Sarikas, and D. D. S. Seixas, Physical Review Letters **111**, 091101 (2013).

[3] I. Izaguirre, G. Raffelt, and I. Tamborra, Physical Review Letters **118**, 021101 (2017), arXiv:1610.01612.