Polarization Equation Hilbert Space

Sunday, January 11, 2015 10:56 PM

① This

Technique to solve equations

Monday, January 12, 2015 2:16 PM

① Put eqns to Hilbert space

$$f(s) = u(s) + \int dt \, \sin k(t-s) \, u(t)$$

$\sin k(t-s) = \sin kt \cos ks - \sin ks \cos kt$

$$|f\rangle = |u\rangle + |\cos k\square\rangle\langle \sin k\square|u\rangle - |\sin k\square\rangle\langle\cos k\square|u\rangle$$

$$\boxed{\text{So what about } \omega B_a + \mu \int P \, d\omega = P(\omega), \text{ solve } P(\omega)}$$

$$\omega B_a + \mu \int P \, d\omega = P$$

$$B_a|\omega\rangle + \mu|1\rangle\langle 1|P\rangle = |P\rangle$$

Just need $\langle 1|P\rangle$

$$B_a\langle 1|\omega\rangle + \mu\langle 1|1\rangle\langle 1|P\rangle = \langle 1|P\rangle$$

Find $\langle 1|P\rangle$ then solved

② $u(s) + \lambda \int u \cdot t \, dt = f(s)$

$u_0 = f(s) \quad u = u_0 + u_1 + \cdots$

Keep only

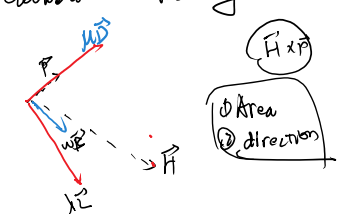
$\cancel{u_1 =}$ $\cancel{u_0 + u_1} + \lambda \int (u_0 + u_1) t \, dt = \cancel{f(t)} \; 0$

$$u_1 = -\lambda \int u_0 \, t \, dt$$

③ How do ω understand

$$\dot{\vec{P}} = \vec{H} \times \vec{P}$$

Geometrical meaning



To Do

Monday, January 26, 2015 2:15 PM

☐ Read Huaiyu's PRD

☒ means only half of the work done

☒ Read about Rabi oscillation or transition

☐ Kneller's paper more

☐ List of systems analyzed

* Spectra
* Geometry
* Interactions included

☐ Parametric resonance

☒ Read Gilmati's book

Basics

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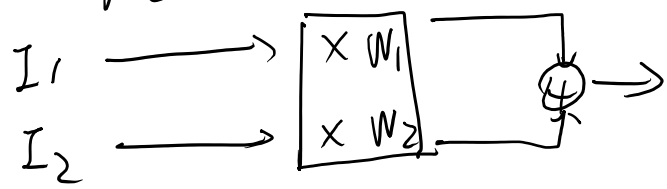
1. neuron $\quad f = \dfrac{\exp\left(\frac{\beta x}{2}\right)}{2\cosh\left(\frac{\beta x}{2}\right)} = \dfrac{1}{1+\exp(-\beta x)}$

2. Example



3. Backward propagation of error

Delta rule $\begin{cases} \text{"speed"} & (\beta\text{-function}) \\ \text{"momentum"} & \text{of learning} \end{cases}$ warning:

① Calculate $I = \vec{w}\cdot\vec{x}$, $\vec{x}$ input, w

<u>Scipy . optimize minimize</u> <u>Newton</u> <u>conj.</u>

Computational example

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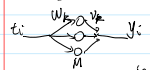
For a simple example $\frac{dy}{dt} = -y$, $y(0)=1$

① specify a vector of t : t_i

② Construct the network



To calculate y_i ~~to~~ this network

$M_k = t_i \cdot w_k + u_k$

$y'_{k,i} = f(M_k) \cdot v_k$

Then $y_i' = v_k f(t w_k + u_k)$

But we need initial condition

$y(0) = 1$

Then we have

$y_i = 1 + t_i \cdot y_i'$

$= 1 + t_i \cdot v_k f(t_i w_k + u_k)$

③ Calculate the cost

$I = \sum_i \left(y \frac{dy_i}{dt} + y_i\right)$

Here $\frac{dy}{dt}$ is something we can do analytically

known $\frac{df}{dt} = f(1-f)$

$\frac{dy}{dt} = v_k f(t w_k + u_k) + t v_k \cdot \frac{df(\quad)}{dt}$

$= v_k f(t w_k + u_k) + t \cdot v_k f(t w_k + u_k)(1 - f(t w_k + u_k)) \cdot \frac{d}{dt}(t \cdot w_k + u_k)$

$= v_k f(t \cdot w_k + u_k) + t v_k f(t \cdot w_k + u_k)(1 - f(t w_k + u_k)) \cdot w_k$

So I can write down this I explicitly

$I = \sum_i (\cdot \cdots \cdot)^2$

④ Then I minimize it using scipy optimize minimize

need Jacobian $\frac{dI}{dw_k}, \frac{dI}{du_k}, \frac{dI}{dv_k}$

cost $[w_k, u_k, v_k, t]$ ⇐ the cost I

cost P $[w_k, u_k, v_k, t]$ ⇐ the Jacobians

After minimize I have all the parameters

$w_1, w_2, w_3 \cdots$

$u_1, u_2, \cdots$

$v_1, v_2, \cdots$

---

scipy.optimize.minimize(function, initial guess, method.)

a- write a function that can be used in this optimize.

a function lambda x :

initial guess can be arbitrary

method? . . .

Example: Rosenbrock function

$$f(\vec{x}) = \sum_{i=1}^{N/2} [100 (x_{2i-1}^2 - x_{2i})^2 + (x_{2i-1} - 1)^2]$$

two variable case:

$$f(x,y) = (a-x)^2 + b(y-x^2)^2 \quad , a=1, b=100$$

To do list:

- [ ] 1. Minimize using jac method.
- [ ] 2. Plot the graph
- [ ] 3. Clean up notebook & functions
- [ ] 4. Remove unnessary test of numpy & scipy

1. How to do the Jac automatically? Sympy?

$y = 1 + t \cdot v_k f(t w_k + u_k)$

Derivitives: $I = \sum_i \left(v_k f(t w_k + u_k) + t v_k f(t w_k + u_k)(1 - f(t w_k + u_k)) w_k + y\right)^2$

∂_{v_k}: $\frac{\partial}{\partial y} I = \sum_i 2 M_i \partial_x M_i$ $\quad \triangleq \sum_i M_i M_i$

$\partial_{v_k} M_i = \partial_{v_{k'}} \left[v_k f(t \cdot w_k + u_k) + t v_k f(t \cdot w_k + u_k)(1 - f(t \cdot w_k + u_k)) w_k + y \right]$

$= f(t \cdot w_{k'} + u_{k'}) + t \cdot f(t \cdot w_{k'} + u_{k'})(1 - f(t \cdot w_k + u_k)) w_k + t f(t w_{k'} + u_{k'})$

$\partial_{w_{k'}} M_i = v_{k'} \partial_{w_{k'}} f(t \cdot w_k + u_k) + t v_k (\partial_{w_{k'}} f(t \cdot w_k + u_k))(1 - f(t \cdot w_k + u_k)) w_k + t \cdot v_k f(t \cdot w_k + u_k)(-\partial_{w_{k'}} f(t \cdot w_k + u_k)) w_k$

$+ t v_k f(t \cdot w_k + u_k)(1 - f(t \cdot w_k + u_k)) + t \cdot v_k \partial_{w_{k'}} f(t w_k + u_k)$

$\partial_{w_{k'}} f(t \cdot w_k + u_k) = \frac{\partial f(\square)}{\partial \square} \cdot t\, \delta(k, k') = f(\square)(1 - f(\square))\, t$

$\partial_{u_{k'}} M_i = v_{k'} f'(\square') + t v_{k'} f'(t w_{k'} + u_{k'})(1 - f(t \cdot w_{k'} + u_{k'})) w_{k'} + t v_{k'} f(\square')(1 - f'(\square')) w_{k'}$

$+ t v_{k'} f'(\square')$

A Simple Neutrino Problem

Monday, January 12, 2015 4:38 PM

This is a page for neutrino problem

Vanson

Simple Problem Revisted

Monday, January 26, 2015 10:29 AM

Problem: $\frac{d}{dt} y = y$, $y(0)=1$ $\longrightarrow$ $\frac{dy}{y} = dt$

$d\ln y = dt$

$\ln y = t$

$y = e^t$

$y = 1 + t\, v_k f(t w_k + u_k)$

$f(x) = \frac{1}{1+e^{-x}}$

Cost:

$I = \left(\frac{d}{dt} y - y\right)^2$

So I need $\frac{df(x)}{dx} = \frac{+e^{-x}}{(1+e^{-x})^2} = f \cdot \frac{e^{-x}}{1+e^{-x}} = f(1-f)$

$\frac{d}{dt} y = v_k f(t \cdot w_k + u_k) + t \cdot v_k \frac{df(t w_k + u_k)}{dt}$

$= v_k f(t \cdot w_k + u_k) + t \cdot v_k f(t w_k + u_k)(1 - f(t w_k + u_k)) \cdot w_k$

$$I = \sum_k \left(v_k f(t \cdot w_k + u_k) + t\, v_k f(t w_k + u_k)(1 - f(t \cdot w_k + u_k)) \cdot w_k \right. - \left(1 + t \cdot v_k f(t \cdot w_k + u_k)\right)^2$$

Define $f_k = f(t \cdot w_k + u_k)$. $f_k^{(t)} = \frac{d}{dt} f_k$ $f_k^{(w)} = \frac{d}{dw} f_k = \frac{df(t w_k + u_k)}{d(t w_k + u_k)} \cdot t$

no sum: $\frac{\partial}{\partial w_k} f = f_k(1-f_k) \cdot t$, $\frac{\partial}{\partial u} f = f(1-f)$, $\frac{\partial}{\partial v} f = 0$

Derivatives of $I = \sum_i \left[v_k f_k + t v_k f_k (1-f_k) \cdot w_k - (1 + t \cdot v_k f_k) \right]^2 \equiv \sum_i M_i^2$

$\frac{\partial}{\partial w_k} I = \sum_i 2 M_i M_i^{(w)}$,

$M_i^{(w)} = v_k f_k (1-f_k) \cdot t + t\, v_k f_k (1-f_k) t (1-f_k) w_k - t\, v_k f_k f(1-f) t \cdot w_k$
$+ t \cdot v_k f_k (1-f_k) - t\, v_k f(1-f) t$

$\frac{\partial}{\partial u} I = \sum_i 2 M_i M_i^{(u)}$

$M_i^{(u)} = v_k f_k (1-f_k) + t\, v_k f(1-f)(1-f) \cdot w_k - t\, v_k f_k f(1-f) w_k - t\, v_k f(1-f)$

$\frac{\partial}{\partial v} I = \sum_i 2 M_i M_i^{(v)}$

$M_i^{(v)} = f + t f(1-f) w_k - t f$

dx

Derived by hand, the jac is a list of the following for v

$$2M_i(f(tw_{k'} + u_{k'}) + tf(tw_{k'} - u_{k'})(1 \quad f(tw_{k'} + u_{k'}))w_{k'} + tf(tw_{k'} + u_{k'})),$$

for w:

$$2M_i(v_{k'} tf(') (1 - f(')) + tv_{k'} (tf(')(1 - f(')))w_{k'} - tv_{k'} f(')f(')(1 - f(')) tw_{k'} - tv_{k'} f(')(1 - f(')) + tv_{k'} f(')(1 - f(')) t),$$

for u:

$$v_{k'} f(')(1 - f(')) + tv_{k'} f(')(1 - f('))(1 - f(')) w_{k'} - tv_{k'} f(')f(')(1 - f(')) w_{k'} - tv_{k'} f(')(1 - f(')).$$

where k' is not summed over.

Redo the Jac.

$I = \sum M_i^2 = M_i M_i$ $\frac{\partial f_k}{\partial} = f_k(1-f_k) \cdot t$

Redo the Jac.

$I = \sum M_i^2 = M_i M_i$

$$\frac{\partial f_k t}{\partial w_\alpha} = f_\alpha (1 - f_\alpha) \cdot t$$

$$\frac{\partial I}{\partial w_\alpha} = \frac{\partial I}{\partial M_i} \frac{\partial M_i}{\partial w_\alpha} = 2 M_i \frac{\partial M_i}{\partial w_\alpha}$$

$$M_i = v_k f_k + t v_k f_k (1 - f_k) w_k \cancel{= f(t)} + y$$

$$\frac{\partial M_i}{\partial v_\alpha} = f_\alpha + t f_\alpha (1 - f_\alpha) w_\alpha + \frac{\partial}{\partial v_\alpha} y$$

$$\frac{\partial M_i}{\partial w_\alpha} = v_\alpha f_\alpha (1 - f_\alpha) \cdot t + t^2 v_\alpha f_\alpha (1 - f_\alpha)(1 - f_\alpha) w_\alpha$$
$$+ t^2 v_\alpha f_\alpha f_\alpha (1 - f_\alpha) w_\alpha + \frac{\partial}{\partial w_\alpha} y + \cdots$$
$$= v_\alpha f_\alpha (1 - f_\alpha) t + t^2 v_\alpha f_\alpha (1 - f_\alpha) w_\alpha$$
$$- 2 t^2 v_\alpha f_\alpha f_\alpha (1 - f_\alpha) w_\alpha + t v_\alpha f_\alpha (1 - f_\alpha)$$
$$+ \frac{\partial}{\partial w_\alpha} y$$

$$\frac{\partial M_i}{\partial u_\alpha} = v_\alpha f_\alpha (1 - f_\alpha) + t v_\alpha f_\alpha (1 - f_\alpha)(1 - f_\alpha) w_\alpha$$
$$+ t v_\alpha f_\alpha f_\alpha (1 - f_\alpha) w_\alpha + \frac{\partial y}{\partial u_\alpha}$$

code
compactificatioin

Tuesday, February 10, 2015 11:39 AM

Objectives

* Takes input (v, w, u, t) all four array, output the optimization result.
* All generally used functions move to one section called $\boxed{\text{General function}}$
* Make a general test module
* Generally used plot functions
* $\boxed{\text{Mark}}$ all functions that is problem dependent.

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1. Cost function of any 1st order differential function $\frac{d}{dt}y + f(t) = 0$

$$I = \sum_i \Big[\underbrace{v_k f_k + t v_k f_k (1 - f_k) w_k}_{\frac{d}{dt}y} + f(t) \Big]^2$$

For a single time t_i calculate I_i where

$$I = \sum_i I_i^2$$

$$I_i = \left[\frac{d}{dt}y + f(t)\right]$$

⊕ use $I_i^2 = I_i \cdot I_i = (\,^2, \,^2, \,^2, \,^2)$

then sum over np.sum(I_i^2) to get I

In this case, define. I_i only.

$I_i(v, w, u, t)$

in this case, define. I_i only.

$I_i(v, w, u, t)$

where t is an array.

2. Now I have a function

costODE (x, t, initialCondition, $f(t)$)

parameters — time linspace — initial condition for $y(t)$ — $\frac{d}{dt}y + f(t) = 0$

$\boxed{\text{need to write the minimization of cost function}}$ which should only use x as the variables

very important

① Minimize

② Take the result as input

③ To calculate the function values using output

④ plot the result.

Homogeneous Gas

Wednesday, March 4, 2015 9:19 AM

1. Need the eqns.

$$i\dot{\rho} = [M_+ + \sqrt{2}\, G_F(\rho - \bar{\rho}), \rho]$$

$$i\dot{\bar{\rho}} = [M_- + \sqrt{2}\, G_F(\rho - \bar{\rho}), \bar{\rho}]$$

$$M_\pm = \lambda L \pm \frac{\delta m^2}{2E} B = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pm \frac{\delta m^2}{2E} \frac{1}{2} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \pm \frac{\delta m^2}{2E} \cdot \frac{-\cos 2\theta}{2} & \pm \frac{\delta m^2}{2E} \frac{1}{2} \sin 2\theta \\ \pm \frac{\delta m^2}{2E} \cdot \frac{1}{2} \sin 2\theta & \pm \frac{\delta m^2}{2E} \cdot \frac{1}{2} \cos 2\theta \end{pmatrix}$$

MSW

Monday, August 3, 2015 11:07 AM

1. Matter suppression

$$|\nu_e\rangle = \cos\theta_v |\nu_1\rangle - \sin\theta_v |\nu_2\rangle$$

$$|\nu_\mu\rangle = \sin\theta_v |\nu_1\rangle + \cos\theta_v |\nu_2\rangle$$

Initially $|\nu_e\rangle$, At t

$$|\psi(t)\rangle = \cos\theta_v |\nu_1\rangle e^{-iH_1 t} - \sin\theta_v |\nu_2\rangle e^{-iH_2 t}$$

$$= e^{-iH_1 t}\left(\cos\theta_v |\nu_1\rangle - \sin\theta_v |\nu_2\rangle e^{-iH_2 t + iH_1 t}\right)$$

$$|\psi(t)\rangle = \cos\theta_v |\nu_1\rangle - \sin\theta_v |\nu_2\rangle e^{\frac{it}{\frac{1}{H_2 - H_1}}}$$

$$H_1 - H_2 = \frac{m_1^2}{2E} - \frac{m_2^2}{2E} = \frac{\Delta m_{12}}{2E}$$

$$i\hbar\partial_t \psi = H\psi$$

$$i\hbar\int \frac{\partial\psi}{\partial t}dt = \int H\psi\, dt$$

$$\int d\psi = -i\frac{1}{\hbar}\int H\psi\, dt$$

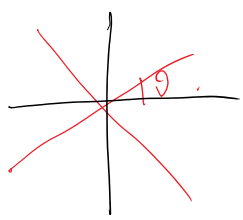
$$\frac{d\psi}{\psi} = -\frac{i}{\hbar} H\, dt$$

$$\ln\psi = -\frac{iH}{\hbar} Et$$

$$\psi = e^{-\frac{iHt}{\hbar}}$$

Define a vacuum oscillation length

L_i



Calculations are easy

$$H' = \begin{pmatrix} & 0 \\ 0 & \end{pmatrix}$$

$$|\nu_e\rangle = \cos\theta_m |\nu_1\rangle + \sin\theta_m |\nu_2\rangle$$

$$a = \cos\theta_m(x_0) \quad b = \sin\theta_m(x_0)$$

① Find instantaneous eigenstates $|\nu_1\rangle, |\nu_2\rangle$

② Project $|\nu_e\rangle$ on $|\nu_1\rangle, |\nu_2\rangle$: $|\psi(x_0)\rangle = a|\nu_1\rangle + b|\nu_2\rangle$

③ Evolution $|\psi(x)\rangle = a e^{-i\int \varepsilon_1 dx}|\nu_1\rangle + b e^{-i\int \varepsilon_2 dx}|\nu_2\rangle$

$$= e^{-i\int \varepsilon_1 dx}\left(a|\nu_1\rangle + b e^{-i\int(\varepsilon_2 - \varepsilon_1)dx}|\nu_2\rangle\right)$$

④ Project $|\psi(x)\rangle$ on $|\nu_e\rangle$

$$\langle\nu_e|\psi(x)\rangle$$

$$= a\langle\nu_e|\nu_1\rangle + b e^{-i\int(\varepsilon_2 - \varepsilon_1)dx}\langle\nu_e|\nu_2\rangle$$

$$= \cos\theta_m(x_0)\cos\theta_m + \sin\theta_m(x_0)\sin\theta_m e^{i\phi}$$

$$|\ \ |^2 = \cos^2\theta_m(x_0)\cos^2\theta_m + \sin^2\theta_m(x_0)\sin^2\theta_m + 2\cos\theta_m(x_0)\cos\theta_m\sin\theta_m(x_0)\sin\theta_m(e^{-i\phi} + e^{i\phi})$$

$$= \cos\theta_m(x_0)\left(\cos^2\theta_m - \sin^2\theta_m\right) + \sin^2\theta_m + \frac{1}{4}\sin 2\theta_m(x_0)\sin 2\theta_m(2\cos\phi)$$

$$= \cos\theta_m(x_0)\cos 2\theta_m + \sin^2\theta_m + \frac{1}{2}\sin 2\theta_m(x)\sin 2\theta_m \cos\phi$$

$$= \frac{\cos 2\theta_m(x_0)+1}{2}\cos 2\theta_m + \frac{1-\cos 2\theta_m}{2} + \frac{1}{2}\sin 2\theta_m(x)\sin 2\theta_m \cos\phi$$

$$= \frac{1}{2} - \cancel{\frac{1}{2}\cos 2\theta_m} + \cancel{\frac{1}{2}\cos 2\theta_m} + \frac{1}{2}\cos 2\theta_m(x_0)\cos 2\theta_m$$

$$+ \frac{1}{2}\sin 2\theta_m(x_0)\sin 2\theta_m \cos\phi$$

Solar netrino problem

Tuesday, August 11, 2015 10:13 AM

$$+\frac{1}{2}\sqrt{-\frac{1+\cos 2\theta}{\sin\theta}} + \frac{1}{8}\sqrt{-\frac{1+\cos 2\theta}{\sin\theta}}\,(1+2)(\delta-1)$$

Doesn't go to the vacuum

$$= +\frac{1}{2}\sqrt{\frac{-(1+2\cos^2\theta-1)}{\sin\theta}} + \frac{3}{8}\sqrt{-\frac{2\sin}{\sin\theta}}$$

$$+\frac{1}{2}\sqrt{\frac{1-\cos 2\theta}{\sin\theta}}$$

~~is it going to change~~ $\frac{\sqrt{2}}{2}\sin\theta\left(-1+\frac{1}{4}(\delta-1)\right)$

$$1-\cos 2\theta = 1-1+2\sin^2\theta = 2\sin^2\theta$$

$$+\frac{1}{2}\sqrt{2\sin\theta}\,\sin\theta + \frac{\sqrt{2}}{8}\sqrt{\sin\theta}\,(\delta-1) = \sin(\theta_m)$$

$$\cos\theta(x) = \frac{\sin(\theta)}{2\sin\theta\sqrt{\sqrt{2}\sin\theta}} + \frac{\left(\frac{3}{\sin\theta}\cdot\sin 2\theta + \frac{1}{2}\frac{-3}{\sin\theta}\sin 4\theta\right)(\delta-1)}{8\sqrt{2}\sin\theta\,(\cos 2\theta-1)}$$

$$= \frac{1}{\sqrt{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} + \frac{6\cos\theta - \frac{3}{2}\cdot 4\cos\theta\cos 2\theta}{-16\sqrt{2}\sin^2\theta\sqrt{\sin\theta}}(\delta-1)$$

$$1-\cos 2\theta = 1-1+2\sin^2\theta$$

$$= \frac{1}{\sqrt{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} + \frac{6\cos\theta\cdot 2\sin^2\theta}{-16\sqrt{2}\sin^2\theta\sqrt{\sin\theta}}(\delta-1)$$

$$= \frac{1}{\sqrt{2}}\frac{\cos\theta}{\sqrt{\sin\theta}} + \frac{3\cos\theta}{-4\sqrt{2}\sqrt{\sin\theta}}(\delta-1)$$

$$= \frac{1}{\sqrt{2}}\frac{\cos\theta}{\sqrt{\sin\theta}}\left(1-\frac{3}{4}(\delta-1)\right)$$

Landau-Zener Transition

Monday, January 12, 2015 3:14 PM

$$\ddot{C}_L - i\delta\omega_m \dot{C}_L + v^2 C_L = 0$$

$$\delta\omega_m = \delta\omega_m(x_r) + \alpha(x - x_r)$$
$$= 2\omega\sin\theta_v + \alpha(x - x_r)$$

$$\delta\omega_m = \omega\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}$$

at x_r

$$= \omega\sqrt{2 - 2\cos 2\theta_v}$$
$$= \omega\sqrt{2}\sqrt{1 - 1 + 2\sin^2\theta_v}$$
$$= \sqrt{2}\,\omega\sqrt{2}\sin\theta_v$$
$$= 2\omega\sin\theta_v$$

$$\frac{d\delta\omega_m}{dx}\Big|_{x_r} = \frac{d\hat{\Delta}}{dx}\,\frac{d\delta\omega_m}{d\hat{\Delta}}\Big|_{x_r}$$
$$= \hat{\Delta}'(x_r)\,\omega\,\frac{2\hat{\Delta} - 2\cos 2\theta}{2\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta}}\Big|_{x_r}$$
$$= \hat{\Delta}'(x_r)\,\omega$$

$$v^2 - \frac{1}{4}\delta\omega^2 =$$

$$\ddot{C}_L - i\left(2\omega\sin\theta_v + \alpha(x - x_r)\right)\dot{C}_L + v^2 C_L = 0 \quad ; \quad C_L = e^{\frac{i}{2}\int\delta\omega_m dx'} W$$

$$\begin{cases} \dot{C}_L = e^{\frac{i}{2}\int\delta\omega_m dx'}\dot{W} + \frac{i}{2}\delta\omega_m e^{\frac{i}{2}\int\delta\omega_m dx'} W \\ \ddot{C}_L = e^{\Omega}\ddot{W} + \frac{i}{2}\delta\omega_m e^{\Omega}\dot{W} + \frac{i}{2}\delta\dot{\omega}_m e^{\Omega} W + \left(\frac{i}{2}\delta\omega\right)^2 e^{\Omega} W + \frac{i}{2}\delta\omega_m e^{\Omega}\dot{W} \\ \quad = e^{\Omega}\left(\ddot{W} + i\delta\omega_m\dot{W} + \left(\frac{i}{2}\alpha - \frac{1}{4}\delta\omega_m^2\right)W\right) \end{cases}$$

$$\ddot{W} + (i\delta\omega_m - i\delta\omega_m)\dot{W} + \left(\frac{i}{2}\alpha - \frac{1}{4}\delta\omega_m^2\right)W + v^2 W = 0$$

$$\left(\frac{i}{2}A\right)^2 + \frac{i}{2}A + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

The coefficient $\frac{i}{2}\alpha \cancel{+ \frac{i}{2}(2\omega\sin\theta + \alpha(x - x_r))} - \frac{1}{4}\left(2\omega\sin\theta + \alpha(x - x_r)\right)^2 + v^2$

Define new $x' = x - x_r \Rightarrow \frac{i}{2}\alpha \cancel{\theta} \cancel{-\frac{1}{4}} + \left(\frac{i}{2}(2\omega\sin\theta + \alpha x + \cancel{\theta})\right)^2 + v^2$

Define $x' = x \cancel{+ \frac{2}{\alpha}} + \frac{2\omega\sin\theta}{\alpha} \Rightarrow \frac{i}{2}\alpha \cancel{\theta} - \frac{\alpha^2 x^2}{4} + v^2 \cancel{- \frac{1}{4}}$

So switch out $\cancel{\theta^2 \to v^2 + 1}$, $\left[\frac{\alpha^2 x^2}{4} \to -\frac{\alpha^2 x^2}{4}\right]$

$$V_m = \begin{pmatrix} 0 & iv \\ -iv & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\gamma = \frac{v^2}{\left|\frac{d\omega_m}{x dx}\right|_{x_r}} \qquad v = \hat{\Delta}'(x_r)\frac{\cot\theta}{4}$$

$$= \frac{\hat{\Delta}'(x_r)\cot\theta}{4}\Big|_x$$

$$\ddot{W} + \left(-\frac{1}{4}\delta\omega_m^2\right)W + v^2 W = 0$$

$$\ddot{W} + \left(v^2 - \frac{1}{4}\delta\omega_m^2\right)W = 0$$

$$W = A e^{iax}$$
$$\dot{W} = A\,i a e^{iax}$$
$$\ddot{W} = -A a^2 e^{iax}$$

$$-A a^2 + \left(v^2 - \frac{1}{4}\delta\omega_m(x_r)^2\right)A = 0$$
$$a^2 = v^2 - \frac{1}{4}\delta\omega_m(x_r)^2$$

In the calculation I have made a mistake

The resonance point is $\frac{\hat{\Delta}}{2} - \frac{\omega}{2}\cos 2\theta_v = 0 \Rightarrow \hat{\Delta} = \cos 2\theta_v$

Do expansion at $\hat{\Delta} - \cos 2\theta = 0$

$$U^{-1}\partial_x U = \begin{pmatrix} 0 & \frac{\sin 2\theta}{1 + 2\alpha^2 - \cos(4\theta)} \\ \frac{\sin 2\theta}{1 - 2\alpha^2 + \cos 4\theta} & 0 \end{pmatrix} = \frac{\sin 2\theta}{1 + 2\alpha^2 - \cos(4\theta)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{\sin 2\theta}{1 + 2\alpha^2 - \cos 4\theta}(\pm i)\underbrace{\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}}_{\sigma_2}$$

$$\frac{\sin 2\theta}{1 + 2\alpha^2 - \cos(4\theta)} = \frac{\sin 2\theta}{1 - \cos(4\theta)} + O(\alpha)^2$$

$$-i\,U^{-1}\partial_x U = (-i)\,i\,\frac{\sin 2\theta}{1+2\alpha^2-\cos(4\theta)}\,\sigma_2 = \frac{\sin 2\theta}{1+2\alpha^2-\cos(4\theta)}\,\sigma_2 = \frac{\sin 2\theta}{1-\cos(4\theta)}\,\sigma_2$$

$$\frac{\sin 2\theta}{1-\cos 4\theta} = \frac{\sin 2\theta}{1-(1-2\sin^2 2\theta)} = \frac{\sin 2\theta}{2\sin^2 2\theta} = \frac{1}{2\sin 2\theta}$$

$$V = -i\left(U^{-1}\partial_x U\right)\frac{d\hat{\Delta}}{dx}\Big|_{x_r}$$ using $\hat{\Delta}(x) = \hat{\Delta}(x_r) + \frac{d\hat{\Delta}}{dx}\Big|_{x_r}(x-x_r)$

$$= \frac{\hat{\Delta}'(x_r)}{2\sin 2\theta}\begin{pmatrix} 0 & -\hat{\Delta} \\ \hat{\Delta} & 0 \end{pmatrix} = -i\,\frac{\hat{\Delta}(x_r)}{2\sin 2\theta}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$v = -\frac{\hat{\Delta}'(x_r)}{2\sin 2\theta} \qquad \alpha = \delta\omega_m^2(x_r)$$

$$\gamma = \frac{v^2}{\alpha} = \frac{1}{|\delta\omega_m^2(x_r)|}\,\frac{\hat{\Delta}'^2(x_r)}{4\sin^2 2\theta} = \frac{1}{|2\omega\sin\vartheta|}\,\frac{\hat{\Delta}'^2(x_r)}{4\sin^2 2\theta}$$

$$\frac{\hat{\Delta}'^2(x_r)}{4\sin^2 2\theta} - 4\omega^2\sin^2\vartheta = \cos$$

Calculation again

The equation of motion is

$$i\partial_x\psi = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\psi + \left(-i\,U_m^{-1}\partial_x U_m\right)\psi$$

Calculate $-i\,U_m^{-1}\partial_x U_m$ first

$$U_m = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \qquad U_m^{-1} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

$$\partial_x U_m = \partial_x\theta\begin{pmatrix} -\sin\theta_m & \cos\theta_m \\ -\cos\theta_m & -\sin\theta_m \end{pmatrix}$$

$$= -\dot{\theta}\begin{pmatrix} \sin\theta_m & -\cos\theta_m \\ \cos\theta_m & \sin\theta_m \end{pmatrix}$$

$$-i\,U_m^{-1}\partial_x U = +i\dot{\theta}\begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{pmatrix}\begin{pmatrix} \sin\theta_m & -\cos\theta_m \\ \cos\theta_m & \sin\theta_m \end{pmatrix}$$

$$= i\dot{\theta}\begin{pmatrix} 0 & -c^2-s^2 \\ s^2+c^2 & 0 \end{pmatrix}$$

$$= i\dot{\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\dot{\theta}\,\sigma_2$$

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad E_1 = -E_2 = \frac{1}{2}\omega\sqrt{\hat{\Delta}^2+1-2\hat{\Delta}\cos 2\theta}$$

Straightforward :

$\begin{pmatrix} 0 & E_2 \end{pmatrix}$,

Straightforward :

$$i\partial_x \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} E_1 & -iv \\ iv & E_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$i\partial_x c_1 = E_1 c_1 - iv c_2$$

$$i\partial_x c_2 = iv c_1 + E_2 c_2$$

$$i\partial_x^2 c_1 = E_1 \partial_x c_1 - v\,(i v c_1 + E_2 c_2)$$

$$c_2 = \frac{1}{iv}(E_1 c_1 - i\partial_x c_1)$$

$$i\ddot{c}_1 = E_1 \dot{c}_1 - v\left(i v c_1 + E_2 \frac{1}{iv}(E_1 c_1 - i\dot{c}_1)\right)$$

$$i\ddot{c}_1 = E_1 \dot{c}_1 - v\left(i v c_1 + \frac{E_2}{iv} E_1 c_1 - \frac{E_2 \dot{c}_1}{v}\right)$$

$$= E_1 \dot{c}_1 - i v^2 c_1 + i E_2 E_1 c_1 + E_2 \dot{c}_1$$

$$i\ddot{c}_1 - (E_1 + E_2)\dot{c}_1 - i(E_2 E_1 - v^2) c_1 = 0$$

$$v^2 = \left(\frac{\delta' \sin 2\theta}{1 + 2(\delta - \cos 2\theta)^2 - \cos 4\theta} \right)^2$$

$$E_1 + E_2 = 0$$

$$\ddot{c}_1 - (E_1 E_2 - v^2) c_1 = 0$$

$$\delta = \delta_0 e^{\alpha x} \quad \text{at a point } x_r$$

$$= \delta_0 e^{\alpha \frac{x}{x_r} \cdot x_r} = \delta_0 e^{\beta \hat{x}}$$

$$\alpha x_r = \beta$$

$\hat{x} \sim 1$ to $\hat{x}^2$ order I have

$$\frac{\beta^2 \Delta_0^2 e^{2\beta} \sin^2(2\theta)}{(2 + 2\delta_0^2 e^{2\beta} - 4\delta_0 e^{\beta} \cos 2\theta)^2}$$

$$- \frac{4(\beta^3 \delta_0^2 e^{2\beta}(-1 + \delta_0^2 e^{2\beta}) \sin^2 2\theta)(X-1)}{(2 + 2\delta_0^2 e^{2\beta} - 4\delta_0 e^{\beta} \cos 2\theta)^3}$$

$$+ \frac{8\beta^4 \delta_0^2 e^{2\beta}(1 - 4\delta_0^2 e^{2\beta} + \delta_0^4 e^{4\beta} + \delta_0 e^{\beta}(1 + \delta_0^2 e^{2\beta}) \cos 2\theta) \sin^2 2\theta\,(X-1)^2}{(2 + 2\delta_0^2 e^{2\beta} - 4\delta_0 e^{\beta} \cos 2\theta)^4}$$

Define $\quad v \doteq v_0 + v_1(\hat{x} - 1) + v_2(\hat{x} - 1)^2$

where $\quad \hat{x} = \dfrac{X}{X_r}$, $\beta = \alpha X_r$

The eqn :

$$\ddot{C}_1 - \left(E_1 E_2 - v_0 - v_1(\tilde{x}-1) - v_2(\tilde{x}-1)^2\right) C_1 = 0$$

① A $x \to -\infty$ and $x \to \infty$

In fact

$$v_1(\tilde{x}-1) + v_2(\tilde{x}-1)^2$$
$$= \left(\sqrt{v_2}(\tilde{x}-1)\right)^2 + v_1(\tilde{x}-1) + \left(\frac{v_1}{2\sqrt{v_2}}\right)^2 - \frac{v_1^2}{2v_2}$$

$$\ddot{C}_1 - \Big(\underbrace{E_1E_2 - v_0 + \frac{v_1^2}{2v_2}}_{A} \underbrace{- \left(\sqrt{v_2}(\tilde{x}-1) + \frac{v_1}{2\sqrt{v_2}}\right)^2}_{-(Bx+D)^2}\Big) C_1 = 0$$

$x = \tilde{x}\, x_r$

Compare Theory and Numerical

Monday, August 17, 2015 9:05 AM

① Theory:

$$P(\nu_e, x) = \frac{1}{2} + \frac{1}{2}(1 - 2P_{tr})\cos 2\theta_m(x_0)\cos 2\theta_v$$

$$\cos 2\theta_m(x_0) = \frac{\cos 2\theta_v - \hat{\delta}}{\sqrt{\hat{\delta}^2 + 1 - 2\hat{\delta}\cos 2\theta_v}}$$

At core $\hat{\delta}(x_0) = 10^{-13}$

$$P_{tr} = \exp\left(-\frac{\pi}{2}\frac{\sin^2 2\theta}{\cos 2\theta}\frac{\omega}{\hat{\delta}'(x_r)}\right) \sim \text{Calculate } -\frac{\pi}{2}\frac{\sin^2 2\theta}{\cos 2\theta}\frac{\omega}{\hat{\delta}'(x_r)}$$

Finally, at $\hat{x} = 1$, for ν_e.

$$P(E) = \frac{1}{2} + \frac{1}{2}(1 - 2P_{tr}(E))\frac{\cos 2\theta_v - \hat{\delta}(x)}{\sqrt{\hat{\delta}^2 + 1 - 2\hat{\delta}\cos 2\theta_v}}\cos 2\theta_v$$

Numerical change $\omega = \frac{\Delta m^2}{2E}$ ⇐ make function as a function of $\underline{\omega, x}$

$$\frac{dn}{dx} = \frac{dn}{R_c\, d\hat{x}}$$

Wolfenstein Paper

Wednesday, August 26, 2015 10:12 PM

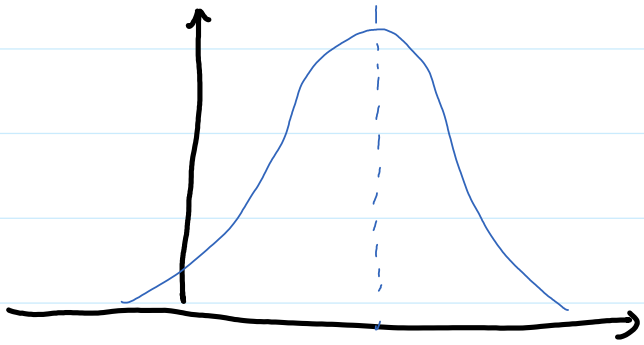
Neutrino Oscillations in Matter

PRD Volume 17 Number 9, 1978

Resonance

Monday, February 8, 2016 10:35 AM

* How to describe the resonance?



↑ should be a "thing" that characterize the resonance

where and what shape

$$i\frac{d}{dx}\begin{pmatrix}\psi_{(a)}\\ \psi_{(b)}\end{pmatrix} = \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}\begin{pmatrix}\psi_{(a)}\\ \psi_{(b)}\end{pmatrix}$$

$$x = \frac{2\pi}{L}$$

Neutrino Matrix

Friday, August 21, 2015 12:15 AM

Key :

A Simple Approach to the L Z formula

Thursday, August 27, 2015 5:53 PM

* Perturbative Hamiltonian

$$V = \begin{pmatrix} 0 & \Omega^{+} e^{i\omega t} \\ \Omega e^{-i\omega t} & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix}$$

1. $\Delta = \omega - \omega_0$
2. State: $|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle$
3. Initial t_i: $|\psi(t=t_i)\rangle = |0\rangle$

* Time dependent H transition rate Γ. out of $|0\rangle$

$$\Gamma = \frac{\Omega^2 \gamma}{\Delta^2 + \gamma^2/4}$$

γ: Decay rate of Rabi oscillations $\gamma = \frac{1}{\tau_D} \sim \sqrt{\frac{\Delta}{4\pi}}$

τ_D : Dephasing time $\Rightarrow$ time to accumulate a phase difference 2π

* At the probality in $|0\rangle$ state is $P_0(t)$

After a time segment

$$P_0(t+\delta t) \doteq e^{-\Gamma(t)\delta t} P_0(t)$$

* From t_i to t_f

$$P_0(t_f) = \exp\left(-\int_{t_i}^{t_f} \Gamma(t)\, dt\right) P_0(t_i)$$

$$= \exp\left(-\int_{t_i}^{t_f} dt\, \Omega^2 \frac{\gamma}{\Delta^2 + \gamma^2/4}\right)$$

$$= \exp\left(-\frac{2\Omega^2}{\gamma}\left[\arctan\left(\frac{\Delta(t_f)}{\gamma/2}\right) - \arctan\left(\frac{\Delta(t_i)}{\gamma/2}\right)\right]\right)$$

Rabi flopping

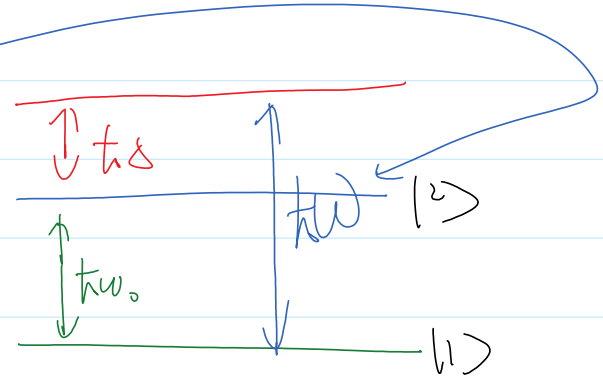
Thursday, August 27, 2015 7:30 PM

Rabi oscillations

we have photon $\hbar\omega$

Detuning δ

Resonance ω_0



$|1\rangle, |2\rangle$: eigenstates of $\hat{H}_0$

$\hat{H}_0 |1\rangle = 0 \cdot |1\rangle \qquad \hat{H}_0 |2\rangle = \hbar\omega_0 |2\rangle$

Initial $\quad |\psi(t_i)\rangle = \sum_n c_n |n\rangle \qquad \frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\psi\rangle$

$$H_0 \Longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_0 \end{pmatrix}, \quad V \Longrightarrow \hbar \begin{pmatrix} 0 & \Omega\cos(\omega t) \\ \Omega^*\cos(\omega t) & 0 \end{pmatrix}$$

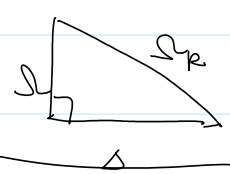
$|1\rangle$ as initial, $\quad \psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ evolve,

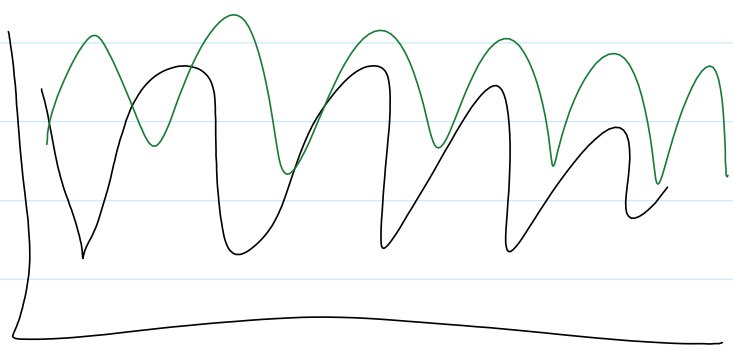
$c_1 = 1, \; c_2 = 0$

$$|c_1|^2 = \frac{\Omega^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right)$$

$$|c_2|^2 = \frac{\delta^2}{\Omega_R^2} + \frac{\Omega^2}{\Omega_R^2}\cos^2\left(\frac{\Omega_R t}{2}\right)$$

$$\Omega_R^2 = \Omega^2 + \delta^2$$





Same frequency

arXiv:1509-0897

Monday, October 5, 2015 1:34 PM

Phenomenon :

ν-ν interaction matter potential

different sign
can cancel

Thus suppress matter suppression of oscillation

Called Matter-Neutrino Resonances
MNRs

*NFIS : How do u define?
Meaning of H ?

1. Origin of NFIS

Start from Schrödinger Eqn:

$$i\frac{d}{dt}\psi_{\nu,i} = (H_\nu + H_e + H_{\nu\nu,i})\psi_{\nu,i}$$

In flavor basis $\psi_{\nu,i} = \begin{pmatrix} a_{\nu_e} \\ a_{\nu_\tau} \end{pmatrix}$

Vacuum : $H_\nu = \frac{\omega}{2}\begin{pmatrix} -\cos 2\theta_\nu & \sin 2\theta_\nu \\ \sin 2\theta_\nu & \cos 2\theta_\nu \end{pmatrix}$

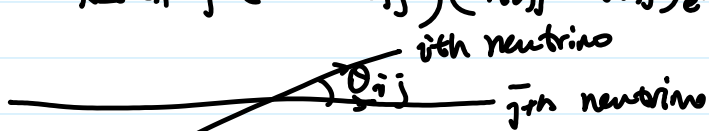
e-ν interacting : $H_e = \frac{\sqrt{2}G_F n_e}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

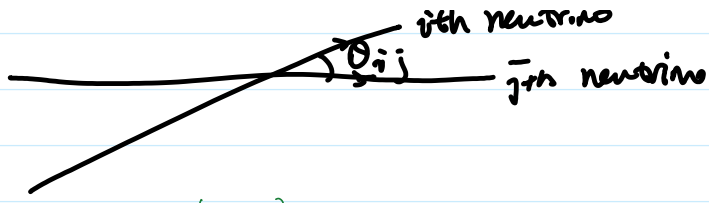
$$H_{\bar{e}} = -\frac{\sqrt{2}G_F n_e}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H_{\nu\nu,i} = \frac{1}{2}\begin{pmatrix} B & B_{e\tau} \\ B_{e\tau}^* & -B \end{pmatrix}$$

$$B = \sqrt{2}G_F \sum_j (1-\cos\theta_{ij})\{n_{\nu,j}((\rho_{\nu,i})_{ee} - (\rho_{\nu,j})_{\tau\tau}) - n_{\bar{\nu},j}((\rho_{\bar{\nu},i})_{ee} - (\rho_{\bar{\nu},j})_{\tau\tau})\}$$

$$B_{e\tau} = 2\sqrt{2}G_F \sum_j (1-\cos\theta_{ij})(n_{\nu,j}(\rho_{\nu,j})_{e\tau} - n_{\bar{\nu},j}(\rho_{\bar{\nu},j})_{e\tau})$$





$$\rho_\nu = \begin{pmatrix} |a_{\nu e}|^2 & a_{\nu e} a^*_{\nu \tau} \\ a^*_{\nu e} a_{\nu \tau} & |a_{\nu \tau}|^2 \end{pmatrix}$$

$$\rho_{\bar{\nu}} = \begin{pmatrix} |a_{\bar{\nu} e}|^2 & a^*_{\bar{\nu} e} a_{\bar{\nu} \tau} \\ a^*_{\bar{\nu} \tau} a_{\nu e} & |a_{\bar{\nu} \tau}|^2 \end{pmatrix}$$

Introduce StatMech concept of polarization

$$\rho = \frac{1}{2}(1 + \vec{P} \cdot \vec{\sigma})$$

$$\vec{P}_\nu \Rightarrow \begin{pmatrix} 2\,\mathrm{Re}(a^*_{\nu e} a_{\nu \tau}) \\ 2\,\mathrm{Im}(a^*_{\nu e} a_{\nu \tau}) \\ |a_{\nu e}|^2 - |a_{\nu \tau}|^2 \end{pmatrix}$$

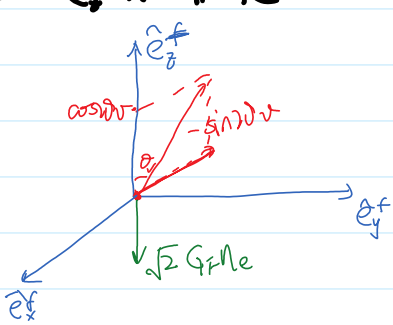
Please Read about Stokes parameters and optical polarization in

$$\vec{P}_{\bar{\nu}} \Rightarrow \begin{pmatrix} 2\,\mathrm{Re}(a_{\bar{\nu} e} a^*_{\bar{\nu} \tau}) \\ 2\,\mathrm{Im}(a_{\bar{\nu} e} a^*_{\bar{\nu} \tau}) \\ |a_{\bar{\nu} e}|^2 - |a_{\bar{\nu} \tau}|^2 \end{pmatrix}$$

The Hamiltonian is

$$H = H_\nu + H_e$$
$$= -\frac{\vec{\sigma}}{2} \cdot (\mu_\nu \vec{H}_\nu + \vec{H}_e)$$

$$\vec{H}_\nu = -\hat{e}^f_x \overset{\omega}{\sin 2\theta_\nu} + \hat{e}^f_z \cos 2\theta_\nu$$
$$\vec{H}_e = -\hat{e}^f_z \sqrt{2}\, G_F n_e$$



Define $\vec{S}_\nu = \psi_\nu^\dagger \frac{\vec{\sigma}}{2} \psi_\nu = \frac{\vec{P}_\nu}{2}$

E.o.M.

$$\frac{d}{dt}\vec{S}_\nu = \vec{S}_\nu \times \vec{H}_{eff}$$

$$\vec{H}_{eff} = \mu_\nu \vec{H}_\nu + \vec{H}_0$$

* Eqn for Homogeneous $\nu, \bar{\nu}$ gas

* Monochromatic neutrino spectrum

$$f_\omega = \alpha\,\delta(-\omega_0) + \delta(\omega_0)$$

"Exact Solutions for Matter-Enhanced Neutrino Oscillations

Monday, March 21, 2016 6:59 PM

This is the reading notes for
hep-ph/9712304

Fast Pairwise Conversion of supernova Neutrinos

Monday, October 10, 2016 6:10 PM

Dispersion-relation approach

* Flavor ration!

* Matter suppression :

$$\nu_e(\mathbf{p}) \rightarrow \nu_x(\mathbf{p})$$

suppress
due to matter large density

* Matter stimulation ineffective during accretion phase by Kneller 1! → $\boxed{\text{read it}}$

* $\nu_e(p) + \nu_x(k) \leftrightarrow \nu_x(p) + \nu_e(k)$

eg $\underbrace{\nu_e(p) + \bar{\nu}_e(k) \leftrightarrow \nu_x(k) + \bar{\nu}_x(k)}_{\text{flavor conversion}}$

* Conventional idea :

$$\nu_e \bar{\nu}_e \rightarrow \nu_x \bar{\nu}_x$$

happens at large radii well beyond the shock wave.

* Sawyer's idea

1. $\nu_e \bar{\nu}_e$ angle distribution
2. backward ν_e $\bar{\nu}_e$ flux

Q: growth rate

$$\Im_0 = \sqrt{2} G_F (n_{\nu_e} - n_{\bar{\nu}_e})$$

Fast compared to vacuum
ω_V

* Our concentration: flavor matrix
$$\rho(t, \vec{r}, \vec{p})$$

Flavor waves!

* Mean field

Q why $\rho = \frac{f_{\nu_e} + f_{\nu_x}}{2} + \frac{f_{\nu_e} - f_{\nu_x}}{2} \begin{pmatrix} s & S \\ S^* & -s \end{pmatrix}$

$S_{\vec{p}}(t, \vec{r})$ flavor coherence for mode $\vec{p}$
complex

$s_p(t, \vec{r})$ $\quad s_p^2 = 1 - |S_p|^2$
real

Flavor oscillation modes in dense neutrino media

Wednesday, November 2, 2016 5:13 PM

arxiv: 1309.7377

Using flavor isospin EoM

$$\frac{d}{dt}\vec{S}_{\omega,\hat{v}} = -2\sqrt{2}\,G_F\,\vec{S}_{\omega,\hat{v}} \times \sum_{\omega',\hat{v}'}(1-\hat{v}\cdot\hat{v}')\,n_{\omega',\hat{v}'}\,\vec{S}_{\omega',\hat{v}'}$$

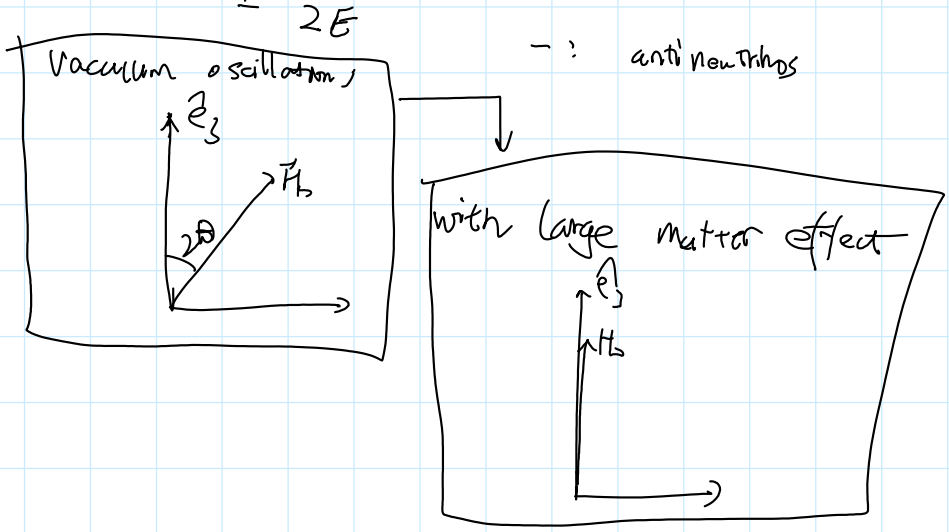
$$+\ \vec{S}_{\omega,\hat{v}} \times \omega\vec{H}_0$$

$$\vec{S}_{\omega,\hat{v}} = (\sigma_2\psi)^\dagger\left(\frac{\vec{\sigma}}{2}\right)(\sigma_2\psi)$$

$$\omega = \pm\frac{\Delta m^2}{2E}$$

$+$: neutrinos

$-$: antineutrinos



① Assumptions: 1. $n_{\omega_0,\hat{v}} = n_{-\omega_0,\hat{v}}$: neutrinos & antineutrinos with same energy and velocity has same number density

2. Flavor isospin of neutrinos & antineutrino with same energy & velocity $\boxed{\text{Initially}}$

$$S_{\omega_0,\hat{v}}(0) = S_{-\omega_0,\hat{v}}(0)$$

$\boxed{\text{This is called bipolar}}$

3. Suppose we have total # density of neutrinos n_{tot}

spherically symmetric emmission means we have

$$n_{\omega_0,\hat{v}} = \frac{n_{tot}}{4\pi}$$ where $n_{\omega_0,\hat{v}}$ also carries

the meaning of # density in each solid angle

But in general we define a distribution

$$f_{\hat{v}} = \frac{n_{\omega_0,\hat{v}}}{n_{tot}}$$

Think about the equation of motion,

$$\dot{\vec{S}}_{\omega,\hat{v}} = -2\sqrt{2}\,G_F\,\vec{S}_{\omega,\hat{v}} \times \int (1-\hat{v}\cdot\hat{v}')\,n_{\omega',\hat{v}'}\,\vec{S}_{\omega',\hat{v}'}\,d\Omega_{\hat{v}'}\,d\omega'$$
$$+\ \vec{S}_{\omega,\hat{v}} \times \omega\vec{H}_0$$

For single energy case, $\pm\omega_0$

$$\dot{\vec{S}}_{\omega_0,\hat{v}} = -2\sqrt{2}\,G_F\,\vec{S}_{\omega,\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,n_{\omega_0,\hat{v}'}\,S_{\omega_0,\hat{v}'}$$
$$+\ \omega_0\,\vec{S}_{\omega_0,\hat{v}} \times \vec{H}_0$$

$$= -2\sqrt{2}\,G_F\,\vec{S}_{\omega_0,\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,n_{tot}\,\vec{S}_{\omega_0,\hat{v}'}$$
$$+\ \omega_0\,\vec{S}_{\omega_0,\hat{v}} \times \vec{H}_0$$

$$\dot{\vec{S}}_{-\omega_0,\hat{v}} = -2\sqrt{2}\,G_F\,\vec{S}_{-\omega_0,\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,n_{tot}\,\vec{S}_{-\omega_0,\hat{v}'}$$
$$-\ \omega_0\,\vec{S}_{-\omega_0,\hat{v}} \times \vec{H}_0$$

Define

$$\vec{S}_{\omega_0,\hat{v}} = \frac{\vec{d}_{\hat{v}} + \vec{g}_{\hat{v}}}{2}$$

$$\vec{S}_{-\omega_0,\hat{v}} = \frac{\vec{d}_{\hat{v}} - \vec{g}_{\hat{v}}}{2}$$

$$\frac{1}{2}\left(\dot{\vec{d}}_{\hat{v}} + \dot{\vec{g}}_{\hat{v}}\right) = -2\sqrt{2}\,G_F\,\frac{\vec{d}_{\hat{v}} + \vec{g}_{\hat{v}}}{2} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,n_{tot}\,\frac{\vec{d}_{\hat{v}'} + \vec{g}_{\hat{v}'}}{2}$$

$$\frac{1}{2}\left(\dot{\vec{d}}_{\hat{v}} + \dot{\vec{g}}_{\hat{v}}\right) = -2\sqrt{2}\,G_F\,\frac{\vec{d}_{\hat{v}} + \vec{g}_{\hat{v}}}{2} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,n_{tot}\,\frac{\vec{d}_{\hat{v}'} + \vec{g}_{\hat{v}'}}{2}$$
$$+\ \omega_0\,\frac{\vec{d}_{\hat{v}} + \vec{g}_{\hat{v}}}{2} \times \vec{H}_0$$

$$\frac{1}{2}\left(\dot{\vec{d}}_{\hat{v}} - \dot{\vec{g}}_{\hat{v}}\right) = -2\sqrt{2}\,G_F\,\frac{\vec{d}_{\hat{v}} - \vec{g}_{\hat{v}}}{2} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,n_{tot}\,\frac{\vec{d}_{\hat{v}'} - \vec{g}_{\hat{v}'}}{2}$$
$$-\ \omega_0\,\frac{\vec{d}_{\hat{v}} - \vec{g}_{\hat{v}}}{2} \times \vec{H}_0$$

$$\left(\dot{\vec{d}}_{\hat{v}} + \dot{\vec{g}}_{\hat{v}}\right) = -\sqrt{2}\,G_F\,n_{tot}\,\vec{d}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,\cancel{n_{tot}}\,(\vec{d}_{\hat{v}'} + \vec{g}_{\hat{v}'})$$
$$+\ \omega_0\left(\vec{d}_{\hat{v}} \times \vec{H}_0 + \vec{g}_{\hat{v}} \times \vec{H}_0\right)$$
$$-\sqrt{2}\,G_F\,n_{tot}\,\vec{g}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,(\vec{d}_{\hat{v}'} + \vec{g}_{\hat{v}'})$$

$$\left(\dot{\vec{d}}_{\hat{v}} - \dot{\vec{g}}_{\hat{v}}\right) = -\sqrt{2}\,G_F\,n_{tot}\,\vec{d}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,(\vec{d}_{\hat{v}'} - \vec{g}_{\hat{v}'})$$
$$-\ \omega_0\left(\vec{d}_{\hat{v}} \times \vec{H}_0 - \vec{g}_{\hat{v}} \times \vec{H}_0\right)$$
$$+\sqrt{2}\,G_F\,n_{tot}\,\vec{g}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,(\vec{d}_{\hat{v}'} - \vec{g}_{\hat{v}'})$$

① + ②

$$2\,\dot{\vec{d}}_{\hat{v}} = -2\sqrt{2}\,G_F\,n_{tot}\,\vec{d}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,\vec{d}_{\hat{v}'}$$
$$+2\omega_0\,\vec{g}_{\hat{v}} \times \vec{H}_0$$
$$-2\sqrt{2}\,G_F\,n_{tot}\,\vec{g}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,\vec{g}_{\hat{v}'}$$

$$2\,\dot{\vec{g}}_{\hat{v}} = -2\sqrt{2}\,G_F\,n_{tot}\,\vec{d}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,\vec{g}_{\hat{v}'}$$
$$+2\omega_0\,\vec{d}_{\hat{v}} \times \vec{H}_0$$
$$-2\sqrt{2}\,G_F\,n_{tot}\,\vec{g}_{\hat{v}} \times \int d\Omega_{\hat{v}'}\,(1-\hat{v}\cdot\hat{v}')\,f_{\hat{v}'}\,\vec{d}_{\hat{v}'}$$

Master Equation

Friday, August 21, 2015 7:27 PM

Neutrino density matrix can be generally written as

$\rho =$

1

$(1-\hat{D})L\rho_d =$

$$\rho_d = \begin{pmatrix} \rho_1 & a \\ a^* & \rho_2 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} \rho_1-\rho_2 & 0 \\ 0 & \rho_2-\rho_1 \end{pmatrix} + \begin{pmatrix} \rho_1+\rho_2 & 0 \\ 0 & \rho_1+\rho_2 \end{pmatrix}\right)$$

$$= \frac{1}{2}\left((\rho_1-\rho_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \rho_1+\rho_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{2}\left((\rho_1-\rho_2)\sigma_3 + (\rho_1+\rho_2) I\right)$$

$$L\rho_d = [H, \rho_d]$$

$$= \frac{1}{2}(\rho_1-\rho_2)[H,\sigma_3] + \frac{1}{2}(\rho_1+\rho_2)[H, I]$$

$$\cancel{= \frac{1}{2}(\rho_1-\rho_2)(-2i\frac{\omega}{2}c\,\sigma_2)}$$

$$\cancel{= \frac{1}{2}(\rho_2-\rho_1)\, i\frac{\omega}{2}\sigma_2}$$

$$L\sigma_3 = [H, \sigma_3]$$

$$= -\frac{\omega}{2}c[\sigma_3,\sigma_1] - \frac{\omega}{2}s[\sigma_1,\sigma_3]$$

$$= -\frac{\omega}{2}s \cdot 2i\sigma_2$$

$$= -i\omega s\sigma_2$$

$$L\rho_d = \frac{1}{2}(\rho_1-\rho_2)(-i\omega s\sigma_2)$$

$$= \frac{\omega}{2} i(\rho_2-\rho_1) s\,\sigma_2$$

(2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$$-iDL\rho_d = 0$$

$$\partial_t \rho_d = -DL\int_0^t dt'\, e^{-i(1-D)L(t-t')} \cdot \underline{(1-D)L\,\rho_d(t')}$$

$$(1-D)L\,\rho_d = \frac{\omega}{2} i(\rho_2-\rho_1)s\,\sigma_2$$

$$e^{\hat{A}} = \hat{1} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \frac{1}{3!}\hat{A}^2 + \cdots$$

$$\partial_t \rho_d = -\left(DL\int_0^t dt'\, \underbrace{e^{-i(1-D)L(t-t')}\sigma_2}\, \frac{\omega}{2} i(\rho_2-\rho_1)s\right)$$

Find out $e^{-i(1-D)L(t-t')}\sigma_2$

$$= \left[1 - i(1-D)\hat{L}(t-t') + \frac{1}{2!}\left(-i(1-D)L(t-t')\right)^2 + \frac{1}{3!}\left(-i(1-D)L(t-t')\right)^3 \cdots\right]\sigma_2$$

$$T_0 = \sigma_2$$

$$T_1 = -i(1-D)\hat{L}(t-t')\,\sigma_2$$

$$= -i(t-t')(1-D)\hat{L}\,\sigma_2$$

$$= -i(t-t')(1-D)\,\omega i(c\sigma_1 + s\sigma_3)$$

$$= -i(t-t')\,\omega i\, c\sigma_1 = (t-t')\omega c\sigma_1$$

$$T_2 = \frac{1}{2!}\left(-i(1-D)L(t-t')\right)\sigma_2$$

$$= \frac{1}{2}\left(-i(1-D)L(t-t')\left(\omega c(t-t')\sigma_1\right)\right)$$

$$= \frac{1}{2}\omega c(t-t')^2\left(-i(1-D)L\right)\sigma_1$$

$$= \frac{1}{2}\omega c(t-t')^2\left(-i(1-D)\right)(-\omega c i\sigma_2)$$

$$= \frac{1}{2}\omega^2 c^2 (t-t')^2(-\sigma_2)$$

$$L\sigma_2 = -\frac{\omega}{2}c[\sigma_3,\sigma_2] + \frac{\omega}{2}s[\sigma_1,\sigma_2]$$

$$= +\frac{\omega}{2}c\,2i\sigma_1 + \frac{\omega}{2}s\,2i\sigma_3$$

$$= \omega i c\,\sigma_1 + \omega s i\,\sigma_3$$

$$L\sigma_1 = -\frac{\omega}{2}c[\sigma_3,\sigma_1] + 0$$

$$= -\frac{\omega}{2}c\,2i\,\sigma_2$$

$$= -\omega c i\,\sigma_2$$

③

$$T_3 = \frac{1}{3!}\left(-i\,(1-D)L\,(t-t')\right)^3 \sigma_2$$
$$= \frac{1}{3!}(t-t')^3\, i\,\left((1-D)L\right)^3 \sigma_2$$
$$= \frac{1}{3!}(t-t')^3\, i\,\left((1-D)L\right)^2 (wic\,\sigma_1)$$
$$= \frac{1}{3!}(t-t')^3\, wc(-1)\,\left((1-D)L\right)\left((1-D)L\right)\sigma_1$$
$$= \frac{1}{3!}(t-t')^3\, wc\,(-1)\,(1-D)L\,(-i\,wc\,\sigma_2)$$
$$= \frac{1}{3!}(t-t')^3\, w^2c^2\, i\,(1-D)L\,\sigma_2$$
$$= \frac{1}{3!}(t-t')^3\, w^2c^2\, i\,(wic\,\sigma_1)$$
$$= \frac{1}{3!}(t-t')^3\, w^3c^3\,(-1)\,\sigma_1$$

$$\begin{cases} \sin(x) = x - \frac{1}{3!}x^3 + \cdots \\ \cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \end{cases}$$

$$\begin{cases} T_1 + T_3 + \cdots = \sin\left((t-t')\,wc\right)\sigma_1 \\ T_2 + T_4 + \cdots = \cos\left((t-t')\,wc\right)\sigma_2 \end{cases}$$

Now master eqn becomes $\quad M = (t-t')\,wc$

$$\partial_t \rho_d = -DL\int_0^t dt'\left(\left(\sin(M)\sigma_1 + \cos M\,\sigma_2\right)\frac{w}{2}i(\rho_2-\rho_1)s\right)$$
$$\partial_t \rho_d = \cancel{(\rho_1-\rho_2)}\,\frac{w}{2}\,is\int_0^t dt'\left(\sin(M)DL\,\sigma_1 + \cos M\,DL\,\sigma_2\right)(\rho_1-\rho_2)$$
$$\partial_t \rho_d = \cancel{(\rho_1-\rho_2)}\,\frac{w}{2}\,is\int_0^t\left(\cos M\;ws\,i\,\sigma_3\right)(\rho_1-\rho_2)\,dt'$$
$$= \frac{w}{2}\,is\;ws\,\sigma_3\int_0^t \cos M\,(\rho_1-\rho_2)\,dt'$$
$$\cancel{\int_0^t \cos M\,dt' = \int_0^t \cos\left(wc\,(t-t')\right)dt' = \frac{\sin(wct)}{cw}}$$

The equation becomes

$$\partial_t \rho_d = -\frac{w^2}{2}s^2\,\sigma_3\int_0^t (\cos M)\,(\rho_1-\rho_2)\,dt'$$

where $M = wc\,(t-t')$

$$\partial_t^2 \rho_d = -\frac{w^2}{2}s^2\,\sigma_3\left\{(\rho_1-\rho_2)\cos(wct)\right.$$

should be 1

$$\boxed{\frac{1}{2}(\rho_1-\rho_2)\sigma_3 = \rho_d - \frac{1}{2}(\rho_1+\rho_2)I}$$

We have

$$\partial_t^2 \rho_d = -w^2 s^2\left(\rho_d - \frac{1}{2}(\rho_1+\rho_2)I\right)\cos(wct)$$

✓

$\because$ $\rho_1+\rho_2$ is conserved, ($\rho_1+\rho_2=1$ here)

$\therefore$ $\partial_t^2 \rho_d = -\omega^2 s^2 \left(\rho_d - \frac{1}{2} I\right) \cos(\omega c t)$

The equation becomes

$$\partial_t^2 \rho_d + \omega^2 s^2 \rho_d \cos(\omega c t) - \frac{\omega^2 s^2}{2} I \cos(\omega c t) = 0$$

All elements decoupled!

First element: ρ_1 $\quad t \to x$

$$\partial_x^2 \rho_1 + \omega^2 s^2 \rho_1 \cos(\omega c x) - \frac{\omega^2 s^2}{2} \cos(\omega c x) = 0$$

ρ_1 should be $\rho = P = 1 - \sin^2\theta \sin^2\left(\frac{\omega}{2} x\right) = 1 - s^2 \sin^2\left(\frac{\omega}{2} x\right)$

$$= 1 - s^2 \left(\frac{2\sin^2(\frac{\omega}{2}x) - 1 + 1}{2}\right) = 1 - s^2 \frac{1 - \cos(\omega x)}{2}$$

$\partial_x P = -s^2 \frac{1}{2} \sin(\omega x)\, \omega$ $\quad$ $\partial_x^2 P = -s^2 \frac{1}{2} \cos(\omega x)\, \omega^2 c^2$

$$\boxed{\omega^2 s^2 P \cos(\omega x) = \omega^2 s^2 \cos(\omega x)\left(1 - s^2 \frac{1}{2}(1 - \cos(\omega x))\right) - \frac{\omega^2 s^2}{2} \cos(\omega x)}$$

$$-s^2 \frac{\omega^2}{2} c^2 \cos(\omega x) + \omega^2 s^2 \cos(\omega x) - \omega^2 s^2 s^2 \frac{1}{2}(1 - \cos(\omega x)) - \frac{\omega^2 s^2}{2} \cos(\omega x)$$

$$= +s^2 \frac{\omega^2 c^2}{2} \cos(\omega c x) - \frac{\omega^2 s^4}{2} + \frac{\omega^2 s^4}{2} \cos(\omega c x) - \frac{\omega^2 s^2}{2} \cos(\omega c x)$$

$$= \cos(\omega c x) \frac{\omega^2 c^2}{2}(1 - s^2) + \frac{\omega^2 s^2}{2} s^2 \cos(\omega c x) - \frac{\omega^2 s^4}{2}$$

$$= \cos(\omega c x) \frac{\omega^2 c^2}{2} - \frac{\omega^2 s^4}{2}$$

$\square$

$$= \cos(\omega c x)\frac{\omega^2 c^2}{2} - \frac{\omega^2 s^4}{2}$$

$$= \frac{\omega^2 c^2}{2}\left(\cos(\omega c x) - s^2\right)$$

not 0 which is required

$$\partial_t^2 P_d = -\frac{\omega^2 s^2}{2}\sigma_3 (P_1 - P_2) \quad , \text{ with } \quad P = 1 - s^2 \sin^2\left(\frac{\omega}{2}x\right)$$

$$\partial_x P = 2s^2 \sin\left(\frac{\omega}{2}x\right)\cos\left(\frac{\omega}{2}x\right)\frac{\omega}{2}$$

$$\partial_x^2 P = \omega s^2 \cos^2\left(\frac{\omega}{2}x\right)\frac{\omega}{2} - \omega c^2 \sin\left(\frac{\omega}{2}x\right)\frac{\omega}{2}$$

$$= \frac{\omega^2 s^2}{2}\left(\cos^2\left(\frac{\omega}{2}x\right) - \sin\left(\frac{\omega}{2}x\right)^2\right)$$

$$= \frac{\omega^2 s^2}{2}\left(1 - 2\sin^2\left(\frac{\omega}{2}x\right)\right)$$

$$RHS = -\frac{\omega^2 s^2}{2}\left(P - (1-P)\right)$$

$$= -\frac{\omega^2 s^2}{2}(2P - 1)$$

$$= -\frac{\omega^2 s^2}{2}\left(2 - 2s^2 \sin^2\left(\frac{\omega}{2}x\right) - 1\right)$$

$$\partial_t \rho_d(t) = \frac{\omega}{4} i \cos 2\theta_v \sigma_3 + \frac{\omega^2 \sin^2 2\theta_v}{2}\int_0^t dt' (\mathbf{I} - 2\rho_d(t')) \cos(2\cos 2\theta_v (t - t')) .$$

$$\partial_t^2 P_d = \frac{\omega^2 s^2}{2}(1 - 2P_d)$$

$$RHS = \frac{1}{2}\omega^2 s^2 \left(1 - 2 + 2s^2 \sin^2\left(\frac{\omega}{2}x\right)\right)$$

Master Eqn in Papers

Thursday, August 27, 2015 7:43 PM

Master eqn in papers

Neutrino Propagation Through Helioseismic Waves

Readings

Monday, August 24, 2015 9:14 AM

- [x] Parametric Resonances (29-33, 12)
- [ ] A simple approach : Rabi oscillation → L-Z transition
- [ ] Read about Rabi transition

McLaughlin new paper uncovering matter ...

SUSY analytical solutions for

1. MSW effect find all the possible matter profile that can be solved

2. SUSY quantum mechanics

physics report

Avinash Khare, Cooper

Notes

Monday, August 24, 2015 9:17 AM

Scales not clear:

1. neutrino mass splitting ?
2. Scale corresponds to amplitude of density fluctuation

$$\delta V_{ee}(r) = V_0 \left\{ \sum_{a=1}^{\infty} C_a \sin(q_a r + \eta_a) \right\}$$

perturbation ~~around~~ ← Look at the potential

Reversability of processes

- [ ] Understand Jacobi-Anger
- [ ] understand the role of this expansion
- [ ] then try to understand the ~~sec~~ RWA
- [ ] a better expansion?
- [ ] Compare analytical with numerical.

Effective Potential

Thursday, August 27, 2015 9:33 PM

Equation of motion in general

$$i\partial_x \psi = H\psi$$

$$\begin{pmatrix} \nu_L \\ \nu_H \end{pmatrix} = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

Using flavour basis

Define $U^{-1} = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix}$

$$H = H_0 + V_f$$

$$\Rightarrow \frac{\omega}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{\delta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$U = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

Diagonalize H_0 using U

$$H_d = U^{-1} H U$$

$$\Rightarrow \frac{\omega}{2}\sqrt{\delta^2+1-2\delta\cos 2\theta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

However to diagonalize the Hamiltonian we need

$$U^{-1} i(\partial_x \psi) = U^{-1} H_0 \psi$$

$$U^{-1}U = I$$

$$i U^{-1}(\partial_x \psi) = (U^{-1} H_0 U)\, U^{-1}\psi$$

We have already shown that $\psi_{in} = U^{-1}\psi$

$$i U^{-1}(\partial_x (U\psi_{in})) = H_d \psi_{in}$$

$$i U^{-1} U \partial_x \psi_{in} + i\, U^{-1}(\partial_x U)\, \psi_{in} = H_d \psi_{in}$$

$$i\partial_x \psi_{in} = H_d \psi_{in} \underbrace{- i U^{-1}\partial_x U}\, \psi_{in}$$

So $V = -iU^{-1}\partial_x U$ is effective potential

What is interesting is that

$U^{-1}\partial_x U$ is always off-diagonal

$U\,\partial_x U$ is always off-diagonal

$$U = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

$$\partial_x U = \frac{\partial\theta}{\partial x}\partial_\theta U = \frac{\partial\theta}{\partial x}\begin{pmatrix} -\sin\theta(x) & \cos\theta(x) \\ -\cos\theta(x) & -\sin\theta(x) \end{pmatrix}$$

$$= -\frac{\partial\theta}{\partial x}\begin{pmatrix} \sin\theta(x) & -\cos\theta(x) \\ \cos\theta(x) & \sin\theta(x) \end{pmatrix}$$

$$U^{-1} = U^T = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ +\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

Thus

$$U^{-1}\partial_x U = -\frac{\partial\theta}{\partial x}\begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ +\sin\theta(x) & \cos\theta(x) \end{pmatrix}\begin{pmatrix} \sin\theta(x) & -\cos\theta(x) \\ \cos\theta(x) & \sin\theta(x) \end{pmatrix}$$

$$= -\frac{\partial\theta}{\partial x}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Find out $\frac{\partial\theta}{\partial x}$

$$\frac{\partial\theta(x)}{\partial x} = \frac{\partial\sin\theta(x)}{\partial x}\Big/\frac{\partial\theta(x)}{\partial\sin\theta(x)} = \frac{1}{\cos\theta(x)}\frac{\partial\sin\theta(x)}{\partial x} \qquad \alpha = \hat{\delta} - \cos 2\theta$$

$\sin\theta(x)$ can be solved as $\sin 2\theta(x)$ & $\cos 2\theta(x)$

$$\sin\theta(x) = \frac{\sqrt{1 + \frac{\hat{\delta} - \cos 2\theta_v}{\sqrt{1 + (\hat{\delta} - \cos 2\theta_v)^2 - \cos^2 2\theta_v}}}}{\sqrt{2}}$$

$$\cos\theta(x) = \frac{1}{2\sqrt{2}}\left(\sqrt{1 + \frac{\hat{\delta} - \cos 2\theta_v}{\sqrt{1 + (\hat{\delta} - \cos 2\theta_v)^2 - \cos 2\theta_v}}}\right.$$

$$\left. - 2(\hat{\delta} - \cos 2\theta) + \sqrt{2 + 4(\hat{\delta} - \cos 2\theta_v)^2 - 2\cos(4\theta_v)}\ \csc(2\theta)\right)$$

Rabi oscillation

Friday, September 4, 2015 10:26 AM

A simple Rabi oscillation is

$$H = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} + \underbrace{\cos(\omega t)\begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix}}_{V}$$

Schrödinger picture:

$$H = \begin{pmatrix} \varepsilon_1 & \Omega(e^{i\omega t}+e^{-i\omega t}) \\ \Omega^*(e^{i\omega t}+e^{-i\omega t}) & \varepsilon_2 \end{pmatrix}$$

Define time-independent unitary operator

$$H_1 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

Then the Hamiltonian in interaction picture is

$$H_{Int} = e^{iH_1 t}(V)e^{-iH_1 t}$$

Since H_1 is diagonalized,

$$= \begin{pmatrix} e^{i\epsilon t} & 0 \\ 0 & e^{i\epsilon t} \end{pmatrix}\begin{pmatrix} 0 & \cos(\omega t)\Omega \\ \cos(\omega t)\Omega^* & 0 \end{pmatrix}\begin{pmatrix} e^{-i\epsilon t} & 0 \\ 0 & e^{-i\epsilon t} \end{pmatrix}$$

$=$

\# Hamiltonian $\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z - A\cos(\omega t)\sigma_x$

Ansatz $|\psi\rangle = C_g(t)e^{-iE_g t}|g\rangle + C_e(t)e^{-iE_e t}|e\rangle$

Schrödinger eqn $\sigma_z|g\rangle = |g\rangle \quad \sigma_z|e\rangle = -|e\rangle$

$i\partial_t|\psi\rangle = H|\psi\rangle \qquad E_g = -\frac{\hbar\omega_0}{2}$

$$i\partial_t\left(C_g e^{-iE_g t}|g\rangle + C_e e^{-iE_e t}|e\rangle\right) = \left(-\frac{\hbar\omega_0}{2}\sigma_z - A\cos(\omega t)\sigma_x\right)\left(\qquad\right)$$

$$i e^{-iE_g t}\dot{C}_g|g\rangle \cancel{+(-iE_g)i e^{-iE_g t}C_g|g\rangle}$$
$$+ i e^{-iE_e t}\dot{C}_e|e\rangle \cancel{+(-iE_e)i e^{-iE_e t}C_e|e\rangle}$$
$$= \cancel{-\frac{\hbar\omega_0}{2}\sigma_z C_g e^{-iE_g t}|g\rangle - \frac{\hbar\omega_0}{2}\sigma_z C_e e^{-iE_e t}|e\rangle}$$
$$- A\cos(\omega t)\sigma_x C_g e^{-iE_g t}|g\rangle - A\cos(\omega t)\sigma_x C_e e^{-iE_e t}|e\rangle$$

I already have

$$-\frac{\hbar\omega_0}{2}\sigma_z C_g e^{-iE_g t}|g\rangle = -\frac{\hbar\omega_0}{2}C_g e^{-iE_g t}|g\rangle$$

$\left\{\begin{matrix}|g\rangle \\ |e\rangle\end{matrix}\right.$ basis $\quad |g\rangle \Longrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad |e\rangle \Longrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$$\begin{cases} |g\rangle \\ |e\rangle \end{cases} \text{basis} \qquad |g\rangle \Longrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad |e\rangle \Longrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\sigma_x |g\rangle \Longrightarrow \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |e\rangle$$

So $\quad \sigma_x |g\rangle = |e\rangle, \quad \sigma_x |e\rangle = |g\rangle$

$$i e^{-iE_g t}\dot{C}_g = -A\cos(\omega t) C_e e^{-iE_e t} \qquad : |g\rangle$$

$$i e^{-iE_e t}\dot{C}_e = -A\cos(\omega t) C_g e^{-iE_g t} \qquad : |e\rangle$$

So $\quad \dot{C}_g = iA\cos(\omega t) C_e e^{-iE_e t + iE_g t}$

$$\dot{C}_g = iA\cos(\omega t) C_e e^{i(E_g - E_e)t}$$

$$\dot{C}_e = iA\cos(\omega t) C_g e^{-i(E_g - E_e)t}$$

We have $\quad \cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}), \quad E_g - E_e = -\omega_0$

$$\begin{cases} \dot{C}_g = \frac{iA}{2} C_e e^{-i\omega_0 t}(e^{i\omega t} + e^{-i\omega t}) \\ \dot{C}_e = \frac{iA}{2} C_g e^{i\omega_0 t}(e^{i\omega t} + e^{-i\omega t}) \end{cases}$$

Doing approximation : Drop $e^{-i(\omega + \omega_0)t}$

rapid oscillations

$$\dot{C}_g = \frac{iA}{2} C_e e^{-i(\omega_0 - \omega)t}$$

$$\dot{C}_e = \frac{iA}{2} C_g e^{i(\omega_0 - \omega)t}$$

~~Remove~~ Eliminate C_g :

$$\ddot{C}_e = \frac{iA}{2}\dot{C}_g e^{i(\omega_0 - \omega)t} + \frac{A}{2} i C_g e^{i(\omega_0 - \omega)t}(i(\omega_0 - \omega))$$

$$\ddot{C}_e = \frac{iA}{2} e^{i(\omega_0 - \omega)t} \cdot \frac{iA}{2} C_e e^{-i(\omega_0 - \omega)t} + \cancel{\frac{iA}{2} e^{i(\omega_0 - \omega)t}} i(\omega_0 - \omega) \cdot \dot{C}_e \frac{1}{\cancel{\frac{iA}{2} e^{i(\omega_0 - \omega)t}}}$$

Then we have

$$\ddot{C}_e - i(\omega_0 - \omega)\dot{C}_e + \frac{A^2}{4} C_e = 0$$

Characteristic eqn: $C_e = e^{i\lambda t}$

$$-\lambda^2 - i(\omega_0-\omega)\, i\lambda + \frac{A^2}{4} = 0$$

$$\lambda^2 - (\omega_0-\omega)\lambda - \frac{A^2}{4} = 0$$

$$\lambda = \frac{(\omega_0-\omega) \pm \sqrt{(\omega_0-\omega)^2 + A^2}}{2}$$

Define detuning $\boxed{\Delta = \omega_0 - \omega}$

$$C_e = C_+ e^{i\lambda_+ t} + C_- e^{i\lambda_- t}$$

Use initial condition to find C_+, C_-.

Initial condition: $C_g(0) = 1$, $C_e(0) = 0$

Then we have

$$C_+ = -C_-$$

$$C_e(t) = i\frac{A}{\Omega_R} e^{i\Delta t/2} \sin(\Omega_R t/2)$$

$$C_g(t) = e^{-i\Delta t/2}\left(\cos\left(\frac{\Omega_R t}{2}\right) + i\frac{\Delta}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\right)$$

C_e: $\frac{\omega}{2}$, $\begin{pmatrix}0\\1\end{pmatrix}$

C_g: $-\frac{\omega}{2}$, $\begin{pmatrix}1\\0\end{pmatrix}$

$\Omega_R^2 = \Delta^2 + A^2$

Rabi frequency

$$P = \frac{A^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R}{2}t\right)$$

For a system: $\hbar\begin{pmatrix}0 & 0\\ 0 & \omega_0\end{pmatrix} + \hbar\begin{pmatrix}0 & \Omega\cos\omega t\\ \Omega^*\cos\omega t & 0\end{pmatrix}$, In RF:

$$\begin{pmatrix}\dot{C}_1\\ \dot{C}_2\end{pmatrix} = -\frac{i}{2}\begin{pmatrix}0 & \Omega\\ \Omega^* & -2\Delta\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}$$

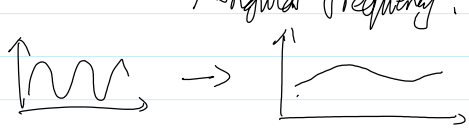
Use $C_1(0) = 1$, $C_2(0) = 0$ $\quad \Omega_R^2 = \Omega^2 + \Delta^2$

$$\begin{pmatrix}C_1\\ C_2\end{pmatrix} = e^{i\Delta t/2}\begin{pmatrix}\cos\left(\frac{\Omega_R t}{2}\right) - i\frac{\Delta}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\\ -i\frac{\Omega}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\end{pmatrix}$$

Probability to be on $|2\rangle$ is

$$P \simeq \left(\frac{\Omega}{\Omega_R}\right)^2 \sin^2\left(\frac{\Omega_R t}{2}\right)$$

$\Delta \uparrow$: $\begin{cases}\text{Amplitude}: \searrow\\ \text{Angular frequency}: \uparrow\end{cases}$



Matter in general

Tuesday, September 8, 2015 5:18 PM

The most general form is (in instantaneous energy eigen basis)

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \dot{\theta} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} E_1 &= -\frac{1}{2}\sqrt{\hat{\delta}^2 + 1 - 2\hat{\delta}\cos 2\theta_v} \\ E_2 &= -E_1 \end{aligned}}$$

$$\hat{\delta}'(x) = \frac{d\hat{\delta}}{dx}$$

$$v = \dot{\theta} = \frac{\hat{\delta}'(x)\sin 2\theta_v}{1 + 2(\hat{\delta} - \cos 2\theta_v)^2 - \cos 4\theta_v}$$

$$V = v \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_2}$$

To solve the problem, I assume

$$|\psi\rangle = C_1 e^{-iE_1 t}|1\rangle + C_2 e^{-iE_2 t}|2\rangle$$

$$E_1 = -\sqrt{}, \quad E_2 = \sqrt{}$$

$$i\partial_t |\psi\rangle = H|\psi\rangle$$

$$H = E\sigma_3 + v\sigma_2$$

$$i\partial_t\left(C_1 e^{-iE_1 t}|1\rangle + C_2 e^{-iE_2 t}|2\rangle\right)$$

$$= (-E\sigma_3 + v\sigma_2)\left(C_1 e^{-iE_1 t}|1\rangle + C_2 e^{-iE_2 t}|2\rangle\right)$$

$$LHS = i\dot{C}_1 e^{-iE_1 t}|1\rangle + E_1 C_1 e^{-iE_1 t}|1\rangle + \cancel{iC_1 e^{-iE_1 t}\partial_t|1\rangle} \approx 0$$

$$+ i\dot{C}_2 e^{-iE_2 t}|2\rangle + E_2 C_2 e^{-iE_2 t}|2\rangle + \cancel{iC_2 e^{-iE_2 t}\partial_t|2\rangle} \approx 0$$

$$RHS = C_1 E\sigma_3 e^{-iE_1 t}|1\rangle - C_2 E\sigma_3 e^{-iE_2 t}|2\rangle$$

$$+ C_1 v\sigma_2 e^{-iE_1 t}|1\rangle + C_2 e^{-iE_2 t} v\sigma_2|2\rangle$$

I know

$i\partial_t$

No! I should use a time independent

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In a basis $|1\rangle$, $|2\rangle$ $\quad |1\rangle \Rightarrow \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |2\rangle \Rightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$$-c_1 E\, e^{-iE_1 t}|1\rangle + c_2 E\, e^{-iE_2 t}|2\rangle$$

$$= c_1 E_1 e^{-iE_1 t}|1\rangle + c_2 E_2 e^{-iE_2 t}|2\rangle$$

So the equation reduces to

$$i\dot{c}_1 e^{-iE_1 t}|1\rangle + i\dot{c}_2 e^{-iE_2 t}|2\rangle$$

$$= c_1 v\, \sigma_2 e^{-iE_1 t}|1\rangle + c_2 v\, \sigma_2 e^{-iE_2 t}|2\rangle$$

I know $\sigma_2|1\rangle \Rightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix} = i|2\rangle$

$\sigma_2|2\rangle \Rightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -i\begin{pmatrix}1\\0\end{pmatrix} = -i|1\rangle$

$$\boxed{\begin{aligned} i\dot{c}_1 e^{-iE_1 t} &= -ic_2 v\, e^{-iE_2 t} \\ i\dot{c}_2 e^{-iE_2 t} &= i c_1 v\, e^{-iE_1 t} \end{aligned}}$$

So All

$$\dot{c}_1 = -c_2 v\, e^{-i(E_2 - E_1)t} = -c_2 v e^{-i2Et}$$

Define $E_2 - E_1 = 2E$

$$\dot{c}_2 = c_1 v\, e^{+i2Et}$$

Fourier decomposition of v

$$v = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i\omega t}\, d\omega$$

So I have

$$\dot{c}_1 = -c_2 \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{-iE\cdot 2\cdot t} e^{i\omega t}\, d\omega$$

$$= -c_2 \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i(\omega - 2E)t}\, d\omega$$

Also $\dot{c}_2 = c_1 \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i(\omega + 2E)t}\, d\omega$

Since c_1, c_2 are independant of ω, these eqns are valid for each mode.

$$\dot{c}_1 + c_2 \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i(\omega - 2E)t}\, d\omega = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \left(\dot{c}_1 + c_2 \frac{1}{\sqrt{2\pi}} \hat{v}(\omega)\, e^{i(\omega - 2E)t}\right) d\omega = 0$$

$$\Rightarrow \cancel{\int_{-\infty}^{\infty}\left(\dot{C}_1 + C_2 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i(\omega-2E)t}\right) d\omega = 0}$$

$$\Rightarrow \dot{C}_1 + C_2 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i(\omega-2E)t} = 0$$

$$\boxed{\int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i\omega t}\, d\omega = \int_0^{\infty} \hat{v}(\omega)\, e^{i\omega t}\, d\omega + \int_0^{\infty} \hat{v}(-\omega)\, e^{-i\omega t}\, d(-\omega)}$$

$$\dot{C}_2 = C_1 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{v}(\omega)\, e^{i(\omega+2E)t}\, d\omega$$

$$\cancel{\dot{C}_2 = C_1 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i(\omega+2E)t}}$$

$$\cancel{\dot{C}_1 = -C_2 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i(\omega-2E)t}}$$

$\Rightarrow$ ~~Can be solved in different cases~~

But Now I use a trick: or $\dot{C}_2 = C_1 \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \hat{v}(-\omega)\, e^{-i(\omega-2E)t}\, d(-\omega)$

$$\cancel{\int_{-\infty}^{\infty}\left(\dot{C}_2 - C_1 \frac{1}{\sqrt{2\pi}}\hat{v}(-\omega)\, e^{i(-\omega+2E)t}\right) d(-\omega) = 0}$$

$$\Downarrow$$

$$\dot{C}_2 = -C_1 \frac{\hat{v}(-\omega)}{\sqrt{2\pi}}\, e^{-i(\omega-2E)t}$$

$$\Rightarrow \dot{C}_2 = C_1 \frac{1}{\sqrt{2\pi}}\hat{v}(-\omega)\, e^{-i(\omega-2E)t}$$

Combine it with $\dot{C}_1 = -C_2 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i(\omega-2E)t}$

Define $\delta = \omega - 2E$

$$\dot{C}_1 = -C_2 \frac{1}{\sqrt{2\pi}}\hat{v}(\omega)\, e^{i\delta t}$$

$$\dot{C}_2 = C_1 \frac{1}{\sqrt{2\pi}}\hat{v}(-\omega)\, e^{-i\delta t}$$

Define –

Compare with Rabi!

$$\dot{C}_g = \frac{iA}{2} C_e\, e^{-i(\omega_0-\omega)t}$$

$$\dot{C}_e = \frac{iA}{2} C_g\, e^{i(\omega_0-\omega)t}$$

$$\Rightarrow$$

$$C_e(t) = i\frac{A}{\Omega_R} e^{i\delta t/2} \sin(\Omega_R t/2)$$

$$C_g(t) = e^{-i\delta t/2}\left(\cos\left(\frac{\Omega_R t}{2}\right) + i\frac{\delta}{\Omega_R}\sin\left(\frac{\Omega_R t}{2}\right)\right)$$

$$P = \frac{A^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R}{2}t\right)$$

Vacuum energy basis

Wednesday, September 9, 2015 3:33 PM

Suppose we are working in vacuum energy eigenstates:

$$H = -\frac{\omega}{2}\sigma_3 + U^\dagger V U$$

$$V = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \lambda = \sqrt{2}G_F n_e$$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c\lambda & 0 \\ s\lambda & 0 \end{pmatrix} \begin{matrix} c \\ \\ \end{matrix}$$

$$= \begin{pmatrix} c^2\lambda & cs\lambda \\ sc\lambda & s^2\lambda \end{pmatrix}$$

Remove trace:

$$V_m = \lambda \begin{pmatrix} \frac{c^2-s^2}{2} & cs \\ cs & \frac{s^2-c^2}{2} \end{pmatrix}$$

$$= \frac{\lambda}{2}\begin{pmatrix} \cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & -\cos 2\theta_v \end{pmatrix}$$

$$= \frac{\lambda}{2}\cos 2\theta_v \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\lambda}{2}\sin 2\theta_v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{\lambda}{2}\cos 2\theta_v\, \sigma_3 + \frac{\lambda}{2}\sin 2\theta_v\, \sigma_1$$

So we have the Hamiltonian:

$$H = -\frac{\omega}{2}\sigma_3 \underbrace{+ \frac{\lambda}{2}\cos 2\theta_v \sigma_3}_{\text{new term}} + \frac{\lambda}{2}\sin 2\theta_v\, \sigma_1$$

So Hamiltonian vector $(I, \sigma_1, \sigma_2, \sigma_3)$

$$\boxed{\vec{H} = \left(0, \frac{\lambda}{2}\sin 2\theta_v, 0, -\frac{\omega}{2} + \frac{\lambda}{2}\cos 2\theta_v\right)}$$

$$i\partial_t |\psi\rangle = H|\psi\rangle \qquad |1\rangle : -\frac{\hbar\omega}{2}$$

$$|2\rangle : \frac{\hbar\omega}{2}$$

Assume $|\psi\rangle = c_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle + c_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle$

I also have $\sigma_3|1\rangle = |1\rangle$, $\sigma_3|2\rangle = -|2\rangle$

$\sigma_1|1\rangle = |2\rangle$, $\sigma_1|2\rangle = |1\rangle$

So the Schrödinger eqn is

$$\left(i\dot{c}_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle - \frac{\hbar\omega}{2}c_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle + i\dot{c}_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle + \frac{\hbar\omega}{2}c_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle\right)$$

$$= \left(\frac{\lambda}{2}\cos 2\theta_v - \frac{\omega}{2}\right)c_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle - \left(\frac{\lambda}{2}\cos 2\theta - \frac{\omega}{2}\right)c_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle$$

$$+ \frac{\lambda}{2}\sin 2\theta_v\, c_1 e^{i\frac{\hbar\omega}{2}t}|2\rangle + \frac{\lambda}{2}\sin 2\theta_v\, c_2 e^{-i\frac{\hbar\omega}{2}t}|1\rangle$$

Collect the terms

$$\left(i\dot{c}_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle \cancel{- \frac{\hbar\omega}{2}i c_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle} + i\dot{c}_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle \cancel{+ \frac{\hbar\omega}{2}i c_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle}\right)$$

$$= \left(\frac{\lambda}{2}\cos 2\theta_v \cancel{- \frac{\omega}{2}}\right)c_1 e^{i\frac{\hbar\omega}{2}t}|1\rangle - \left(\frac{\lambda}{2}\cos 2\theta \cancel{- \frac{\omega}{2}}\right)c_2 e^{-i\frac{\hbar\omega}{2}t}|2\rangle$$

$$+ \frac{\lambda}{2}\sin 2\theta_v\, c_1 e^{i\frac{\hbar\omega}{2}t}|2\rangle + \frac{\lambda}{2}\sin 2\theta_v\, c_2 e^{-i\frac{\hbar\omega}{2}t}|1\rangle$$

$\boxed{\text{In fact } \hbar = 1}$

$$|1\rangle : \quad i\dot{c}_1 e^{i\frac{\omega}{2}t} = \frac{\lambda}{2}\cos 2\theta_v\, c_1 e^{i\frac{\omega}{2}t} + \frac{\lambda}{2}\sin 2\theta_v\, c_2 e^{-i\frac{\omega}{2}t}$$

$$|2\rangle : \quad i\dot{c}_2 e^{-i\frac{\omega}{2}t} = -\frac{\lambda}{2}\cos 2\theta_v\, c_2 e^{-i\frac{\omega}{2}t} + \frac{\lambda}{2}\sin 2\theta_v\, c_1 e^{i\frac{\omega}{2}t}$$

$\boxed{\lambda \text{ is time dependant: So}}$

$$\lambda_c = \lambda\cos 2\theta_v, \quad \lambda_s = \lambda\sin 2\theta_v$$

$$i\dot{c}_1 = \frac{\lambda_c}{2}c_1 + \frac{\lambda_s}{2}c_2 e^{-i\omega t}$$

$$i\dot{c}_2 = -\frac{\lambda_c}{2} c_2 + \frac{\lambda_s}{2} c_1 e^{i\omega t}$$

$$c_1 = \frac{i\dot{c}_2 + \frac{\lambda_c}{2} c_2}{\frac{\lambda_s}{2} e^{i\omega t}} \quad , \quad c_2 = \frac{i\dot{c}_1 - \frac{\lambda_c}{2} c_1}{\frac{\lambda_s}{2} e^{-i\omega t}}$$

$$\dot{c}_1 = \frac{\left(i\ddot{c}_2 + \frac{\dot{\lambda}_c}{2} c_2 + \frac{\lambda_c}{2}\dot{c}_2\right)\left(\frac{\lambda_s}{2} e^{i\omega t}\right) - \left(i\dot{c}_2 + \frac{\lambda_c}{2} c_2\right)\left(\frac{\dot{\lambda}_s}{2} e^{i\omega t} + i\omega \frac{\lambda_s}{2} e^{i\omega t}\right)}{\left(\frac{\lambda_s}{2} e^{i\omega t}\right)^2}$$

So the equation for c_2 is

$$i\left(\frac{i\ddot{c}_2 + \frac{\dot{\lambda}_c}{2} c_2 + \frac{\lambda_c}{2}\dot{c}_2}{\frac{\lambda_s}{2} e^{i\omega t}} - \frac{\left(i\dot{c}_2 + \frac{\lambda_c}{2} c_2\right)\left(\frac{\dot{\lambda}_s}{2} e^{i\omega t} + i\omega \frac{\lambda_s}{2} e^{i\omega t}\right)}{\left(\frac{\lambda_s}{2} e^{i\omega t}\right)^2}\right)$$

$$= \frac{\lambda_c}{2} \frac{i\dot{c}_2 + \frac{\lambda_c}{2} c_2}{\frac{\lambda_s}{2} e^{i\omega t}} + \frac{\lambda_s}{2} c_2 e^{-i\omega t}$$

Simplify:

$$i\left(i\ddot{c}_2 + \frac{\dot{\lambda}_c}{2} c_2 + \frac{\lambda_c}{2}\dot{c}_2 - \left(i\dot{c}_2 + \frac{\lambda_c}{2} c_2\right)\left(\frac{\dot{\lambda}_s}{\lambda_s} + i\omega\right)\right)$$

$$= \frac{\lambda_c}{2}\left(i\dot{c}_2 + \frac{\lambda_c}{2} c_2\right) + \frac{\lambda_s}{2} c_2 e^{-i\omega t} \cdot \frac{\lambda_s}{2} e^{i\omega t}$$

$$-\ddot{c}_2 + i\frac{\dot{\lambda}_c}{2} c_2 + i\frac{\lambda_c}{2}\dot{c}_2 - i\left(i\dot{c}_2 + \frac{\lambda_c}{2} c_2\right)\left(\frac{\dot{\lambda}_s}{\lambda_s} + i\omega\right)$$

$$= i\frac{\lambda_c}{2}\dot{c}_2 + \frac{\lambda_c^2}{4} c_2 + \frac{\lambda_s^2}{4} c_2$$

$$\frac{\dot{\lambda}_s}{\lambda_s} = \frac{\dot{\lambda}}{\lambda}$$

$$-\ddot{c}_2 + \left(+i\frac{\lambda_c}{2} + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) - i\frac{\lambda_c}{2}\right)\dot{c}_2$$

$$+ \left(-i\frac{\lambda_c}{2}\left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) - \left(\frac{\lambda_c^2}{4} + \frac{\lambda_s^2}{4}\right) + i\frac{\dot{\lambda}_c}{2}\right) c_2 = 0$$

$$-\ddot{c}_2 + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)\dot{c}_2 + \left(i\frac{\dot{\lambda}_c}{2} - i\frac{\lambda_c}{2}\left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) - \frac{\lambda^2}{4}\right) c_2 = 0$$

$$\boxed{\frac{\lambda_c^2}{4} - \frac{\lambda_s^2}{4} = \frac{1}{4}\lambda^2 \cos 4\theta_v}$$

$$-\ddot{c}_2 + \left(-i\frac{\lambda_c}{2} + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) - i\frac{\lambda_c}{2}\right)\dot{c}_2$$

$$+ \left(i\frac{\lambda_c}{2}\left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) + \frac{\lambda^2}{4}\cos 4\theta_v\right) c_2 = 0$$

$$-\ddot{c}_2 + \left(-i\lambda_c + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)\right)\dot{c}_2 + \left(\frac{i}{2}\lambda_c\left(\frac{\dot{\lambda}}{\lambda} + i\omega\right) + \frac{\lambda^2}{4}\cos 4\theta_v\right) c_2 = 0$$

$$\lambda_c = \lambda \cos 2\theta_v$$

Solve the problem: So we have the Hamiltonian:

$$H = -\frac{\omega}{2}\sigma_3 + \underbrace{\frac{\lambda}{2}\cos 2\theta_v \sigma_3}_{\text{new term}} + \frac{\lambda}{2}\sin 2\theta_v \sigma_1$$

Question 1: Can we go back to the Rabi Oscillation we worked out before?

Because $\left|\frac{\lambda}{2}\cos 2\theta_v\right| \sim \left|\frac{\lambda}{2}\sin 2\theta_v\right|$ or even larger!
That means it doesn't make sense to drop σ_3 term alone

But to make things clear, I can describe the meaning of the two terms

$$\frac{\lambda}{2}\cos 2\theta_v \sigma_3 \quad \text{and} \quad \frac{\lambda}{2}\sin 2\theta_v \sigma_1$$

Seperately:

$\boxed{\lambda = \lambda_0 \cos(\omega_\lambda t)}$

For the case that $\left|\frac{\lambda}{2}\cos 2\theta_v\right| < \left|\frac{\omega}{2}\right|$, I have (if λ is periodic)

$\frac{\lambda}{2}\cos 2\theta_v\, \sigma_3$ — added a period change in eigenenergies

$\frac{\lambda}{2}\sin 2\theta_v\, \sigma_1$ — Flip the states with a Rabi flopping mechanism. flopping rate is

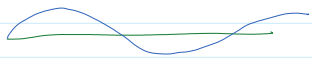
$A \sim \frac{\lambda_0}{2}\sin 2\theta_v$

$$\Omega_R = \sqrt{\Delta^2 + A^2}$$

$$P = \frac{A}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right)$$

$$\Delta = \omega - \omega_\lambda$$

$$A = \frac{\lambda_0}{2}\sin 2\theta_v$$



$$\Rightarrow P = \frac{\frac{\lambda_0}{2}\sin 2\theta_v}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right)$$

Now the interesting part is that

$$\Omega_R \sim \sqrt{(\omega' - \omega_\lambda)^2 + A^2}$$

$$\omega' \sim \omega - \frac{\lambda(t)}{2}\cos 2\theta_v$$

Probably not a very good approximation.

$$P \sim \frac{\frac{\lambda_0}{2}\sin 2\theta_v}{\Omega_R(t)^2}\sin^2\left(\frac{\Omega_R(t)}{2}\cdot t\right)$$

So period $\Omega_R(t)$ is changing periodically
Amplitude is also changing periodically.

IF The Approximation is True

Numerical Verification or Not.

Consider a system

$$H = -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\cos 2\theta_v\, \sigma_3 + \frac{\lambda}{2}\sin 2\theta_v\, \sigma_1$$

Solve it numerically and compare with the predication

Numbers to be used:

$$\omega = \frac{\Delta m^2}{2E} \sim \frac{10^{-4}\,eV^2}{2\cdot 10\,MeV} \sim \frac{10^{-4}}{2\times 10^7}\,eV = 5\times 10^{-12}\,eV$$

$\lambda = \alpha\omega\cos(\omega_\lambda x)$, ω_λ can be something $[0.1\omega, 10\omega]$

$\theta \sim \theta_{12}$, so use $\sin^2\theta = 0.3$ $\Rightarrow$ $\sin 2\theta = 0.917$, $\cos 2\theta = 0.4$

Make the equation dimensionless

$$i\frac{d}{dx}\psi = \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\cos 2\theta\,\sigma_3 + \frac{\lambda}{2}\sin 2\theta\,\sigma_1\right)\psi$$

$$i\frac{d}{\omega dx}\psi = \frac{1}{2}\left(-\sigma_3 + \frac{\lambda}{\omega}\cos 2\theta\,\sigma_3 + \frac{\lambda}{\omega}\sin 2\theta\,\sigma_1\right)\psi$$

Predication:

Scale the x with ω

$$P \sim \frac{\frac{\alpha\omega}{2}\sin 2\theta_v}{\Omega_R(x)^2}\sin^2\left(\frac{\Omega_R(x)}{2\omega}\,\omega x\right)$$

Define $\bar{x} = \omega x$

$$\Omega_R^2 = (\omega' - \omega_\lambda)^2 + \left(\frac{\lambda_0}{2}\sin 2\theta\right)^2$$

Now the interesting part is that

$$\Delta \sim \sqrt{(\omega' - \omega_\lambda)^2 + A^2}$$

$$\omega' \sim \omega + \frac{\lambda(t)}{2}\cos 2\theta_v$$

Probably not a very good approximation.

$$P \sim \frac{\frac{\lambda_0}{2}\sin 2\theta_v}{\Omega_R(x)^2}\sin^2\left(\frac{\Omega_R(x)}{2}\cdot x\right)$$

Define $\bar{x} = \omega x$

$$\Omega_R^2 = (\omega' - \omega_\lambda)^2 + \left(\frac{\lambda_0}{2}\sin 2\theta\right)^2$$

$$= \left(\omega - \frac{\lambda(\omega)}{2}\cos 2\theta - \omega_\lambda\right)^2 + \left(\frac{\alpha\omega}{2}\sin 2\theta\right)^2$$

$$= \left(\omega - \omega_\lambda - \frac{\alpha\omega\cos(\omega_\lambda x)}{2}\cos 2\theta\right)^2 + \left(\frac{\alpha\omega}{2}\sin 2\theta\right)^2$$

$$= \left(\omega - \omega_\lambda - \frac{\alpha\omega}{2}\cos 2\theta\cos\left(\frac{\omega_\lambda}{\omega}\bar{x}\right)\right)^2 + \left(\frac{\alpha\omega}{2}\sin 2\theta\right)^2$$

$$= \omega^2\left[\left(1 - \frac{\omega_\lambda}{\omega} - \frac{\alpha}{2}\cos 2\theta\cos\left(\frac{\omega_\lambda}{\omega}\bar{x}\right)\right)^2 + \left(\frac{\alpha}{2}\sin 2\theta\right)^2\right]$$

$$= \omega^2\left(\hat{\Omega}_R(\bar{x})\right)^2$$

$$P = \frac{\frac{\alpha\omega}{2}\sin 2\theta_v}{\omega^2\left(\hat{\Omega}_R(\bar{x})\right)^2}\sin^2\left(\frac{\hat{\Omega}_R(x)}{2}\bar{x}\right)$$

$$\hat{\Omega}_R(x) = \sqrt{\left(1 - \frac{\omega_\lambda}{\omega} - \frac{\alpha}{2}\cos 2\theta\cos\left(\frac{\omega_\lambda}{\omega}\bar{x}\right)\right)^2 + \left(\frac{\alpha}{2}\sin 2\theta\right)^2}$$

$$P \sim \frac{\frac{\lambda_0}{2}\sin 2\theta_v}{\Omega_R(x)^2}\sin^2\left(\frac{\Omega_R(x)}{2}\cdot x\right)$$

So period $\Omega_R(t)$ is changing periodically
Amplitude is also changing periodically.

IF The Approximation is True

Understanding the system

$$\mathrm{I}\left[\begin{aligned} i\dot{C}_1 &= \boxed{\frac{\lambda_c}{2}C_1} + \boxed{\frac{\lambda_0}{2}C_2 e^{-i\omega t}} \\ i\dot{C}_2 &= \boxed{\frac{\lambda_c}{2}C_2} + \boxed{\frac{\lambda_0}{2}C_1 e^{i\omega t}} \end{aligned}\right.$$

I:

If we have no blue terms, the we have exactly Rabi oscillation, which works as flopping.

Now we have blue terms

$$i\dot{C}_1 = \frac{\lambda_c}{2}C_1$$

$$\frac{dC_1}{C_1} = -i\frac{\lambda_c}{2}dx$$

$$\ln C_1 = -\frac{i}{2}\int_0^x \lambda(x')\cos 2\theta_v\, dx'$$

$$\left[\begin{aligned} C_1(x) &= e^{-\frac{i}{2}\cos 2\theta_v\int_0^x \lambda(x')dx'}C_1(0) \\ C_2(x) &= C_2(0)\, e^{+\frac{i}{2}\cos 2\theta_v\int_0^x \lambda(x')dx'} \end{aligned}\right.$$

from the new term

So blue terms works as oscillations with amplitude $C_2(0)$

If we have $C_1 \to C_2$ the C_2 can be populated.
otherwise this doesn't work since $C_2(0) = 0$

What is interesting is the double oscillation structure

$\cancel{e^{-\frac{i}{2}\cos 2\theta_v\int_0^x}}$ Suppose $\lambda(x') = \lambda_0\cos(\omega_\lambda x')$

$$\int_0^x \lambda_0 \cos(\omega_\lambda x')\, dx' = \lambda_0 \frac{1}{\omega_\lambda} \sin(\omega_\lambda x)$$

Thus

$$c_2 \sim \underbrace{\cos\left(\frac{\lambda_0}{\omega_\lambda} \sin(\omega_\lambda x)\right)}$$

$$\text{II} \quad \begin{cases} c_1 = \dfrac{i\dot{c}_2 \boxed{+ \frac{\dot{\lambda}}{2} c_2}}{\frac{\lambda}{2} e^{i\omega t}}, & c_2 = \dfrac{i\dot{c}_1 \boxed{- \frac{\dot{\lambda}}{2} c_1}}{\frac{\lambda}{2} e^{-i\omega t}} \end{cases}$$

Another way of understanding it is

Suppose we don't have blue term.

~~then~~

blue can either suppress or enhance not trivial.

The new introduce term is only

$$-\frac{\alpha \cos 2\theta_v}{\sin 2\theta_v} \frac{1}{e^{i\omega t}} c_2$$

which doesn't depend on $|\lambda|$!!!

$$-\ddot{c}_2 + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)\dot{c}_2 + \left(\overbrace{i\frac{\dot{\lambda}}{2} - i\frac{\lambda}{2}\left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)}^{0} - \frac{\lambda^2}{4}\right) c_2 = 0$$

$$-\ddot{c}_2 + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)\dot{c}_2 + \left(\frac{\lambda}{2}\omega - \frac{\lambda^2}{4}\right) c_2 = 0$$

Solve the correct equation for c_2

$$\lambda = \lambda_0 \cos(\omega_\lambda x) = \alpha\omega \cos(\omega_\lambda x) = \alpha\omega \cos(\beta \bar{x})$$

$$\frac{\lambda}{\omega} = \alpha \cos(\omega_\lambda x) = \alpha \cos(\beta \bar{x})$$

$$\omega_\lambda = \beta\omega$$

$$\frac{d^2 c_2}{dx^2} = \omega^2 \frac{d^2 c_2}{d\bar{x}^2}, \quad \frac{dc_2}{dx} = \omega \frac{dc_2}{d\bar{x}},$$

$$-\frac{d^2 c_2}{dx^2} + \left(\frac{1}{\lambda}\frac{d\lambda}{dx} + i\omega\right)\frac{dc_2}{dx} + \left(\frac{i}{2}\frac{d\lambda}{dx}\cos 2\theta_v - \frac{i}{2}\lambda \cos 2\theta_v \left(\frac{1}{\lambda}\frac{d\lambda}{dx} + i\omega\right) - \frac{\lambda^2}{4}\right) c_2 = 0$$

$$-\frac{d^2 c_2}{d\bar{x}^2} + \left(\underbrace{\frac{1}{\lambda}\frac{d\lambda}{d\bar{x}}}_{\frac{-\beta\alpha\omega\sin(\beta\bar{x})}{\alpha\omega\cos(\beta\bar{x})}} + i\right)\frac{dc_2}{d\bar{x}} + \left(\underbrace{\frac{i}{2}\frac{1}{\omega}\frac{d\lambda}{d\bar{x}}\cos 2\theta_v}_{-\frac{i}{2}\alpha\beta\sin(\beta\bar{x})\cos 2\theta_v} - \underbrace{\frac{i}{2}\frac{\lambda\cos 2\theta_v}{\omega}}_{\frac{i}{2}\alpha\cos(\beta\bar{x})\cos 2\theta_v}\left(\frac{1}{\lambda}\frac{d\lambda}{d\bar{x}} + i\right) - \underbrace{\frac{\lambda^2}{4}\frac{1}{\omega^2}}_{\frac{1}{4}(\alpha\cos(\beta\bar{x}))^2}\right) c_2 = 0$$

$$= -\beta\tan(\beta\bar{x})$$

A singularity here

Equation becomes

$$-\frac{d^2 c_2}{d\bar{x}^2} + \left(i - \beta\tan(\beta\bar{x})\right)\frac{dc_2}{d\bar{x}} + \left(\cancel{-\frac{i}{2}\alpha\beta\sin(\beta\bar{x})\cos 2\theta_v} - \frac{i}{2}\alpha\cos(\beta\bar{x})\cos 2\theta_v\left(\cancel{i - \beta\tan(\beta\bar{x})}\right) - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right) c_2 = 0$$

Do the analysis again. We don't have exact rabi oscillation!

In the case that $\left|\frac{\lambda}{2}\cos 2\theta_v\right| \ll \left|\frac{\omega}{2}\right|$, I have

Do the analysis again. We don't have exact Rabi oscillation:

In the case that $\left|\frac{\lambda}{2}\cos 2\theta_v\right| \ll \left|\frac{\omega}{2}\right|$, I have

$\frac{\lambda}{2}\cos 2\theta_v\,\sigma_3$

which adds a periodic change to C_2 (or C_1) due to the non zero value of C_2 (or C_1)

$\frac{\lambda}{2}\sin 2\theta_v\,\sigma_1$

which is Rabi flopping mechanism

The flopping rate is given by

$$P = \frac{\left(\frac{\lambda_0}{2}\sin 2\theta_v\right)^2}{\Omega_R^2}\sin^2\left(\frac{1}{2}\Omega_R X\right)$$

$$\Omega_R = \sqrt{\left(\omega - \frac{\lambda(t)}{2}\cos 2\theta_v - \omega_x\right)^2 + \left(\frac{\lambda_0}{2}\sin 2\theta_0\right)^2}$$

But this approximation is probably not good.

To do:

1. Try to solve the equation for C_2 using ~~[illegible]~~ series:
2. Revise the program! Fix all the wrong equations.

$$i\dot{C}_1 = \boxed{\frac{\lambda_c}{2}C_1} + \boxed{\frac{\lambda_s}{2}C_2 e^{-i\omega t}}$$
$$i\dot{C}_2 = \boxed{\frac{\lambda_c}{2}C_2} + \boxed{\frac{\lambda_s}{2}C_1 e^{i\omega t}}$$

For numerical calculation, I need four real equations

$$i\frac{d}{d\bar{x}}(C_{1,R} + iC_{1,I}) = \frac{\lambda_c}{2\omega}(C_{1,R} + iC_{1,I}) + \frac{\lambda_s}{2\omega}(C_{2,R} + iC_{2,I})(\cos\bar{x} - i\sin\bar{x})$$

$$i\frac{d}{d\bar{x}}(C_{2,R} + iC_{2,I}) = -\frac{1}{2}\frac{\lambda_c}{\omega}(C_{2,R} + iC_{2,I}) + \frac{1}{2}\frac{\lambda_s}{\omega}(C_{1,R} + iC_{1,I})(\cos\bar{x} + i\sin\bar{x})$$

$$\cancel{i}\frac{d}{d\bar{x}}C_{1,R} = \cancel{i}\frac{1}{2}\frac{\lambda_c}{\omega}C_{1,I} + \frac{1}{2}\frac{\lambda_s}{\omega}\left(-\cancel{i}C_{2,R}\sin\bar{x} + \cancel{i}C_{2,I}\cos\bar{x}\right)$$

$$-\frac{d}{d\bar{x}}C_{1,I} = \frac{\lambda_c}{2\omega}C_{1,R} + \frac{\lambda_s}{2\omega}\left(C_{2,R}\cos\bar{x} + C_{2,I}\sin\bar{x}\right)$$

$$\cancel{i}\frac{d}{d\bar{x}}C_{2,R} = -\cancel{i}\frac{1}{2}\frac{\lambda_c}{\omega}C_{2,I} + \frac{1}{2}\frac{\lambda_s}{\omega}\left(\cancel{i}C_{1,I}\cos\bar{x} + \cancel{i}C_{1,R}\sin\bar{x}\right)$$

$$-\frac{d}{d\bar{x}}C_{2,I} = -\frac{1}{2}\frac{\lambda_c}{\omega}C_{2,R} + \frac{1}{2}\frac{\lambda_s}{\omega}\left(C_{1,R}\cos\bar{x} - C_{1,I}\sin\bar{x}\right)$$

$$\frac{d}{d\bar{x}}\begin{pmatrix} C_{1,R} \\ C_{1,I} \\ C_{2,R} \\ C_{2,I} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\lambda_s}{\omega} & -\frac{\lambda_s}{\omega}\sin\bar{x} & \frac{\lambda_s}{\omega}\cos\bar{x} \\ -\frac{\lambda_c}{\omega} & 0 & -\frac{\lambda_s}{\omega}\cos\bar{x} & -\frac{\lambda_s}{\omega}\sin\bar{x} \\ \frac{\lambda_s}{\omega}\sin\bar{x} & \frac{\lambda_s}{\omega}\cos\bar{x} & 0 & -\frac{\lambda_c}{\omega} \\ -\frac{\lambda_s}{\omega}\cos\bar{x} & \frac{\lambda_s}{\omega}\sin\bar{x} & \frac{\lambda_c}{\omega} & 0 \end{pmatrix}\begin{pmatrix} C_{1R} \\ C_{1I} \\ C_{2R} \\ C_{2I} \end{pmatrix}$$

$$-\frac{d^2C_2}{d\bar{x}^2} + \left(i - \beta\tan(\beta\bar{x})\right)\frac{dC_2}{d\bar{x}} + \left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right)C_2 = 0$$

has

$$\frac{d^2}{d\bar{x}^2} + (i - \beta\tan(\beta\bar{x}))\frac{d}{d\bar{x}} + \left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right)C_2 = 0$$

has

$$-\frac{d^2(C_{2,R}+iC_{2,I})}{d\bar{x}^2} + \left(i - \beta\tan(\beta\bar{x})\right)\frac{d}{d\bar{x}}(C_{2R}+iC_{2I}) + \left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{\alpha^2}{4}\cos^2(\beta\bar{x})\right)(C_{2R}+iC_{2I})$$

Real part:

$$-\frac{d^2}{d\bar{x}}C_{2R} + \left(-\beta\tan(\beta\bar{x})\right)\frac{d}{d\bar{x}}C_{2R} - \frac{d}{d\bar{x}}C_{2I} + (\qquad\qquad)C_{2R} = 0$$

Imaginary part

$$-\frac{d^2}{d\bar{x}}C_{2I} + \frac{d}{d\bar{x}}C_{2R} - \beta\tan(\beta\bar{x})\frac{d}{d\bar{x}}C_{2I} + (\qquad\qquad)C_{2I} = 0$$

Define $C_{2p} = \frac{d}{d\bar{x}}C_2 \Rightarrow C_{2p,R} = \frac{d}{d\bar{x}}C_{2,R}$, $C_{2p,I} = \frac{d}{d\bar{x}}C_{2I}$

$$\begin{cases} -\frac{d}{d\bar{x}}C_{2p,R} + (-\beta\tan(\beta\bar{x}))C_{2p,R} - C_{2p,I} + (\qquad\qquad)C_{2,R} = 0 \\ -\frac{d}{d\bar{x}}C_{2p,I} + C_{2p,R} - \beta\tan(\beta\bar{x})C_{2p,I} + (\qquad\qquad)C_{2,I} = 0 \end{cases}$$

$$\frac{d}{d\bar{x}}\begin{pmatrix} C_{2,R} \\ C_{2,I} \\ C_{2p,R} \\ C_{2p,I} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\qquad) & 0 & +\beta\tan(\beta\bar{x}) & +1 \\ 0 & -(\qquad) & -1 & +\beta\tan(\beta\bar{x}) \end{pmatrix}\begin{pmatrix} C_{2R} \\ C_{2I} \\ C_{2p,R} \\ C_{2p,I} \end{pmatrix}$$

$$(\qquad\qquad) = \frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{\alpha^2}{4}\cos^2(\beta\bar{x})$$

$$-\ddot{C}_2 + \left(\frac{\dot{\lambda}}{\lambda} + i\omega\right)\dot{C}_2 + \left(\frac{\lambda_c}{2}\omega - \frac{\lambda^2}{4}\right)C_2 = 0$$

$$-\frac{d^2}{d\bar{x}^2}C_2 + \left(\frac{1}{\lambda}\frac{d\lambda}{d\bar{x}} + i\omega\right)$$

Solve

$$-\frac{d^2C_2}{d\bar{x}^2} + (i - \beta\tan(\beta\bar{x}))\frac{dC_2}{d\bar{x}} + \left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right)C_2 = 0$$

Using series solution

$$C_2(\bar{x}) = \sum_{n=0}^{\infty} a_n\bar{x}^n$$

$$\frac{d}{d\bar{x}}C_2(\bar{x}) = \sum_{n=1}^{\infty} a_n n\bar{x}^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)\bar{x}^n$$

$$\frac{d^2}{d\bar{x}^2}C_2(\bar{x}) = \sum_{n=2}^{\infty} a_n n(n-1)\bar{x}^{n-2}$$

$$= \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)\bar{x}^n$$

Plug it into the equation

$$-\frac{d^2C_2}{d\bar{x}^2} + (i - \beta\tan(\beta\bar{x}))\frac{dC_2}{d\bar{x}} + \left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right)C_2 = 0$$

$$-\sum_{n=2}^{\infty} a_n n(n-1)\bar{x}^{n-2} + \overbrace{(i - \beta\tan(\beta\bar{x}))}^{B}\sum_{n=1}^{\infty} a_n n\bar{x}^{n-1}$$

$$+\underbrace{\left(\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta - \frac{1}{4}\alpha^2\cos^2(\beta\bar{x})\right)}_{A}\sum_{n=0}^{\infty} a_n\bar{x}^n = 0$$

$$-\sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)\bar{x}^n + B\sum_{n=0}^{\infty} a_{n+1}(n+1)\bar{x}^n + A\sum_{n=0}^{\infty} a_n\bar{x}^n = 0$$

$$-a_{n+2}(n+1)(n+2) + B\,a_{n+1}(n+1) + A\,a_n = 0$$

Numerical WRONG SIGN

Tuesday, September 22, 2015 2:01 PM

This is Wrong

* Decompose the equation into 4 real equations.
* Arrange the vector to be solved

$$C = (C_{1R}, C_{1I}, C_{2R}, C_{2I})$$

Find the matrix equation

$$\frac{d}{d\bar{x}}\begin{pmatrix} C_{1R} \\ C_{1I} \\ C_{2R} \\ C_{2I} \end{pmatrix} = \underbrace{\begin{pmatrix} 0, & \frac{\alpha\cos 2\theta_v}{2}\cos(\beta\bar{x}), & -\frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\sin\bar{x}, & \frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\cos\bar{x} \\ -\frac{\alpha\cos 2\theta_v}{2}\cos(\beta\bar{x}), & 0, & -\frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\cos\bar{x}, & -\frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\sin\bar{x} \\ \frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\sin\bar{x}, & \frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\cos\bar{x}, & 0, & \frac{\alpha\cos 2\theta_v}{2}\cos(\beta\bar{x}) \\ -\frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\cos\bar{x}, & \frac{\alpha\sin 2\theta_v}{2}\cos(\beta\bar{x})\sin\bar{x}, & -\frac{\alpha\cos 2\theta_v}{2}\cos\beta\bar{x}, & 0 \end{pmatrix}}_{A \text{ antisymmetric matrix} \to K} \begin{pmatrix} C_{1R} \\ C_{1I} \\ C_{2R} \\ C_{2I} \end{pmatrix}$$

$$K = \frac{\alpha\cos 2\theta_v}{2}\cos(\beta\bar{x})\begin{pmatrix} 0, & 1, & -\frac{\sin 2\theta_v}{\cos 2\theta_v}\sin\bar{x}, & \frac{\sin 2\theta_v}{\cos 2\theta_v}\cos\bar{x} \\ -1, & 0, & -\frac{\sin 2\theta_v}{\cos 2\theta_v}\cos\bar{x}, & -\frac{\sin 2\theta_v}{\cos 2\theta_v}\sin\bar{x} \\ \frac{\sin 2\theta_v}{\cos 2\theta_v}\sin\bar{x}, & \frac{\sin 2\theta_v}{\cos 2\theta_v}\cos\bar{x}, & 0, & 1 \\ -\frac{\sin 2\theta_v}{\cos 2\theta_v}\cos\bar{x}, & \frac{\sin 2\theta_v}{\cos 2\theta_v}\sin\bar{x}, & -1, & 0 \end{pmatrix}$$

Solving the RWA eqn.

$$C_2'' + \left(i\lambda\cos 2\theta - \left(\frac{\lambda}{\lambda_1} + i\omega\right)\right)C_2' - \left(\frac{i\lambda\cos 2\theta_v}{2}\left(\frac{\lambda}{\lambda_1} + i\omega\right) + \frac{\lambda^2}{4}\cos 4\theta_v\right)C_2 = 0$$

Make it dimensionless:

$$-\frac{d^2C}{d\bar{x}^2} - \left(-i\alpha\cos(\beta\bar{x})\cos 2\theta_v + i - \beta\tan(\beta\bar{x})\right)\frac{dC_2}{d\bar{x}} + \left(\frac{i}{2}\alpha\cos(\beta\bar{x})\cos 2\theta_v\left(i - \beta\tan(\beta\bar{x})\right) + \frac{\alpha^2}{4}\cos^2(\beta\bar{x})\cos 4\theta_v\right)C_2 = 0$$

Define $C2P = C_2'$, C_2

We have

$$\begin{cases} \frac{dC2P}{d\bar{x}} = \overbrace{-\left(-i\alpha\cos(\beta\bar{x})\cos 2\theta_v + i - \beta\tan(\beta\bar{x})\right)}^{A_{RWA}}C2P \\ \qquad + \underbrace{\left(\frac{i}{2}\alpha\cos(\beta\bar{x})\cos 2\theta_v\left(i - \beta\tan(\beta\bar{x})\right) + \frac{\alpha^2}{4}\cos^2(\beta\bar{x})\cos 4\theta_v\right)}_{B_{RWA}}C_2 \\ \frac{dC_2}{d\bar{x}} = C2P \end{cases}$$

Make it real: $(C_{2,R}), (C_{2,I}), C_{2P,R}, C_{2P,I})$

$$\frac{dC_{2R}}{d\bar{x}} = C_{2P,R}, \quad \frac{dC_{2I}}{d\bar{x}} = C_{2P,I}$$

$$\frac{dC_{2P,R}}{d\bar{x}} = \beta\tan(\beta\bar{x})\,\overset{rwa(2[0])}{C_{2P,R}} + \left(-\alpha\cos(\beta\bar{x})\cos 2\theta_v + 1\right)\overset{rwa(2[1])}{C_{2P,I}}$$
$$+ \left(-\frac{\alpha}{2}\cos(\beta\bar{x})\cos 2\theta_v + \frac{\alpha^2}{4}\cos^2(\beta\bar{x})\cos 4\theta_v\right)\overset{rwa(2[0])}{C_{2,R}}$$

$$\frac{dC_{2P,R}}{d\tilde{x}} = \beta \tan(\beta\tilde{x})\, C_{2P,R} + \left(-\alpha\cos(\beta\tilde{x})\cos 2\theta_v + 1\right) C_{2P,I}$$
$$+ \left(-\frac{\alpha}{2}\cos(\beta\tilde{x})\cos 2\theta_v + \frac{\alpha^2}{4}\cos^2(\beta\tilde{x})\cos 4\theta_v\right) C_{2,R}$$
$$+ \left(\frac{\alpha}{2}\cos(\beta\tilde{x})\,\beta\tan(\beta\tilde{x})\right) C_{2,I}$$

$$\frac{dC_{2P,I}}{d\tilde{x}} = -\left(-\alpha\cos(\beta\tilde{x})\cos 2\theta_v + 1\right) C_{2P,R} + \beta\tan(\beta\tilde{x})\, C_{2P,I}$$
$$+ \left(-\frac{\alpha}{2}\cos(\beta\tilde{x})\cos 2\theta_v\,\beta\sin(\beta\tilde{x})\right) C_{2,R}$$
$$+ \left(-\frac{\alpha}{2}\cos(\beta\tilde{x})\cos 2\theta_v + \frac{\alpha^2}{4}\cos^2(\beta\tilde{x})\cos 4\theta_v\right) C_{2,I}$$

Define $a = \frac{\alpha}{2}\cos 2\theta_v$, $b = \frac{\alpha}{2}\sin 2\theta_v$

$$\frac{dC_{2P,R}}{d\tilde{x}} = \left(-a + \left(\frac{a}{\cos 2\theta_v}\right)^2 \overbrace{\left(\cos^2 2\theta_v - \sin^2 2\theta_v\right)}^{(a^2-b^2)} \cos(\beta\tilde{x})\right)\cos(\beta\tilde{x})\, C_{2,R} + \frac{b}{\cos 2\theta_v}\,\beta\sin(\beta\tilde{x})\, C_{2,I}$$
$$+ \beta\tan(\beta\tilde{x})\, C_{2P,R} + \left(-2a\cos(\beta\tilde{x}) + 1\right) C_{2P,I}$$

$$\frac{dC_{2P,I}}{d\tilde{x}} = \left(-a\,\beta\sin(\beta\tilde{x})\right) C_{2,R} + \left(-a + \overbrace{\left(\frac{a^2}{\cos^2 2\theta}\left(\cos^2 2\theta_v - \sin^2 2\theta_v\right)\right)}^{a^2-b^2}\cos(\beta\tilde{x})\right)\cos(\beta\tilde{x})\, C_{2,I}$$
$$-\left(-2a\cos(\beta\tilde{x}) + 1\right) C_{2P,R} + \beta\tan(\beta\tilde{x})\, C_{2P,I}$$

Initial condition:

$$\begin{pmatrix} C_{2,R} \\ C_{2,I} \\ C_{2P,R} \\ C_{2P,I} \end{pmatrix} \xrightarrow{\text{initially}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -a \end{pmatrix}$$

initially

$$i\,\frac{dC_2}{d\tilde{x}} = \frac{\alpha\sin 2\theta_v}{2} = a$$

$$\frac{dC_2}{d\tilde{x}} = -ia$$

$$C_{2P,R} = 0$$
$$C_{2P,I} = -a$$

Constant matter plus perturbation

Wednesday, September 30, 2015 9:12 AM

$$+\left(\frac{i}{2}\frac{1}{\omega}\frac{d\lambda}{d\bar{x}}\cos 2\theta_v - \frac{i}{2}\frac{\cos 2\theta_v}{\omega}\frac{d\lambda}{d\bar{x}} + \frac{1}{2}\frac{\lambda}{\omega}\cos 2\theta_v - \frac{1}{4}\left(\frac{\lambda}{\omega}\right)^2\right)C_2 = 0$$

$$-\frac{d^2C_2}{d\bar{x}^2} + \left(\frac{1}{\lambda}\frac{d\lambda}{d\bar{x}} + i\right)\frac{dC_2}{d\bar{x}} + \cancel{\left(\frac{i}{2}\frac{1}{\omega}\frac{d\lambda}{d\bar{x}}\cos 2\theta_v - \frac{i}{2}\frac{\lambda\cos 2\theta_v}{\omega}\left(\frac{1}{\lambda}\frac{d\lambda}{d\bar{x}} + i\right) - \frac{\lambda^2}{4}\frac{1}{\omega^2}\right)C_2 = 0}$$

Assuming $\lambda = \lambda_0 + \lambda_1\cos(\beta\bar{x})$

$$\frac{d\lambda}{d\bar{x}} = -\lambda_1\beta\sin(\beta\bar{x})$$

$$-\frac{d^2C_2}{d\bar{x}^2} + \left(\frac{-\lambda_1\beta\sin(\beta\bar{x})}{\lambda_0 + \lambda_1\cos(\beta\bar{x})} + i\right)\frac{dC_2}{d\bar{x}} + \left(\frac{i}{2}\frac{-\lambda_1\beta\sin(\beta\bar{x})}{\omega}\cos 2\theta_v\right.$$

$$-\frac{i}{2}\frac{(\lambda_0 + \lambda_1\cos(\beta\bar{x}))\cos 2\theta_v}{\omega}\left(i + \frac{-\lambda_1\beta\sin(\beta\bar{x})}{\lambda_0 + \lambda_1\cos(\beta\bar{x})}\right)$$

$$\left.-\frac{1}{4}\frac{1}{\omega^2}(\lambda_0 + \lambda_1\cos(\beta\bar{x}))^2\right)C_2 = 0$$

Simplify

$$-\frac{d^2C_2}{d\bar{x}^2} + \left(\frac{-\lambda_1\beta\sin(\beta\bar{x})}{\lambda_0 + \lambda_1\cos(\beta\bar{x})} + i\right)\frac{dC_2}{d\bar{x}} + \left(\cancel{-\frac{i}{2}\frac{\lambda_1\beta\sin(\beta\bar{x})\cos 2\theta_v}{\omega}} + \frac{1}{2}\frac{(\lambda_0 + \lambda_1\cos(\beta\bar{x}))\cos 2\theta_v}{\omega}\right.$$

$$\left.\cancel{+\frac{i}{2}\frac{\lambda_1\beta\sin(\beta\bar{x})\cos 2\theta_v}{\omega}} - \frac{1}{4\omega^2}(\lambda_0 + \lambda_1\cos(\beta\bar{x}))^2\right)C_2 = 0$$

$$\frac{d^2C_2}{d\bar{x}^2} + \left(\frac{\lambda_1\beta\sin(\beta\bar{x})}{\lambda_0 + \lambda_1\cos(\beta\bar{x})} - i\right)\frac{dC_2}{d\bar{x}} + \frac{1}{4}\left(-2\left(\frac{\lambda_0}{\omega} + \frac{\lambda_1}{\omega}\cos(\beta\bar{x})\right)\cos 2\theta_v\right.$$

$$\left.+\left(\frac{\lambda_0}{\omega} + \frac{\lambda_1}{\omega}\cos(\beta\bar{x})\right)^2\right)C_2 = 0$$

Define $\alpha_0 = \frac{\lambda_0}{\omega}$, $\alpha_1 = \frac{\lambda_1}{\omega}$ $\quad \alpha_0 > \alpha_1$

$$\frac{d^2C_2}{d\bar{x}^2} + \left(\frac{\frac{\alpha_1}{\alpha_0}\beta\sin(\beta\bar{x})}{1 + \frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})} - i\right)\frac{dC_2}{d\bar{x}} + \frac{1}{4}\left(-2(\alpha_0 + \alpha_1\cos(\beta\bar{x}))\cos 2\theta_v\right.$$

$$\left.+(\alpha_0 + \alpha_1\cos(\beta\bar{x}))^2\right)C_2 = 0$$

Approximations: Assuming $\alpha_1 \ll \alpha_0$, α_1 is pertubative

$$\frac{\frac{\alpha_1}{\alpha_0}\beta\sin(\beta\bar{x})}{1 + \frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})} \simeq \frac{\alpha_1}{\alpha_0}\beta\sin(\beta\bar{x})\left(1 - \frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right) \simeq \frac{\alpha_1}{\alpha_0}\beta\sin(\beta\bar{x})$$

$$-\frac{1}{4}\left(2\left(\alpha_0+\alpha_1\cos(\beta\bar{x})\right)\cos 2\theta_v-\left(\alpha_0+\alpha_1\cos(\beta\bar{x})\right)^2\right)$$

$$=-\frac{1}{4}\left(2\alpha_0\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\cos 2\theta_v-\alpha_0\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)^2\right)$$

$$\doteq-\frac{1}{4}\left(2\alpha_0\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\cos 2\theta_v-\alpha_0\left(1+2\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\right)$$

$$=-\frac{1}{4}\left(2\alpha_0\cos 2\theta_v-\alpha_0+2\alpha_0\left(\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\cos 2\theta_v-2\alpha_0\left(\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\right)$$

$$=-\frac{1}{4}\left(\alpha_0\left(2\cos 2\theta_v-1\right)+2\alpha_0\left(\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\left(\cos 2\theta_v-1\right)\right)$$

$$=-\frac{1}{4}\alpha_0\left(2\cos 2\theta_v-1+2\left(\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\left(\cos 2\theta_v-1\right)\right)$$

$$=-\frac{1}{4}\alpha_0\left(2\left(\cos 2\theta_v-1\right)\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)+1\right)$$

$$=-\frac{1}{2}\alpha_0\left(\cos 2\theta_v-1\right)\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)\cancel{+}\frac{1}{4}\alpha_0$$

The approximated eqn becomes

$$\frac{d^2C_2}{d\bar{x}^2}+\left(\frac{\alpha_1}{\alpha_0}\sin(\beta\bar{x})-i\right)\frac{dC_2}{d\bar{x}}$$

$$+\left(\frac{1}{2}\alpha_0\left(1-\cos 2\theta_v\right)\left(1+\frac{\alpha_1}{\alpha_0}\cos(\beta\bar{x})\right)-\frac{1}{4}\alpha_0\right)C_2=0$$

Check consistancy : set $\alpha_0\to 0$

$$\frac{d^2C_2}{d\bar{x}^2}+\left(\beta\tan(\beta\bar{x})-i\right)\frac{dC_2}{d\bar{x}}+\frac{1}{4}\left(-2\alpha_1\cos(\beta\bar{x})\cos 2\theta_v+\alpha_1^2\cos^2(\beta\bar{x})\right)C_2=0$$

99

* Eqns for C_1 & C_2.

$$i\frac{dC_1}{d\bar{x}}=\frac{\cos 2\theta_v}{2}\frac{\lambda}{\omega}C_1+\frac{\sin 2\theta_v}{2}\frac{\lambda}{\omega}C_2e^{-i\omega t}$$

$$i\frac{dC_2}{d\bar{x}}=-\frac{\cos 2\theta_v}{2}\frac{\lambda}{\omega}C_2+\frac{\sin 2\theta_v}{2}\frac{\lambda}{\omega}C_1e^{i\omega t}$$

$\Rightarrow$ $i\,dC_1 = \cos 2\theta_v\,(\quad)\quad \sin 2\theta_v\,(\quad)$

$$\frac{d}{dx} - \frac{\sin 2\theta}{2}\left(\alpha_0 + \alpha_1 \cos(\beta x)\right) C_1 + \frac{\sin 2\theta}{2}\left(\alpha_0 + \alpha_1 \cos(\beta x)\right) C_2$$

---

* Wait, what are we really talking about when we mention "stimulated"

$$\boxed{\text{Matter perturbation} \rightarrow \text{flavour transition}}$$

how well is this transfer.

Do we assume a adiabatic?

* Try adiabatic

How do u do with adiabatic?

Adiabatic solution to transition

* Initial probability: $|\nu_e|^2 = 1$

* Try fast perturbation

#1. Adabatic / follow instantaneous matter basis?

How to change from vacuum basis to instantaneous basis? $\omega = \frac{\Delta m^2}{2E}$

$$\begin{pmatrix} \nu_L \\ \nu_H \end{pmatrix} = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

$$\begin{pmatrix} \nu_e \\ \nu_L \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \nu_L \\ \nu_H \end{pmatrix} = \begin{pmatrix} \cos\theta(x) & \\ & \end{pmatrix} \begin{pmatrix} \cos\theta_v & \\ & \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta - \theta(x)) & \sin(\theta - \theta(x)) \\ -\sin(\theta - \theta(x)) & \cos(\theta - \theta(x)) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos(\theta-\theta(x)) & \sin(\theta-\theta(x)) \\ -\sin(\theta-\theta(x)) & \cos(\theta-\theta(x)) \end{pmatrix}}_{U(\theta-\theta(x))} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

So the wave function in vacuum basis is

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = U(\theta-\theta(x)) \begin{pmatrix} \nu_L \\ \nu_H \end{pmatrix}$$

How to calculate $\theta(x)$?

$$\sin 2\theta(x) = \frac{\omega \sin 2\theta_v}{\sqrt{\omega^2+\delta^2-2\omega\Delta\cos 2\theta_v}} = \frac{\sin 2\theta_v}{\sqrt{1+(\alpha_0+\alpha_1\cos(\beta x))^2-2(\alpha_0+\alpha_1\cos(\beta x))\cos 2\theta_v}}$$

$$\Delta = \sqrt{2}\, G_F n_e$$

[This is going to be a reference that
how is it going to deviate from adiabatic

* Do the numerical calculation

This is a reference : to see how things deviate from adiabatic

Adiabatic, in matter instantaneous basis

$$\psi_m(x) = \begin{pmatrix} \nu_L e^{-iE_L x} \\ \nu_H e^{-iE_H x} \end{pmatrix}$$

to vacuum basis

$$\psi_v(x) = U(\theta-\theta(x))^{-1} \psi_m(x)$$

$$= U(\theta-\theta(x))^{T} \psi_m(x)$$

$$= \begin{pmatrix} \cos(\theta-\theta(x)) & -\sin(\theta-\theta(x)) \\ \sin(\theta-\theta(x)) & \cos(\theta-\theta(x)) \end{pmatrix} \begin{pmatrix} \nu_L(0) e^{-iE_L x} \\ \nu_H(0) e^{-iE_H x} \end{pmatrix}$$

I only need $\nu_1(x) = \cos(\theta-\theta(x))\, \nu_L(0) e^{-iE_L x}$
$- \sin(\theta-\theta(x))\, \nu_H(0) e^{-iE_H x}$

$$= e^{-iE_L x}\left(\cos(\theta-\theta(x))\, \nu_L(0) - \sin(\theta-\theta(x))\, \nu_H(0)\, e^{-i(E_H-E_L)x} \right)$$

$$P = |\cos(\theta-\theta(x))\nu_L(0)|^2 + |\sin(\theta-\theta(x))\nu_H(0)|^2$$
$$- \cos(\theta-\theta(x))\nu_L(0) \sin(\theta-\theta(x))\, \nu_H^*(0)\, e^{+i(E_H-E_L)x}$$

$$- \cos(\theta-\theta(x))\, v_L(0)\, \sin(\theta-\theta(x))\, v_H^*(0)\, e^{+i(E_H-E_L)x}$$
$$- \cos(\theta-\theta(x))\, v_L^*(0)\, \sin(\theta-\theta(x))\, v_H(0)\, e^{-i(E_H-E_L)x}$$

$- \boxed{}/2\omega s$

Calculate initial condition

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}^{-1}\begin{pmatrix} v_L(0) \\ v_H(0)\end{pmatrix}$$

vacuum basis

$$\begin{pmatrix} v_L(0) \\ v_H(0)\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\theta-\theta(0)) \\ -\sin(\theta-\theta(0))\end{pmatrix}$$

$$E_L = -E_H = \frac{\omega}{2}\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cdot\cos 2\theta_v}$$
$$= \frac{\omega}{2}\sqrt{(\alpha_0+\alpha_1\cos(\beta x))^2 + 1 - 2(\alpha_0+\alpha_1\cos(\beta x))\cos 2\theta_v}$$

* Completely non-adiabatic

(no way)

* Look at the equations

* Small perturbation approximation

$$i\frac{dc_2}{dx} = \frac{\sin 2\theta_v}{2}\frac{\lambda}{\omega}\cancel{(\alpha+\alpha)}$$

$$c_2'(0) = -i\frac{\sin 2\theta_v}{2}\left(\alpha_0 \cancel{+\alpha_1\cos(\beta x)}\right)$$

**** Using parameters used in

20 MeV, ν_1

θ : 0.573

Δm^2 : $3\times10^{-3}\,eV^2$

perturbation $c_x = 0.1$

$\alpha_1 \approx 0.1\,\alpha_0$

wavelength of perturbation is

$\lambda_x = 5.27$ km

$k_x = \frac{2\pi}{\lambda_x} = \frac{2\pi}{5.27\,km}$

correspondingly

$\omega_1 = k_x = \frac{2\pi}{1}$

197 MeV · fm = 1

197 eV · km · $10^6 \cdot 10^{-18}$ = 1

eV · km = $\frac{1}{197}\times10^{12}$

λ_*

$$\beta = \frac{\omega_\lambda}{\omega} = \frac{2\pi}{\lambda_* \omega}$$

$$= \frac{2\pi}{5.27\ \text{km} \cdot \frac{8\pi R_\odot}{2G}}$$

$$= \frac{2\pi}{5.27 \cdot \frac{3\times10^{-3}}{2\times2.4\times10^{6}}\ \text{eV}\cdot\text{km}}$$

$$= \frac{2\pi}{5.27 \times \frac{3}{4}\, 10^{-10} \times \frac{1}{(a)}\times10^{12}}$$

Initial condition

In instantaneous basis

$\begin{pmatrix}1\\0\end{pmatrix}$ in vacuum basis

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\theta_v-\theta_\lambda) & \sin \\ & \cos\theta_v\end{pmatrix}\begin{pmatrix}\nu_\beta\\ \nu_2\end{pmatrix}$$

$\begin{pmatrix}\nu\end{pmatrix}$

Work in averaged matter basis

Hamiltonian in this basis

ϕ $\psi_f = U\psi_v$

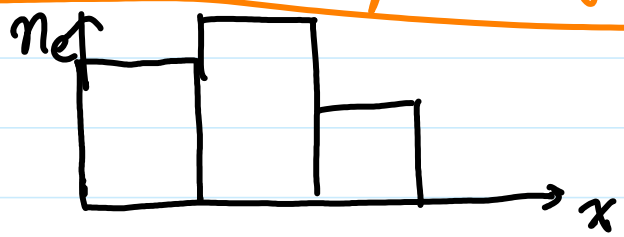
$\psi_m =$

Parametric resonance

Monday, October 12, 2015 1:58 PM

Slab Approximation

$\boxed{\text{What is the requirement for this to be good}}$?



* In a constant matter potential $V(n_e) = \sqrt{2}\, G_F n_e$

$$\begin{pmatrix} \nu_L \\ \nu_H \end{pmatrix} = [U^T] \begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

In matter the propagation is

$$\begin{pmatrix} \nu_L \exp(i E_L (x - x_0)) \\ \nu_H \exp(i E_H (x - x_0)) \end{pmatrix}$$

Define evolution operator

$$\tilde{U} = \begin{pmatrix} \exp(i E_L (x - x_0)) & 0 \\ 0 & \exp(i E_H (x - x_0)) \end{pmatrix}$$

Then we can evolve any wave function in matter basis

$$\tilde{U} \cdot \psi_m$$

In the language of states

$$\hat{\tilde{U}} |\psi\rangle$$

For flavor basis

$$\underbrace{\langle a|\hat{U}|b\rangle}\langle b|\psi\rangle$$

This is the matrix in flavor basis

The one easy to get is matter basis

So we rotate it to flavor basis

$$\boxed{\underbrace{\langle a|m\rangle}_{U_m}\underbrace{\langle m|\hat{U}|n\rangle}_{\tilde{U}}\underbrace{\langle n|b\rangle}_{U_m^+}\underbrace{\langle b|\psi\rangle}_{\psi_m}}$$

$\Uparrow$

Now we know how to evolve a state in flavor basis

Define a new evolution operator for flavor basis

$$\tilde{U}_f = U_m \tilde{U} U_m^+$$

which depends on $x_0, \Delta x$, (or n_e)

$$\tilde{U}_f = \begin{cases} \cdots & [x_0, x_0+\Delta x_1) \\ \cdots & [x_0+\Delta x_1, x_0+\Delta x_1+\Delta x_2) \\ \cdots & [\quad , \quad) \\ \vdots & \end{cases}$$

So we can evolve a wavefunction $\psi_e(x_0)$ to

$$\psi_e(x) = \cdots \tilde{U}_f(x_0+\Delta x_1, x_0+\Delta x_1+\Delta x_2)\tilde{U}_f(x_0, \Delta x_1)\psi_e(x_0)$$

$$\begin{pmatrix} e^{ia} & e^{ic} \\ e^{-ic} & e^{ib} \end{pmatrix}\begin{pmatrix} e^{ia} & e^{ic} \\ e^{-ic} & e^{ib} \end{pmatrix}$$

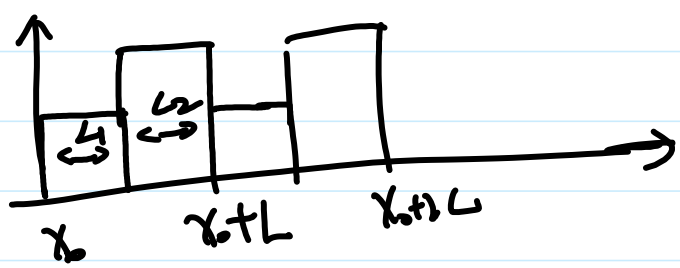
$$\begin{pmatrix} e^{-ic} & e^{ic} \\ e^{-ic} & e^{ib} \end{pmatrix} \begin{pmatrix} e^{ia} & e^{-ia} \\ e^{-ic} & e^{ib} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i2a}+1 & e^{i(a+c)}+e^{i(b+c)} \\ e & e^{i2b}+1 \end{pmatrix}$$

$$\downarrow \qquad b=-a$$

$$= \begin{pmatrix} e^{i2a}+1 & e^{ic}(e^{ia}+e^{-ia}) \\ & e^{i2b}+1 \end{pmatrix}$$

Castle wall



$$\psi_f(x_0+PL) = [\hat{U}_f]^P \psi_f$$

$\downarrow$

the operator for the whole period L which is composed of 2 parts.

basis

Tuesday, October 13, 2015 5:51 PM

$\boxed{\text{If } \lambda \text{ is constant}}$

In vacuum eigenenergy basis

$$H = -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\cos(2\theta_v)\,\sigma_3 + \frac{\lambda}{2}\sin(2\theta_v)\,\sigma_1$$

Schrödinger equation

$$i\frac{d}{dx}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = H\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

A transformation that removes the second term.

Define

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

where $\lambda_c = \lambda\cos 2\theta_v$

Plug into Schrödinger equation

$$i\frac{d}{dx}\left[\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}\right] = H\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x}\end{pmatrix} i\frac{d}{dx}\begin{pmatrix} g_1\\ g_2\end{pmatrix} + i\left[\frac{d}{dx}\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\right]\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

$$= \begin{pmatrix} & \\ & \end{pmatrix} i\frac{d}{dx}\begin{pmatrix} g_1\\ g_2\end{pmatrix} + i\begin{pmatrix} -i\frac{\lambda_c}{2}e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & i\frac{\lambda_c}{2}e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

$$= \begin{pmatrix} & \\ & \end{pmatrix} \cdots + \frac{\lambda_c}{2}\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & -e^{i\frac{\lambda_c}{2}x}\end{pmatrix}\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x}\end{pmatrix}\text{LHS} = i\frac{d}{dx}\begin{pmatrix} g_1\\ g_2\end{pmatrix} + \frac{\lambda_c}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}}_{\sigma_3}\begin{pmatrix} g_1\\ g_2\end{pmatrix}$$

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x}\end{pmatrix} = \begin{pmatrix} \cos\frac{\lambda_c}{2}x + i\sin\frac{\lambda_c}{2}x & 0 \\ 0 & \cos\frac{\lambda_c}{2}x - i\sin\frac{\lambda_c}{2}x\end{pmatrix}$$

$$= \cos\frac{\lambda_c}{2}x\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + i\sin\frac{\lambda_c}{2}x\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$$

$$= \cos\frac{\lambda_c}{2}x\; I + i\sin\frac{\lambda_c}{2}x\;\sigma_3$$

Also using $\sigma_i\sigma_j = i\sum_k \epsilon_{ijk}\sigma_k + \delta_{ij} I$

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x}\end{pmatrix} H = \left(\cos\frac{\lambda_c}{2}x\; I + i\sin\frac{\lambda_c}{2}x\;\sigma_3\right)\cdot \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda_c}{2}\sigma_3 + \frac{\lambda_s}{2}\sigma_1\right)$$

$$= -\frac{\omega}{2}\cos\frac{\lambda_c}{2}x\;\sigma_3 + i\left(\sin\frac{\lambda_c}{2}x\right)\left(-\frac{\omega}{2}\right) + \frac{\lambda_c}{2}\cos\frac{\lambda_c}{2}x\;\sigma_3$$

$$= -\frac{\omega}{2}\cos\frac{\lambda_c}{2}x\,\sigma_3 + i\left(\sin\frac{\lambda_c}{2}x\right)\left(-\frac{\omega}{2}\right) + \frac{\lambda_c}{2}\cos\frac{\lambda_c}{2}x\,\sigma_3$$

$\therefore$ - - - - not simplifying

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x} \end{pmatrix} H \begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x} \end{pmatrix} \sigma_3 \begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$= \sigma_3 \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x} \end{pmatrix} \sigma_1 \begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{-i\frac{\lambda_c}{2}x} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\frac{\lambda_c}{2}x} \\ e^{-i\frac{\lambda_c}{2}x} & 0 \end{pmatrix}\begin{pmatrix} e^{-i\frac{\lambda_c}{2}x} & 0 \\ 0 & e^{i\frac{\lambda_c}{2}x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\lambda_c x} \\ e^{-i\lambda_c x} & 0 \end{pmatrix}$$

So the equation of motion becomes

$$i\frac{d}{dx}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \frac{\lambda_c}{2}\sigma_3\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda_c}{2}\sigma_3 + \frac{\lambda_s}{2}\begin{pmatrix} 0 & e^{i\lambda_c x} \\ e^{-i\lambda_c x} & 0 \end{pmatrix}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

Then

$$i\frac{d}{dx}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda_s}{2}\begin{pmatrix} 0 & e^{i\lambda_c x} \\ e^{-i\lambda_c x} & 0 \end{pmatrix}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

Turn it into dimensionless eqn

$$i\frac{d}{d(x\omega)}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(-\frac{1}{2}\sigma_3 + \frac{1}{2}\frac{\lambda_s}{\omega}\begin{pmatrix} 0 & e^{i\frac{\lambda_c}{\omega}\omega x} \\ e^{-i\frac{\lambda_c}{\omega}\omega x} & 0 \end{pmatrix}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & e^{i\alpha_c \bar{x}} \\ e^{-i\alpha_c \bar{x}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos\alpha_c\bar{x} + i\sin\alpha_c\bar{x} \\ \cos\alpha_c\bar{x} - i\sin\alpha_c\bar{x} & 0 \end{pmatrix}$$

$$= \cos\alpha_c\bar{x}\,\sigma_1 - \sin\alpha_c\bar{x}\,\sigma_2$$

$$\begin{pmatrix} e^{i\Omega x} & 0 \\ 0 & e^{-i\Omega x} \end{pmatrix} \sigma_2 \begin{pmatrix} e^{-i\Omega x} & 0 \\ 0 & e^{i\Omega x} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\Omega x} & 0 \\ 0 & e^{-i\Omega x} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} e^{-i\Omega x} & 0 \\ 0 & e^{i\Omega x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i e^{i\Omega x} \\ i e^{-i\Omega x} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\Omega x} & 0 \\ 0 & e^{i\Omega x} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i e^{2i\Omega x} \\ i e^{-2i\Omega x} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\cos 2\Omega x + \sin 2\Omega x \\ i\cos 2\Omega x + \sin 2\Omega x & 0 \end{pmatrix}$$

$$= \sin 2\Omega x \, \sigma_1 + \cos 2\Omega x \, \sigma_2$$

$\boxed{\text{If } \lambda \text{ is a function of } X}$

In vacuum eigenenergy basis

$$H = -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\cos(2\theta_v)\,\sigma_3 + \frac{\lambda}{2}\sin(2\theta_v)\,\sigma_1$$

Schrödinger equation

$$i\frac{d}{dx}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Assuming a transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)x} & 0 \\ 0 & e^{i\eta(x)x} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

Plug into Schrödinger equation

$$\text{LHS} = i\frac{d}{dx}\left[\begin{pmatrix} e^{-i\eta(x)x} & 0 \\ 0 & e^{i\eta(x)x} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right]$$

$$= i\begin{pmatrix} -i\eta\, e^{-i\eta x} - ix\frac{d\eta}{dx} e^{-i\eta x} & 0 \\ 0 & i\eta\, e^{i\eta x} + ix\frac{d\eta}{dx} e^{i\eta x} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$+i\begin{pmatrix} & \\ & \end{pmatrix}\frac{d}{dx}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\begin{pmatrix} e^{+i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} LHS = i\begin{pmatrix} -i\eta - ix\frac{d\eta}{dx} & 0 \\ 0 & i\eta + ix\frac{d\eta}{dx} \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + i\frac{d}{dx}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\begin{pmatrix} e^{+i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} RHS = \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\cos 2\theta\,\sigma_3 + \frac{\lambda}{2}\sin 2\theta_v \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-2i\eta x} & 0 \end{pmatrix}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

The final EoM we have is

$$i\frac{d}{dx}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(-\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\sin 2\theta_v \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-i2\eta x} & 0 \end{pmatrix}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

if we use the condition that

$$\boxed{\eta + x\frac{d\eta}{dx} = \frac{\lambda}{2}\cos 2\theta_v}$$

The general solution is

$$\eta(x) = \frac{C_1}{x} + \frac{1}{x}\int^x \frac{1}{2}\cos(2\theta_v)\,\lambda(\tau)\,d\tau$$

We can chose any C_1 we want, set to 0.

$$\eta(x) = \frac{1}{2}\cos(2\theta_v)\,\frac{1}{x}\int_1^x \lambda(\tau)\,d\tau$$

If λ is constant

$$\eta(x) = \frac{1}{2}\cos 2\theta_v\,\frac{1}{x}\left(\lambda x - \lambda\right)$$

In this new basis our Hamiltonian is

$$H = -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\sin 2\theta_v \begin{pmatrix} 0 & e^{i2\eta x} \\ e^{-i2\eta x} & 0 \end{pmatrix}$$

$$= -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\sin 2\theta_v\left(\cos(2\eta x)\,\sigma_1 - \sin(2\eta x)\,\sigma_2\right)$$

$$= -\frac{\omega}{2}\sigma_3 + \frac{\lambda}{2}\sin 2\theta_v\cos(2\eta x)\,\sigma_1 - \frac{\lambda}{2}\sin 2\theta_v\sin(2\eta x)\,\sigma_2$$

In order to do rotations quickly, we can precalculate

$$\mathcal{U}\,\sigma_i\,\mathcal{U}^\dagger$$

$$\mathcal{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cos\theta\, I + i\sin\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= I\cos\theta + i\sin\theta\,\sigma_2$$

We know that

$$\sigma_i\sigma_j = i\sum_c \epsilon_{abc}\,\sigma_c + \delta_{ab} I$$

$$\begin{cases} \sigma_2\sigma_1 = i\,\epsilon_{213}\,\sigma_3 = -i\sigma_3 \\ \sigma_2\sigma_2 = I \\ \sigma_2\sigma_3 = i\,\epsilon_{231}\,\sigma_1 = i\sigma_1 \end{cases}$$

So for a Hamiltonian in flavor basis

$$\frac{\omega}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In vacuum basis

$$\frac{\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}^\dagger \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix}}_{\parallel}$$

$$\lambda \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

$$= \frac{\lambda}{2}\begin{pmatrix} 2\cos^2\theta & \sin 2\theta \\ \sin 2\theta & \end{pmatrix}$$

Check code

Tuesday, November 10, 2015 8:44 PM

1. Check vacuum:

$$i\partial_x \psi = -\frac{1}{2}\sigma_3 \psi$$

Frequency 2π

Amplitude 1

2. Matter (constant)

* See initial to be at one of the eigenstates

probability : constant

frequency

real part: oscillating of

frequency : $2\pi \cdot 293$

3. parametric resonance:

$$c_1 = \cos\varphi_1 = \cos\left(\frac{\omega}{2} x\right)$$

thickness of slab

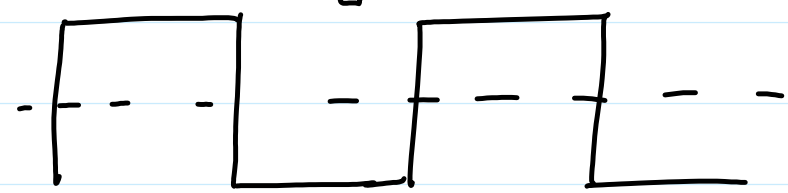
$$R = c_1 c_2 - s_1 s_2 (\vec{n}_1 \cdot \vec{n}_2)$$

$$\vec{n}_1 = (\sin 2\theta_m, 0, -\cos 2\theta_m)$$

$$\vec{n}_1 \cdot \vec{n}_2 = \xi \sin 2\theta_{m1} \cdot \sin 2\theta_{m2} + \cos 2\theta_{m1} \cos 2\theta_{m2}$$

$\leftarrow L \rightarrow \leftarrow L \rightarrow$

$\hat{\lambda} = \frac{\omega_{3}2\theta_v}{2}$



$$\sin 2\theta_{m1} = \frac{\sin 2\theta_v}{\sqrt{\hat{\lambda}_1 + 1 - 2\hat{\lambda}_1 \cos 2\theta_v}} = \frac{\sin 2\theta_v}{\sqrt{1.21\hat{\lambda}^2 + 1 - 2\times 1.1\hat{\lambda} \cos 2\theta_v}}$$

$$\hat{\lambda}_1 = (\hat{\lambda} + 0.1\hat{\lambda}) = 1.1^2\hat{\lambda} = 1.21\hat{\lambda}^2$$

$$\hat{\lambda}_2 = (\lambda - 0.1\hat{\lambda}) = 0.9^2\hat{\lambda} = 0.81\hat{\lambda}$$

$$\sin 2\theta_{m1} = \frac{\sin 2\theta_v}{\sqrt{1.21\frac{\cos^2 2\theta_v}{4} + 1 - 2\times 1.1\times \frac{\cos^2 2\theta_v}{2}}}$$

$$= \frac{\sin 2\theta_v}{\sqrt{1.21 \times \frac{0.476^2}{4} + 1 - 1.1 \times 0.476^2}}$$

$\cos 2\theta_v \sim 0.476$

$$R = \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 \, (\vec{n}_1 \cdot \vec{n}_2)$$

$$\cos\varphi_1 = \cos\left(\frac{\omega_{m1}}{2} L\right)$$

$$\vec{n}_1 \cdot \vec{n}_2 = \sin 2\theta_{m1} \cdot \sin 2\theta_{m2} + \cos 2\theta_{m1} \cos 2\theta_{m2}$$

Parametric resonance

$$P_{\nu_e \to \nu_x}(PX) = \left(1 - \frac{I_3^2}{I_1^2}\right) \sin^2 \xi$$

$$I_3 = -S_1 C_2 \cos 2\theta_{M1} - C_1 S_2 \cos 2\theta_{M2}$$

$$I_1 = S_1 C_2 \sin 2\theta_{M1} + C_1 S_2 \sin 2\theta_{M2}$$

$$I_2 = -S_1 S_2 \sin 2(\theta_{M1} - \theta_{M2})$$

$$\xi = \arccos R$$

$$R = C_1 C_2 - S_1 S_2 (\vec{n}_1 \cdot \vec{n}_2)$$

$$\vec{n}_1 = (\sin 2\theta_{M1}, 0, -\cos 2\theta_{M1})$$

$$C_k = \cos \varphi_k, \quad S_k = \sin \varphi_k$$

$$\varphi_k = \frac{\omega_k}{2} X_k$$

$$\omega_M = \omega \sqrt{\lambda^2 + 1 - 2\lambda \cos 2\theta_V}$$

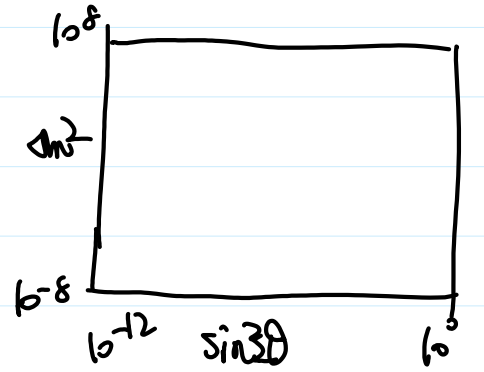
MSW

Wednesday, December 9, 2015 2:48 PM

Parameters : $\omega(E)$, angle ( ) , matter profile

Survival probability

$$\frac{2E \cdot \sqrt{2}G_F N_e}{\Delta m^2} = \frac{\sqrt{2}G_F N_e}{\omega}$$

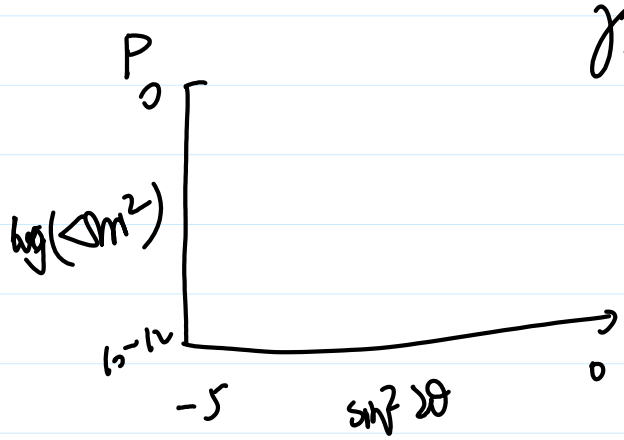


Energy

energy

A each distance

for a matter profile



$$\gamma = 2.54 \times 10^3 \frac{\sin^2 2\theta}{\cos 2\theta} \frac{1}{\lambda}$$

$$\lambda = \omega \frac{1}{\omega} \cdot \frac{0.76 \times 10^{-5}\, eV^2}{1\, MeV}$$

$$= \frac{1}{\omega}\, 0.76 \times 10^{-11}\, eV$$

$$\gamma = \frac{2.554 \times 10^3 \frac{\sin^2 2\theta}{\cos 2\theta}}{\frac{1}{\omega}\, 0.76 \times 10^{-11}}$$

$$= \omega \frac{\sin^2 2\theta}{\cos 2\theta} \cdot \frac{2.554 \times 10^3}{0.76 \times 10^{-11}}$$

$$\gamma = \omega \frac{\sin^2 2\theta}{\cos 2\theta} \Big/ \left| \frac{1}{\lambda} \frac{d\lambda}{dx} \right|$$

$$P = \frac{1}{2} + \left(\frac{1}{2} - P_f\right) \cos 2\theta_v \frac{\cos 2\theta_v - \lambda}{}$$

$$p_2 \quad \frac{1}{2} + \left(\frac{1}{2} - P_f\right) \cos 2\theta_v \, \frac{\cos 2\theta_v - \hat{\lambda}}{\sqrt{\hat{\lambda}^2 + 1 - 2\hat{\lambda} \cos 2\theta_v}} \quad \frac{\cos 4\lambda p}{\cos 2\theta_v} \cdot \frac{\cos 4\lambda p}{0.76 \times 10^{-11}}$$

$$= \omega \frac{\sin^2 2\theta}{\cos 2\theta} \quad 3.36 \times 10^{14}$$

Stimulated Oscillation

Thursday, December 10, 2015 6:28 PM

Review of Kneller's work

$$i\frac{d}{dt}S = HS$$

$$\delta\check{k}_{ij} = \check{k}_i - \check{k}_j$$

$$\delta\xi_{ij} = \xi_i - \xi_j$$

$$z_{ij} = \frac{C_x V_*}{k_*}\left(|\check{u}_{ei}|^2 - |\check{u}_{ej}|^2\right)$$

$$\lambda_{ij,n} = (-i)^{n+1}\frac{n C_x V_*}{z_{ij}} J_n(z_{ij})\, \check{u}_{ei}^* \check{u}_{ej}\, e^{i(n\eta + z_{ij}\cos\eta)}$$

$$\frac{\lambda_n^2}{q_n^2} = 1 \qquad \lambda_n^2 = q_n^2$$

$$\Rightarrow \quad k_n^2 = k_N^2 + k_n^2$$

$$\Rightarrow \quad k_N^2 = 0$$

$$0 = k_n^2 = \frac{1}{4}\left(\delta\check{k}_{12} + n k_*\right)^2$$

$$= \frac{1}{4}\left(\check{k}_1 - \check{k}_2 + n k_*\right)^2$$

$\check{k}_i$ eigen values of $\check{H}$

In 2 flavor case

$$\check{k}_1 - \check{k}_2 = -\omega_N$$

We require

$$n k_* = \omega_m \quad \text{for a resonance,}$$

$$k_* = \frac{1}{n}\,\omega\sqrt{\lambda^2+1-2\lambda\cos 2\theta}$$

For small angle θ,

$$k_* \approx \frac{1}{n}\,\omega\sqrt{\lambda+1-2\lambda}$$

$$= \frac{\lambda-1}{n}\,\omega$$

n	k_*		
1	$(\lambda-1)\omega$	$=$	$\left(\frac{1}{m}-1\right)\omega$
2	$\frac{\lambda-1}{2}\,\omega$	$=$	

Note that we use $\lambda = \frac{\cos 2\theta_v}{m}$

Derive RWA

Monday, January 18, 2016 5:26 PM

Rotation wave approximation for

$$\delta V_\alpha = V_0 \left\{ \sum_{\alpha=1}^{N_\alpha} C_\alpha \sin(q_\alpha r + \eta_\alpha) \right\}$$

- Start with the equation of motion

Flavor basis

$$H^{(f)} = U_0 K_0^{(m)} U_0^\dagger + V^{(f)}$$

↳ vacuum Hamiltonian in vacuum basis

↳ U_0 transform a state from vacuum basis to flavor basis

Hamiltonian is rewritten

$$H^{(f)} = \check{H}^{(f)} + \delta V^{(f)}$$

$$\check{H}^{(f)} = U_0 K_0^{(m)} U_0^\dagger + V_0^{(f)}$$

Now plug in the perturbation we had in the beginning.

Calculate transition probability between eigenstates of $\check{H}$

$$\check{H}^{(f)} = \check{U} K^{(\check{m})} \check{U}^\dagger$$

↳ diagonalized in average matter basis

↳ rotate state from averaged matter basis to flavor basis

So $K^{(\check{m})} = \check{U}^\dagger \underbrace{\check{H}^{(f)}}_{\text{only the average \& constant part}} \check{U}$

So we can write down the Hamiltonian in averaged matter basis

$$\boxed{H^{(\check{m})} = \check{K}^{(\check{m})} - i\check{U}^{\dagger}\frac{\partial}{\partial t}\check{U} + \check{U}^{\dagger}\delta V^{(f)}\check{U}}$$

The equation we are going to solve is

$$\boxed{i\frac{d}{dx}S = HS}$$

where S is the matrix that evolves a state at x_0 to a state at x

$$|\psi(x)\rangle = S(x, x_0)|\psi(x_0)\rangle$$

Decompose $S^{(\check{m})} = \check{S}A$

where $\check{S}$ satisfies

$$i\frac{d}{dx}\check{S} = \left(\check{K}^{(\check{m})} - i\check{U}^{\dagger}\frac{d}{dx}\check{U}\right)\check{S}$$

Since $\check{U}$ is <u>not</u> space dependent,

$$i\frac{d}{dx}\check{S} = \check{K}^{(\check{m})}\check{S}$$

which means

$$\check{S} = e^{-i\check{K}^{(\check{m})}x}$$

Since

$$i\frac{\partial}{\partial x}(\check{S}A) = H^{(\check{m})}\check{S}A$$

$$i\left(\frac{\partial}{\partial x}\check{S}\right)A + i\check{S}\frac{\partial}{\partial x}A = \left(\check{K}^{(\check{m})} + \check{U}^{\dagger}\delta V^{(f)}\check{U}\right)\check{S}A$$

$$= \check{K}^{(\check{m})}\check{S}A + \check{U}^{\dagger}\delta V^{(f)}\check{U}\check{S}A$$

$$i\check{S}\frac{\partial}{\partial x}A = \check{U}^{+}\delta V^{(f)}\check{U}\,\check{S}A$$
$$i\frac{\partial}{\partial x}A = \check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\,\check{S}A$$

Seperate out off diagonal part of $A = WB$

$$W = \exp(-i\Xi_1) \qquad \Xi_1 = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$$

$$i\frac{\partial}{\partial x}(WB) = \check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\,\check{S}\,WB$$
$$\underbrace{i\left(\frac{\partial}{\partial x}W\right)B}_{\downarrow} + iW\frac{\partial}{\partial x}B = \check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\check{S}WB$$

$$i\frac{\partial}{\partial x}\left(\exp\left(-i\Xi_1\right)\right)B = i(-i)\left(\frac{\partial}{\partial x}\Xi_1\right)WB$$
$$= \left(\frac{\partial}{\partial x}\Xi_1\right)WB$$

$$iW\frac{\partial}{\partial x}B = \check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\,\check{S}WB - \left(\frac{\partial}{\partial x}\Xi_1\right)WB$$
$$iW\frac{\partial}{\partial x}B = \left(\check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\check{S}W - \left(\frac{\partial}{\partial x}\Xi_1\right)W\right)B$$
$$i\frac{\partial}{\partial x}B = W^{+}\left(\check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\check{S}W - \left(\frac{\partial}{\partial x}\Xi_1\right)W\right)B$$

$$i\frac{\partial}{\partial x}B = W^{+}\Big(\underbrace{\check{S}^{+}\check{U}^{+}\delta V^{(f)}\check{U}\check{S} - \frac{\partial}{\partial x}\Xi_1}\Big)WB$$

We choose Ξ_1 so that this term has only off diagonal elements.

Recall that

$$\delta V_{ee}^{(f)} = V_0\sum_{a=1}^{N_f} C_a \sin(q_a r + \eta_a)$$

$$\check{U} = \begin{pmatrix} u_{e1} & u_{e2} \\ u_{x1} & u_{x2} \end{pmatrix} \qquad \check{U}^{+} = \begin{pmatrix} u_{e1}^{*} & u_{x1}^{*} \\ u_{e2}^{*} & u_{x2}^{*} \end{pmatrix}$$

$$\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{-ib} \end{pmatrix}\begin{pmatrix} m & n \\ q & p \end{pmatrix}\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{ib} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-ia} m & n e^{-ia} \\ e^{-ib} q & e^{-ib} p \end{pmatrix}\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{ib} \end{pmatrix}$$

$$= \begin{pmatrix} m & n e^{i(b-a)} \\ e^{i(b+a)} q & n \end{pmatrix}$$

$\Rightarrow$ A diagonal matrix $\check{S}$.

$\check{S}^{+} M \check{S}$ doesn't change the diagonal elements

So we only need to know the $\check{U}^{+} \delta V^{(f)} \check{U}$ term

$$\delta V^{(f)} = \begin{pmatrix} \delta V_{ee}^{(f)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\check{U}^{+} \delta V^{(f)} \check{U}$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} & \check{U}_{x1}^{*} \\ \check{U}_{e2}^{*} & \check{U}_{x1}^{*} \end{pmatrix}\begin{pmatrix} \delta V_{ee}^{(f)} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \check{U}_{e1} & \check{U}_{e2} \\ \check{U}_{x1} & \check{U}_{x2} \end{pmatrix}$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} \delta V_{ee}^{(f)} & 0 \\ \check{U}_{e2}^{*} \delta V_{ee}^{(f)} & 0 \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} \check{U}_{e1} \delta V_{ee}^{(f)} & \check{U}_{e1}^{*} \check{U}_{e2} \delta V_{ee}^{(f)} \\ \check{U}_{e2}^{*} \check{U}_{e1} \delta V_{ee}^{(f)} & \check{U}_{e2}^{*} \check{U}_{e2} \delta V_{ee}^{(f)} \end{pmatrix}$$

$$= \delta V_{ee}^{(f)} \begin{pmatrix} |\check{U}_{e1}|^2 & \check{U}_{e1}^{*} \check{U}_{e2} \\ \check{U}_{e2}^{*} \check{U}_{e1} & |\check{U}_{e2}|^2 \end{pmatrix}$$

So we require

$$\frac{d}{\ldots} = \delta V_{ee}^{(f)} \begin{pmatrix} |U_{e1}|^2 & 0 \\ & \end{pmatrix}$$

so we require

$$\frac{d}{dx}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \delta V_{ee}^{(1)} \begin{pmatrix} |u_{e1}|^2 & 0 \\ 0 & |u_{e2}|^2 \end{pmatrix}$$

that is

$$\xi_1 = V_0 |\check{u}_{e1}|^2 \sum_{a=1}^{N_k} \frac{C_a}{q_a}\left(\cos\eta_a - \cos(q_a x + \eta_a)\right)$$

why this?

To make sure that $x=0 \Rightarrow \xi=0$

So write down the EoM for B

$$i\frac{d}{dx}B = V_0\left(\sum_{a=1}^{N_k} C_a \sin(q_a x + \eta_a)\right)\begin{pmatrix} 0 & \check{u}_{e1}^* u_{e2} e^{i(\delta k_{12}x + \delta\xi_{12})} \\ \check{u}_{e2}^* u_{e1} e^{-i(\delta k_{12}x + \delta\xi_{12})} & 0 \end{pmatrix} B$$

$$\delta k_{12} = k_1 - k_2 \qquad \delta\xi_{12} = \xi_1 - \xi_2$$

$$\check{\zeta} = \exp(-i\check{k}^{(\check{a})}x)$$

$$= \begin{pmatrix} \exp(-ik_1 x) & 0 \\ 0 & \exp(-ik_2 x) \end{pmatrix}$$

Then Jacobi-Anger expansion of exponential. $e^{i\delta\xi_{12}}$

$$e^{i\delta\xi_{12}} = \prod_{a=1}^{N_k}\sum_{n_a=-\infty}^{\infty} (-i)^{n_a} J_{n_a}(Z_a) \exp\left(iZ_a\cos\eta_a + in_a(q_a x + \eta_a)\right)$$

$$Z_a = \frac{C_a V_0}{\hbar c\, q_a}\left(|\check{u}_{e1}|^2 - |\check{u}_{e2}|^2\right)$$

$$\gamma_a = (-i)^{n_a} J(Z_a) \exp\left(i(n_a\eta + Z_a\cos\eta)\right)$$

$$\chi_{a,n_a} = (-i)^{n_a} J_{n_a}(z_a) \exp\left(i(n_a \eta_a + z_a \cos \eta_a)\right)$$

$$2p = \delta k_\nu + \sum_a n_{*a} q_a$$

$$n_{*a} \Leftarrow \sum_a n_a q_a + \delta k_\nu = 0$$

$$Q^2 = p^2 + k^2$$

$$k = \frac{\check{U}_{e1}^* \check{U}_{e2}}{(\check{U}_{e1})^2 - |\check{U}_{e2}|^2} \left(\sum_a n_{*a} q_a\right) \prod_{a=1}^{N_a} \chi_{a,n_a}$$

In general, if we keep only the off diagonal terms, the equation for B can be derived using a general $\delta V_{ee}^{(f)}$

Suppose the potential is

$$\delta V^{(f)} = \begin{pmatrix} \delta V_{ee}^{(f)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$i \frac{\partial}{\partial x} B = W^* \Big(\underbrace{\check{S}^+ \check{U}^+ \delta V^{(f)} \check{U} \check{S} - \frac{\partial}{\partial x} \check{\Omega}}_{} \Big) W B$$

We choose $\check{\Omega}$ so that this term has only off diagonal elements.

I only need to calculate the off diagonal elements, since the diagonal can always be removed.

$$\check{U}^+ \delta V^{(f)} \check{U} = \delta V_{ee}^{(f)} \begin{pmatrix} |\check{U}_{e1}|^2 & \check{U}_{e1}^* \check{U}_{e2} \\ \check{U}_{e2}^* \check{U}_{e1} & |\check{U}_{e2}|^2 \end{pmatrix}$$

The off diagonal part of $\check{S}^+ \check{U}^+ \delta V^{(f)} \check{U} \check{S}$ is $\delta V^{(f)} \begin{pmatrix} 0 & \check{U}_{e1}^* \check{U}_{e2} e^{i(\varepsilon_1 - \varepsilon_2)} \end{pmatrix}$

The off diagonal part of

$$\check{S}^\dagger \check{U}^\dagger \delta V^{(f)} \check{U} \check{S} \Big|_{\text{off-diagonal}} \text{ is } \delta V_{ee}^{(f)} \begin{pmatrix} 0 & \check{U}_{e1}^* \check{U}_{e2} e^{i(\varepsilon_1-\varepsilon_2)} \\ \check{U}_{e2}^* \check{U}_{e1} e^{-i(\varepsilon_1-\varepsilon_2)} & 0 \end{pmatrix}$$

for

$$\check{S} = \begin{pmatrix} e^{-i\varepsilon_1} & 0 \\ 0 & e^{-i\varepsilon_2} \end{pmatrix}$$

So

$$i\frac{d}{dx}B = W^\dagger \delta V_{ee}^{(f)} \begin{pmatrix} 0 & \check{U}_{e1}^* \check{U}_{e2} e^{i(\varepsilon_1-\varepsilon_2)x} \\ \check{U}_{e2}^* \check{U}_{e1} e^{-i(\varepsilon_1-\varepsilon_2)x} & 0 \end{pmatrix} WB$$

$$W = \begin{pmatrix} \exp(-i\xi_1) & 0 \\ 0 & \exp(-i\xi_2) \end{pmatrix}$$ doesn't change off-diagonal

$$\delta V_{ee}^{(f)} \longrightarrow \xi_1$$

But the form of ~~solution~~ equation is given by

$$i\frac{d}{dx}B = \begin{pmatrix} 0 & \delta V_{ee}^{(f)} \check{U}_{e1}^* \check{U}_{e2} e^{i(\varepsilon_1-\varepsilon_2)x} e^{i(\xi_1-\xi_2)} \\ \delta V_{ee}^{(f)} \check{U}_{e2}^* \check{U}_{e1} e^{-i(\varepsilon_1-\varepsilon_2)x} e^{-i(\xi_1-\xi_2)} & 0 \end{pmatrix} B$$

$$WB = A$$

$$S^{(m)} = \check{S} A$$

For an equation

$$i\frac{d}{dx}B = H_B B = \delta V_{ee}^{f} \begin{pmatrix} 0 & U_{e1}^* U_{e2} e^{i\delta\omega_{12}x} e^{i\delta\xi_{12}} \\ U_{e2}^* U_{e1} e^{-i\delta\omega_{12}x} e^{-i\delta\xi_{12}} & 0 \end{pmatrix} B$$

$$= \begin{pmatrix} 0 & h_b \\ h_b^* & 0 \end{pmatrix} B$$

For $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$.

$$i\frac{d}{dx}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} 0 & h_b \\ h_b^* & 0 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$= \begin{pmatrix} b_3 h_b & b_4 h_b \\ b_1 h_b^* & b_2 h_b^* \end{pmatrix}$$

$$i\frac{d}{dx} b_2 = b_4 h_b$$

$$i\frac{d}{dx} b_3 = b_1 h_b^* \quad , \quad i\frac{d}{dx}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & h_b & 0 \\ 0 & 0 & 0 & h_b \\ h_b^* & 0 & 0 & 0 \\ 0 & h_b^* & 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Equations of motion

Wednesday, January 20, 2016 10:30 PM

1. System to be solved

$$|\psi(x)\rangle = S(x, x_0)\,|\psi(x_0)\rangle$$

We work in averaged matter basis

S statisfies (from Schrödinger equation)

$$i\frac{d}{dx}S = HS$$

The Hamiltonian is $H^{(m)} = K^{(m)} - i\tilde{U}^{+}\frac{\partial}{\partial x}\tilde{U} + \tilde{U}^{+}\delta V^{(f)}\tilde{U}$

$$H = H_0 - iU^{+}\frac{\partial}{\partial x}U + U^{+}\delta V^{f}U$$

$$H_0 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$ in averaged matter basis and only for non-perturbed part of matter potential

$-iU^{+}\frac{\partial}{\partial x}U$ is 0 if we have constant potential U_0

$U^{+}\delta V^{f}U$ is perturbation and rotated from flavor basis to averaged mateor basis

We decompose $S = S_0 A$ where S_0 satisfies

$$i\frac{d}{dx}S_0 = H_0 S_0 \qquad \left(-iU^{+}\frac{d}{dx}U = 0\right)$$

Since $H_0 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$,

$$S_0 = \begin{pmatrix} \exp(-i\omega_1 x) & 0 \\ 0 & \exp(-i\omega_2 x) \end{pmatrix}$$

The equation to be solved is

$$i\frac{d}{dx}(S_0 A) = H(S_0 A)$$

$$i\frac{d}{dx}A = S_0^\dagger U^\dagger \delta V^\dagger U S_0 A$$

Decompose $A = WB$,

$$i\frac{d}{dx}(WB) = S_0^\dagger U^\dagger \delta V^\dagger U S_0 WB$$

choose $W = \begin{pmatrix} \exp(-i\xi_1) & 0 \\ 0 & \exp(-i\xi_2) \end{pmatrix}$

So that the effective Hamiltonian for B is off diagon

$$W \cdot i\frac{d}{dx}B = S_0^\dagger U^\dagger \delta V^\dagger U S_0 WB - i\left(\frac{d}{dx}W\right)B$$

$$\frac{d}{dx}W = -i\left(\frac{d}{dx}\underbrace{\begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}}_{\text{defined to be } \Xi}\right)W$$

$$i\frac{d}{dx}B = \underbrace{W^\dagger\left(S_0^\dagger U^\dagger \delta V^\dagger U S_0 - \frac{d}{dx}\Xi\right)W}_{\text{diagonal elements can be removed by choosing } \xi_1, \xi_2 \text{ carefully}} B$$

Useful relations & facts

$$\boxed{\begin{pmatrix} 0 & m^* \\ m & 0 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}m & b_{12}m \\ b_{21}m^* & b_{22}m^* \end{pmatrix}}$$

$$\begin{pmatrix} e^{-ia} & 0 \\ 0 & e^{-ib} \end{pmatrix}\begin{pmatrix} m & n \\ q & p \end{pmatrix}\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{ib} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-ia} m & n e^{-ia} \\ e^{-ib} q & e^{-ib} p \end{pmatrix}\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{ib} \end{pmatrix}$$

$$= \begin{pmatrix} m & n e^{i(b-a)} \\ e^{i(b+a)} q & n \end{pmatrix}$$

$\Rightarrow$ A diagonal matrix $\check{S}$.

$\check{S}^{+} M \check{S}$ doesn't change the diagonal elements

$$\check{U}^{+} \delta V^{(f)} \check{U}$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} & \check{U}_{x1}^{*} \\ \check{U}_{e2}^{*} & \check{U}_{x2}^{*} \end{pmatrix}\begin{pmatrix} \delta V_{ee}^{(f)} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \check{U}_{e1} & \check{U}_{e2} \\ \check{U}_{x1} & \check{U}_{x2} \end{pmatrix}$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} \delta V_{ee}^{(f)} & 0 \\ \check{U}_{e2}^{*} \delta V_{ee}^{(f)} & 0 \end{pmatrix}\Big(\quad \Big)$$

$$= \begin{pmatrix} \check{U}_{e1}^{*} \check{U}_{e1} \delta V_{ee}^{(f)} & \check{U}_{e1}^{*} \check{U}_{e2} \delta V_{ee}^{(f)} \\ \check{U}_{e2}^{*} \check{U}_{e1} \delta V_{ee}^{(f)} & \check{U}_{e2}^{*} \check{U}_{e2} \delta V_{ee}^{(f)} \end{pmatrix}$$

$$= \delta V_{ee}^{(f)} \begin{pmatrix} |\check{U}_{e1}|^2 & \check{U}_{e1}^{*} \check{U}_{e2} \\ \check{U}_{e2}^{*} \check{U}_{e1} & |\check{U}_{e2}|^2 \end{pmatrix}$$

The equation for B is

$$i \frac{d}{dx} B = W^{+} \delta V_{ee}^{f} \begin{pmatrix} 0 & U_{e1}^{*} U_{e2} e^{i(\omega_1-\omega_2)x} \\ U_{e2}^{*} U_{e1} e^{-i(\omega_1-\omega_2)x} & 0 \end{pmatrix} W B$$

$$i\frac{d}{dx}B = W\,\delta V_{ee}\begin{pmatrix} 0 & U_{e1}^{*}U_{e2}e^{\cdots} \\ U_{e2}^{*}U_{e1}e^{-i(\omega_1-\omega_2)x} & 0 \end{pmatrix}WB$$

$$\delta V^{f} = \begin{pmatrix} \delta V_{ee}^{f} & 0 \\ 0 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} \exp(-i\xi_1) & 0 \\ 0 & \exp(-i\xi_2) \end{pmatrix}$$

$$i\frac{d}{dx}B = \delta V_{ee}^{f}\begin{pmatrix} 0 & U_{e1}^{*}U_{e2}e^{i\delta\omega_{12}x}e^{i\delta\xi_{12}} \\ U_{e2}^{*}U_{e1}e^{-i\delta\omega_{12}x}e^{-i\delta\xi_{12}} & 0 \end{pmatrix}B$$

We only need to determine $\delta\xi_{12}$

and matter effect comes in from approximations of the matrix

We require

$$\frac{d}{dx}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \delta V_{ee}^{f}\begin{pmatrix} |U_{e1}|^2 & 0 \\ 0 & |U_{e2}|^2 \end{pmatrix}$$

Fourier series of δV_{ee}^{f}

$$\delta V_{ee}^{f} = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$\frac{d\xi_i}{dx} = \delta V_{ee}^{f}|U_{ei}|^2$$

$$\xi_i = |U_{ei}|^2\sum_{n=-\infty}^{\infty}\frac{C_n}{in}e^{inx}$$

$$= -i\,|U_{ei}|^2\sum_{n=-\infty}^{\infty}\frac{C_n}{n}e^{inx}$$

The term to be investigated is

$$\delta\omega_{12} X + \delta\xi_{12}$$

and figure out which to be dropped?
and solving the equation for B

2. The second formalize is to have the Hamiltonian in non-perturbed basis. first the use ~~traditional~~ wave function method (or density matrix)

* Wavefunction

In background matter basis

$$H = H_0 + U^\dagger \delta\lambda^f U$$

$$H_0 = \frac{1}{2}\begin{pmatrix} -\omega_m & 0 \\ 0 & \omega_m \end{pmatrix} = -\frac{\omega_m}{2}\sigma_3 \qquad \omega_m = \omega\sqrt{\hat{\lambda}_0^2 + 1 - 2\hat{\lambda}\cos 2\theta_v}$$

U rotate states from flavor to non-perturbed matter basis

$$U = \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{pmatrix} \qquad \lambda = \lambda_0 + \lambda_1 \qquad \hat{\lambda}_0 = \frac{\lambda_0}{\omega}$$

$$\sin 2\theta_m = \frac{\sin 2\theta_v}{\sqrt{\hat{\lambda}_0^2 + 1 - 2\hat{\lambda}_0\cos 2\theta_v}}$$

$$\cos 2\theta_m = \frac{\cos 2\theta_v - \hat{\lambda}_0}{\sqrt{\cdots\cdots}}$$

$$\delta\lambda^f = \begin{pmatrix} \delta\lambda & 0 \\ 0 & 0 \end{pmatrix} - \frac{\delta\lambda}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\delta\lambda & 0 \\ & -1 \end{pmatrix} = \frac{\delta\lambda}{2}\sigma_3$$

$$= \begin{pmatrix} \frac{1}{2}\delta\lambda & 0 \\ 0 & -\frac{\delta\lambda}{2} \end{pmatrix} = \frac{\delta\lambda}{2}\sigma_3$$

So the Hamiltonian would be

$$H = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2} U^\dagger \sigma_3 U$$

$$U^\dagger \sigma_3 U = \cos 2\theta_m \, \sigma_3 - \sin 2\theta_m \, \sigma_1$$

$$H = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2}\cos 2\theta_m \, \sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m \, \sigma_1$$

Use the trick to remove $\frac{\delta\lambda}{2}\cos 2\theta_m \sigma_3$ term

* wave function in non-perturbed matter basis is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

We can define a transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix}}_{Y} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

Schrödinger equation

$$LHS = i\frac{d}{dx}\left[\begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}\right]$$

$$= i\left[\frac{d}{dx}\begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix}\right]\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i\begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= i\begin{pmatrix} -i\eta e^{-i\eta x} & -ix\frac{d\eta}{dx}e^{-i\eta x} \\ 0 & i\eta e^{i\eta x} + ix\frac{d\eta}{dx}e^{i\eta x} \end{pmatrix}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$+ \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\eta x} & 0 \\ \vdots & \vdots \end{pmatrix} LHS = i\begin{pmatrix} e^{i\eta x} & 0 \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} -i\eta e^{-i\eta x} - ix\frac{d\eta}{dx}e^{-i\eta x} & 0 \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} LHS = i \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} \begin{pmatrix} -i\eta e^{-i\eta x} - ix\frac{d\eta}{dx} e^{-i\eta x} & 0 \\ 0 & i\eta e^{i\eta x} + ix\frac{d\eta}{dx} e^{i\eta x} \end{pmatrix} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= i \begin{pmatrix} -i\eta - ix\frac{d\eta}{dx} & 0 \\ 0 & i\eta + ix\frac{d\eta}{dx} \end{pmatrix} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} RHS = \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} \left(-\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2}\cos 2\theta_m \sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m \sigma_1 \right) \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= \left[-\frac{\omega_m}{2} \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= \left[-\frac{\omega_m}{2} \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & -e^{-i\eta x} \end{pmatrix} \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} - \frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{i\eta x} \\ e^{-i\eta x} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x} \end{pmatrix} \right] \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= \left[-\frac{\omega_m}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-i2\eta x} & 0 \end{pmatrix} \right] \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

We equate

$$\begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} LHS = \begin{pmatrix} e^{i\eta x} & 0 \\ 0 & e^{-i\eta x} \end{pmatrix} RHS$$

$$i\begin{pmatrix} -i\eta - ix\frac{d\eta}{dx} & 0 \\ 0 & i\eta + ix\frac{d\eta}{dx} \end{pmatrix}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$= \left[-\frac{\omega_m}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-i2\eta x} & 0 \end{pmatrix}\right]\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

To Remove these terms, we require

$$-\frac{\omega_m}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i\begin{pmatrix} -i\eta - ix\frac{d\eta}{dx} & 0 \\ 0 & i\eta + ix\frac{d\eta}{dx} \end{pmatrix}$$

$$-\frac{\omega_m}{2} + \frac{\delta\lambda}{2}\cos 2\theta_m = -\eta - x\frac{d\eta}{dx}$$

$$\left(\frac{\omega_m}{2} - \frac{\delta\lambda}{2}\cos 2\theta_m = \eta + x\frac{d\eta}{dx}\right)$$

So to solve η,

$$\eta + x\frac{d\eta}{dx} = \frac{\omega_m}{2} - \frac{\delta\lambda}{2}\cos 2\theta_m$$

$\cos 2\theta_m$ → constant

$\frac{\delta\lambda}{2}$ → x dependent

$\frac{\omega_m}{2}$ → constant

Then the equation of motion becomes

$$i\frac{d}{dx}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} = -\frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-2i\eta x} & 0 \end{pmatrix}\begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

In a general case

$$\delta\lambda = \sum_{n=-\infty}^{\infty} f_n e^{in\omega x}$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \underbrace{-\frac{1}{2}\sin 2\theta_m \begin{pmatrix} 0 & \sum_{n=-\infty}^{\infty} f_n e^{in\omega x+2i\eta x} \\ \sum_{n=-\infty}^{\infty} f_n e^{in\omega x-2i\eta x} & 0 \end{pmatrix}}_{\text{Effective Hamiltonian in this new basis}}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

Can u solve it directly? $\boxed{\text{Regarding } \eta = \eta(x)}$

* Try:

$$i\frac{d}{dx}\psi_{b1} = -\frac{1}{2}\sin 2\theta_m \sum_{n=-\infty}^{\infty} f_n e^{in\omega x+2i\eta x}\psi_{b2}$$

$$i\frac{d}{dx}\psi_{b2} = -\frac{1}{2}\sin 2\theta_m \sum_{n=-\infty}^{\infty} f_n e^{in\omega x-2i\eta x}\psi_{b1}$$

Method 1.

Combine to form an equation of ψ_{b1}

$$i\frac{d^2}{dx^2}\psi_{b1} = -\frac{1}{2}\sin 2\theta_m \sum_{n=-\infty}^{\infty} f_n e^{in\omega x+2i\eta x}\frac{d\psi_{b2}}{dx}$$

$$-\frac{1}{2}\sin 2\theta_m \sum_{n=-\infty}^{\infty} f_n \Big[(in\omega+2i\eta)e^{in\omega x+2i\eta x}$$

$$+ ix\,2\frac{d\eta}{dx}e^{in\omega x+2i\eta x}\Big]\psi_{b2}$$

$$\left((in\omega+2i\eta)+2ix\frac{d\eta}{dx}\right)e^{in\omega x+2i\eta x}$$

$$= \left(in\omega + 2i\left(\eta + x\frac{d\eta}{dx}\right)\right)e^{in\omega x+2i\eta x}$$

$$= \left(in\omega + 2i\left(\eta + x\frac{d\eta}{dx}\right)\right) e^{in\omega x + 2i\eta x}$$

$$= \left(in\omega + 2i\left(\frac{\omega_m}{2} - \frac{\delta\lambda}{2}\cos 2\theta_m\right)\right) e^{in\omega x + 2i\eta x}$$

$$= \left(in\omega + i\left(\omega_m - \delta\lambda \cos 2\theta_m\right)\right) e^{in\omega x + 2i\eta x}$$

Method 2

The equations are also written as

$$i\frac{d}{dx}\psi_{b1} = -\frac{\delta\lambda}{2}\sin 2\theta_m e^{2i\eta x}\psi_{b2}$$

$$-\frac{2}{\delta\lambda \sin 2\theta_m} e^{-2i\eta x} \quad i\frac{d}{dx}\psi_{b1} = \psi_{b2}$$

$$-i\frac{2}{\sin 2\theta_m} \quad \frac{1}{\delta\lambda} e^{-2i\eta x} \quad \frac{d}{dx}\psi_{b1} = \psi_{b2}$$

$$i(-i)\frac{2}{\sin 2\theta_m}\left[\frac{d}{dx}\left(\frac{1}{\delta\lambda}\right) e^{-2i\eta x}\frac{d}{dx}\psi_{b1} + \frac{1}{\delta\lambda}\left(\frac{d}{dx}e^{-i2\eta x}\right)\frac{d}{dx}\psi_{b1} + \frac{1}{\delta\lambda}e^{-2i\eta x}\frac{d^2}{dx^2}\psi_{b1}\right] = i\frac{d}{dx}\psi_{b2} = -\frac{\delta\lambda}{2}\sin 2\theta_m e^{-2i\eta x}\psi_{b1}$$

Then I have an equation for ψ_{b1}

$$\frac{2}{\sin 2\theta_m}\left[\left(-\frac{1}{(\delta\lambda)^2}\frac{d}{dx}\delta\lambda - 2i\left(\eta + x\frac{d\eta}{dx}\right)\right)\frac{d}{dx}\psi_{b1} + \frac{1}{\delta\lambda}\frac{d^2}{dx^2}\psi_{b1}\right] = -\frac{\delta\lambda}{2}\sin 2\theta_m \psi_{b1}$$

A simple case

$$\delta\lambda = A\sin(kx)$$

$$\frac{d^2}{dx^2}\psi_{b1} - \left(\frac{1}{\delta\lambda}\frac{d}{dx}\delta\lambda + 2i\left(\eta + x\frac{d\eta}{dx}\right)\delta\lambda\right)\frac{d}{dx}\psi_{b1} + \frac{\delta\lambda^2}{2}\cdot\frac{\sin 2\theta_m}{2}\sin 2\theta_m \psi_{b1} = 0$$

$$\eta + x\frac{d\eta}{dx} = \frac{\omega_m}{2} - \frac{\delta\lambda}{2}\cos 2\theta_m$$

$$\frac{d^2}{dx^2}\psi_{b1} - \left(\frac{1}{\delta\lambda}\frac{d}{dx}\delta\lambda + i\left(\omega_m - \delta\lambda\cos 2\theta_m\right)\delta\lambda\right)\frac{d}{dx}\psi_{b1} + \frac{\sin^2 2\theta_m}{4}\delta\lambda^2\psi_{b1} = 0$$

$$\delta\lambda = Ak\cot(kx)$$

$$\frac{d^2}{dx^2}\psi_{b1} - \left(A\cdot k\cdot\cot(kx) + i\left(\omega_m - A\cdot\sin(kx)\cos(2\theta_m)\right)A\sin(kx)\right)\frac{d}{dx}\psi_{b1} + \left(\frac{\sin 2\theta_m}{2}\right)^2 A^2\sin^2(kx)\,\psi_{b1} = 0$$

This could not be solved by Mathematica

Method 3. Try the expansion for $\exp(2i\eta(x)\cdot x)$

In a general case

$$\delta\lambda = \sum_{n=-\infty}^{\infty} f_n e^{in\omega x}$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_b\\ \psi_{b2}\end{pmatrix} = \underbrace{-\frac{1}{2}\sin 2\theta_m\begin{pmatrix}0 & \sum_{n=-\infty}^{\infty} f_n e^{in\omega x+2i\eta x}\\ \sum_{n=-\infty}^{\infty} f_n e^{in\omega x-2i\eta x} & 0\end{pmatrix}}_{\text{Effective Hamiltonian in this new basis}}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

Using Jacobi-Anger expansion (or any other?)

η is not the η I used before

$$\exp(i\xi_n) = \exp\left(i V_0\left(|u_{e1}|^2 - |u_{e2}|^2\right)\left\{\sum_{a=1}^{N_k}\frac{C_a}{q_a}\left(\cos\eta_a - \cos(q_a x+\eta_a)\right)\right\}\right)$$

$$= \exp\left(i\sum_{a=1}^{N_k} Z_a\left(\cos\eta_a - \cos(q_a x+\eta_a)\right)\right)$$

$$= \prod_{a=1}^{N_k}\exp\left(i Z_a\left(\cos\eta_a - \cos(q_a x+\eta_a)\right)\right)$$

$$= \prod_{a=1}^{N_k}\exp\left(i Z_a\cos\eta_a\right)\exp\left(-i Z_a\cos(q_a x+\eta_a)\right)$$

Jacobi-Anger

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z)\, e^{in\theta}$$

$$\Rightarrow \exp(-i Z_a\cos(q_a x+\eta_a)) = \sum_{n=-\infty}^{\infty} i^n J_n(-Z_a)\, e^{in(q_a x+\eta_a)}$$

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}$$

$$\Rightarrow \exp(-i z_a \cos(q_a x + \eta_a)) = \sum_{n=-\infty}^{\infty} i^n J_n(-z_a) e^{in(q_a x + \eta_a)}$$

$$J_n(-z_a) = (-1)^n J_n(z_a) \quad \text{for } n: \text{integer}$$

$$\Rightarrow \exp(-i z_a \cos(q_a x + \eta_a)) = \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_a) e^{in(q_a x + \eta_a)}$$

$$= \prod_{a=1}^{N_a} \sum_{n_a=-\infty}^{\infty} (-i)^{n_a} J_{n_a}(z_a) \exp\left[i z_a \cos\eta_a + i n_a (q_a x + \eta_a)\right]$$

I have to proceed on a specific choice of matter profile. $\delta\lambda = \sum_{n=-\infty}^{\infty} A_n e^{inx}$ (not really specific

$$\eta + x\frac{d\eta}{dx} = \frac{\omega_m}{2} - \frac{\delta\lambda}{2}\sin 2\theta_m$$

Effective Hamiltonian

$$\begin{pmatrix} 0 & \sum_{n=-\infty}^{\infty} A_n e^{inx + 2i\eta x} \\ \sum_{n=-\infty}^{\infty} A_n e^{inx - 2i\eta x} & 0 \end{pmatrix}$$

$$\sum_{n=-\infty}^{\infty} A_n e^{inx + 2i\eta x}$$

To remove a complicated $\eta(x)$ in the exponential I solve η from $\delta\lambda$,

$$\eta(x) = \frac{C_1}{x} + \frac{1}{x}\int_1^x \frac{1}{2}\left(\omega_m - \delta\lambda(\tau)\sin 2\theta_m\right) d\tau$$

$$= \frac{C_1}{x} + \frac{1}{2x}\left(\int_1^x \omega_m \, d\tau - \sum_{n=-\infty}^{\infty} A_n \int_1^x e^{in\tau} d\tau \, \sin 2\theta_m\right)$$

$$= \frac{C_1}{x} + \frac{1}{2x}\left(\omega_m \tau\Big|_1^x - \sum_{n=-\infty}^{\infty} A_n \frac{1}{in} e^{in\tau}\Big|_1^x \sin 2\theta_m\right)$$

$$= \frac{C_1}{x} + \frac{\omega_m x - \omega_m}{2x} - \frac{1}{2x}\sum_{n=-\infty}^{\infty} A_n \frac{1}{in}\sin 2\theta_m \left(e^{inx} - e^{in}\right)$$

Choose $C_1 = \frac{\omega_m}{2}$.

$$\eta(x) = \frac{\omega_m}{2} + \frac{i}{2x}\sin 2\theta_m \sum_{n=-\infty}^{\infty}\left(\frac{A_n}{n} e^{inx} - \frac{A_n}{n} e^{in}\right)$$

$$= \frac{\omega_m}{2} + \frac{i}{2x}\sin 2\theta_m \left[\sum_{n=-\infty}^{\infty} \frac{A_n}{n}\left(\cos(nx) + i\sin(nx)\right) - \sum_{n=-\infty}^{\infty} \frac{A_n}{n}\left(\cos n - i\sin n\right)\right]$$

From ESF Highly Oscillatory Problems: Approximation by plane waves Andrea Moiola

Jacobi-Anger: $z \in \mathbb{C}$, $\vartheta \in \mathbb{R}$

$$e^{iz\cos\vartheta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\vartheta}$$

$$e^{iz\sin\vartheta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\vartheta}$$

$$e^{2i\eta(x)x} = e^{2ix\left(\frac{\omega_m}{2} + \frac{i}{2x}\sin 2\vartheta_m\left[\sum_{n=-\infty}^{\infty}\frac{A_n}{n}(\cos(nx)+i\sin(nx)\right]\right)} \cdot e^{-2ix\left(\frac{i}{2x}\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}(\cos n - i\sin n)\right)}$$

$$= e^{i\omega_m x} e^{-\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}\cos(nx)} e^{-i\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}\sin(nx)} \cdot e^{\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}(\cos(n) - i\sin(n))}$$

I can expand

$$e^{-\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}\cos(nx)}$$

$$= \prod_{n=-\infty}^{\infty} e^{-\sin 2\vartheta_n \frac{A_n}{n}\cos(nx)}$$

$$= \prod_{n=-\infty}^{\infty} e^{i z_n \cos(nx)}$$

$$= \prod_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^m J_m(z_n) e^{im\cdot nx}$$

Define $z_n = i\sin 2\vartheta_m \frac{A_n}{n}$

$$e^{-i\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}\sin(nx)}$$

$$= \prod_{n=-\infty}^{\infty} e^{-i\sin 2\vartheta_m \frac{A_n}{n}\sin(nx)}$$

$$= \prod_{n=-\infty}^{\infty} e^{i y_n \sin(nx)}$$

$$= \prod_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m(y_n) e^{im\cdot nx}$$

$$y_n = -\sin 2\vartheta_m \frac{A_n}{n} = i z_n$$

Combine the 2 terms

$$e^{2i\eta(x)\cdot x} = e^{i\omega_m x} \prod_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} i^m J_m(z_n) e^{im\cdot nx} \cdot \prod_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_m(y_n) e^{im\cdot nx} \cdot e^{\sin 2\vartheta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{n}(\cos(n) - i\sin(n))}$$

* It seems that this form of matter profile is

not easy to work with So I'll turn to use

$$\delta\lambda = \sum_{n=-\infty}^{\infty} A_n \sin(k_n X + \phi_n)$$

A_n corresponds to $V_0 G_a$ in Kelly Patten's paper

Using this profile / expansion, I have

$$\eta(x) = \frac{C_1}{X} + \frac{1}{X}\int_1^X \frac{1}{2}\left(\omega_m - \delta\lambda(\tau)\sin 2\theta_m\right) d\tau$$

$$= \frac{C_1}{X} + \frac{1}{2X}\left[\int_1^X (\omega_m)\, d\tau - \int_1^X \delta\lambda(\tau)\sin 2\theta_m\, d\tau\right]$$

$$= \frac{C_1}{X} + \frac{1}{2X}\left[\omega_m \tau\Big|_1^X - \sum_{n=-\infty}^{\infty} A_n \int_1^X \sin(k_n \tau + \phi_n)\, d\tau \;\sin 2\theta_m\right]$$

$$= \frac{C_1}{X} + \frac{(\omega_m X - \omega_m)}{2X} - \sum_{n=-\infty}^{\infty} A_n \frac{\sin 2\theta_m}{2X} \frac{-1}{k_n}\cos(k_n \tau + \phi_n)\Big|_1^X$$

Choose $C_1 = \frac{\omega_m}{2}$

$$\eta(x) = \frac{\omega_m}{2} + \frac{\sin 2\theta_m}{2X}\sum_{n=-\infty}^{\infty} A_n \frac{1}{k_n}\left(\cos(k_n X + \phi_n) - \cos(k_n + \phi_n)\right)$$

$$= \frac{\omega_m}{2} + \frac{\sin 2\theta_m}{2X}\sum_{n=-\infty}^{\infty} \frac{A_n}{k_n}\cos(k_n X + \phi_n) - \frac{\sin 2\theta_m}{2X}\sum_{n=-\infty}^{\infty} \frac{A_n}{k_n}\cos(k_n + \phi_n)$$

$$e^{2i\eta X} = e^{2iX\frac{\omega_m}{2}}\, e^{2iX \cdot \frac{\sin 2\theta_m}{2X}\sum_{n=-\infty}^{\infty}\frac{A_n}{k_n}\cos(k_n X + \phi_n)} \cdot e^{2iX \cdot \frac{\sin 2\theta_m}{2X}\sum_{n=-\infty}^{\infty}\frac{A_n}{k_n}\cos(k_n + \phi_n)}$$

$$= e^{i\omega_m X}\, \underbrace{e^{i\sin 2\theta_m \sum_{n=-\infty}^{\infty}\frac{A_n}{k_n}\cos(k_n + \phi_n)}}_{e^{i\sum_{n=-\infty}^{\infty} Z_n \cos(k_n + \phi_n)}} \prod_{n=-\infty}^{\infty} e^{i\sin 2\theta_m \frac{A_n}{k_n}\cos(k_n X + \phi_n)}$$

I expand

$$e^{i\sin 2\theta_m \frac{A_n}{k_n}\cos(k_n X + \phi_n)}$$

$$= e^{i Z_n \cos(k_n X + \phi_n)}$$

$$= \sum_{a=-\infty}^{\infty} i^a J_a(Z_n)\, e^{ia\cdot(k_n X + \phi_n)}$$

$$\boxed{\text{define } Z_n = \sin 2\theta_m \frac{A_n}{k_n}}$$

So

$$e^{2i\eta X} = e^{i\omega_m X}\, e^{i\sum_{n=-\infty}^{\infty} Z_n \cos(k_n + \phi_n)} \prod_{n=-\infty}^{\infty}\sum_{a=-\infty}^{\infty} i^a J_a(Z_n)\, e^{ia(k_n X + \phi_n)}$$

So the 12 element of Effective Hamiltonian is

$$\sum_{n=-\infty}^{\infty} A_n e^{inx+2i\eta x} = \sum_{n=-\infty}^{\infty} A_n e^{inx} e^{2i\eta x}$$

$$= \sum_{n=-\infty}^{\infty} A_n e^{inx} e^{i\omega_m x} \underbrace{e^{i\sum_{d=-\infty}^{\infty} z_d \cos(k_d+\phi_d)}}_{F} \prod_{b=-\infty}^{\infty} \sum_{a=-\infty}^{\infty} i^a J_a(z_b) e^{ia(k_b x+\phi_b)}$$

$$= \sum_{n=-\infty}^{\infty} A_n F e^{inx+i\omega_m x} \prod_{b=-\infty}^{\infty} \sum_{a=-\infty}^{\infty} i^a J_a(z_b) e^{ia(k_b x+\phi_b)}$$

I can NOT move $e^{inx+i\omega_m x}$ inside the second summation (over a)

Physics:

I have to remove the rapid oscillations

Math:

remove the summations

I could not move on anymore.

Now move to a single frequency

$$\delta\lambda = A\sin(kx+\phi)$$

$$\eta = \frac{c_1}{x} + \frac{1}{x}\int_1^x \frac{1}{2}\left(\omega_m - \delta\lambda(\tau)\sin 2\theta_m\right) d\tau$$

$$= \frac{c_1}{x} + \frac{1}{2x}\int_1^x \left(\omega_m - A\sin(kx+\phi)\sin 2\theta_m\right) d\tau$$

$$= \frac{c_1}{x} + \frac{\omega_m x - \omega_m}{2x} + \frac{1}{2x} A\sin 2\theta_m \frac{1}{k}\left(\cos(kx+\phi) - \cos(k+\phi)\right)$$

define $c_1 = \frac{\omega_m}{2}$

$$\eta = \frac{\omega_m}{2} + \frac{A\sin 2\theta_m}{2x}\cdot\frac{1}{k}\cos(kx+\phi) - \frac{A\sin 2\theta_m}{2x}\frac{1}{k}\cos(k+\phi)$$

$$e^{2i\eta x} = e^{i\omega_m x} e^{-2ix\frac{A\sin 2\theta_m}{2x}\frac{1}{k}\cos(k+\phi)} e^{2ix\frac{A\sin 2\theta_m}{2x}\frac{1}{k}\cos(kx+\phi)}$$

$$= e^{i\omega_m x} e^{-iA\sin 2\theta_m \frac{1}{k}\cos(k+\phi)} e^{iA\sin 2\theta_m \frac{1}{k}\cos(kx+\phi)}$$

Jacobi - Anger Expansion

$$e^{i\frac{A}{k}\sinh 2\theta_m \cos(kx+\phi)} \qquad \text{define } Z_k = \frac{A}{k}\sinh 2\theta_m$$

$$= e^{i Z_k \cos(kx+\phi)}$$

$$= \sum_{n=-\infty}^{\infty} i^n J_n(Z_k) e^{in(kx+\phi)}$$

So

$$e^{2i\eta x} = e^{-iZ_k\cos(k+\phi)} e^{i\omega_m x} \sum_{n=-\infty}^{\infty} i^n J_n(Z_k) e^{in(kx+\phi)}$$

$$= e^{-iZ_k\cos(k+\phi)} \sum_{n=-\infty}^{\infty} i^n J_n(Z_k)\, e^{i(\omega_m+nk)x + in\phi}$$

$$= e^{-iZ_k\cos(k+\phi)} \sum_{n=-\infty}^{\infty} e^{in\phi}\, i^n J_n(Z_k)\, e^{i(\omega_m+nk)x}$$

When $nk+\omega_m \sim 0$ ~~(minimize~~

or $|nk+\omega_m|$ is minimized.

Solve out $n = n_0$

$$e^{i2\eta x} \doteq \underbrace{e^{-iZ_k\cos(k+\phi)}\, e^{in_0\phi}\, i^{n_0} J_{n_0}(Z_{n_0})}\, e^{i(\omega_m+n_0k)x}$$

$$= G_{n_0}\, e^{i(\omega_m+n_0k)x}$$

Then the equation of motion becomes

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix} 0 & e^{2i\eta x} \\ e^{-2i\eta x} & 0 \end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$\boxed{\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} e^{-i\eta x} & 0 \\ 0 & e^{i\eta x}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}}$$

In this period basis I could solve ψ_{b1}, ψ_{b2}

$$i\frac{d}{dx}\psi_{b_1} = G_{n_0} e^{i(\omega_m + \eta_0 k)x}\,\psi_{b_2}$$
$$i\frac{d}{dx}\psi_{b_2} = G_{n_0}^{*} e^{-i(\omega_m + \eta_0 k)x}\,\psi_{b_1}$$

Solution is

$$\psi_{b_1} = e^{\frac{1}{2} i\left(\eta_0 k + \omega_m + i\sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right)x} G_1 + e^{\frac{1}{2}\left[i(\eta_0 k + \omega_m) + \sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right]} G_2$$

$$\psi_{b_2} = \frac{i}{2G_{n_0}^{*}} e^{-i(\eta_0 k + \omega_m)x + \frac{1}{2} i\left(\eta_0 k + \omega_m + i\sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right)x} \cdot \left(\eta_0 k + \omega_m + i\sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right) G_1$$
$$+ \frac{1}{2G_{n_0}^{*}} e^{-i(\eta_0 k + \omega_m)x + \frac{1}{2}\left(i(\eta_0 k + \omega_m) + \sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right)x} \cdot \left(i(\eta_0 k + \omega_m) + \sqrt{-4|G_{n_0}|^2 - (\eta_0 k + \omega_m)^2}\right) G_2$$

Using Simpler Transformation

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$$H = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2}\cos 2\theta_m\,\sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m\,\sigma_1$$

background matter basis

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

Solving Schrödinger equation $\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$

$$i\frac{d}{dx}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \left(-\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2}\cos 2\theta_m\,\sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m\,\sigma_1\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

Apply the transformation

$$LHS = \left[i\frac{d}{dx}\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix}\right]\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix}\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$= i\begin{pmatrix} -i\frac{d\eta}{dx}e^{-i\eta} & 0 \\ 0 & i\frac{d\eta}{dx}e^{i\eta}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\begin{pmatrix} & \\ & \end{pmatrix}\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix} LHS = i\cdot i\frac{d\eta}{dx}\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\begin{pmatrix} -e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix} RHS = \left[-\frac{\omega_m}{2}\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\sigma_3\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix} + \frac{\delta\lambda}{2}\cos 2\theta_m\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\sigma_3\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix}\right.$$

$$\left. - \frac{\delta\lambda}{2}\sin 2\theta_m\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\sigma_1\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix}\right]\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

I know

$$\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{+i\eta}\end{pmatrix}\sigma_3\begin{pmatrix} e^{+i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix} = \begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{+i\eta}\end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} e^{+i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}$$

$$= \begin{pmatrix} +e^{-i\eta} & 0 \\ 0 & -e^{+i\eta}\end{pmatrix}\begin{pmatrix} e^{+i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix} = \begin{pmatrix} +1 & 0 \\ 0 & -1\end{pmatrix} = \sigma_3$$

In general

$$\begin{pmatrix} e^{i\eta_1} & 0 \\ 0 & e^{-i\eta_1}\end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} e^{i\eta_2} & 0 \\ 0 & e^{-i\eta_2}\end{pmatrix}$$

$$= \begin{pmatrix} a_{11}e^{i\eta_1} & a_{12}e^{i\eta_1} \\ & \end{pmatrix}\begin{pmatrix} e^{i\eta_2} & 0 \\ & \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} e^{i\eta_1} & a_{12} e^{i\eta_1} \\ a_{21} e^{-i\eta_1} & a_{22} e^{-i\eta_1} \end{pmatrix} \begin{pmatrix} e^{i\eta_2} & 0 \\ 0 & e^{-i\eta_2} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} e^{i(\eta_1+\eta_2)} & a_{12} e^{i(\eta_1-\eta_2)} \\ a_{21} e^{-i(\eta_1-\eta_2)} & a_{22} e^{-i(\eta_1+\eta_2)} \end{pmatrix}$$

If $-\eta_2 = +\eta_1$, then it becomes → Hermitian conjugate matrices

$$\begin{pmatrix} a_{11} & a_{12} e^{-2i\eta_2} \\ a_{21} e^{+2i\eta_2} & a_{22} \end{pmatrix}$$

So

$$\underbrace{\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{+i\eta} \end{pmatrix}}_{T} \underbrace{\sigma_1}_{\sigma_1} \underbrace{\begin{pmatrix} e^{+i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix}}_{T^\dagger} = \begin{pmatrix} 0 & e^{-2i\eta} \\ e^{+2i\eta} & 0 \end{pmatrix}$$

$$i \begin{pmatrix} -i\frac{d\eta}{dx} & 0 \\ 0 & i\frac{d\eta}{dx} \end{pmatrix} \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix} + i \frac{d}{dx} \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix} = \left(-\frac{\omega_m}{2} + \frac{\delta\lambda}{2} \cos 2\theta_m \right) \sigma_3$$

$$- \frac{\delta\lambda}{2} \sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta} \\ e^{-2i\eta} & 0 \end{pmatrix} \begin{pmatrix} \psi_{01} \\ \psi_{02} \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Remove σ_3 term requires

$$i \begin{pmatrix} -i\frac{d\eta}{dx} & 0 \\ 0 & i\frac{d\eta}{dx} \end{pmatrix} = \left(-\frac{\omega_m}{2} + \frac{\delta\lambda}{2} \cos 2\theta_m \right) \sigma_3$$

$$\frac{d\eta}{dx} = -\frac{\omega_m}{2} + \frac{\delta\lambda}{2} \cos 2\theta_m$$

The solution is

$$\int_0^{\eta(x)} d\eta(\tau) = \int_0^x \left(-\frac{\omega_m}{2} \right) d\tau + \int_0^x \left(+\frac{\delta\lambda(\tau)}{2} \cos 2\theta_m \right) d\tau$$

$$\eta(x) - \eta(0) = -\frac{\omega_m}{2} x + \frac{\cos 2\theta_m}{2} \int_0^x \delta\lambda \, d\tau$$

Choose $\eta(0) = 0$

$$\eta(x) = -\frac{\omega_m}{2} x + \frac{\cos 2\theta_m}{2} \int_0^x \delta\lambda(\tau) \, d\tau$$

or I can choose other initial condition

* For Single frequency

$$\delta\lambda(x) = A\sin(kx+\phi)$$

$$\int_0^x \delta\lambda(\tau)\,d\tau = A\int_0^x \sin(k\tau+\phi)\,d\tau$$

$$= -A\frac{1}{k}\left(\cos(kx+\phi) - \cos(\phi)\right)$$

I use $\eta(0)$ to cancel the constant

$$\eta(0) = \frac{\cos 2\theta_m}{2}\frac{A}{k}(+\cos\phi)$$

$$\eta(x) = -\frac{\omega_m}{2}x + \frac{\cos 2\theta_m}{2}\cdot\frac{A}{k}\cos(kx+\phi)$$

Now the equation of motion becomes

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} \cong \frac{\delta\lambda}{2}\sin 2\theta_m\begin{pmatrix}0 & e^{2i\eta}\\ e^{-2i\eta} & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \frac{\sin 2\theta_m}{2}\begin{pmatrix}0 & \delta\lambda(x)\,e^{2i\eta(x)}\\ \delta\lambda\, e^{-2i\eta(x)} & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

When it comes to the wavefunction in background matter basis, it should be

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$= \begin{pmatrix}e^{-i\eta(x)}\psi_{b1}\\ e^{i\eta(x)}\psi_{b2}\end{pmatrix}$$

Given initial

$$\begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}\psi_{b1}(0)\\ \psi_{b2}(0)\end{pmatrix}$$

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix}e^{-i\eta(x)}\psi_{b1}(x)\\ e^{i\eta(x)}\psi_{b2}(x)\end{pmatrix} \Rightarrow$$

The probability from $\psi_1 \to \psi_2$

$$|\psi_{b2}(x)|^2$$

So we solve

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b_1}\\ \psi_{b_2}\end{pmatrix} = \frac{\sin 2\theta_m}{2}\begin{pmatrix} 0 & \delta\lambda(x)\, e^{2i\eta(x)} \\ \delta\lambda\, e^{-2i\eta(x)} & 0\end{pmatrix}\begin{pmatrix}\psi_{b_1}\\ \psi_{b_2}\end{pmatrix}$$

$$\frac{\sin 2\theta_m}{2}\lambda\, e^{2i\eta(x)} = \frac{\sin 2\theta_m}{2} A \sin(kx+\phi)\, e^{2i\left(-\frac{\omega_m}{2}x \overset{-}{+} \frac{\cos 2\theta_m}{2}\frac{A}{k}\cos(kx+\phi)\right)}$$

$$= \frac{\sin 2\theta_m}{2} A \sin(kx+\phi)\, e^{-i\omega_m x}\, e^{-i\cos 2\theta_m \frac{A}{k}\cos(kx+\phi)}$$

Jacobi - Anger expansion

$$e^{-i\frac{\cos 2\theta_m A}{k}\cos(kx+\phi)} \qquad z_k = \frac{\cos 2\theta_m A}{k}$$

$$= e^{i z_k \cos(kx+\phi)}$$

$$= \sum_{n=-\infty}^{\infty} i^n J_n(-z_k)\, e^{in(kx+\phi)} \qquad J_n(-z_m) = (-1)^n J_n(z_m)$$

$$= \frac{\sin 2\theta_m}{2} A \sin(kx+\phi)\, e^{-i\omega_m x}\sum_{n=-\infty}^{\infty} i^n J_n(z_k)\, e^{in(kx+\phi)}\,(-1)^n$$

$$\sin(kx+\phi) = \frac{1}{2i}\left(e^{i(kx+\phi)} - e^{-i(kx+\phi)}\right)$$

$$= \frac{A\sin 2\theta_m}{2\cdot 2i}\left[\sum_{n=-\infty}^{\infty} i^n J_n(z_k)\, e^{i(kx+\phi-\omega_m x+n(kx+\phi))}\,(-1)^n - \sum_{n=-\infty}^{\infty} i^n J_n(z_k)\, e^{i(-kx-\phi-\omega_m x+n(kx+\phi))}\,(-1)^n\right]$$

which terms are going to make a difference

$$= \frac{A\sin 2\theta_m}{4i}\left[\sum_{n=-\infty}^{\infty} e^{i(n+1)\phi}\, i^n J_n(z_k)\, e^{i((n+1)k-\omega_m)x}\,(-1)^n - \sum_{n=-\infty}^{\infty} e^{i(n-1)\phi}\, i^n J_n(z_k)\, e^{i((n-1)k-\omega_m)x}\,(-1)^n\right]$$

whenever $e^{i((n+1)k-\omega_m)x}$ and $e^{i((n-1)k-\omega_m)x}$ are almost 1

a resonance happens solve

$$n_- = \mathrm{Int}\left(\frac{\omega_m}{k}\right) - 1 \quad \text{and} \quad n_+ = \mathrm{Int}\left(\frac{\omega_m}{k}\right) + 1$$

for each summation $\rightarrow$ define $\mathrm{Int}\left(\frac{\omega_m}{k}\right) = n_0$

$$n_- = n_0 - 1, \quad n_+ = n_0 + 1$$

$$\frac{A\sin 2\theta_m}{\ldots}\left[\ldots e^{i n_0\phi}\ldots e^{i(n_0 k-\omega_m)x}\ldots\right]$$

$$n_- = n_0 - 1 \quad , \quad n_+ = n_0 + 1$$

$$h_{12} = \frac{A\sin 2\theta_m}{4i}\Big[e^{i n_0 \phi}\, i^{n_0-1} J_{n_0-1}(z_k)\, e^{i(n_0 k - \omega_m)x}\,(-1)^{n_0-1}$$
$$- e^{i n_0 \phi}\, i^{n_0+1} J_{n_0+1}(z_k)\, e^{i(n_0 k - \omega_m)x}\,(-1)^{n_0+1}\Big]$$
$$= \frac{A\sin 2\theta_m}{4i} e^{i n_0 \phi}\Big((-i)^{n_0-1} J_{n_0-1}(z_k) - (-i)^{n_0+1} J_{n_0+1}(z_k)\Big) e^{i(n_0 k - \omega_m)x}$$
$$= \frac{A\sin 2\theta_m}{4i} e^{i n_0 \phi}\,(-i)^{n_0-1}\underbrace{\Big(J_{n_0-1}(z_k) + J_{n_0+1}(z_k)\Big)}_{\Downarrow} e^{i(n_0 k - \omega_m)x}$$

$$J_{n_0-1}(z_k) + J_{n_0+1}(z_k) = \frac{2n+1}{z} J_{n_0}(z_k)$$

$$= \underbrace{\frac{A\sin 2\theta_m}{4} e^{i n_0 \phi}\,(-i)^{n_0}\,\frac{2n_0+1}{z} J_{n_0}(z_k)}_{F(n_0, z_k)}\, e^{i(n_0 k - \omega_m)x}$$

$$= F\, e^{i(n_0 k - \omega_m)x}$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix} 0 & F e^{i(n_0 k - \omega_m)x} \\ F^* e^{-i(n_0 k - \omega_m)x} & 0 \end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

General solution

$$\psi_{b1} = C_1 e^{\frac{1}{2} i \left(n_0 k + i\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} - \omega_m\right)x}$$
$$+ C_2 e^{\frac{1}{2}\left(\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} + i(k n_0 - \omega_m)\right)x}$$

$$\psi_{b2} = \frac{C_1}{2F^*}\, i\, e^{-i(n_0 k - \omega_m)x + \frac{1}{2} i\left(n_0 k + i\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} - \omega_m\right)x}$$

$$\left(n_0 k + i\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} - \omega_m\right)$$
$$+ C_2 \frac{1}{2F^*} e^{\frac{1}{2}\left(\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} + i(n_0 k - \omega_m)\right)x - i(n_0 k - \omega_m)x}$$
$$\cdot \left(\sqrt{-4|F|^2 - (n_0 k - \omega_m)^2} + i(n_0 k - \omega_m)\right)$$

Initial condition :

$$\begin{pmatrix}\psi_{b1}(0)\\ \psi_{b2}(0)\end{pmatrix} = \begin{pmatrix} e^{i\eta(0)} \\ 0 \end{pmatrix} \Leftarrow \begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)}\psi_{b1}(x) \\ \psi_{b2}(x)\end{pmatrix}$$

$$\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \begin{pmatrix} -i\eta(x)\,\psi_{b_1}(x) \\ e^{i\eta(x)}\,\psi_{b_2}(x)\end{pmatrix}$$

$$\eta(0) = \frac{\cos 2\theta_m}{2}\, \Delta_k \cos\phi = \frac{z_k}{2}\cos\phi$$

So the constants can be solved:

$$\begin{cases} C_1 + C_2 = e^{i\frac{z_k}{2}\cos\phi} & ① \\ \frac{C_1}{2F^*}\, i\left(n_0 k + i\sqrt{-4|F|^2 - (n_0 k - w_m)^2} - w_m\right) \\ + \frac{C_2}{2F^*}\cdot\left(\sqrt{-4|F|^2 - (n_0 k - w_m)^2} + i(n_0 k - w_m)\right) = 0 & ② \end{cases}$$

$$② : \quad C_1\left(i(n_0 k - w_m) - \sqrt{-4|F|^2 - (n_0 k - w_m)^2}\right) + C_2\left(i(n_0 k - w_m) + \sqrt{}\right) = 0$$

$$C_2 = e^{i\frac{z_k}{2}\cos\phi} - C_1$$

$$C_1\left(i(n_0 k - w_m) - \sqrt{}\right) + e^{i\frac{z_k}{2}\cos\phi}\left(i() + \sqrt{}\right) - C_1\left(i() + \sqrt{}\right) = 0$$

$$+2C_1\sqrt{} \quad ④ = e^{i\frac{z_k}{2}\cos\phi}\left(i() + \sqrt{}\right)$$

$$C_1 = \frac{e^{i\frac{z_k}{2}\cos\phi}\left(i() + \sqrt{}\right)}{2\sqrt{}}$$

Define $\quad l = 4|F|^2 + (n_0 k - w_m) = 4|F|^2 + g^2$

$\quad\quad\quad\quad g = n_0 k - w_m$

$$C_1 = \frac{e^{i\frac{z_k}{2}\cos\phi}\left(ig + i\sqrt{l}\right)}{i2\sqrt{l}} = e^{i\frac{z_k}{2}\cos\phi}\,\frac{g + \sqrt{l}}{2\sqrt{l}}$$

$$i2\sqrt{L} \qquad -e \qquad 2\sqrt{L}$$

$$C_2 = e^{i\frac{z_k}{2}\cos\phi} - C_1$$

$$= e^{i\frac{z_k}{2}\cos\phi}\left(\frac{2i\sqrt{L} - ig - i\sqrt{L}}{2i\sqrt{L}}\right)$$

$$= e^{i\frac{z_k}{2}\cos\phi}\,\frac{i\sqrt{L} - ig}{2i\sqrt{L}} = e^{i\frac{z_k}{2}\cos\phi}\,\frac{\sqrt{L} - g}{2\sqrt{L}}$$

$$\psi_{bz} = \frac{C_1}{2F^*}\, i\, e^{-i(n_0k - \omega_m)x + \frac{1}{2}i\left(n_0k + i\sqrt{-4|F|^2 - (n_0k - \omega_m)^2} - \omega_m\right)x}$$

$$\left(n_0k + i\sqrt{-4|F|^2 - (n_0k - \omega_m)^2} - \omega_m\right)$$

$$+ C_2\,\frac{1}{2F^*}\, e^{\frac{1}{2}\left(\sqrt{-4|F|^2 - (n_0k - \omega_m)^2} + i(n_0k - \omega_m)\right)x - i(n_0k - \omega_m)x}$$

$$\cdot\left(\sqrt{-4|F|^2 - (n_0k - \omega_m)^2} + i(n_0k - \omega_m)\right)$$

$$= \frac{1}{2F^*}\Big[e^{i\frac{z_k}{2}\cos\phi}\,\frac{g + \sqrt{L}}{2\sqrt{L}}\, i e^{-igx + \frac{1}{2}(ig - i\sqrt{L})x}$$

$$(g - \sqrt{L}) + e^{i\frac{z_k}{2}\cos\phi}\,\frac{g + \sqrt{L}}{2\sqrt{L}}\, e^{\frac{1}{2}(i\sqrt{L} + ig)x - igx}$$

$$\cdot\left(i\sqrt{L} + ig\right)\Big]$$

$$= \frac{e^{i\frac{z_k}{2}\cos\phi}}{2F^*}\,\frac{1}{2\sqrt{L}}\Big(i(g + \sqrt{L})(g - \sqrt{L})\, e^{-\frac{i}{2}gx - \frac{1}{2}i\sqrt{L}x}$$

$$+ i(\sqrt{L} - g)(\sqrt{L} + g)\, e^{-\frac{i}{2}gx + \frac{i}{2}\sqrt{L}x}\Big)$$

$$= \frac{e^{i\frac{z_k}{2}\cos\phi}}{4\sqrt{L}F^*}\, i\,(g^2 - L)\, e^{-\frac{i}{2}gx}\left[e^{-\frac{1}{2}i\sqrt{L}x} - e^{\frac{1}{2}i\sqrt{L}x}\right]$$

$$g = n_0 k - \omega_m, \qquad L = 4|F|^2 + g^2$$

$$\hookrightarrow -4|F|^2$$

$$2n_0+1 = n$$

$$F = \frac{A\sin 2\theta_m}{4}\, e^{in_0\phi}\, (-i)^{n_0}\, \frac{2n_0+1}{z}\, J_{n_0}(z_k)$$

$$i^{n_0}(i^{n_0})^* \left| \begin{array}{l} i\cdot(-i) = 1 \\ 1\cdot 1 = 1 \end{array} \right.$$

$$|F|^2 = F F^* = \left(\frac{A\sin 2\theta_m}{4}\right)^2 \frac{(2n_0+1)^2}{z_k^2}\, J_{n_0}(z_k)\, J_{n_0}(z_k)^*$$

$$z_k \text{ real} \rightarrow J_{n_0}(z_k) \text{ real}$$

$$\cancel{e^{-is} = e^{-is}}$$

$$\psi_{bz} = \frac{e^{i\frac{z_k}{2}\cos\phi}}{4\sqrt{L}\, F^*}\, i\,(-4|F|^2)\, e^{-\frac{i}{2}gx}\left[-2i\sin\left(\frac{1}{2}\sqrt{L}\,x\right)\right]$$

$$= \frac{1}{\sqrt{L}}\, e^{i\frac{z_k}{2}\cos\phi}\, e^{-\frac{i}{2}gx}(-F)\, 2\sin\left(\frac{1}{2}\sqrt{L}\,x\right)$$

$$P = |\psi_{bz}|^2 = \frac{|F|^2}{L}\, 4\sin^2\left(\frac{\sqrt{L}}{2}x\right)$$

* Compare with Kneller's result

Kneller et al ~~[illegible]~~ have

$$P_{12} = \frac{k_n^2}{q_n^2}\sin^2(q_n x) \qquad \omega_m = \delta k_{21}$$

matter perturbation

$$q_n^2 = k_n^2 + \kappa_n^2, \quad 2k_n = \delta k_{21} + \Delta k_* = n_0 k - \omega_m$$

$$= g$$

$$|F|^2 = |\kappa_n|^2$$

$$q_n^2 = \frac{g^2}{4} + |F|^2 = \frac{1}{4}L$$

$$q_n = \frac{\nu}{2}$$

So my result is the same as Kimleng's

* Understand the physics

-[ ] The meaning of the J-A expansion

What if $\delta\lambda$ is constant?

is it $e^{in\delta x}$ form?

The effective Hamiltonian would be

$$H_e = \frac{\sin 2\theta_m}{2}\delta\lambda(x)\begin{pmatrix} 0 & e^{2i\Delta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix}$$

$$\eta(x) = -\frac{W_m}{2}x + \frac{\cos 2\theta_m}{2}\int_0^x \delta\lambda(\bar{x})d\bar{x}$$

$$= -\frac{W_m}{2}x + \frac{\cos 2\theta_m}{2}\delta\lambda\, x$$

choose $\eta(0)=0$

$$\frac{\sin 2\theta_m}{2}\delta\lambda\, e^{2i\left(-\frac{W_m}{2} + \frac{\cos 2\theta_m}{2}\delta\lambda\right)x}$$

This is the result of constant matter potential

So J-A expansion indeed expand the period matter potential to constant potential

$$e^{iz_k\cos(kx+\phi)}$$

$$= \sum_{n=-\infty}^{\infty} i^n J_n(z_k)\, e^{in(kx+\phi)}$$

like Fourier modes

$$e^{in(kx+\phi)} = e^{in\phi}\underbrace{e^{inkx}}_{\text{is a term of constant matter potential}}$$

$$inkx \sim i\cos 2\theta_m \delta\lambda\, x$$
$$\frac{nk}{\cos 2\theta_m} \sim \delta\lambda$$

[ ] What is the picture / meaning of this transformation?

$$\begin{pmatrix} e^{-in\omega x} & 0 \\ 0 & e^{in\omega x} \end{pmatrix}$$

adds a different phase to each element the wave function

First of all what does each term do in Hamiltonian?

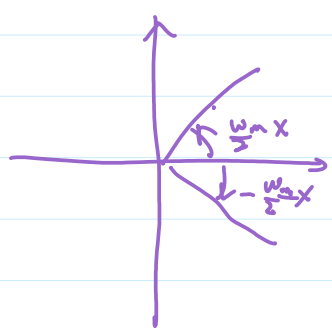
$$H = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda}{2}\cos 2\theta_m\, \sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m \sigma_1$$

* $-\frac{\omega_m}{2}\sigma_3$ is the linear term and provides harmonic oscillatory solutions

$$i\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = -\frac{\omega_m}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\begin{cases} i\frac{\partial}{\partial x}\psi_1 = -\frac{\omega_m}{2}\psi_1 \\ i\frac{\partial}{\partial x}\psi_2 = \frac{\omega_m}{2}\psi_2 \end{cases} \Rightarrow \begin{cases} \frac{d\psi_1}{\psi_1} = i\frac{\omega_m}{2}dx \\ \frac{d\psi_2}{d\psi_2} = -i\frac{\omega_m}{2}dx \end{cases}$$

$$\psi_1 \sim e^{i\frac{\omega_m}{2}x}$$
$$\psi_2 \sim e^{-i\frac{\omega_m}{2}x}$$



*The $\frac{\delta\lambda}{2}\cos 2\theta_m \sigma_3$ term

$$i\frac{d}{dx}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{\cos 2\theta_m}{2}\begin{pmatrix}\delta\lambda & 0\\ 0 & -\delta\lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\delta\lambda = A\sin(kx+\phi)$$

$$\begin{cases} i\frac{d}{dx}\psi_1 = \frac{A\cos 2\theta_m}{2}\sin(kx+\phi)\,\psi_1 \\ i\frac{d}{dx}\psi_2 = -\frac{A\cos 2\theta_m}{2}\sin(kx+\phi)\,\psi_2 \end{cases}$$

$$-i\,\frac{A\cos 2\theta_m}{\ }\left(-\cos(kx+\phi)\right)$$

$$\begin{cases} i\frac{d}{dx}\psi_1 = -\frac{A\cos 2\theta_m}{2}\sin(kx+\phi)\,\psi_2 \\ \psi_1 \sim e^{-i\frac{A\cos 2\theta_m}{2}\left(-\frac{\cos(kx+\phi)}{k}\right)} \end{cases}$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \frac{\delta\lambda\sin 2\theta_m}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\begin{cases} i\frac{d}{dx}\psi_1 = \frac{\sin 2\theta_m}{2}\delta\lambda\,\psi_2 \\ i\frac{d}{dx}\psi_2 = \frac{\sin 2\theta_m}{2}\delta\lambda\,\psi_1 \end{cases}$$

$$\frac{2i}{\sin 2\theta_m}\frac{1}{\delta\lambda}\frac{d}{dx}\psi_1 = \psi_2$$

$$\frac{2i}{\sin 2\theta_m}\frac{d}{dx}\left(\frac{1}{\delta\lambda}\right)\frac{d}{dx}\psi_1 + \frac{2i}{\sin 2\theta_m}\frac{1}{\delta\lambda}\frac{d^2}{dx^2}\psi_1 = \frac{d}{dx}\psi_2$$

$$\frac{2i}{\sin 2\theta_m}\frac{-1}{(\delta\lambda)^2}\frac{d}{dx}\delta\lambda\frac{d}{dx}\psi_1 + \frac{2i}{\sin 2\theta_m}\frac{1}{\delta\lambda}\frac{d^2}{dx^2}\psi_1 = -i\frac{\sin 2\theta_m}{2}\delta\lambda\,\psi_1$$

$$\frac{d^2}{dx^2}\psi_1 - \frac{\frac{d}{dx}\delta\lambda}{\delta\lambda}\frac{d}{dx}\psi_1 = -\frac{\sin 2\theta_m}{2i}\delta\lambda\cdot\frac{i\sin 2\theta_m}{2}\delta\lambda\,\psi_1$$

$$\frac{d^2}{dx^2}\psi_1 - \frac{\delta\lambda'}{\delta\lambda}\frac{d}{dx}\psi_1 = -\frac{\sin^2 2\theta_m}{4}\delta\lambda^2\,\psi_1$$

$$\psi_1 \approx C_1\cos\left(\frac{A\cos(kx+\phi)\sin 2\theta_m}{2k}\right) - C_2\sin\left(\frac{a\cos(kx+\phi)\sin 2\theta_m}{2k}\right)$$
$$= C_1' e^{i\frac{A}{2k}\sin 2\theta_m\cos(kx+\phi)} + C_2' e^{-i\frac{A}{2k}\sin 2\theta_m\cos(kx+\phi)}$$

So when I remove the σ_3 terms, I remove the simple harmonic oscillatory term $-\frac{\omega_m}{2}\sigma_3$ ~~an~~ which means ~~the~~ the reference frame is co-oscillating with the system and also the $\frac{\delta\lambda}{2}\cos 2\theta_m\,\sigma_3$ term, which means

Understand stimulated

Monday, February 8, 2016 1:23 PM

* The resonance shape for
1. "parametric" resonance!
2. the result we had!

1. "parametric" constants $\delta\lambda$ in our case

$\delta\lambda = \lambda_c \Rightarrow$ solution to

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix}0 & h\\ h^* & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$h = \frac{\sin 2\theta_m}{2}\lambda_c e^{i\cos 2\theta_m \lambda_c x - i\frac{\omega_m}{v}x}$$

$$\psi_{b2} = -e^{\frac{1}{2}i\lambda_c(-1+\cos 2\theta_m)x - i\lambda_c x\cos 2\theta_m}\, C_1(-1+\cos 2\theta_m)$$

$$\csc(2\theta_m) - C_2 e^{-i\lambda_c x\cos 2\theta_m + \frac{1}{2}i\lambda_c(\cos 2\theta_m + 1)x}$$

$$C_2\left(1+\cos 2\theta_m \csc(2\theta_m)\right)$$

$$\csc 2\theta_m = \frac{1}{\sin 2\theta_m}$$

$$\eta(x) = \frac{\cos 2\theta_m}{2}\lambda_c x$$

Initial condition:

$$\begin{pmatrix}\psi_{b1}(0)\\ \psi_{b2}(0)\end{pmatrix} = \begin{pmatrix}e^{i\eta(0)}\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$C_1(\cot 2\theta_m - 1) - C_2(\cot 2\theta_m + 1) = 0$$

$$C_1 = C_2\frac{\cot 2\theta_m + 1}{\cot 2\theta_m - 1} = C_2\frac{\cos 2\theta_m + \sin 2\theta_m}{\cos 2\theta_m - \sin 2\theta_m}$$

$$|\psi_{b2}|^2 = |G|^2$$

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

h now is

$$h = \frac{\sin 2\theta_m}{2}\lambda_c$$

2. How to describe in this problem

Amplitude is $$\frac{\Phi|F|^2}{g^2} = \frac{\Phi\left|\frac{1}{4}A\sin 2\theta_m \cancel{e^{i\phi}(-1)^{n_b}} \frac{2n_b+1}{z_k} J_{n_b}(z_b)\right|^2}{4|F|^2+g^2}$$

$$g = n_0 k - \omega_m$$

$$= \frac{\Phi|F|^2}{4|F|^2+g^2}$$

$g^2 = 0 \rightarrow$ resonance

Amplitude as a function of what? k?

$$\delta x = A\sin(kx+\phi)$$

$$z_b = \frac{D}{k}(\omega^2)_m$$

Make dimensionless quantities

$$n_0 = J_{n_b}\left(\frac{\omega_m}{\ldots}\right)$$

Scale everything with ω_m (background)

$$\hat{k} = \frac{k}{\omega_m} \qquad \Rightarrow \qquad n_0 = I_m\left(\frac{1}{\hat{k}}\right)$$

$$\frac{g}{\omega_m} = \hat{g} = n_0 \hat{k} - 1 \quad , \quad z_k = \frac{A}{k}\cos 2\theta_m = \frac{1}{\omega_m}\frac{A}{k}\cos 2\theta_m$$

$$= \frac{\hat{A}}{\hat{k}}\cos 2\theta_m$$

$$|F|^2 = \left|\frac{1}{4} A \sin 2\theta_m \frac{2n_0+1}{z_k} J_n(z_k)\right|^2$$

$$= \left|\frac{1}{4} A \sin 2\theta_m \frac{2n_0+1}{z_k} J_n(z_k)\right|^2$$

$$\frac{|F|^2}{\omega_m^2} = \left|\frac{1}{4}\frac{A}{\omega_m}\sin 2\theta_m \frac{2n_0+1}{z_k} J_n(z_k)\right|^2 = \left|\frac{1}{4}\frac{\hat{A}\sin 2\theta_m}{z_k}(2n_0+1) J_n(z_k)\right|^2$$

Amplitude is

$$\frac{4|F|^2}{4|F|^2 + g^2} = \frac{4\frac{|F|^2}{\omega_m^2}}{\frac{4|F|^2}{\omega_m^2} + \hat{g}^2} = \frac{4\left|\frac{1}{4}\frac{\hat{A}}{z_k}\sin 2\theta_m (2n_0+1) J_n(z_k)\right|^2}{4\left|\frac{1}{4}\frac{\hat{A}}{z_k}\sin 2\theta_m (2n_0+1) J_n(z_k)\right|^2 + \hat{g}^2}$$

Two frequencies

Sunday, February 14, 2016 10:19 AM

$$\delta\lambda = A_1 \sin(k_1 x + \phi_1) + A_2 \sin(k_2 \phi + \phi_2)$$

$$y = -\frac{\omega_m}{\Sigma} x + \frac{\cos\theta_m}{2} \int_0^x \delta\lambda(x)\, dx + \eta(0)$$

$$= -\frac{\omega_m}{\Sigma} x + \frac{\cos\theta_m}{2} \underbrace{\int_0^x \left(A_1 \sin(k_1 x + \phi_1) + A_2 \sin(k_2 x + \phi_2) \right) dx}_{-\frac{A_1}{k_1}\cos(k_1 x + \phi_1) - \frac{A_2}{k_2}\cos(k_2 x + \phi_2) + \underbrace{\cdots}} + \eta(0)$$

choose $\eta(0)$ to cancel the constants

$$y = -\frac{\omega_m}{\Sigma} x - \sum_n \frac{A_n}{k_n} \cos(k_n x + \phi_n)$$

$$h = \frac{\sin\theta_m}{2} \delta\lambda(x)\, e^{2i\eta(x)}$$

$$= \frac{\sin\theta_m}{2} \sum A_n \sin(k_n x + \phi_n)\; e^{-i\omega_m x - i\cos\theta_m \sum_n \frac{A_n}{k_n}\cos(k_n x + \phi_n)}$$

$$= \frac{\sin\theta_m}{2} \sum_n A_n \sin(k_n + \phi_n)\; e^{-i\omega_m x}\, \underbrace{e^{-\sum_n i\cos\theta_m \frac{A_n}{k_n}\cos(k_n x + \phi_n)}}_{\underbrace{e^{-\sum_n i z_{k_n}\cos(k_n x + \phi_n)}}}$$

$$z_{k_n} = \frac{A_n \cos\theta_m}{k_n}$$

Jacobi - Anger:

$$e^{iz\cos(\phi)} = \sum_{n=-\infty}^{\infty} i^n J_n(z)\, e^{in\phi}$$

$$\prod_a e^{-iz_{k_a}\cos(k_a x + \phi_a)} = \prod_a \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_a})\, e^{in(k_a + \phi_a)}$$

$$h = \frac{\sin\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n)\, e^{-i\omega_m x} \prod_a \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_a})\, e^{in(k_a x + \phi_a)}$$

If I can not rewrite Π to Σ, I have to solve ~~a=1~~ two frequency first.

* Using 2 frequencies

$$\delta\lambda = A_1 \sin(k_1 x + \phi_1) + A_2 \sin(k_2 x + \phi_2)$$

$$h = \frac{\sin\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n)\, e^{-i\omega_m x} \underbrace{\sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{in(k_1 x + \phi_1)} \sum_{m=-\infty} (-i)^m J_m(z_{k_2}) e^{im(k_2 x + \phi_2)}}$$

$$\sum a_n \sum b_m$$

$$= \sum_{N=-\infty}^{\infty} \sum_{m+n=N} a_n b_m$$

$$= \sum_{N=-\infty}^{\infty} \sum_{n=-\infty}^{N} a_n b_{N-n}$$

So

$$h = \frac{\sin\theta_m}{2}\sum_{a=1,2} A_a \sin(k_a x + \phi_a)\, e^{-i\omega_m x} \sum_{N=-\infty}^{\infty}\sum_{n=-\infty}^{N} (-i)^n J_n(z_{k_1}) e^{in(k_1 x+\phi_1)}\, (-i)^{N-n} J_{N-n}(z_{k_2})\, e^{i(N-n)(k_2 x+\phi_2)}$$

$$= \cdots \quad - \quad e^{-i\omega_m x}\sum_{N=-\infty}^{\infty}\sum_{n=-\infty}^{N} (-i)^N J_n(z_{k_1}) J_{N-n}(z_{k_2})\, e^{in((k_1-k_2)x+\phi_1-\phi_2)+iN(k_2 x+\phi_2)}$$

The resonance condition is

$$n(k_1-k_2) - \omega_m + N k_2 \approx 0$$

n_{0N} which makes it small.

Then

$$h \approx \frac{\sin\theta_m}{2}\sum_{a=1,2} A_a \sin(k_a x+\phi_a)\, e^{-i\omega_m t}\sum_{N=-\infty}^{\infty} (-i)^N J_{n_{0N}}(z_{k_1}) J_{N-n_{0N}}(z_{k_2})\, e^{i n_{0N}(\phi_1-\phi_2)+iN\phi_2}$$

Oh wait! Kelly doesn't apply this! She just use the conditions to satisfies each

I need to expand $A_a \sin(k_a x+\phi_a)$

$$A_1 \sin(k_1 x+\phi_1) + A_2 \sin(k_2 x+\phi_2)$$
$$= A_1 \frac{1}{2i}\left(e^{i(k_1 x+\phi_1)} - e^{-i(k_1 x+\phi_1)}\right)$$
$$+ \cdots$$

h becomes ~~four~~ two parts

$$h_1 = \frac{\sin\theta_m}{2}\frac{A_1}{2i}\Big[\sum_{N=-\infty}^{\infty}\sum_{n=-\infty}^{N} (-i)^N J_n(z_{k_1}) J_{N-n}(z_{k_2})\, e^{in((k_1-k_2)x+\phi_1-\phi_2)+iN(k_2 x+\phi_2)-i\omega_m x+i(k_1 x+\phi_1)}$$
$$- \sum_{N=-\infty}^{\infty}\sum_{n=-\infty}^{N} (-i)^N J_n(z_{k_1}) J_{m-n}(z_{k_2})\, e^{i(n-1)(k_1 x+\phi_1)+i(N-n)(k_2 x+\phi_2)-i\omega_m x}\Big]$$

$$h_2 = \frac{\sin\theta_m}{2}\frac{A_2}{2i}\Big[\cdots\cdots\cdots\cdots\cdots\; e^{in(k_1 x+\phi_1)+i(N-n+1)(k_2 x+\phi_2)-i\omega_m x}$$
$$- \Sigma\Big(\cdots\cdots\cdots\cdots\cdots\; e^{in(k_1 x+\phi_1)+i(N-n-1)(k_2 x+\phi_2)-i\omega_m x}\Big]$$

$h = h_1 + h_2$

We require each $e^{i(\;)x} \sim 1$ for the resonance to happen

$$\begin{cases} i(n+1)(k_1 x\;) + i(N-n)(k_2 x\;) - i\omega_m x \sim 0 \Rightarrow n_{11}(N) \\ i(n-1)k_1 x + i(N-n)k_2 x - i\omega_m x \sim 0 \Rightarrow n_{12}(N) \\ i n k_1 x + i(N-n+1)k_2 x - i\omega_m x \sim 0 \Rightarrow n_{21}(N) \\ i n k_1 x + i(N-n-1)k_2 x - i\omega_m x \sim 0 \Rightarrow n_{22}(N) \end{cases}$$

$$n(k_1-k_2) + k_1 + Nk_2 - \omega_m \approx 0 \Rightarrow n \approx \frac{\omega_m - k_1 - Nk_2}{k_1 - k_2}$$

$$(n+1)k_1 + (N-n)k_2 - \omega_m \approx 0$$

$n \le N$

$\omega_m - k_1 - Nk_2 \le (k_1 - k_2)N$

$\omega_m - k_1 \le N k_1$

these n_{11}'s depend on N, but each n_{ij} has a upper limit

What's left

$$h_1 = \frac{\sin\theta_m}{2}\frac{A_1}{2i}\Big[\sum_{N=-\infty}^{\infty} (-i)^N J_{n_{11}}(z_{k_1}) J_{N-n_{11}}(z_{k_2})\, e^{i(n_{11}+1)\phi_1+i(N-n_{11})\phi_2}$$
$$- \sum_{N=-\infty} (-i)^N J_{n_{12}}(z_{k_1}) J_{N-n_{12}}(z_{k_2})\, e^{i(n_{12}-1)\phi_1+i(N-n_{12})\phi_2}\Big]$$

So we have a cut-off for N sum

$$h_1 = \frac{\cancel{\text{[illegible]}}}{2}\frac{1}{2i}\Big[\sum_{N=-\infty}^{\infty} (-1)^N J_{n_{11}}(z_{k_1}) J_{N-n_{11}}(z_{k_2}) e^{i(n_{11}-1)\phi_1 + i(N-n_{11})\phi_2}$$
$$- \sum_{N=-\infty}^{\infty} (-1)^N J_{n_{12}}(z_{k_1}) J_{N-n_{12}}(z_{k_2}) e^{i(n_{12}-1)\phi_1 + i(N-n_{12})\phi_2}\Big]$$

$$h_2 = \frac{\cancel{\text{[illegible]}}}{2i}\frac{A_2}{2i}\Big[\sum_{N=-\infty}^{\infty} (-1)^N J_{n_{21}}(z_{k_1}) J_{N-n_{21}}(z_{k_2}) e^{i n_{21}\phi_1 + i(N-n_{21}+1)\phi_2}$$
$$- \sum_{N=-\infty}^{\infty} (-1)^N J_{n_{22}}(z_{k_1}) J_{N-n_{22}}(z_{k_2}) e^{i n_{22}\phi_1 + i(N-n_{22}-1)\phi_2}\Big]$$

for N sum

define it as N_0

$$\sum_{N=N_0}^{\infty}$$

In principle, these are constants that doesn't depend on X
I would like to simplify them. but I can't

Anyway h is a constant now. even though
I can't calculate them directly

$\boxed{\text{This is the most exact method}}$

However, like what Kelly did in her paper.

I could keep the multiplication and require a
different resonance condition.

$$n_{11,N} k_1 + N k_2 - n_{11,N} k_2 \sim \omega_m$$
$$n_{11,N}(k_1 - k_2) + N k_2 + k_1 \sim \omega_m$$
$$n_{11,N} \sim \frac{\omega_m - N k_2 - k_1}{k_1 - k_2}$$
$$n_{11,N} \leq N \Rightarrow$$

$$n_{12,N}(k_1 - k_2) \sim \omega_m - N k_2 + k_1$$
$$n_{12,N} \sim \frac{\omega_m - N k_2 + k_1}{k_1 - k_2}$$

$$n_{21,N} k_1 + (N+1) k_2 - n_{21,N} k_2 \sim \omega_m$$

$$n_{21,N}(k_1 - k_2) \sim \omega_m - (N+1) k_2$$
$$n_{21,N} \sim \frac{\omega_m - (N+1) k_2}{k_1 - k_2}$$
$$k_1 > k_2$$

$$n_{12,N} \leq N \Rightarrow \quad \omega_m - N k_2 + k_1 \leq (k_1 - k_2) N$$
$$N \geq \frac{\omega_m + k_1}{k_1 - \cancel{k_2} + \cancel{k_2}}$$

$$\omega_m - (N+1) k_2 \leq N(k_1 - k_2)$$

$$N(k_1 - \cancel{k_2} + \cancel{k_2}) \geq \omega_m - k_2$$

* Concentrating on 2 frequencies

$$h = \frac{\sin^2\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n)\, e^{-i\omega_m x} \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{in(k_1 x+\phi_1)} \sum_{m=-\infty} (-i)^m J_m(z_{k_2}) e^{im(k_2 x+\phi_2)}$$

$$= \frac{\sin^2\theta_m}{2} \left[\frac{A_1}{2i}\left(e^{i(k_1 x+\phi_1)} - e^{-i(k_1 x+\phi_1)}\right) + \frac{A_2}{2i}\left(e^{i(k_2 x+\phi_2)} - e^{-i(k_2 x+\phi_2)}\right)\right] e^{-i\omega_m x}$$

$$\cdot \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{in(k_1 x+\phi_1)} \sum_{m=-\infty} (-i)^m J_m(z_{k_2}) e^{im(k_2 x+\phi_2)}$$

$$= \frac{\sin^2\theta_m}{2} \frac{A_1}{2i}\Big[\sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{in(k_1 x+\phi_1) + i(k_1 x+\phi_1) - i\omega_m x} \sum_{j=-\infty}^{\infty} (-i)^j J_j(z_{k_2}) e^{ij(k_2 x+\phi_2)}$$

$$\sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{i(n-1)(k_1 x+\phi_1) - i\omega_m x} \; \cancel{\sum} \cdots \quad \cdots\cdots \Big]$$

$$+ \frac{\sin^2\theta_m}{2} \frac{A_2}{2i}\Big[\cdots\cdots \quad \cdots\cdots \quad \sum_{j=-\infty}^{\infty} (-i)^j J_j(z_{k_2}) e^{i(j+1)(k_2 x+\phi_2) - i\omega_m x}$$

$$\cdots\cdots \quad - \quad \sum_{j=-\infty}^{\infty} (-i)^j J_j(z_{k_2}) e^{i(j-1)(k_2 x+\phi_2) - i\omega_m x} \Big]$$

Wasted so much time on the Math !!!

2 frequencies

$$h = \frac{\sin^2\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n)\, e^{-i\omega_m x} \sum_{n=-\infty}^{\infty} (-i)^n J_n(z_{k_1}) e^{in(k_1 x+\phi_1)} \sum_{m=-\infty} (-i)^m J_m(z_{k_2}) e^{im(k_2 x+\phi_2)}$$

Assuming $n = n_1$ is the most important term,

$$h \approx \frac{\sin^2\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n) e^{-i\omega_m x} (-i)^{n_1} J_{n_1}(z_{k_1}) e^{in_1(k_1 x+\phi_1)} \cdot$$

$$\sum_{n_b=-\infty}^{\infty} (-i)^{n_b} J_{n_b}(z_{k_2}) e^{in_b(k_2 x+\phi_2)}$$

$$= \frac{\sin^2\theta_m}{2} \sum_n A_n \sin(k_n x + \phi_n) \; \cancel{(-i)^{n_1} J_{n_1}(z_{k_1})}$$

$$\sum_{n_b=-\infty}^{\infty} (-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2})\, e^{in_b(k_2 x+\phi_2) + in_1(k_1 x+\phi_1) - i\omega_m x}$$

$$\sum_n A_n \sin(k_n x + \phi_n) = \frac{A_1}{2i}\left(e^{i(k_1 x+\phi_1)} - e^{-i(k_1 x+\phi_1)}\right) + \frac{A_2}{2i}\left(e^{i(k_2 x+\phi_2)} - e^{-i(k_2 x+\phi_2)}\right)$$

$$h = \frac{A_1 \sin^2\theta_m}{4i}\Big(\sum_{n_b=-\infty}^{\infty} (-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2})\, e^{in_b(k_2 x+\phi_2) + i(n_1+1)(k_1 x+\phi_1) - i\omega_m x}$$

$$- \cdots\cdots\cdots\cdots + i(n_1-1)(k_1 x+\phi_1) - i\omega_m x \Big)$$

$$\cdots + i(n_1-1)(k_1x+\phi_1) - i\omega_m x \Big)$$

$$+ \frac{A_2 \sin(\theta_m)}{4i}\Bigg(\sum_{n_b=-\infty}^{\infty}(-i)^{n_b+n_1} J_1(z_{k_1}) J_{n_b}(z_{k_2}) e^{i(n_b+1)(k_2x+\phi_2)+i(n_1)(k_1x+\phi_1)-i\omega_m x}$$

$$\cdots - i(n_b-1)(k_2x+\phi_2) \cdots \Big)$$

$n_{b_{11}}$: $\quad n_{b_{11}}$: $\quad n_b k_2 + (n_1+1)k_1 - \omega_m \sim 0$

$$n_{b_{11}} = \mathrm{Round}\left[\frac{(n_1+1)k_1 - \omega_m}{k_2}\right]$$

Define $\hat{k}_1 = \frac{k_1}{\omega_m}$, $\hat{k}_2 = \frac{k_2}{\omega_m}$, $\hat{A}_1 = \frac{A_1}{\omega_m}$, $\hat{A}_2 = \frac{A_2}{\omega_m}$, $\hat{x} = x\omega_m$

$$\hat{h} = \frac{h}{\omega_m} = \frac{\hat{A}_1 \sin(\theta_m)}{4i}\Bigg(\sum_{n_b=-\infty}^{\infty}(-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2}) e^{i[(n_b\hat{k}_2+(n_1+1)\hat{k}_1-1)\hat{x}+n_b\phi_2+(n_1+1)\phi_1]}$$

$$- \sum_{n_b=-\infty}^{\infty}(-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2}) e^{i[(n_b\hat{k}_2+(n_1-1)\hat{k}_1-1)\hat{x}+n_b\phi_2+(n_1-1)\phi_1]}\Bigg)$$

$$+ \frac{\hat{A}_2 \sin(\theta_m)}{4i}\Bigg(\sum_{n_b=-\infty}^{\infty}(-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2}) e^{i[(n_b+1)\hat{k}_2+n_1\hat{k}_1-1)\hat{x}+(n_b+1)\phi_2+n_1\phi_1]}$$

$$- \sum_{n_b=-\infty}^{\infty}(-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2}) e^{i[(n_b-1)\hat{k}_2+n_1\hat{k}_1-1)\hat{x}+(n_b-1)\phi_2+n_1\phi_1]}\Bigg)$$

Define $B = B_{n_1,n_b}(z_{k_1}, z_{k_2}) = (-i)^{n_b+n_1} J_{n_1}(z_{k_1}) J_{n_b}(z_{k_2})$

$$\Phi_{n_1,n_b}(\phi_1,\phi_2) = e^{i[n_b\phi_2+n_1\phi_1]}$$

$$\Phi_{n_1,n_b\pm1}(\phi_1,\phi_2) = e^{i[(n_b\pm1)\phi_2+n_1\phi_1]}$$

$$\hat{h} = \frac{h}{\omega_m} = \frac{\hat{A}_1 \sin(\theta_m)}{4i}\Bigg[\sum_{n_b=-\infty}^{\infty} B\,\Phi_{n_1+1,n_b} e^{i[n_b\hat{k}_2+(n_1+1)\hat{k}_1-1]\hat{x}}$$

$$- \sum_{n_b=-\infty}^{\infty} B\,\Phi_{n_1-1,n_b} e^{i[n_bk_2+(n_1-1)k_1-1]\hat{x}}\Bigg]$$

$$+ \frac{\hat{A}_2 \sin(\theta_m)}{4i}\Bigg[\sum_{n_b=-\infty}^{\infty} B\,\Phi_{n_1,n_b+1} e^{i[n_1\hat{k}_1+(n_b+1)\hat{k}_2-1]\hat{x}}$$

$$- \sum_{n_b=-\infty}^{\infty} B\,\Phi_{n_1,n_b-1} e^{i[n_1\hat{k}_1+(n_b-1)\hat{k}_1-1]\hat{x}}\Bigg]$$

$$\Big[h = \frac{\sin(\theta_m)}{2}\sum_n A_n \sin(k_nx+\phi_n)\, e^{-i\omega_m x}\sum_{n_a=-\infty}^{\infty}(-i)^n J_n(z_{k_1}) e^{in(k_1x+\phi_1)}\sum_{n_b=-\infty}^{\infty}(-i)^m J_m(z_{k_2}) e^{im(k_2x+\phi_2)}$$

$$\hat{h} = \frac{h}{\omega_m} = \frac{\hat{A}_1 \sin(\theta_m)}{4i}\Big(e^{i(k_1x+\phi_1)} - e^{-i(k_1x+\phi_1)}\Big)\sum_{n_a=-\infty}^{\infty}(-i)^{n_a} J_{n_a}(z_{k_1}) e^{in_a(k_1x+\phi_1)}\sum_{n_b=-\infty}^{\infty}(-i)^{n_b} J_{n_b}(z_{k_2}) e^{in_b(k_2x+\phi_2)}$$

$$+ \frac{\hat{A}_2 \sin(\theta_m)}{4i}\Big(e^{i(k_2x+\phi_2)} - e^{-i(k_2x+\phi_2)}\Big) e^{-i\omega_m x}\sum(-i)^{n_a} J_{n_a}(z_{k_1}) e^{in_a(k_1x+\phi_1)}\sum(-i)^{n_b} J_{n_b}(z_{k_2}) e^{in_b(k_2x+\phi_2)}$$

$$+\frac{A_1 \sin 2\theta_m}{4i}\left(e^{i(k_1x+\phi_1)}-e^{-i(k_1x+\phi_1)}\right)e^{-i\omega_m x}\sum_{n_a=-\infty}^{\infty}(-i)^{n_a}J_{n_a}(z_{k_1})e^{in_a(k_1x+\phi_1)}\sum_{n_b=-\infty}^{\infty}(-i)^{n_b}J_{n_b}(z_{k_2})e^{in_b(k_2x+\phi_2)}$$

$$h_1=\frac{A_1\sin 2\theta_m}{4i}\left(\sum_{n_a=-\infty}^{\infty}(-i)^{n_a}J_{n_a}(z_{k_1})e^{i(n_a+1)(k_1x+\phi_1)-i\omega_m x}-\sum_{n_a=-\infty}^{\infty}(-i)^{n_a}J_{n_a}(z_{k_1})e^{i(n_a-1)(k_1x+\phi_1)+i\omega_m x}\right)\sum_{n_b=-\infty}^{\infty}(-i)^{n_b}J_{n_b}(z_{k_2})e^{in_b(k_2x+\phi_2)}$$

$$=\frac{A_1\sin 2\theta_m}{4i}\left(\sum_{n_a=-\infty}^{\infty}(-i)^{n_a-1}J_{n_a-1}(z_{k_1})e^{in_a(k_1x+\phi_1)-i\omega_m x}-\sum_{n_a=-\infty}^{\infty}(-i)^{n_a+1}J_{n_a+1}(z_{k_1})e^{in_a(k_1x+\phi_1)-i\omega_m x}\right)\sum \cdots$$

$$=\frac{A_1\sin 2\theta_m}{4i}\sum_{n_a=-\infty}^{\infty}(-i)^{n_a-1}\left(J_{n_a-1}(z_{k_1})-(-i)^2J_{n_a+1}(z_{k_1})\right)e^{in_a(k_1x+\phi_1)-i\omega_m x}\sum_{n_b=-\infty}^{\infty}(-i)^{n_b}J_{n_b}(z_{k_2})e^{in_b(k_2x+\phi_2)}$$

$$=\frac{2n_a+1}{z_{k_1}}J_{n_a}(z_{k_1}) \Rightarrow \frac{A_1\sin 2\theta_m}{4}\left(-(-i)^{n_a+n_b}\right)\frac{2n_a+1}{z_{k_1}}J_{n_a}(z_{k_1})J_{n_b}(z_{k_2})e^{\cdots}$$

$$=\frac{A_1\sin 2\theta_m}{4}\sum_{n_a=-\infty}^{\infty}\left(-(-i)^{n_a}\right)\frac{2n_a+1}{z_{k_1}}J_{n_a}(z_{k_1})e^{in_a(k_1x+\phi_1)-i\omega_m x}\sum_{n_b=-\infty}^{\infty}(-i)^{n_b}J_{n_b}(z_{k_2})e^{in_b(k_2x+\phi_2)}$$

$$h_2=\frac{A_1\sin 2\theta_m}{4}\sum_{n_a=-\infty}^{\infty}(-i)^{n_a}J_{n_a}(z_{k_1})e^{in_a(k_1x+\phi_1)}\sum_{n_b=-\infty}^{\infty}\left(-(-i)^{n_a}\right)\frac{2n_b+1}{z_{k_2}}J_{n_b}(z_{k_2})e^{in_b(k_2x+\phi_2)-i\omega_m x}$$

$h=h_1+h_2$

Assume we only have k_1 profile,

$h_1 \Rightarrow$ only part is present!

are due to the second frequency.

Assume we only have k_2 profile, we can determine which is due to the second frequency

Now

1. Should I use the resonance for k_1 only?
2. Or should I use resonance for k_1 for h_1 the resonance condition for k_2 for h_2?

I use $\boxed{2.}$

Resonance condition for k_1 frequence

n_1

- - - - k_2 -

n_2

small phase $n_1k_1-\omega_m \sim 0$

$$h_1=\frac{A_1\sin 2\theta_m}{4}\left(-(-i)^{n_1}\right)\frac{2n_1+1}{z_{k_1}}J_{n_1}(z_{k_1})e^{in_1(k_1x+\phi_1)-i\omega_m x}\sum_{n_b=-\infty}^{\infty}(-i)^{n_b}J_{n_b}(z_{k_2})e^{in_b(k_2+\phi_2)}$$

We define $\frac{\sin 2\theta_m}{4}\left(-(-i)^{n_1+n_b}\right)\frac{2n_1+1}{z_{k_1}}J_{n_1}(z_{k_1})J_{n_b}(z_{k_2}) = B_{n_1,n_b}(z_{k_1},z_{k_2})$

$$h_1=A_1\sum_{n_b=-\infty}^{\infty}B_{n_1,n_b}(z_{k_1},z_{k_2})e^{i(n_1k_1+n_bk_2-\omega_m)x}\,e^{i(n_1\phi_1+n_b\phi_2)}$$

$$n_b'=\mathrm{Round}\left[\frac{n_1k_1-\omega_m}{\ldots}\right]$$

$$\hat{h}_1 \approx \hat{A}_1 B_{n_1, n_2'}(z_{k_1}, z_{k_2})\, e^{i(n_1 k_1 + n_2' k_2 - \omega_m)x}\, e^{i(n_1\phi_1 + n_2'\phi_2)}$$

I test:

$$B_{n_2, n_a}(z_{k_2}, z_{k_1}) = \frac{\pi\omega_m}{4}\left(-(-i)^{n_2+n_a}\right)\frac{2n_2+1}{z_{k_2}} J_{n_2}(z_{k_2})\, J_{n_a}(z_{k_1})$$

$$\hat{h}_2 \approx \hat{A}_2 \sum_{n_a=0}^{\infty} B_{n_2, n_a}(z_{k_2}, z_{k_1})\, e^{i(n_2 k_2 + n_a k_1 - \omega_m)x}\, e^{i(n_2\phi_2 + n_a\phi_1)}$$

$$\approx \hat{A}_2 B_{n_2, n_1'}(z_{k_2}, z_{k_1})\, e^{i(n_2 k_2 + n_1' k_1 - \omega_m)x}\, e^{i(n_2\phi_2 + n_1'\phi_1)}$$

$$n_1' = \text{Round}\left[\frac{n_2 k_2 - \omega_m}{k_1}\right] = \text{Round}\left[\frac{n_2 \hat{k}_2 - 1}{\hat{k}_1}\right]$$

Make dimensionless

$$\Phi(n_1, n_2) = e^{i(n_1\phi_1 + n_2\phi_2)}$$

$$\hat{h} = \hat{h}_1 + \hat{h}_2$$
$$= \hat{A}_1 B_{n_1, n_2'}(z_{k_1}, z_{k_2})\, e^{i(n_1\hat{k}_1 + n_2'\hat{k}_2 - 1)\hat{x}}\, \Phi(n_1, n_2')$$
$$+ \hat{A}_2 B_{n_2, n_1'}(z_{k_2}, z_{k_1})\, e^{i(n_1'\hat{k}_1 + n_2\hat{k}_2 - 1)\hat{x}}\, \Phi(n_1', n_2)$$

Equation of motion

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix} 0 & h \\ h^* & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$i\frac{d}{d\hat{x}}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix} 0 & \hat{h} \\ \hat{h}^* & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$n_1 k_1 + n_2 k_2 \sim 1$$

Now include higher order terms from the series

	k_1	k_2
0th order k_1	n_1	$n_2 = \text{Round}\left[\frac{1 - n_1 k_1}{k_2}\right]$
0th order k_2	$n_1 = \text{Round}\left[\frac{1 - n_2 k_2}{k_1}\right]$	n_2
1st order k_1	$n_1 + 1$	$n_2 =$

Jacobi Anger Expansion

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$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}$$

$$= \sum_{n=1}^{\infty} i^n J_n(z) e^{in\theta} + \sum_{n=-\infty}^{-1} i^n J_n(z) e^{in\theta} + J_0(z)$$

$$= \sum_{n=1}^{\infty} i^n J_n(z) e^{in\theta} + \sum_{n=1}^{\infty} i^{-n} J_{-n}(z) e^{-in\theta} + J_0(z)$$

$$J_{-n}(z) = (-1)^n J_n(z)$$

$$= \sum_{n=1}^{\infty} \left(i^n J_n(z) e^{in\theta} + i^{-n} (-1)^n J_n(z) e^{-in\theta} \right) + J_0(z)$$

$$= \sum_{n=1}^{\infty} \left(\left(i^n e^{in\theta} + (-1)^n (i)^{-n} e^{-in\theta} \right) J_n(z) \right) + J_0(z)$$

$$i^{-n} = \frac{1}{i^n} = \frac{i^n}{i^n \cdot i^0} = \frac{i^n}{(-1)^n} = (-1)^n i^n$$

$$= \sum_{n=1}^{\infty} \left(\left(i^n e^{in\theta} + i^n e^{-in\theta} \right) J_n(z) \right) + J_0(z)$$

$$= \sum_{n=1}^{\infty} \left(i^n J_n(z) \left(e^{in\theta} + e^{-in\theta} \right) \right) + J_0(z)$$

Kelly's paper

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$$h_{12} = V_0 \sum_{a=1}^{N_k} C_a \sin(q_a x + \eta_a)\, \breve{u}_{e_1}^{*}\, \breve{u}_{e_2}\, e^{i(\delta k_{12} x + \delta\xi_{12})}$$

$$= V_0 \sum_{a=1}^{N_k} C_a \sin(q_a x + \eta_a)\, \breve{u}_{e_1}^{*}\, \breve{u}_{e_2}\, e^{i\delta\breve{k}_{12} x} \prod_{a=1}^{N_k} \sum_{n_a=-\infty}^{\infty} (-i)^{n_a} J_{n_a}(z_a)\, e^{i z_a \cos\theta_a + i n_a (q_a x + \eta_a)}$$

$$= \frac{V_0\, \breve{u}_{e_1}^{*}\, \breve{u}_{e_2}}{|\breve{u}_{e_1}|^2 - |\breve{u}_{e_2}|^2} \left(|\breve{u}_{e_1}|^2 - |u_{e_2}|^2\right) \sum_{a=1}^{N_k} C_a \sin$$

$$= \frac{\breve{u}_{e_1}^{*}\, \breve{u}_{e_2}}{|\breve{u}_{e_1}|^2 - |\breve{u}_{e_2}|^2} \sum_{a=1}^{N_k} z_a \sin(q_a x + \eta_a)\, e^{i\delta\breve{k}_{12} x} \prod_{a=1}^{N_k} \underbrace{\sum_{n_a=-\infty}^{\infty} (-i)^{n_a} J_{n_a}(z_a)\, e^{i z_a \cos\theta_a + i n_a (q_a x + \eta_a)}}_{K_{a,n_a}}$$

$$= \frac{\breve{u}_{e_1}^{*}\, \breve{u}_{e_2}}{|\breve{u}_{e_1}|^2 - |\breve{u}_{e_2}|^2} \sum_{a=1}^{N_k} z_a \sin(q_a x + \eta_a)\, e^{i \frac{\delta\breve{k}_{12}}{N_k} x} \prod_{a=1}^{N_k} \sum_{n_a=-\infty}^{\infty} K_{a,n_a}\, e^{i n_a q_a x}$$

$$= \frac{\cdot\ \cdot}{\cdot\ \cdot} \sum_{a=1}^{N_k} z_a \sin(q_a x + \eta_a) \prod_{a=1}^{N_k} \sum_{n_a=-\infty}^{\infty} K_{a,n_a}\, e^{i n_a q_a x + i \frac{\delta\breve{k}_{12}}{N_k} x}$$

$$\left(\frac{u\ u}{\cdot\ \cdot} \left(\sum_{a=1}^{N_k} n_{xa} q_a\right) \prod_{a=1}^{N_k} K_{a,n_a} \right) e^{i x \left(\delta\breve{k}_{12} + \sum_a n_{xa} q_a\right)}$$

Approximation Around Resonance

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$$z_k = \frac{\hat{A}}{\hat{k}} \cos 2\theta_m$$

$$Amp = \frac{|\hat{F}|^2}{|\hat{F}|^2 + |\hat{g}|^2}$$

$$|\hat{F}|^2 = \left| \frac{1}{2} \hat{k} \tan 2\theta_m \, (2n_0+1) \, J_{n_0}(z_k) \right|^2$$

$$|\hat{g}|^2 = (n_0 \hat{k} - 1)^2$$

Bessel functions

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{m!\,(m+n)!} \left(\frac{z}{2}\right)^{2m+n}$$

$$z_k = \frac{\hat{A}}{\hat{k}} \cos 2\theta_m = \frac{\hat{A}}{\hat{k}_0 + \hat{k}'} \cos 2\theta_m$$

$$\doteq n_0 \hat{A} \cos 2\theta_m \left(1 - n_0 \hat{k}' + (n_0 \hat{k}')^2\right) \equiv z_0 + z_1 + z_2$$

keep only 2nd order terms of $\hat{k}'$

$$J_{n_0}(z_k) \doteq \sum_{m=0}^{\infty} \frac{(-1)^{m+n_0}}{m!\,(n_0+m)!} \left(\frac{z_0}{2}\right)^{2m+n_0} \left(1 - (2m+n_0)\, n_0 \hat{k}' + \frac{1}{2}(2m+n_0)(2m+n_0+1)\, n_0 \hat{k}'^2\right)$$

$$z_0 = n_0 \hat{A} \cos 2\theta_m$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+n_0}}{m!\,(n_0+m)!} \left(\frac{z_0}{2}\right)^{2m+n_0} - \sum_{m=0}^{\infty} \frac{(-1)^{m+n_0}}{m!\,(n_0+m)!} \left(\frac{z_0}{2}\right)^{2m+n_0} (2m+n_0)\, n_0 \hat{k}'$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^{m+n_0}}{m!\,(n_0+m)!} \left(\frac{z_0}{2}\right)^{2m+n_0} \frac{1}{2}(2m+n_0)(2m+n_0+1)\, n_0 \hat{k}'^2$$

$$\equiv J_{n_0}(z_0) - bes1 \cdot \hat{k}' + bes2\, \hat{k}'^2$$

$$|F|^2 = \left| \frac{1}{2} \tan 2\theta_m \, (2n_0+1) \left(\frac{1}{n_0} + \hat{k}'\right) \left(J_0(z_0) - bes1\, \hat{k}' + bes2\, \hat{k}'^2\right) \right|^2$$

$$= f_0 + f_1 \hat{k}' + f_2 \hat{k}'^2$$

$$f_0 = \frac{J_{n_0}(z_0)\,(2n_0+1)\tan 2\theta_m}{2}$$

$$= J_0 + J_1 K_8 + J_2 K^2$$

$$f_0 = \frac{J_{n_0}(z_0)\,(2n_0+1)\tan^2\theta_m}{4n_0^2}$$

$$f_1 = \frac{1}{4}\left(\frac{2J_{n_0}(z_0)}{n_0} - \frac{2J_{n_0}(z_0)\,\mathrm{bes}'}{n_0^2}\right)(2n_0+1)^2\tan^2\theta_m$$

$$f_2 = \frac{1}{4}\left(J_{n_0}(z_0) + \frac{\mathrm{bes}'^2}{n_0^2} + \frac{2\,\mathrm{bes}'\,\mathrm{bes}''}{n_0^2} - \frac{4\,\mathrm{bes}'\,J_{n_0}(z_0)}{n_0}\right)(2n_0+1)^2\tan^2\theta_m$$

So

$$\mathrm{Amp}' \approx \frac{f_0 + f_1\hat{k}' + f_2\hat{k}'^2}{f_0 + f_1\hat{k}' + f_2\hat{k}'^2 + n_0^2\hat{k}'^2}$$

width : $$\frac{\sqrt{f_1^2 - 4f_0(f_2 - n_0^2)}}{f_2 - n_0^2}$$

* Small argument ~~Approximation~~ of Bessel

$$J_n(z) \sim \frac{1}{\Gamma(n+1)}\left(\frac{z}{2}\right)^n$$

n : integer

$$\Gamma(n+1) = n!$$

Using stirling's

$$\ln n! = n\ln n - n$$

$$\ln(n!) \doteq \ln n^n - \ln(e^n)$$

$$n! = n^n/e^n \left.\frac{z}{2}\right)^n$$

$$J_n(z) \sim \frac{e^n}{n^n}\Big|$$

At resonance

$$z \sim n \cdot \frac{A \cos 2\theta_m}{\delta}$$

if $\delta \ll 1$, n not large, we have

$n\delta$ small, then

$$J_n(nA\cos 2\theta_m) \doteq \frac{e^n}{n^n} n^n \left(\frac{A\cos 2\theta_m}{2}\right)$$

$$2F = \frac{\hat{A}}{z_k} \qquad = \left(\frac{A\cos 2\theta_m}{2}\right)^n \cdot e^n \qquad z_k = \frac{\hat{A}}{\hat{k}} \cos 2\theta_m$$

So with ~~th~~ under Lorentzian

$$\sin 2\theta_m (2n_0+1) J_{n_0}(z_k)$$

$$= \hat{k} \tan 2\theta_m (2n_0+1) J_{n_0}(n_0 \hat{A} \cos 2\theta_m)$$

$$= \tan 2\theta_m (2n_0+1) \frac{1}{n_0} \cdot \left(\frac{A\cos 2\theta_m}{2}\right)^n e^n$$

Erdelyi?

Two frequencies Table

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$\{k_1, A_1\}$

	$A_1 > A_2$	$A_1 < A_2$
$k_1 > k_2$	* I_1, I_2	II
$k_1 < k_2$	III	IV

I_1: k_1 at resonance

I_2: k_2 at resonance

	small A_{mp}	large A_{mp}
Resonance k small	RkS-AmpS	RkS-AmpL
. . . . . large	RkL-AmpS	RkL-AmpL

1. Fix the bug
2. do the $\boxed{+}$ ← Second doing a better job
3. If higher order terms solves the problem then good

$$h = \frac{\sin\theta_m}{2}(\delta\lambda_1 + \delta\lambda_2)\, e^{2i\left(-\frac{\omega_m}{2}x - \frac{A_1\cos\theta_m}{2k_1}\cos(k_1x+\phi_1) - \frac{A_2}{2k_2}\cos\theta_m\cos(k_2x+\phi_2)\right)}$$

$= h_1 + h_2$

$$h_1 = \frac{\sin\theta_m}{2}\delta\lambda_1\, e^{2i\left(-\frac{\omega_m}{2}x - \frac{A_1}{2k_1}\cos\theta_m\cos(k_1x+\phi_1)\right)}\, e^{-i\frac{A_2}{k_2}\cos\theta_m\cos(k_2x+\phi_2)}$$

$$= h(k_1)\, e^{-i\frac{A_2}{k_2}\cos\theta_m\cos(k_2x+\phi_2)}$$

Now for each pair of n_1, n_2, we can solve

$$i\frac{d}{dx}\psi = \begin{pmatrix} & h \\ h^* & \end{pmatrix}\psi$$

where $h = \left(B_{n_1,n_2}(k_1,k_2) + B_{n_2,n_1}(k_2,k_1)\right)\xi\, e^{i(n_1k_1+n_2k_2-\omega_m)x}$

Make dimensionless

$$i\frac{d}{d\xi}\psi = \begin{pmatrix} & h \\ h^* & \end{pmatrix}\psi$$

$$i\frac{d}{d\hat{x}}\psi = \begin{pmatrix} & h \\ h^* & \end{pmatrix}\psi$$

$$z_k = \frac{A_k}{\omega_m}\cos 2\theta_m$$

$$-\frac{(-i)^{n_1+n_2}}{2}\tan 2\theta_m\,(n_1k_1+n_2k_2)\,J_{n_1}(z_{k_1})J_{n_2}(z_{k_2})$$

$$e^{i(n_1\phi_1+n_2\phi_2)}e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}}$$

Define $\hat{h} = \frac{b}{\omega_m}$

$$\hat{h} = b\,(n_1\hat{k}_1+n_2\hat{k}_2)\,J_{n_1}\left(\frac{\hat{A}_1}{\hat{k}_1}\cos 2\theta_m\right)J_{n_2}\left(\frac{\hat{A}_2}{\hat{k}_2}\cos 2\theta_m\right)\underbrace{e^{i(n_1\phi_1+n_2\phi_2)}}_{\hat{g}}e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}}$$

$$i\frac{d}{d\hat{x}}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} & \hat{h} \\ \hat{h}^* & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\begin{cases} i\frac{d}{d\hat{x}}\psi_1 = \hat{h}\psi_2 \\ i\frac{d}{d\hat{x}}\psi_2 = \hat{h}^*\psi_1 \end{cases}$$

$$\hat{h} = g\,e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}}$$

$$= (g_R + i g_I)\,e^{i(\quad)\hat{x}}$$

$$M_\pm = n_1\hat{k}_1+n_2\hat{k}_2-1 \pm\sqrt{4|g|^2+(\quad)^2}$$

using $\psi(0) = \begin{pmatrix}1\\0\end{pmatrix}$

$$\psi_{b2} = \frac{e^{-i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}}\left(e^{\frac{i}{2}\left(n_1\hat{k}_1+n_2\hat{k}_2-1-\sqrt{4|g|^2+(n_1\hat{k}_1+n_2\hat{k}_2-1)^2}\right)\hat{x}} - e^{\frac{i}{2}\left(n_1\hat{k}_1+n_2\hat{k}_2-1+\sqrt{4|g|^2+(n_1\hat{k}_1+n_2\hat{k}_2-1)^2}\right)\hat{x}}\right)(-iM_-)(+iM_+)}{-4g\sqrt{+4|g|^2+(n_1\hat{k}_1+n_2\hat{k}_2-1)^2}}$$

$$= \frac{e^{-i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}}\,M_+M_-\left(e^{\frac{i}{2}M_-\hat{x}} - e^{\frac{i}{2}M_+\hat{x}}\right)}{-4g\sqrt{4|g|^2+(n_1\hat{k}_1+n_2\hat{k}_2-1)^2}}$$

$$M_+M_- = g^2-(4|F|^2+g^2)$$
$$= -4|F|^2$$

$$P = |\psi_{b2}|^2 = \frac{|M_+M_-|^2\,\left|e^{\frac{i}{2}M_-\hat{x}} - e^{\frac{i}{2}M_+\hat{x}}\right|^2}{16|g|^2\left(4|g|^2+(n_1\hat{k}_1+n_2\hat{k}_2-1)^2\right)}$$

$g \to F$ to be consistent with before (single freq)

$$P = \frac{16|F|^4\,\left|e^{\frac{i}{2}M_-x} - e^{\frac{i}{2}M_+\hat{x}}\right|^2}{16|F|^2\left(4|F|^2+g^2\right)}$$

$\hat{F} = \ldots$

$\hat{g} = n_1\hat{k}_1+n_2\hat{k}_2-1$

$$h = \hat{F}e^{i\hat{g}\hat{x}}$$

$$M = g \pm\sqrt{4|F|^2+g^2}$$

Amplitude is

$$\frac{4|\hat{F}|^2}{4|F|^2+g^2}$$

$$\overline{4|\vec{F}|^2 + g^2}$$

$$\left| e^{i\frac{M_-}{2}x} - e^{i\frac{M_+}{2}x} \right|^2 = \left| e^{\frac{i}{2}\left(g - \sqrt{4|\vec{F}|^2+g^2}\right)x} - e^{\frac{i}{2}\left(g + \sqrt{4|\vec{F}|^2+g^2}\right)x} \right|^2$$

$$= \left| e^{-i\frac{1}{2}\sqrt{4|\vec{F}|^2+g^2}\,x} - e^{i\frac{1}{2}\sqrt{4|\vec{F}|^2+g^2}\,x} \right|^2$$

$$= 4 \sin^2\left(\frac{\sqrt{4|\vec{F}|^2+g^2}}{2}\,x\right)$$

Now if I actually have

$$h = \frac{\vec{F}}{2} e^{igx}$$

Amplitude is

$$\frac{|\vec{F}|^2}{|\vec{F}|^2 + g^2}$$

Probability is

$$\frac{|\vec{F}|^2}{|\vec{F}|^2+g^2} \sin^2\left(\frac{\sqrt{|\vec{F}|+g^2}}{2}\,x\right)$$

Half width happens when

$$\frac{|\vec{F}|^2}{|\vec{F}|^2+g^2} = \frac{1}{2}$$

$$|\vec{F}|^2 = g^2$$

$$n_1 k_1 + n_2 k_2 - 1 = |\vec{F}|^2$$

3 Frequencies

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$$h = \frac{\sin\theta_m}{2}\left(\delta\lambda_1 + \delta\lambda_2 + \delta\lambda_3\right)\sum_{n_1}(-i)^{n_1} J_{n_1}(z_{n_1}) e^{i(n_1 k_1 x + \phi_1)} \Sigma \qquad \Sigma$$

$$h_1 = \frac{\sin\theta_m}{2}\left(\delta\lambda_1 \sum_{n_1}\right)\left(\sum_{n_2}\right)\left(\sum_{n_3}\right)$$

$$h_2 = \frac{\sin\theta_m}{2}\left(\sum_{n_1}\right)\left(\delta\lambda_2 \sum_{n_2}\right)\left(\sum_{n_3}\right)$$

$$h_3 = \frac{\sin\theta_m}{2}\left(\sum_{n_1}\right)\left(\sum_{n_2}\right)\left(\delta\lambda_3 \sum_{n_3}\right)$$

Phase Portrait

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For single frequency

$$i\frac{d}{dx}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix}0 & h\\ h & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$h = \frac{\sin 2\theta_m}{2} A \sin(kx+\phi)\, e^{2i\left(-\frac{\omega_m}{2}x - \frac{A\cos 2\theta_m}{2k}\cos(kx+\phi)\right)}$$

Rewrite to scaled quantities

$$\hat{x} = x\omega_m, \quad \hat{k} = \frac{k}{\omega_m}, \quad \hat{A} = \frac{A}{\omega_m}$$

$$i\frac{d}{d\hat{x}}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} = \begin{pmatrix}0 & \hat{h}\\ \hat{h}^* & 0\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$\hat{h} = \frac{\sin 2\theta_m}{2}\hat{A}\sin(\hat{k}\hat{x}+\phi)\, e^{-i\left(\hat{x} + \frac{\hat{A}}{\hat{k}}\cos 2\theta_m \cos(kx+\phi)\right)}$$

$\boxed{\text{Need to write down real } \phi \text{ equation } \& \ X}$

$$i\frac{d}{d\hat{x}}\begin{pmatrix}\psi_{b1,R} + i\psi_{b1,I}\\ \psi_{b2,R} + i\psi_{b2,I}\end{pmatrix} = \begin{pmatrix}0 & \hat{h}_R + i\hat{h}_I\\ \hat{h}_R - i\hat{h}_I & 0\end{pmatrix}\begin{pmatrix}\psi_{b1,R} + i\psi_{b1,I}\\ \psi_{b2,R} + i\psi_{b2,I}\end{pmatrix}$$

Constraint $\psi_{b1,R}^2 + \psi_{b1,I}^2 + \psi_{b2,R}^2 + \psi_{b2,I}^2 = 1$

$$i\frac{d}{d\hat{x}}\psi_{b1,R} - \frac{d}{d\hat{x}}\psi_{b1,I} = (\hat{h}_R + i\hat{h}_I)(\psi_{b2,R} + i\psi_{b2,I})$$

$$\Rightarrow i\frac{d}{d\hat{x}}\psi_{b1,R} - \frac{d}{d\hat{x}}\psi_{b1,I} = \hat{h}_R\psi_{b2,R} - \hat{h}_I\psi_{b2,I} + i(\hat{h}_R\psi_{b2,I} + \hat{h}_I\psi_{b2,R})$$

$$+\frac{d}{d\hat{x}}\psi_{b1,I} = -\hat{h}_R\psi_{b2,R} + \hat{h}_I\psi_{b2,I}$$

$$\frac{d}{d\hat{x}}\psi_{b1,R} = \hat{h}_R\psi_{b2,I} + \hat{h}_I\psi_{b2,R}$$

$$\frac{d}{d\hat{x}}\begin{pmatrix}\psi_{b1,R}\\ \psi_{b1,I}\end{pmatrix} = \begin{pmatrix}\hat{h}_I & \hat{h}_R\\ -\hat{h}_R & \hat{h}_I\end{pmatrix}\begin{pmatrix}\psi_{b2,R}\\ \psi_{b2,I}\end{pmatrix}$$

$$\partial_{\hat{x}} \psi_{b1,R} = h_R \psi_{b2,I} + h_I \psi_{b2,R}$$

$$\frac{d}{d\hat{x}} \begin{pmatrix} \psi_{b1,R} \\ \psi_{b1,I} \end{pmatrix} = \begin{pmatrix} \hat{h}_I & \hat{h}_R \\ -\hat{h}_R & \hat{h}_I \end{pmatrix} \begin{pmatrix} \psi_{b2,R} \\ \psi_{b2,I} \end{pmatrix}$$

$$i \frac{d}{d\hat{x}} \psi_{b2,R} - \frac{d}{d\hat{x}} \psi_{b2,I} = (\hat{h}_R - i\hat{h}_I)(\psi_{b1,R} + i\psi_{b1,I})$$

$$+ \frac{d}{d\hat{x}} \psi_{b2,I} = -\hat{h}_R \psi_{b1,R} + \hat{h}_I \psi_{b1,I}$$

$$i \frac{d}{d\hat{x}} \psi_{b2,R} = -i\hat{h}_I \psi_{b1,R} + i\hat{h}_R \psi_{b1,I}$$

$$\frac{d}{d\hat{x}} \psi_{b2,R} = -\hat{h}_I \psi_{b1,R} + \hat{h}_R \psi_{b1,I}$$

$$\frac{d}{d\hat{x}} \begin{pmatrix} \psi_{b2,R} \\ \psi_{b2,I} \end{pmatrix} = \begin{pmatrix} -\hat{h}_I & \hat{h}_R \\ -\hat{h}_R & \hat{h}_I \end{pmatrix} \begin{pmatrix} \psi_{b1,R} \\ \psi_{b1,I} \end{pmatrix}$$

Multi frequency Perturbation

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Hamiltonian

$$h = -\sum_{\gamma}\sum\sum\cdots\sum \overbrace{(-i)^{n_1+n_2\cdots n_N}\left(\sum_a n_a\hat{k}_a\right)\left(\prod_a J_{n_a}\left(\frac{\hat{A}_a}{k_a}\cos\theta_a\right)\right)}^{B_N} \Phi_N e^{i\left(\sum_a n_a\hat{k}_a - 1\right)\hat{x}}$$

What I need to do is to compare $\frac{d}{B_N} = \frac{\left|\sum_a n_a k_a - 1\right|}{B_N}$

and Find the small ones

$$h = \frac{\sin\theta_m}{2}\left(\sum_a A_a \sin(k_a x + \phi_a)\right) e^{i\left(-x - \cos\theta_m\left(\sum_a \frac{A_a}{k_a}\cos(k_a x + \phi_a)\right)\right)}$$

Matter Profile (more general background)

Monday, May 16, 2016 10:41 AM

Suppose we have a most general background

$$\lambda_0(x)$$

as background

In flavor basis

$$H = \left(\frac{\lambda(x)}{2} - \frac{\omega_v}{2}\cos 2\theta_v\right)\sigma_3 + \frac{\omega_v}{2}\sin 2\theta_v\,\sigma_1$$

$$= \frac{1}{2}\left(\lambda_0(x) + \delta\lambda(x) - \frac{\omega_v}{2}\cos 2\theta_v\right)\sigma_3 + \frac{\omega_v}{2}\sin 2\theta_v\,\sigma_1$$

Unitary transformation

$$\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$i\partial_x\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix} = \left(\left(\frac{\lambda_0(x)+\delta\lambda(x)}{2} - \frac{\omega_v}{2}\cos 2\theta_v\right)\sigma_3 + \frac{\omega_v}{2}\sin 2\theta_v\,\sigma_1\right)\begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$\text{LHS} = i\partial_x\left[\begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix}\right]$$

$$= i\left[\partial_x\begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\right]\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\partial_x\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$= -\partial_x\eta(x)\begin{pmatrix} -e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\begin{pmatrix} e^{-i\eta(x)} & 0\\ 0 & e^{i\eta(x)}\end{pmatrix}\partial_x\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$= -\partial_x\eta(x)\,A\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix} + i\,T\begin{pmatrix}\psi_{b1}\\ \psi_{b2}\end{pmatrix}$$

$$= -\partial_x \eta(x)\, A \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i T \partial_x \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\text{RHS} = \left(\left(\frac{\lambda_0(x) + \delta\lambda(x)}{2} - \frac{\omega_v}{2} \cos 2\theta_v \right) \sigma_3 + \frac{\omega_v}{2} \sin 2\theta_v\, \sigma_1 \right) \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\sigma_3 \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & -e^{i\eta(x)} \end{pmatrix} = -A$$

$$\sigma_1 \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{i\eta(x)} \\ e^{-i\eta(x)} & 0 \end{pmatrix} = B$$

$$\text{RHS} = \left(\left(\frac{\lambda_0(x) + \delta\lambda(x)}{2} - \frac{\omega_v}{2} \cos 2\theta_v \right) (-A) + \frac{\omega_v}{2} \sin 2\theta_v\, B \right) \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

$$\text{LHS} = -\partial_x \eta(x)\, A \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix} + i T \partial_x \begin{pmatrix} \psi_{b1} \\ \psi_{b2} \end{pmatrix}$$

Matter Profile (more general background)

Monday, May 16, 2016 10:41 AM

Suppose we have a most general background

$$\lambda_0(X)$$

as background

Parametric resonance

Smirnov View

① oscillation length l_m

$$l_m = \frac{l_v}{\sqrt{\left(\cos 2\theta - \frac{l_v}{l_0}\right)^2 - \sin^2 2\theta}}$$

$$l_v = \frac{2\pi}{\frac{\Delta m^2}{2E}} = \frac{2\pi}{\omega_v}$$

Other useful quantities

Oscillation angular frequency

$$\omega_m = \omega_v \sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v}\cos 2\theta_v}$$

$$= \omega_v \sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 - 2\frac{\lambda}{\omega_v}\cos 2\theta_v + \cos^2 2\theta_v + 1 - \cos^2 2\theta_v}$$

$$= \omega_v \sqrt{\left(\frac{\lambda}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}$$

Parametric resonance condition

$$\int_0^{L_f} \frac{2\pi}{l_m} dx \sim 2\pi k, \quad k = 1, 2, 3, \cdots \Rightarrow \int_0^{L_f} \omega_m \, dx \sim 2\pi k$$

Work out an example

$$\rho = \bar{\rho}(x)\left(1 + A(x)\sin\left(\frac{2\pi}{L_f}x\right)\right)$$

consider $\bar{\rho}(x) = \bar{\rho}$, $A(x) = A$

Let's use $n_e(x)$ and

$$\lambda(x) = \sqrt{2}\, G_F\, n_b(x)$$

$$\lambda(x) = \bar{\rho}\left(1 + A\sin\left(\frac{2\pi}{L_f}x\right)\right)$$

$$\int_0^{L_f} \omega_m \, dx$$

$$= \int_0^{L_f} \omega_v \sqrt{\left(\frac{\lambda(x)}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v} \, dx$$

This is not an easy expression

to integrate

Oscillation Picture

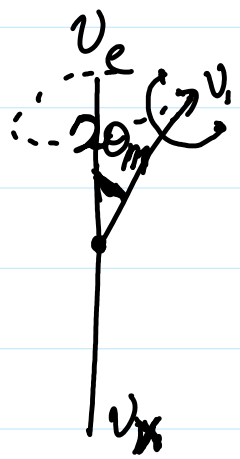
Tuesday, March 29, 2016 10:32 AM

Hamiltonian

$$H = \frac{\omega_m}{2}\begin{pmatrix} -\cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & \cos 2\theta_m \end{pmatrix}$$

$$i\frac{\partial}{\partial x}\begin{pmatrix}\psi_e \\ \psi_x\end{pmatrix} = \frac{\omega_m}{2}\begin{pmatrix} -\cos 2\theta_m & \sin 2\theta_m \\ \sin 2\theta_m & \cos 2\theta_m \end{pmatrix}\begin{pmatrix}\psi_e \\ \psi_x\end{pmatrix}$$

small



$$i\frac{\partial}{\partial x}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} = \frac{\omega_m}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$$

#

How to understand the picture

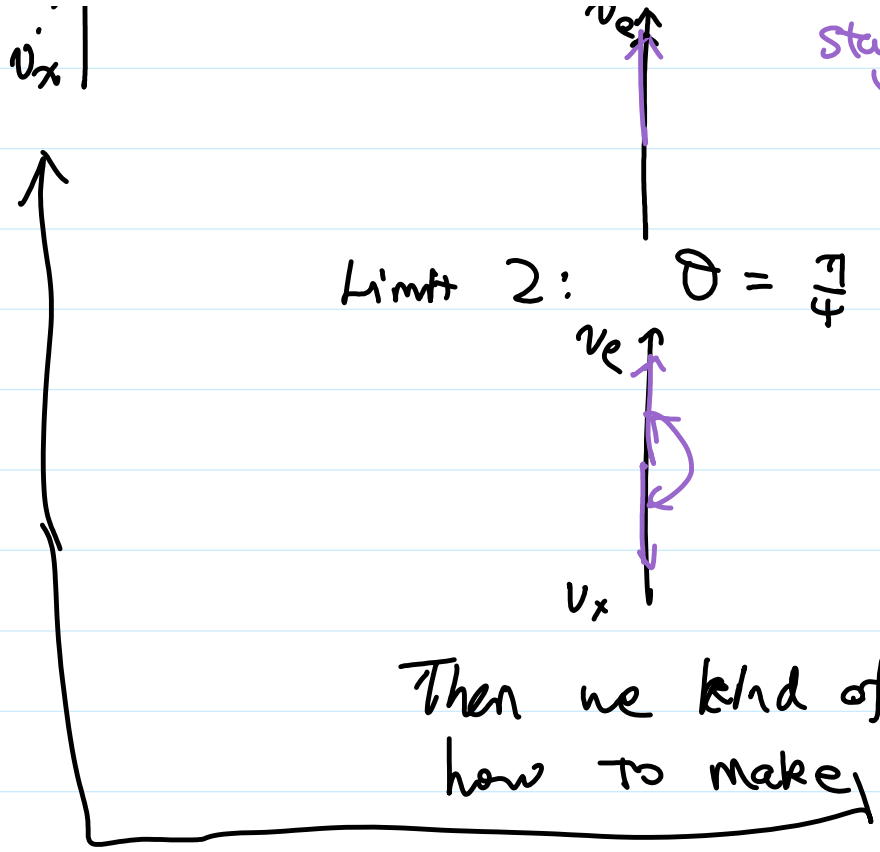
① Vacuum, oscillation

How to understand it?

Look at limits

Limit 1: $\theta_v = 0$

stays here



stays here

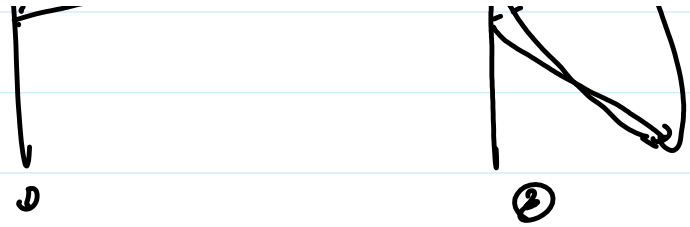
Limit 2: $\theta = \frac{\pi}{4}$ ⇒ Full conversion

Then we kind of figure out how to make

② Matter: We define θ_m then everything is the same as vacuum, we just replace $\theta_v \to \theta_m$

③ Slow varying λ → adiabatic

④ Fast varying λ



A more rigorous way: Flavor Isospin

Equation of motion in flavor basis

$$i\frac{\partial}{\partial x}\psi = \left(\frac{1}{2}(\lambda - \omega\cos 2\theta_v)\sigma_3 + \frac{1}{2}\omega\sin 2\theta_v\,\sigma_1\right)\psi$$

Define $\vec{S} = \psi^\dagger \frac{\vec{\sigma}}{2}\psi$

and also $\vec{H}$, which satisfies

$$H = -\frac{\vec{H}\cdot\vec{\sigma}}{2} =$$

because we don't need this

$$\vec{\sigma} = \sigma_1\hat{e}_1 + \sigma_2\hat{e}_2 + \sigma_3\hat{e}_3$$

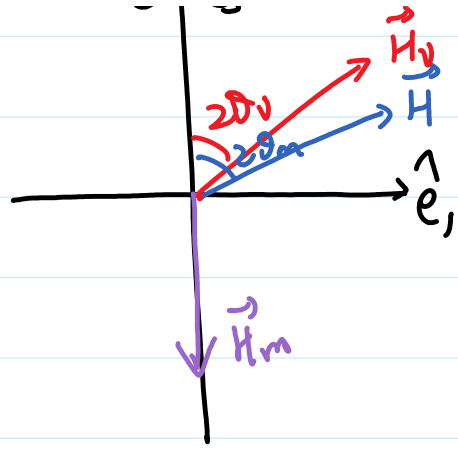
also for vacuum

$$H_v = -\frac{1}{2}\vec{\sigma}\cdot\vec{H}_v = -\frac{\omega}{2}\cos 2\theta_v\,\sigma_3 + \frac{\omega}{2}\sin 2\theta_v\,\sigma_1$$

and matter effect, by

$$H_m = -\frac{1}{2}\vec{\sigma}\cdot\vec{H}_m = \frac{\lambda}{2}\sigma_3$$

$$\vec{H}_v \Rightarrow \begin{pmatrix} -\omega\sin 2\theta_v \\ 0 \\ \omega\cos 2\theta_v \end{pmatrix}, \quad \vec{H}_m \to \begin{pmatrix} 0 \\ 0 \\ -\lambda \end{pmatrix}$$



Equation of motion

$$i\partial_x \psi = -\frac{\vec{\sigma}}{2}\cdot\vec{H}\,\psi$$

$$i\partial_x \psi^\dagger \frac{\vec{\sigma}}{2}\psi = \psi^\dagger \frac{\vec{\sigma}}{2}\left(-\frac{\vec{\sigma}}{2}\cdot\vec{H}\right)\psi$$

$$-\frac{1}{4}\psi^\dagger\,\vec{\sigma}$$

How is this picture useful?

1. MSW conversion
2. Parametric resonance.

$$\begin{pmatrix}\psi_e^* & \psi_x^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix}$$

$$= \begin{pmatrix}\psi_e^* & -\psi_x^*\end{pmatrix}\begin{pmatrix}\psi_e\\ \psi_x\end{pmatrix}$$

$$= \psi_e\psi_e^* - \psi_x^*\psi_x$$

Rabi oscillations

Thursday, June 23, 2016 5:49 PM

$$\begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$h = -\frac{1}{2} \sum_{n_1} (-i)^{n_1+n_2} (n_1 k_1 + n_2 k_2) J_{n_1}\left(\frac{A_1}{k_1}\cos\theta_m\right) J_{n_2}\left(\frac{A_2}{k_2}\cos\theta_m\right) e^{i(n_1 k_1 + n_2 k_2 - \omega_m)X}$$

For one mode it's obvious

For two modes how can another mode change

Suppose we have one mode at resonance $\quad k_1 = 1, \quad k_2 = k_2$

n_1, n_2 ~~$n_1 = 0, n_2 = 0$~~ $\{1, 0\}$

$$h^{(1)} = -\frac{1}{2} J_1\left(\frac{A_1}{k_1}\cos\theta_m\right) J_0\left(\frac{A_2}{k_2}\cos\theta_m\right)$$

Now adding another mode

$$h^{(2)} = -\frac{1}{2} \sum (-i)^{n_1+n_2} (n_1 k_1 + n_2 k_2) J_{n_1}\left(\frac{A_1}{k_1}\cos\theta_m\right) J_{n_2}\left(\frac{A_2}{k_2}\cos\theta_m\right) e^{i(n_1 \cdots)}$$

$$\begin{pmatrix} 0 & h^{(1)} + h^{(2)} \\ & \end{pmatrix}$$

Berry Phase Basics

Friday, June 17, 2016 10:15 AM

For a two level system, with period matter perturbation

$$\lambda = \lambda_0 + \delta\lambda$$

$$\delta\lambda = A \sin(kx + \phi)$$

In background matter basis

$$H =$$

parametric resonance

Monday, July 25, 2016 10:26 AM

parametric resonance can be explained as

$$\int_0^{l_f} dx \frac{2\pi}{l_m} = 2\pi k$$

$$l_m = \frac{2\pi}{\omega_m} \qquad \omega_m = \frac{2\pi}{l_m}$$

$$\int_0^{l_f} dx\, \omega_m \sim 2\pi k$$

$$\omega_m = \omega_v \sqrt{\left(\frac{\lambda}{\omega} - \cos 2\theta_v\right)^2 + \sin 2\theta_v} \qquad \lambda = \lambda_0 + \lambda_1 \cos(kx)$$

$$\omega_v \int_0^{l_f} dx \sqrt{\left(\frac{\lambda}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}$$

$$\approx \omega_v \int_0^{l_f} dx \left| \frac{\lambda}{\omega_v} - \cos 2\theta_v \right|$$

$$\sim \omega_v \int_0^{l_f} dx \left(\cos 2\theta_v - \frac{\lambda}{\omega_v} \right)$$

$$(\omega_v \cos 2\theta_v - \lambda)\, l_f \sim 2\pi n \quad , n = 1, 2, 3, \ldots$$

$$\boxed{l_f = \frac{2\pi}{k} \qquad (\omega_v \cos 2\theta_v - \lambda) \sim nk}$$

$$\Downarrow$$

$$nk \sim \omega_m$$

Akhmedov's method

$$H = -\frac{1}{2}\omega_m \sigma_3 + \frac{1}{2}\lambda \cos 2\theta_m \sigma_3 + \frac{1}{2}\lambda \sin 2\theta_m \sigma_1$$

$$\Lambda = \sum_{q=-\infty,\, q\neq 0}^{\infty} \Lambda_q e^{iq\omega_0 x}, \qquad \omega_0 = \frac{2\pi}{X_1 + X_2}$$

Now rotate to the other frame

$$i\frac{d}{dx}\begin{pmatrix}\psi_{r1}\\ \psi_{r2}\end{pmatrix} = -\frac{\delta\lambda}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta} \\ e^{-2i\eta} & 0\end{pmatrix}\begin{pmatrix}\psi_{r1}\\ \psi_{r2}\end{pmatrix}$$

$$\eta = -\frac{\omega_m}{2}x + \frac{\cos 2\theta_m}{2}\int_0^x \delta\lambda(\tau)\,d\tau$$

$$= -\frac{\omega_m}{2}x + \frac{1}{2}\cos 2\theta_m \int_0^x \sum_{q\neq 0,\, q=-\infty}^{\infty} \Lambda_q e^{i\omega_0 q\tau}\,d\tau$$

$$= -\frac{\omega_m}{2}x + \frac{1}{2}\cos 2\theta_m \sum_{q\neq 0,\, q=-\infty}^{\infty} \Lambda_q \int_0^x e^{i\omega_0 q\tau}\,d\tau$$

$$= -\frac{\omega_m}{2}x + \frac{1}{2}\cos 2\theta_m \sum_{q\neq 0,\, q=-\infty}^{\infty} \Lambda_q \frac{1}{i\omega_0 q} e^{i\omega_0 q x}$$

$$\begin{pmatrix} e^{+i\eta^*} & 0 \\ 0 & e^{-i\eta^*}\end{pmatrix}\begin{pmatrix} e^{-i\eta} & 0 \\ 0 & e^{i\eta}\end{pmatrix} = \begin{pmatrix} e^{i\eta^* - i\eta} & 0 \\ 0 & \end{pmatrix}$$

Using an even Fourier series

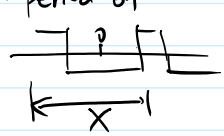
$$\lambda(x) = \frac{\Lambda_0}{2} + \sum_{m=1}^{\infty} \Lambda_q \cos\left(\frac{q2\pi x}{X}\right)$$

$$\Lambda_0 = \frac{2}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} \lambda(\tau)\,d\tau$$

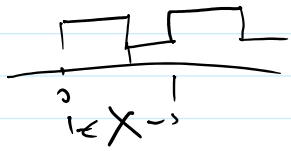
$$\Lambda_q = \frac{2}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} \lambda(\tau)\cos\left(\frac{q\cdot 2\pi\tau}{X}\right)d\tau$$

$$\omega_0 = \frac{2\pi}{X}$$

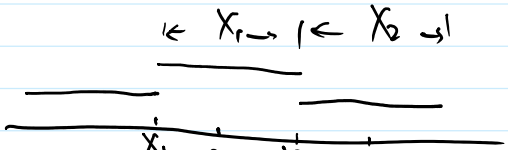
X: period of

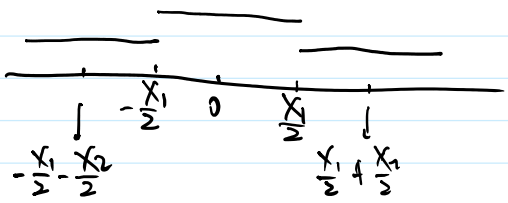


ODD:



no constant term





$$\int_a^b \cos(n\omega_0 x)\,dx = \frac{1}{n\omega_0}\sin(n\omega_0 x)\Big|_a^b$$

$$A_q = \frac{2}{X}\Big(\lambda_2 \frac{1}{n\omega_0}\Big(\sin\big(-\frac{n\omega_0 X_1}{2}\big) - \sin\big(-\frac{n\omega_0 X}{2}\big)\Big)$$
$$+ \lambda_1 \frac{1}{n\omega_0}\Big(\sin\big(n\omega_0 \frac{X_1}{2}\big) - \sin\big(-n\omega_0 \frac{X_1}{2}\big)\Big)$$
$$+ \lambda_2 \frac{1}{n\omega_0}\Big(\sin\big(n\omega_0 \frac{X}{2}\big) - \sin\big(n\omega_0 \frac{X_1}{2}\big)\Big)\Big)$$
$$= \frac{2}{X}\Big(\frac{2\lambda_2}{n\omega_0}\Big(\sin\big(n\omega_0 \frac{X}{2}\big) - \sin\big(n\omega_0 \frac{X_1}{2}\big)\Big)$$
$$+ \frac{2\lambda_1}{n\omega_0}\sin\big(n\omega_0 \frac{X_1}{2}\big)\Big)$$
$$= \frac{4}{n\omega_0 X}\Big(\lambda_2 (\qquad\qquad) + \lambda_1 (\quad)\Big)$$

$$\omega_0 X = \frac{2\pi}{X}\cdot X = 2\pi$$

$$= \frac{2}{n\pi}\Big(\qquad\qquad\qquad\qquad\Big)$$

Set the first mode at resonance

$\omega_0 \sim \omega_m$

$$n \cdot q\,\omega_0 \sim \omega_m$$

$$\frac{\tan(\omega_{m1} X_1/2)}{\tan(\omega_{m2} X_2/2)} = -\frac{\cos 2\theta_{m2}}{\cos 2\theta_{m1}}$$

① For Rabi : choose ~~A_0 so~~

$$q \cdot \omega_0 \sim \omega_m$$

$$\omega_m = \omega_0 \sqrt{\Big(\frac{\Lambda_0}{\omega_0} - \cos 2\theta_0\Big)^2 + \sin^2 2\theta_0}$$

$$\omega_0 = \frac{2\pi}{X} = \frac{2\pi}{X_1 + X_2}$$

using ω_m as unit

$$q\,\omega_0 \sim 1$$

choose $q = 1$

$$\frac{2\pi}{X_1 + X_2} \sim 1$$

$$\overline{X_1 + X_2} \sim 1$$

② In want of ω_m,

$$\frac{\tan(\omega_{m1} X_1/2)}{\tan(\omega_{m2} X_2/2)} = -\frac{\cos 2\theta_{m2}}{\cos 2\theta_{m1}}$$

becomes

$$\omega_{m1} = \frac{\omega_v \sqrt{\left(\frac{\lambda_1}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}}{\omega_v \sqrt{\left(\frac{\lambda_0}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}}$$

$$= \sqrt{\frac{\left(\frac{\lambda_0}{\omega_v} - \cos 2\theta_v + \frac{\lambda_1 - \lambda_0}{\omega_v}\right)^2 + \sin^2 2\theta_v}{\left(\quad\right)^2 + \quad^2}}$$

$$\omega_v\left[\left(\frac{\lambda_0}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v\right] \sim \omega_m$$

$$\frac{\omega_v}{\omega_m}\left[\left(\frac{\lambda_0/\omega_m}{\omega_v/\omega_m} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v\right] \sim 1$$

$$\hat{\omega}_v\left[\left(\frac{\hat{\lambda}_0}{\hat{\omega}_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v\right] \sim 1$$

$$\hat{\omega}_v\left[\left(\frac{F \cdot \cos 2\theta_v\, \hat{\omega}_v}{\hat{\omega}_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v\right] \sim 1$$

$$\hat{\omega}_v\left[(F-1)^2 \cos^2 2\theta_v + \sin^2 2\theta_v\right] \sim 1$$

$$\hat{\omega}_{m1} \simeq \frac{\omega_v \sqrt{\left(\frac{\lambda_1}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}}{\omega_v \sqrt{\left(\frac{\lambda_0}{\omega_v} - \cos 2\theta_v\right)^2 + \sin^2 2\theta_v}} = \sqrt{\frac{(\lambda_0 - \Delta}{}}$$

Vacuum Oscillations

Monday, July 25, 2016 11:06 AM

$$H_0 = \frac{\omega_0}{2}\begin{pmatrix} -\cos 2\theta_0 & \sin 2\theta_0 \\ \sin 2\theta_0 & \cos 2\theta_0 \end{pmatrix}$$

$$= -\frac{\omega_0}{2}\cos 2\theta_0\,\sigma_3 + \frac{\omega_0}{2}\sin 2\theta_0\,\sigma_1$$

is a resonance already, width

$$w = \frac{\omega_0}{2}\sin 2\theta_0$$

oscillation wavelength

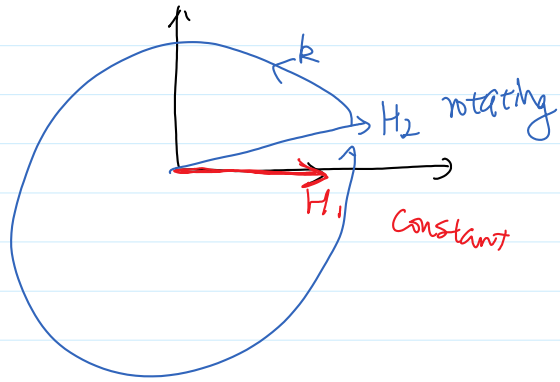
$$\frac{2\pi}{w}$$

Rabi Oscillations

Saturday, September 10, 2016 3:54 PM

Interference

Wednesday, September 21, 2016 10:54 AM



$$H = \begin{pmatrix} 0 & A_1 + A_2 e^{ikx} \\ A_1 + A_2 e^{-ikx} & 0 \end{pmatrix}$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$i \frac{d}{dx} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} (A_1 + A_2 e^{ikx}) \psi_2 \\ (A_1 + A_2 e^{-ikx}) \psi_1 \end{pmatrix}$$

$$\begin{cases} i \frac{d}{dx} \psi_1 = (A_1 + A_2 e^{ikx}) \psi_2 \\ i \frac{d}{dx} \psi_2 = (A_1 + A_2 e^{-ikx}) \psi_1 \end{cases}$$

Solve ψ_2

$$\psi_1 = \frac{1}{A_1 + A_2 e^{-ikx}} \frac{d}{dx} \psi_2$$

$$- \frac{d}{dx} \left(\frac{1}{A_1 + A_2 e^{-ikx}} \frac{d}{dx} \psi_2 \right) = (A_1 + A_2 e^{ikx}) \psi_2$$

$$- \left(\frac{d}{dx} \frac{1}{A_1 + A_2 e^{-ikx}} \right) \frac{d}{dx} \psi_2 - \frac{1}{A_1 + A_2 e^{-ikx}} \frac{d^2}{dx^2} \psi_2 = (A_1 + A_2 e^{ikx}) \psi_2$$

$$0 = (A_1 + A_2 e^{-ikx}) \left(\frac{d}{dx} \frac{1}{A_1 + A_2 e^{-ikx}} \right) \frac{d}{dx} \psi_2 + \frac{d^2}{dx^2} \psi_2 + (A_1 + A_2 e^{-ikx})(A_1 + A_2 e^{ikx}) \psi_2$$

$$\frac{i A_2 e^{-ikx} k}{A_1 + A_2 e^{-ikx}} \frac{d}{dx} \psi_2 + \frac{d^2}{dx^2} \psi_2 +$$

$$\left[(A_1 + A_2 e^{-ikx})(A_1 + A_2 e^{ikx}) = A_1^2 + A_2^2 + A_1 A_2 e^{ikx} + A_1 A_2 e^{-ikx} \right.$$

$$\frac{d^2}{dx^2}\psi_2 + \underbrace{\frac{i k A_2 e^{-ikx}}{A_1 + A_2 e^{-ikx}}}_{\Downarrow} \frac{d}{dx}\psi_2 + \left(A_1^2 + A_2^2 + 2A_1 A_2 \cos(kx)\right) = 0$$

$$\frac{i k A_2}{A_1 e^{ikx} + A_2} \sim i k A_2 \frac{1}{A_1 \cos(kx) + A_2 + i A_1 \sin(kx)}$$

$$\sim i k A_2 \frac{1}{|A_2 + A_1 e^{ikx}|^2} \left(A_1 \cos(kx) + A_2 - i A_1 \sin(kx) \right)$$

$$\psi_2 = C_2 + \frac{1}{2}\left[x\left(\frac{2iA_2^2}{k} - A_1^2 x\right) - \frac{2i A_1 A_2 x \cos(kx)}{k} \right.$$

$$- \frac{2i A_1 (A_1 - A_2) x \, C_1 \sqrt{}}{\sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(kx)}}$$

Define $M = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(kx)}$

$$N = \sqrt{\frac{(A_1 + A_2)^2}{(A_2 - A_1)^2 k^2}} \qquad Y = \csc\left(\frac{kx}{2}\right)$$

$$\psi_2 = \frac{\sin\left(\frac{kx}{2}\right)}{2 N \cdot k^3 M} \cdot \left[-2 A_2^2 N \cdot k \cdot M \cdot Y \right.$$

$$- 2A_2 N \cdot k^2 M \cdot Y + 2i A_2^2 N \cdot k^2 x \cdot M \cdot Y$$

$$- A_1^2 N k^3 x^2 M \cdot Y - 2i A_1 A_2 N k^2 x \cos(kx) M Y$$

$$- 2i A_1 A_2 N \cdot \sqrt{k^2}\, k x M \sqrt{Y^2} - 2i A_1 N \sqrt{k^2}\, k^2 x M \sqrt{Y^2}$$

$$+ 2A_2^2 N \sqrt{k^2} \cos(kx) M \sqrt{Y^2} + 2A_2 N \sqrt{k^2}\, k^2 x \cos(kx) M \cdot \sqrt{Y^2}$$

$$+ 4i A_1 A_2 N M Y k \sin(kx) - 2A_1 A_2 N k^2 x M Y \sin(kx)$$

$$\left. -2i A_2^2 N \sqrt{k^2} M \sqrt{Y^2} \sin(kx) - 2i A_2 N \sqrt{k^2}\, k M \sqrt{Y^2} \sin(kx) \right]$$

Assume $k>0$

$$\mathrm{Re}[\psi_2] = \frac{\sin(kX)}{2Nk^3M}\Big[-2A_2NkMY(A_2+k) - A_1^2Nk^3X^2MY$$
$$+2A_2^2Nk\cos(kX)M\sqrt{Y^2} + 2A_2Nk^3\cos(kX)M\sqrt{Y^2}$$
$$-2A_1A_2Nk^2X\cdot MY\sin(kX)\Big]$$
$$= \frac{\sin(kX/2)}{2\cancel{N}k^3\cancel{M}}\cdot \cancel{N}k\cancel{M}\Big[-2A_2Y(A_2+k) - A_1^2k^2X^2Y$$
$$+2A_2^2\cos(kX)\sqrt{Y^2} + 2A_2k^2\cos(kX)\sqrt{Y^2}$$
$$-2A_1A_2kXY\sin(kX)\Big]$$

$$\psi_2 = \frac{\sin\left(\frac{kX}{2}\right)}{2N\cdot k^3M}\cdot\Big[-2A_2^2N\cdot k\cdot M\cdot Y$$
$$-2A_2N\cdot k^2M\cdot Y + 2iA_2^2N\cdot k^2X\cdot M\cdot Y$$
$$-A_1^2Nk^3X^2M\cdot Y - 2iA_1A_2Nk^2X\cos(kX)MY$$
$$-2iA_1A_2N\cdot\sqrt{k^2}kXM\sqrt{Y^2} - 2iA_1N\sqrt{k^2}k^2XM\sqrt{Y^2}$$
$$+2A_2^2N\sqrt{k^2}\cos(kX)M\sqrt{Y^2} + 2A_2N\sqrt{k^2}k^2\cos(kX)M\cdot\sqrt{Y^2}$$
$$+4iA_1A_2NMYk\sin(kX) - 2A_1A_2Nk^2XMY\sin(kX)$$
$$-2iA_2^2N\sqrt{k^2}M\sqrt{Y^2}\sin(kX) - 2iA_2N\sqrt{k^2}kM\sqrt{Y^2}\sin(kX)\Big]$$

$$\mathrm{Im}(\psi_2) = \frac{\sin(kX/2)}{2Nk^3M}\Big[2A_2^2Nk^2XMY - 2A_1A_2Nk^2X\cos(kX)MY$$
$$-2A_1A_2Nk^2XM\sqrt{Y^2} - 2A_1Nk^3XM\sqrt{Y^2} + 4A_1A_2NMYk\sin(kX)$$
$$-2A_2^2NkM\sqrt{Y^2}\sin(kX) - 2A_2Nk^2M\sqrt{Y^2}\sin(kX)$$

$$= \frac{\sin(kX/2)}{2\cancel{N}k^3\cancel{M}}\cancel{N}k\cancel{M}\Big[2A_2^2kXY - 2A_1A_2kX\cos(kX)Y$$
$$-2A_1A_2kX\sqrt{Y^2} - 2A_1k^2X\sqrt{Y^2} + 4A_1A_2Y\sin(kX)$$
$$-2A_2^2\sqrt{Y^2}\sin(kX) - 2A_2k\sqrt{Y^2}\sin(kX)\Big]$$

Rederive the equation of motion

$$i\frac{d}{dx}\psi_1 = (A_1 + A_2e^{ikX})\psi_2$$
$$i\frac{d}{dx}\psi_2 = (A_1 + A_2e^{-ikX})\psi_1$$

$$\psi_1 = \frac{}{A_1 + A_2 e^{-ikx}} \; i \frac{d}{dx}\psi_2$$

$$-\cancel{i}\frac{d}{dx}\left[\frac{1}{A_1 + A_2 e^{-ikx}} \frac{d}{dx}\psi_2\right] = (A_1 + A_2 e^{ikx})\psi_2$$

$$\frac{1}{A_1 + A_2 e^{-ikx}}\frac{d^2}{dx^2}\psi_2 + \left(\frac{d}{dx}\frac{1}{A_1 + A_2 e^{-ikx}}\right)\frac{d}{dx}\psi_2 + (A_1 + A_2 e^{ikx})\psi_2 = 0$$

$$\frac{d^2}{dx^2}\psi_2 + \frac{i A_2 e^{-ikx} k}{A_1 + A_2 e^{-ikx}}\frac{d}{dx}\psi_2 + \left|A_1 + A_2 e^{ikx}\right|^2 \psi_2 = 0$$

$$\frac{d}{dx}\psi_2 + \frac{i k A_2}{A_1 e^{ikx} + A_2}\frac{d}{dx}\psi_2 + (A_1 + A_2 e^{ikx})^2 \psi_2 = 0$$

$$\frac{i k A_2}{A_1 e^{ikx} + A_2} = \frac{i k A_2}{A_1 \cos kx + i A_1 \sin kx + A_2} = \frac{i k A_2 (A_1 \cos kx + A_2 - i A_1 \sin kx)}{(A_1 \cos kx + A_2)^2 + (A_1 \sin kx)^2}$$

$$= \frac{k A_1 A_2 \sin kx + i k A_2 (A_1 \cos kx + A_2)}{(A_1 \cos kx + A_2)^2 + (A_1 \sin kx)^2}$$

$$= k\frac{A_1 A_2 \sin kx}{A_1^2 + A_2^2 + 2 A_1 A_2 \cos kx} + i k \frac{A_1 A_2 \cos kx + A_2^2}{A_1^2 + A_2^2 + 2 A_1 A_2 \cos kx}$$

$$(A_1 + A_2 e^{ikx})^2 = A_1^2 + A_2^2 + 2 A_1 A_2 \cos kx$$

Pictures

Wednesday, October 5, 2016 4:59 PM

1. Resonance

2. Short wavelength ~~are~~ resnance, long wavelength not

$k_1 \quad > \quad k_2$

① Frame corotate with k_2

$$k_1' = k_1 - k_2$$
$$\omega_m' = \omega_m - k_2$$

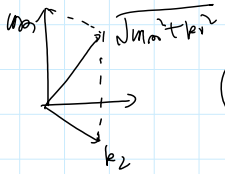
Relative detuning becomes

$$\frac{|k_1' - \omega_m'|}{A_1} = \frac{|k_1 - k_2 - (\omega_m - k_2)|}{A_1}$$
$$= \frac{k_1 - \omega_m}{A_1}$$

② Frame k_1 static

$$k_2' = k_2 - k_1$$
$$\omega_m' = \omega_m - k_1$$

③ Trick invented by ~~Hamiltonian~~



$$\left| k_1 - \sqrt{\omega_m^2 + k_2^2} \right| \geq |A_1|$$

only works for k_1 at/close resonance

$k_1 \gtrsim k_2$

~~thus~~ thus this

this is nice

Rabi Oscillations & Neutrino oscillations

Wednesday, October 19, 2016 12:30 PM

Elevator speech?

1. Rabi oscillations

E_2

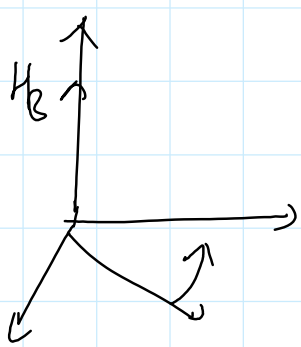
E_1

$$\omega_m = \Delta E = E_2 - E_1$$

$$H = -\frac{\omega_m}{2}\sigma_3 + \begin{pmatrix} 0 & A e^{ikx} \\ A e^{-ikx} & 0 \end{pmatrix}$$

$$= -\frac{\omega_m}{2}\sigma_3 + \begin{pmatrix} 0 & A\cos(kx) + iA\sin(kx) \\ A\cos(kx) - i\ldots & 0 \end{pmatrix}$$

$$= \underbrace{-\frac{\omega_m}{2}\sigma_3}_{} + A\cos(kx)\,\sigma_1 - A\sin(kx)\,\sigma_2$$

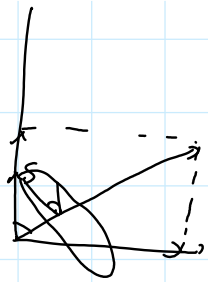


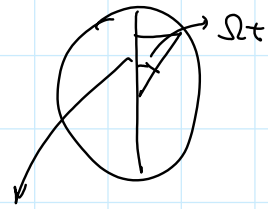
Define vectors

$$-\frac{\vec{\sigma}}{2}\cdot\vec{H} = H$$

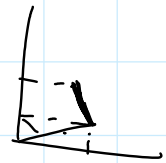
$$\vec{S} = \psi^\dagger \frac{\vec{\sigma}}{2}\psi$$

$$\frac{d}{dt}\psi = H\psi$$





$\frac{1}{2}\cdot\cos(\Omega t)$.

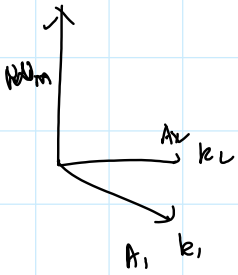


$\frac{1}{2}\cos$

$\frac{\frac{1}{2}}{\pi} = \frac{\sqrt{A_1^2 + \omega_0^2 A_1}}{\omega_n - k_1}$

$P(r) = \frac{1}{2}(1 - 2S_3)$

Interference



Monday, October 24, 2016 12:21 PM

Different modes

Neutrino oscillation & matter

Saturday, August 29, 2015 5:58 PM

* What is neutrino oscillation?

$$\psi_f = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \qquad \text{initial} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{propagate}} \begin{pmatrix} p \\ 1-p \end{pmatrix}$$

* Review of 2-level system

$$H \Longrightarrow \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \longrightarrow \text{two eigenstates } |1\rangle, |2\rangle$$

$$\langle 1|2\rangle = 0, \quad \langle 1|1\rangle = \langle 2|2\rangle = 1$$

1. init $|\psi(t=0)\rangle = |1\rangle$,

The time evolution of the state is

$$|\psi(t)\rangle = |1\rangle e^{-i\epsilon_1 t}$$

What is the probability to find the state in $|1\rangle$?

$$|\langle 1|\psi(t)\rangle|^2 \sim 1$$

2. What if we start with initial state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right) \quad ?$$

Time evolution of the system is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle e^{-i\epsilon_1 t} + |2\rangle e^{-i\epsilon_2 t}\right)$$

$$\approx \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle e^{-i(\epsilon_2-\epsilon_1)t}\right)$$

The probability of find it is remains on original state is

$$|\langle\psi(0)|\psi(t)\rangle|^2$$

$$= \frac{1}{4}\left|\left(\langle 1| + \langle 2|\right)\left(|1\rangle + |2\rangle e^{-i(\epsilon_2-\epsilon_1)t}\right)\right|^2$$

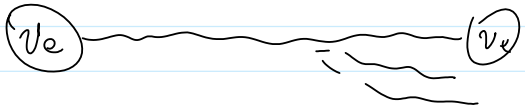
$$= \frac{1}{4}\left|\left(1 + e^{-i(\epsilon_2-\epsilon_1)t}\right)\right|^2$$

$$= \frac{1}{4}\left|1 + \cos((\epsilon_2-\epsilon_1)t) - i\sin((\epsilon_2-\epsilon_1)t)\right|^2$$

$$= \frac{1}{4}\left(\left(1+\cos((\epsilon_2-\epsilon_1)t)\right)^2 - \sin((\epsilon_2-\epsilon_1)t)^2\right)$$

$$= \frac{1}{4}\left(1 + 2\cos((\epsilon_2-\epsilon_1)t) + \cos(2(\epsilon_2-\epsilon_1)t)\right)$$

* Go back to neutrinos



Simple idea How does neutrino propagate?

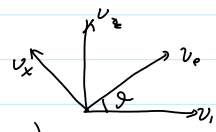
$\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$ at high energy limit $\vec{p} \ll$

we go

$$\frac{1}{2E}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}$$

has eigenstates $|\nu_1\rangle$, $|\nu_2\rangle$,

Suppose $\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$



$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ why? unitary!

Initial condition $|\nu_e\rangle$ in the basis $\{\nu_e, \nu_x\}$
we have

$\Downarrow$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad U^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

In $\{|\nu_1\rangle, |\nu_2\rangle\}$ basis

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{U^{-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$|\psi(t=0)\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$\Rightarrow |\psi(t)\rangle = \cos\theta\,|\nu_1\rangle e^{-iE_1 t} + \sin\theta\,|\nu_2\rangle e^{-iE_2 t}$$

$$|\langle \nu_e | \psi(t)\rangle|^2 =$$

** In mass eigenstates

In flavor basis

$$V_f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Transform back to vacuum mass eigenstates

Transform back to vacuum mass eigenstates

$$V = U^{-1} V_f U$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \Delta\cos\theta & 0 \\ \Delta\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \Delta\cos^2\theta & \Delta\sin\theta\cos\theta \\ \Delta\sin\theta\cos\theta & \Delta\sin^2\theta \end{pmatrix}$$

$$H_0 + V = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \Delta \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

$$E = \sqrt{p^2 + m^2} = -\frac{1}{2}$$

$$i\hbar\partial_t \psi_f = V_f \psi_f$$

$$\psi_f = U\psi_m$$

$$i\hbar\partial_t \underbrace{U^{-1}\psi_f}_{\psi_m} = U^{-1}V_f U \underbrace{U^{-1}\psi_f}$$

In flavor basis

$$\underbrace{\begin{pmatrix} c & -s \\ s & c \end{pmatrix}}_{U^{-1}} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} cE_1 & -sE_2 \\ sE_1 & cE_2 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c^2E_1 + s^2E_2 & csE_1 - scE_2 \\ scE_1 - scE_2 & s^2E_1 + c^2E_2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^2E_1 + s^2E_2 + 0 \end{pmatrix}$$

set $E_2 = 0$

$$\delta E = -\frac{E_2 - E_1}{2} + \frac{E_2 + E_1}{2}$$

$$\frac{E_2 - E_1}{2} + \frac{E_2 + E_1}{2}$$

$$V = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c & s \\ s & -c \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$= \begin{pmatrix} c^2-s^2 & cs+sc \\ cs+sc & s^2-c^2 \end{pmatrix}$$

I am going to talk about the paper by Wolfenstein. We all know neutrino oscillations reveals the fact that neutrinos has mass. What Wolfenstein proposed is that the interactions with matter can play a big role.

Here is the outline. First of all we will have a review of neutrino oscillations, from which we can find out the very nature of neutrino oscillations. Then we add in matter and try to see what is the role of matter.

Neutrino oscillation is a phenomenon that the # of neutrinos for each flavor is not conserved. For example, if all neutrinos produced at a source are electron flavor, we could have all three different flavors at the detection.

Why is that true? Before we start I would like to remind you a trick we used in QM. Suppose we have a Hamiltonian

$$\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix} -(E_2-E_1) & 0 \\ 0 & E_2-E_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} E_2+E_1 & 0 \\ 0 & E_2+E_1 \end{pmatrix}$$

This is a two level system. We can remove the trace without changing the dynamics of the system. Why is that true? The solution to the system is

$$|\psi(t)\rangle = c_1|1\rangle e^{-iE_1 t} + c_2|2\rangle e^{-iE_2 t}$$

$$\Rightarrow e^{\cdot}\Big(\qquad \Big)$$

$$E_1 = \frac{E_2+E_1}{2} - \frac{E_2-E_1}{2}$$

$$E_2 = \frac{E_2+E_1}{2} + \frac{E_2-E_1}{2}$$

So, what is the Hamiltonian of free neutrinos?

…….

So Why do we have this period structure in the survival probability? Suppose we don't have the mixing that is the eigen energies are the eigenvalues of flavor states? Say we have E_1 for electron flavor. What is the solution to the

$$
\begin{aligned}
\mathbf{H}_\alpha &= \mathbf{U}\hat{H}_j\mathbf{U}^T \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \right) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} \cos^2\theta m_1^2 + \sin^2\theta m_2^2 & -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 \\ -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 & \sin^2\theta m_1^2 + \cos^2\theta m_2^2 \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 - \delta^2 m\sin^2\theta & -\frac{1}{2}\sin 2\theta\delta m^2 \\ -\frac{1}{2}\sin 2\theta\delta m^2 & m_2^2 + \delta m^2\sin^2\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\left(\frac{1}{2}(m_1^2 + m_2^2)\mathbf{I} - \frac{1}{2}\begin{pmatrix} -\delta m^2\cos 2\theta & \delta^2 m\sin 2\theta \\ \delta m^2\sin 2\theta & \delta^2 m\cos 2\theta \end{pmatrix} \right) \\
&= \left(p + \frac{m_1^2 + m_2^2}{4p} \right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\delta m^2\cos 2\theta & \delta^2 m\sin 2\theta \\ \delta m^2\sin 2\theta & \delta^2 m\cos 2\theta \end{pmatrix}
\end{aligned}
$$

$$\sin^2\theta \quad \frac{\cancel{\cos^2\theta} - 1 + 1}{2} = \frac{1 - (1 - 2\sin^2\theta)}{2} = \frac{1 - \cos 2\theta}{2}$$

$$\cancel{\times \frac{m_1^2 - m_2^2 + m_1^2 + m_2^2}{2}}$$

$$pI + \frac{1}{2p}\left(m_1^2 - \delta m^2 \frac{1}{2} + \frac{1}{2}\delta m^2 \cos 2\theta \right.$$

$$pI + \frac{1}{2p}\left(\frac{1}{2}(m_1 + m_2) \right.$$

Majorana fermion

Friday, September 11, 2015 6:15 PM

How do you get Dirac eqn?

Lagrangian: ← Real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

The variation:

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$$

$$S = \int d^4x\, \mathcal{L}$$

$$\delta S = \int d^4x\, \delta\mathcal{L}$$

$$= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right)$$

We have $\delta(\partial_\mu\phi) = \underline{\partial_\mu\delta\phi}$

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\phi_{,\mu})}\partial_\mu(\delta\phi) \right)$$

$$= \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right)}_{\text{drop}} - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right)\delta\phi \right)$$

So I have $\delta S = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right)\right)\delta\phi$

So we have

$$\frac{\partial\mathcal{L}}{\partial\phi} - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) = 0$$

Plug in $\mathcal{L} = \frac{1}{2} \phi^*_{,\mu} \phi^{,\mu} - \frac{1}{2} m^2 \phi^2$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \quad , \quad \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \frac{\partial}{\partial \phi_{,\mu}} \left(g^{\mu\nu} \phi_{,\mu} \phi^*_{,\nu} \right)$$

$$= g^{\mu\nu} \phi_{,\mu} + g^{\mu\nu} \phi_{,\mu} \delta_{\mu\nu}$$

$$= \phi^{,\mu} + \phi^{,\mu}$$

$$= 2 \phi^{,\mu}$$

So E-L eqn $\Rightarrow$ $-m^2 \phi + \partial_\mu \phi^{,\mu} = 0$

$$\Rightarrow \quad \partial_\mu \partial^\mu \phi - m^2 \phi = 0$$

$$(\Box - m^2) \phi = 0$$

Complex scalar field:

$$\mathcal{L} = \frac{1}{2} \phi^*_{,\mu} \phi^{,\mu} - \frac{1}{2} m^2 |\phi|^2$$

* Dirac field:

$$\mathcal{L} = \frac{1}{2} i \bar{\psi} \overleftrightarrow{\not{\partial}} \psi - M \bar{\psi} \psi$$

$$= \frac{1}{2} i \bar{\psi} \overrightarrow{\not{\partial}} \psi - \frac{1}{2} i \bar{\psi} \overleftarrow{\not{\partial}} \psi - M \bar{\psi} \psi$$

Since $\not{\partial}(\bar{\psi}\psi) = \bar{\psi} \overrightarrow{\not{\partial}} \psi + (\overrightarrow{\not{\partial}} \bar{\psi}) \psi$

So I can write

$$\mathcal{L} = -\frac{i}{2} \left(\not{\partial}(\bar{\psi}\psi) - \bar{\psi} \not{\partial} \psi \right) + \frac{i}{2} \bar{\psi} \not{\partial} \psi - M \bar{\psi} \psi$$

$$= i\bar{\psi}\partial\psi - m\bar{\psi}\psi$$

So

RMP 87.137

Sunday, September 13, 2015 3:19 PM

First, what is Dirac theory:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

or $(\gamma^\mu p_\mu - m)\psi = 0$

$$p^\mu = (i\partial_t, \vec{p}), \quad p_\mu = (i\partial_t, -\vec{p})$$

$$\left(i\Gamma^{\mu\nu}\partial_\mu - m_\mu I^{\mu\nu}\right)\psi_\nu = 0$$

We have spin coming in because we have $\vec{\sigma}$s

Plus H-M

99

Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \gamma^\mu = (\beta, \beta\vec{\alpha})$$

Majorana said, oh this choice is not unique:

We can find one choice of γ^μ so that

$$i\dot{\psi} = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi$$

has real solution:

$$\tilde{\gamma}^0 = i\begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & \end{pmatrix}, \quad \tilde{\gamma}^1 = i\begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}$$

$$\tilde{\gamma}^2 = i\begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$$

Require

$$(\vec{\alpha}\cdot\vec{p} + \beta m)(\vec{\alpha}\cdot\vec{p} + \beta m) = E^2 = p^2 + m^2$$

$$= (\alpha_i p_i + \alpha_j m \beta)(\alpha_j p_j + \beta m p_i) m$$

$$+ \beta m \alpha_j p_j + \beta^2 m^2$$

$$= \alpha_i^2 p_i^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i) p_i p_j$$

$$+ (\alpha_i\beta + \beta\alpha_i) p_i m$$

$$+ \beta^2 m^2$$

$$= p_i^2 + m^2$$

$\Rightarrow$ - - - -

So that only two equations

Majorana - Dirac eqn

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

use Weyl representation:

$$(i\partial_t - \vec{p}\cdot\vec{\sigma})\psi_R - m_D\psi_L = 0$$

$$(i\partial_t + \vec{p}\cdot\vec{\sigma})\psi_L - m_D\psi_R = 0$$

However in Majorana case

$$(i\partial_t - \vec{p}\cdot\vec{\sigma})\psi_R - i m_R \sigma^2 \psi_R^* = 0$$

$$(i\partial_t + \vec{p}\cdot\vec{\sigma})\psi_L - i m_L \sigma^2 \psi_L^* = 0$$

Majorana states: eigenstates of CP

Charge conjugation: $C = i\gamma^2$: Dirac

$C = I$: Majorana.

CP (Majorana)

If $CP_i\psi_R \to \psi_R$, $CP_i\psi_L \to -\psi_L$ also assuming $m_D = m_R$

Then $\longrightarrow$ Dirac

Summary of 3 basis: using $\gamma^\mu = (\beta, \beta\vec{\alpha})$

Dirac-Pauli	Weyl	Majorana
$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$	$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$	
$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$	$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$	
$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$	$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$	$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$
$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$	$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$	$\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$
	$\gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$	$\gamma^2 = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$
$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$		$\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$
		$\gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$
$\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} I \mp \sigma^2 & 0 \\ 0 & I \pm \sigma^1 \end{pmatrix}$
$\frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$	$\Rightarrow \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & \mp i \\ \mp i & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} \end{pmatrix}$

L: $\frac{1}{2}(1-\gamma^5)$

R: $\frac{1}{2}(1+\gamma^5)$

899999

$$\psi_L = \begin{pmatrix} -i\sigma^2 \psi_L^* \\ \psi_L \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \psi_R \\ i\sigma^2 \psi_R^* \end{pmatrix}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

~~Plug in Majorana rep.~~

$$i(\gamma^0 \partial_t - \gamma^i \partial_i)\psi - m\psi = 0$$

$$\psi_L = \frac{1}{2}(1-\gamma^5)\psi =$$

$$\psi_R = \frac{1}{2}(1+\gamma^5)\psi$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} I \mp \sigma^2 & 0 \\ 0 & I \pm \sigma^1 \end{pmatrix}$$

$$\psi_L = \frac{1}{2}(I - \gamma^5)\psi$$

$$= \frac{1}{2}\begin{pmatrix} \xi \\ \eta \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \sigma^2 \xi \\ -\sigma^2 \eta \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \xi(1-\sigma^2) \\ \eta(1+\sigma^2) \end{pmatrix}$$

$$\psi_R = \frac{1}{2}\begin{pmatrix} \xi(1+\sigma^2) \\ \eta(1-\sigma^2) \end{pmatrix}$$

question

Sunday, September 13, 2015 5:20 PM

1. In GR we have real Tensor equations
However. In Dirac eqn, we
choose representation without specifying
what basis we use

A matrix is representation of rank-2 tensor by specifying a basis

Or in another view, the three different "basis" → Dirac-Pauli, Weyl, Majorana
must mean something. of the equation

Terrestrial

Thursday, September 24, 2015 10:57 AM

Sensitivity

* source-detector distance
* neutrino energy
* power of source
* mass of detector
* detection cross section
* background.

Sensitivity

* Two important parameters:
  - $\sin^2 2\theta$
  - Δm^2

* sensitivity curve

Averaged transition probility

Thursday, September 24, 2015 3:39 PM

$$P_{\nu_\alpha \to \nu_\beta}(L,E) = \frac{1}{2}\sin^2 2\theta\left(1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right)$$

$$= \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

is not possible to obtain in pratice.

We have to do the average!

The only quantity in this experiment is $\frac{L}{E}$ that is controllable

So we have to average over the correct distribution of $\frac{L}{E}$

$$\langle P_{\nu_\alpha \to \nu_\beta}(L,E)\rangle = \frac{1}{2}\sin^2 2\theta_\nu \left(1 - \left\langle \cos\left(\Delta m^2 \frac{L}{2E}\right)\right\rangle\right)$$

as we said

$$\left\langle \cos\left(\Delta m^2 \frac{L}{2E}\right)\right\rangle = \int \cos\left(\Delta m^2 \frac{L}{2E}\right) \overbrace{\phi\left(\frac{L}{E}\right)}^{\text{distribution of } \frac{L}{E}} d\frac{L}{E}$$

* Gaussian distribution $\frac{L}{E} \equiv b$

$$\phi(b) = \frac{1}{\sqrt{2\pi\sigma^2_{L/E}}} \exp\left(\frac{(b-\bar{b})^2}{2\sigma_b^2}\right)$$

$$\left\langle \cos\left(\Delta m^2 \frac{1}{2} b\right)\right\rangle = \cos\left(\Delta m^2 \frac{1}{2}\bar{b}\right) \exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_b\right)^2\right)$$

Independent L & E, $\Rightarrow$

$$\left(\frac{\sigma_{L/E}}{\langle L/E\rangle}\right)^2 = \left(\frac{\sigma_L}{\langle L\rangle}\right)^2 + \left(\frac{\sigma_E}{\langle E\rangle}\right)^2$$

look at Gaussian

$$\exp\left(\frac{1}{2}\left(\frac{b}{\sigma_b} - \frac{\bar{b}}{\sigma_b}\right)^2\right)$$

$$\exp\left(\frac{1}{2}\left(\frac{b}{\sigma_b} - \frac{\bar{b}}{\sigma_b}\right)^2\right)$$

Detector can set limit on $\langle P \rangle \leq P^{max}$

I have [How? I don't understand how to set upperlimit]

$$\frac{1}{2}\sin^2 2\theta \left(1 - \cos\left(\Delta m^2 \frac{1}{2}\bar{b}\right) \exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_b\right)^2\right)\right) \leq P^{max}$$

$$\sin^2 2\theta \leq \frac{2P_{max}}{1 - \cos\left(\Delta m^2 \frac{1}{2}\bar{b}\right) \exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_b\right)^2\right)}$$

* Could use a relation $\sigma_b \propto b$ assuming $\sigma_b = \alpha b$

an example $\sigma_b = 0.2\bar{b}$

So

$$\sin^2 2\theta \leq \frac{2P_{max}}{1 - \cos\left(\Delta m^2 \frac{1}{2}\bar{b}\right) \exp\left(-\frac{1}{2}\left(\frac{1}{2}\Delta m^2 \bar{b}\right)^2\right)}$$

$$\boxed{\sin^2 2\theta \leq f(\Delta m^2 \cdot \bar{b})}$$

is $\frac{1}{2}\Delta m^2 \bar{b} << \pi$ too small

RHS is almost constant

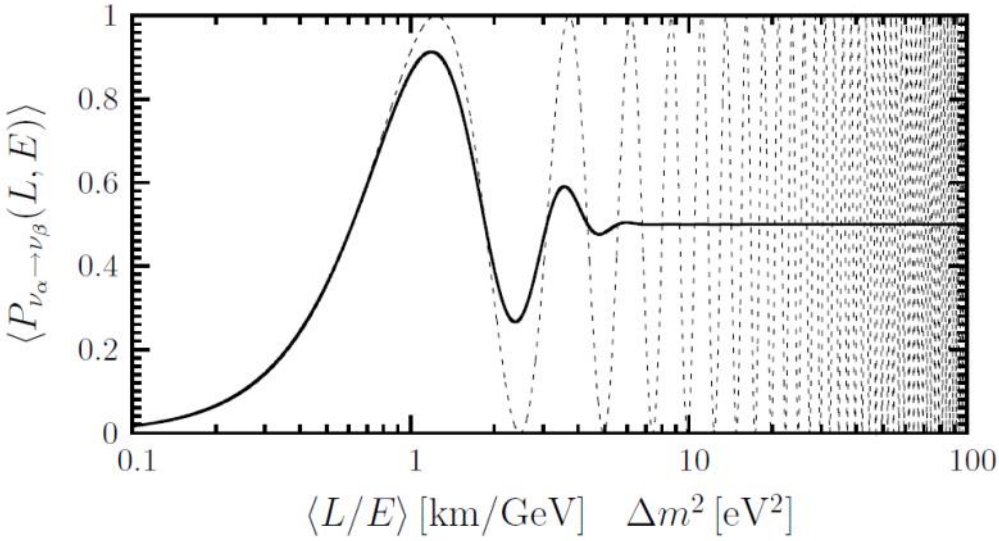
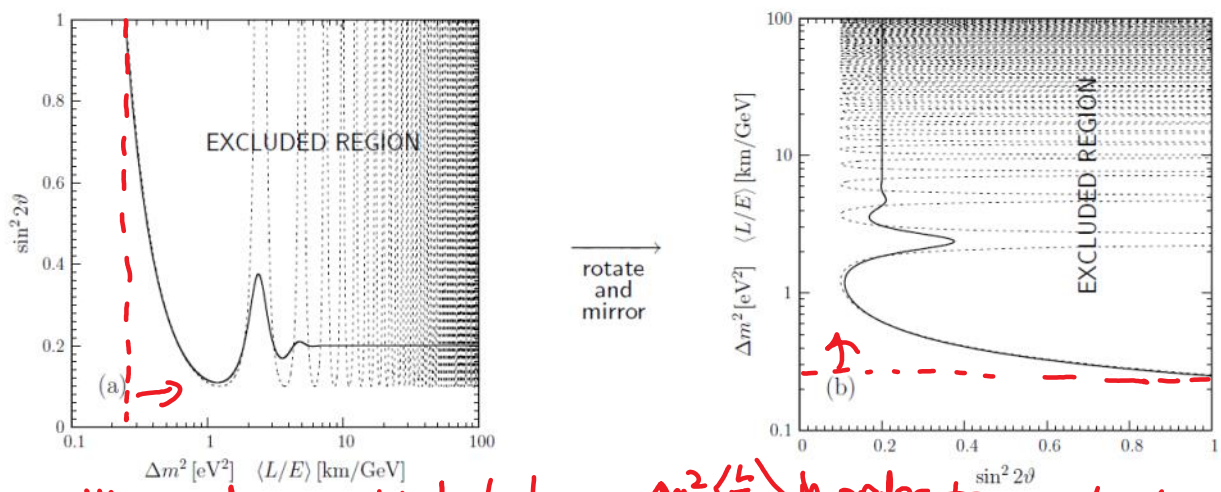


FIG. 7.2. Probability of $\nu_\alpha \to \nu_\beta$ transitions for $\sin^2 2\vartheta = 1$ as a function of $\langle L/E\rangle$ [km/GeV] Δm^2 [eV^2]. The average ratio $\langle L/E\rangle$ can also be expressed in units [m/MeV]. Solid line: transition probability averaged over a Gaussian L/E distribution with $\sigma_{L/E} = 0.2\,\langle L/E\rangle$ (see eqn (7.93)). Dashed line: unaveraged transition probability (see eqn (7.70)), with $L/E = \langle L/E\rangle$.



we need some kind of Large $\Delta m^2 \langle \frac{L}{E} \rangle$ in order to set limits on $\sin^2 2\vartheta$

FIG. 7.3. Upper limit in eqn (7.99) with a Gaussian L/E distribution as a function of Δm^2 [eV^2] $\langle L/E\rangle$ [km/GeV] for $P^{\max}_{\nu_\alpha \to \nu_\beta} = 0.1$. Solid line: $\sigma_{L/E} = 0.2\,\langle L/E\rangle$. Dashed line: $\sigma_{L/E} = 0$, which corresponds to $P_{\nu_\alpha \to \nu_\beta}(L,E) \le P^{\max}_{\nu_\alpha \to \nu_\beta}$ with $L/E = \langle L/E\rangle$, where $P_{\nu_\alpha \to \nu_\beta}(L,E)$ is the unaveraged transition probability in eqn (7.70).

* The best constraint is at

$$\frac{2p^{max}}{1-\langle\cos(\frac{1}{2}\Delta m^2\frac{L}{E})\rangle} \quad \text{max}$$

Which mean

$$\langle\cos(\tfrac{1}{2}\Delta m^2\tfrac{L}{E})\rangle = -1$$

If the range of $\frac{L}{E}$ when doing the averge is fairly small, we can say that

$$\cos(\tfrac{1}{2}\Delta m^2\langle\tfrac{L}{E}\rangle) = -1$$

since $\langle\ \rangle$ $\cos(\ )$ is almost at same argument

which leads to

This is true in experiments

So the most stringent bound occurs at

$$\frac{1}{2}\cdot\Delta m^2\langle\tfrac{L}{E}\rangle \sim \pi$$

correspondingly $\Delta m^2 \langle\frac{L}{E}\rangle \sim 1.24$ if using the corresponding units.

$\Delta m^2 \to eV^2$, $L \to km$, $E \to GeV$

* Another observation is that for large enough $\Delta m^2\langle\frac{L}{E}\rangle$, $\sin^2 2\theta$ doesn't depend on it anymore.

---

basically we have

$$\sin^2 2\theta \leq 2p^{max}$$

the condition

$\to km$

$$\Delta m\left\langle\frac{L}{E}\right\rangle \gg \pi \rightarrow \underset{\text{eV}^2}{\Delta m^2}\underset{\text{GeV}}{\left\langle\frac{L}{E}\right\rangle} \gg 1$$

If we have a really small

$\Delta m^2\left\langle\frac{L}{E}\right\rangle$

if we use units $\Delta m^2 \sim \text{eV}^2$
$L \sim \text{km}$
$E \sim \text{GeV}$

what we should have is

$$1\,\text{eV}^2 \cdot \frac{1\,\text{km}}{1\,\text{GeV}} \sim \frac{1}{1.97}$$

$$\sim 1\,\text{eV}^2 \frac{10^{3+15}}{10^{8}\,\text{eV}\;\; 1.97\,\text{MeV}}$$

$$\sim 1\,\text{eV}^2 \frac{10^{18}}{10^{8+2+6}\;\; 1.97\,\text{eV}^2}$$

So I have

$$\left(\frac{\Delta m^2}{1\,\text{eV}^2}\right)\left(\frac{L/1\,\text{km}}{E/1\,\text{GeV}}\right)\cdot 1.97$$

Example

$$\Delta m^2\left\langle\frac{L}{E}\right\rangle \sim 2\pi$$

$$\Rightarrow \left(\frac{\Delta m^2}{1\,\text{eV}^2}\right)\left(\frac{L/1\,\text{km}}{E/1\,\text{GeV}}\right)\cdot \sim \frac{2\pi}{1.97}$$

* $\Delta m^2 \frac{L}{E} \ll 1$

$$\Rightarrow \left\langle\cos\left(\frac{1}{2}\Delta m^2\frac{L}{E}\right)\right\rangle \sim \left\langle 1-\frac{1}{2}\left(\frac{\Delta m^2 L}{2E}\right)^2\right\rangle$$

$$\sim 1-\frac{1}{4}\Delta m^2\left\langle\left(\frac{L}{E}\right)^2\right\rangle \frac{1}{2}$$

Δ_{max}

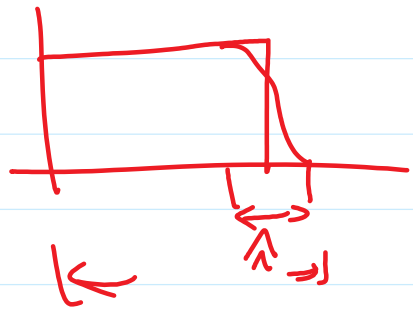
$$\sim 1 - \frac{1}{4}\Delta m^2 \left\langle \left(\frac{L}{E}\right)^2 \right\rangle \frac{1}{2}$$

$$\Rightarrow \quad \sin^2 2\theta \leq \frac{2p_{max}}{1 - 1 + \frac{1}{4}\Delta m^2 \left\langle \left(\frac{L}{E}\right)^2 \right\rangle \frac{1}{2}} \sim \frac{2p^{max}}{\frac{1}{2}\left(\frac{\Delta m}{2}\right)^2 \left\langle \left(\frac{L}{E}\right)^2 \right\rangle}$$

We also have in experiments that

$$\left\langle \left(\frac{L}{E} - \left\langle \frac{L}{E} \right\rangle^2\right) \right\rangle \ll \left(\left\langle \frac{L}{E} \right\rangle \right)^2$$

meaning the region of $\frac{L}{E}$ is small.



So we use

$$\sin^2 2\theta \ll \frac{2p_{max}}{\frac{1}{2}\left(\frac{\Delta m^2}{2}\left\langle \frac{L}{E} \right\rangle\right)^2}$$

We can find the region that no constraint is put on to $\sin^2 2\theta$.

Sensitivity

Thursday, September 24, 2015 9:01 PM

\# Disappearance experiment $\nu_\alpha \to \nu_\beta$

* Large $\Delta m^2 \langle \frac{L}{E} \rangle$ ~ where we can define sensitivity for $\sin^2 2\theta$

We noticed from the exclusive curve that

$$(\sin^2 2\theta)_s = 2\, p^{max}$$

independent of Δm^2

this value is the sensitivity

* Sensitivity of Δm^2 → when $\sin^2 2\theta \sim 1$

In Gaussian $\phi(\frac{L}{E})$ case, it is equivalent to $\boxed{(\Delta m^2)\langle \frac{L}{E} \rangle \text{ smallest}}$

this is why it is called sensitivity for Δm^2

$$(\Delta m^2)_s \sim \frac{4\sqrt{p^{max}}}{\langle \frac{L}{E} \rangle}$$

* How do we determine $P_{\nu_\alpha \to \nu_\beta}$?

$$P_{\nu_\alpha \to \nu_\beta} = 1 - P_{\nu_\alpha \to \nu_\alpha} = 1 - \frac{N_{\nu_\alpha}^{obs}}{N_{\nu_\alpha}^{no-osc}}$$

← observed

← theoretical value assuming no oscillation.

Assuming no uncertainty in $N_{\nu_\alpha}^{no-osc}$:

$$\Rightarrow \quad \Delta P_{\nu_\alpha - \nu_\beta} \sim \frac{\Delta N_{\nu_\alpha}^{obs}}{N_{\nu_\alpha}^{no-osc}}$$

← has a Poisson distribution $(\Delta N_{\nu_\alpha}^{obs})^2 = N_{\nu_\alpha}^{obs}$

Variance

To do an estimation we set

$$N_{\nu_\alpha}^{obs} \sim N_{\nu_\alpha}^{no-osc}$$

$$\Rightarrow \left(P^{max}_{\nu_\alpha \to \nu_\beta}\right)_s \sim \underbrace{\Delta P_{\nu_\alpha \to \nu_\beta}\Big|_{no-signal}}_{} \sim \frac{\left(N_{\nu_\alpha}^{no-osc}\right)^{\frac{1}{2}}}{\left(N_{\nu_\alpha}^{no-osc}\right)} \sim \frac{1}{\sqrt{N_{\nu_\alpha}^{no-osc}}}$$

the uncertainty when no signal and assuming no background

$\Rightarrow$ We infer the sensitivity of $\sin^2 2\theta$:

increase the # of events doesn't really help a lot since it goes as

$$\frac{1}{\sqrt{N_{\nu_\alpha}^{no-osc}}} \sim \frac{1}{\sqrt{\#\text{ of events}}}$$

* # of events

$$N_{\nu_\alpha}^{no-osc} \propto \frac{N_{\nu_\alpha}^{source} \cdot \sigma_{\nu_\alpha}(E) \cdot \eta \cdot M_{det}}{L^2}$$

\# of neutrinos by source; average detection cross-section; detection efficiency; mass of the detector

$$\Rightarrow \left(P^{max}_{\nu_\alpha \to \nu_\beta}\right)_s \sim \frac{L}{\sqrt{N_{\nu_\alpha}^{src} \cdot \sigma_{\nu_\alpha}(E) \cdot \eta \cdot M_{det}}}$$

We need it to be small

* Sensitivity for Δm^2

$$\left(\Delta m^2\right)_s \sim \frac{E}{\sqrt{L}} \cdot \frac{1}{\sqrt[4]{N_{\nu_\alpha}^{src} \cdot \sigma_{\nu_\alpha}(E) \cdot \eta \cdot M_{det}}}$$

* What can we do?

do not understand

Due to Relativity?

* What can we do?

do not understand
$\sigma \searrow$ $(\Delta m^2)_s \nearrow$ Due to Relativity:

or $\omega = \frac{\Delta m^2}{E}$?

$E \uparrow$, $\sigma \uparrow$
$\omega \searrow$

	$(\sin^2 2\theta)_s$	$(\Delta m^2)_s$
$L \searrow$:	$\searrow$	$\nearrow$
$E \searrow$:	$\searrow$	$\searrow$

$\sigma_{\nu_\alpha}(E) \nearrow \leftarrow E \uparrow$ but
$\sigma_{\nu_\alpha}(E) \sim G_F E^2$

* But sometimes we don't have a lot of control of energy. We

$L \nearrow$

which will cause $(\sin^2 2\theta)_s \nearrow$. but
we increase $\underline{N_{\nu_\alpha}^{src} M_{det}}$ to compensate the
effect caused by $L \uparrow$.

To keep $(\sin^2 2\theta)_s$ unchanged, we require

$$N_{\nu_\alpha}^{src} \eta M_{det} \sim L^2$$

So that whenever we increase L
we increase a lot in $N_{\nu_\alpha}^{src} \eta M_{det}$

* Appearance experiment $\nu_\alpha \to \nu_\beta$

$$\left(P^{max}_{\nu_\alpha \to \nu_\beta}\right)_s \sim \Delta P_{\nu_\alpha \to \nu_\beta}\Big|_{no\text{-}signal}$$

$$\sim \frac{\sqrt{N^{bg}_{\nu_\beta}}}{\phi_{\nu_\alpha}\, T_{obs}\, \sigma_{\nu_\beta}(E)\, \eta\, M_{det}}$$

background events estimation

flux of ν_α through detector

observation time

To have $\left(P^{max}_{\nu_\alpha \to \nu_\beta}\right)_s$ in appearance experiment $\searrow$

need:

$N^{bg}_{\nu_\beta}$ $\searrow$ but hard.

like in $\pi^+ \to \mu^+ + \nu_\mu$
always companioned with
$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$
} beam-related bg

or ν_α can also produce events in detector

Presentation of Neutrino experiments

Monday, September 28, 2015 12:12 PM

① What does theory tell us?

The transition rate from $\nu_\alpha \to \nu_\beta$

$$P = \frac{1}{2}\sin^2 2\theta \left(1 - \cos\left(\frac{\Delta m^2}{2}\underbrace{\frac{L}{E}}\right)\right)$$

② What does experiments tell us?

We detect neutrino events. N_{obs}

$$\bar{\nu}_e + p^+ \to n + e^+$$

$$P_{obs} = \frac{N_{obs}}{N_{exp}}$$ → number of neutrinos the would be detected assuming no oscillation

③ Are the 2 probabilities the same?

No!

Because in the detector we don't measure the neutrino events for a specific L, E.

Our detector span a length $[L_1, L_n]$

energy also span a spectral width $[E_1, E_n]$

We are measuring the averaged $\langle P \rangle$

averaging over L, E.

Since P only depends only on $\frac{L}{E}$, we only need to average over $\frac{L}{E}$ for some distribution $\phi(\frac{L}{E})$

④ What distribution do we use?

Gaussian

$$\to \langle P \rangle \sim \frac{1}{2}\sin^2 2\theta\left(1 - \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right)\right)$$

⑤ What can detectors tell us?

A upper limit if we detect no oscillation

mpe because otherwise we will detect the oscillation.

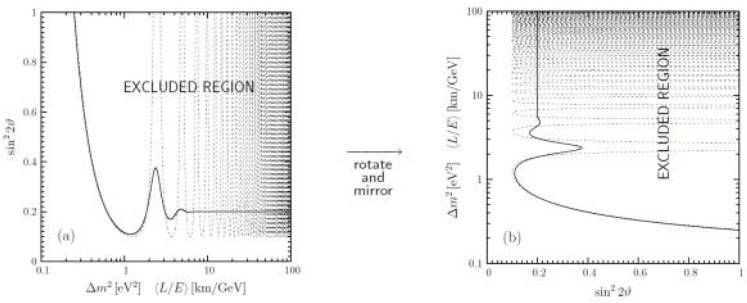
set by exp

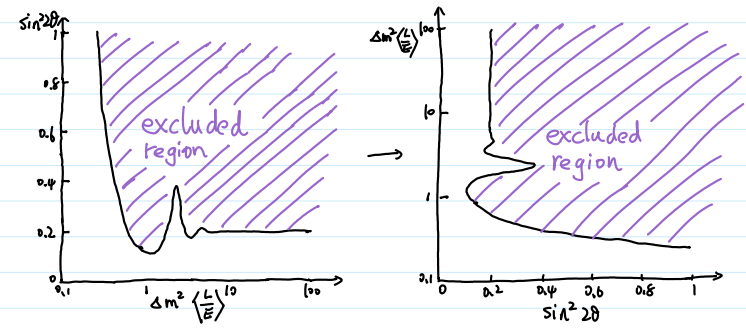
$$\langle P \rangle \sim \frac{1}{2}\sin^2 2\theta\left(1 - \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right)\right) \le P^{max}$$

$$\Rightarrow \underbrace{\sin^2 2\theta \le \frac{2P^{max}}{1 - \cos\left(\frac{\Delta m^2}{2}\left\langle\frac{L}{E}\right\rangle\right)\exp\left(-\frac{1}{2}\left(\frac{\Delta m^2}{2}\sigma_{L/E}\right)^2\right)}}_{\text{This part gives us the exclusion curve}} \cancel{\le 2P^{max}}$$

This is the sensitivity

Section 2: Exclusion Curve





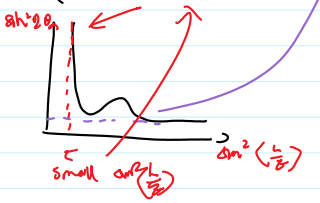
Section 3 : Sensitivity

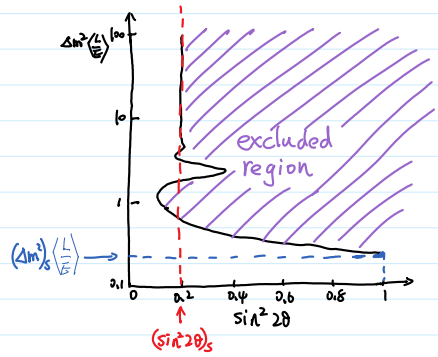
$\sin^2 2\theta \leq 2p^{max}$: most general constraint ← $\boxed{\text{Large } \Delta m^2 \langle \frac{L}{E} \rangle \text{ limits}}$

$\Delta m^2 \sim 4\sqrt{(p^{max})}\,\frac{E}{L}$: when $\sin^2 2\theta \sim 1$

we can use $\boxed{\text{small } \frac{\Delta m^2 L}{2E} \text{ approximation}}$

(seen from graph)





Analytic results

$$(\sin^2 2\theta)_s \propto \frac{L}{\sqrt{N^{src}\,\sigma(E)\,\eta\,M_{det}}}$$

η ↑ detection efficiency

$$(\Delta m^2)_s \propto E \cdot \frac{1}{\sqrt{L}\,\sqrt{N^{src}\,\sigma(E)\,\eta\,M_{det}}}$$

Even if $\sigma \sim E^2$ $\Delta m^2 \searrow$, with $E \searrow$.

We need $L \uparrow$ if we want to have a good sensitivity of $\underline{(\Delta m^2)_s}$ but $\underbrace{(\sin^2 2\theta)_s \uparrow}_{\text{bad!}}$

have to
1. power of source ↑
2. mass of detector

Section 4 Some experiments

SBL
LBL
VLBL

Thursday, October 1, 2015 11:33 AM

high energy neutrinos scattered by CMB

Detecting CNB

Saturday, October 10, 2015 8:18 PM

Tom Weiler: PRD Vol 49, No 3 1994

① Distribution of CNB

$m \ll T$, using F-D

Subscript d : decoupling of neutrinos

probably only for light neutrinos

$$f_{\nu_i}(p) = \frac{1}{e^{p/T_d - \bar{\mu}_i} + 1}$$

Due to expansion $p(p) \sim \frac{1}{R}$ (need to be more specific)

$$p(p_d) = p(p) \cdot \frac{R}{R_d}$$

Define $\beta = \frac{R}{R_d} \frac{1}{T_d}$

$$f_{\nu_i}(p) = \frac{1}{e^{\beta p - \bar{\mu}_i} + 1}$$

The neutrino # density drops as

$$n_{\nu_i}(\bar{\mu}_i) = \frac{1}{2\pi^3} \int d^3p \cdot f_{\nu_i}(p)$$

$$\sim \frac{1}{R^3}$$ (need to be more specific)

* As an example: massless particle

$$T \sim \frac{1}{R}$$

$$T_r = \eta \left(\frac{R_d T_d}{R} \right)$$

$$\eta = \left(\frac{11}{4}\right)^{\frac{1}{3}}$$ ← due to reheating by $e^+e^- \to \gamma\gamma$ when $T \sim m_e$

Thus

$$\beta = \frac{\eta}{T_\gamma}$$

Current value is

$$\beta_0 = \frac{1}{\frac{T_{\gamma 0}}{2.7K} \times 1.66 \times 10^{-4}\ eV}$$

Here comes the problem, mean momentum of neutrinos is $(\bar{u}+1)\beta_0^{-1}$

☆ Outline for the talk

What is CNB?

A soup of particles from early universe

When the environment gets transparent for ν, we say neutrinos decouple from the soup and propagate freely in the universe.

CNB Speculations

Sunday, October 25, 2015 8:27 PM

Speculations on Detection of the Neutrino Sea

L. Stodolsky

* $N_\nu - N_{\bar\nu}$

* e^- in CNB

weak neutral current

$$\sqrt{2}\, G_F \bar\Psi_e \gamma_\mu (1+\gamma_5)\Psi_e \bar\Psi_\nu \gamma_\mu (1+\gamma_5)\Psi_\nu$$

* parity violation in weak interation

two helicity states for electron $\vec\sigma\cdot\vec v = \pm 1$

rest frame of electron

ρ: neutrinos

$\vec v$: electron velocity

neutrino current density

$$\vec j = 2\rho\vec v/(1-v^2)^{\frac{1}{2}}$$

energy

$$\sqrt{2}\, G_F\, \vec\sigma\cdot\vec j$$

For two different spins there are two different energies, the energy difference

$$\delta E = 2\sqrt{2}\, G_F \rho \vec v/(1-v^2)^{\frac{1}{2}}$$

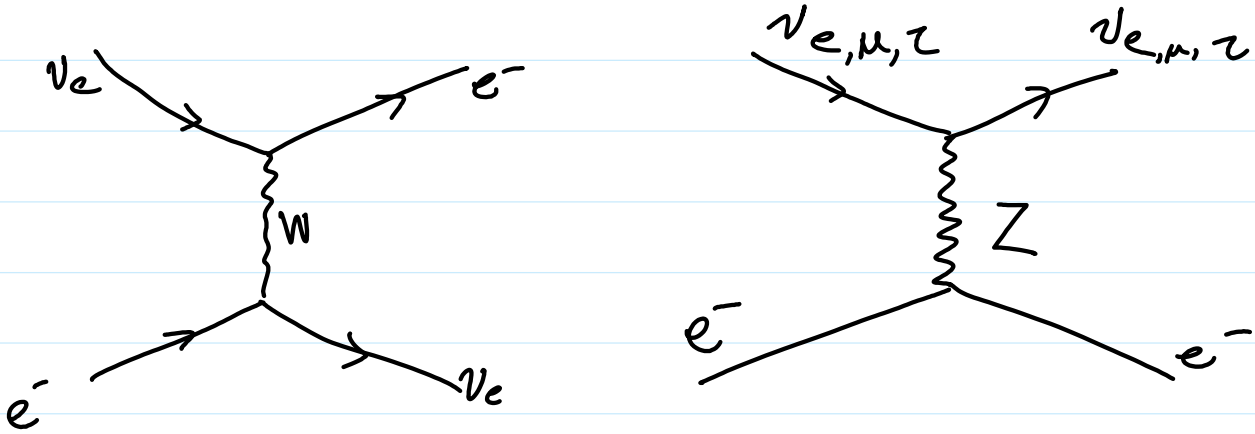
The energy difference has a ρ dependent number density of neutrinos

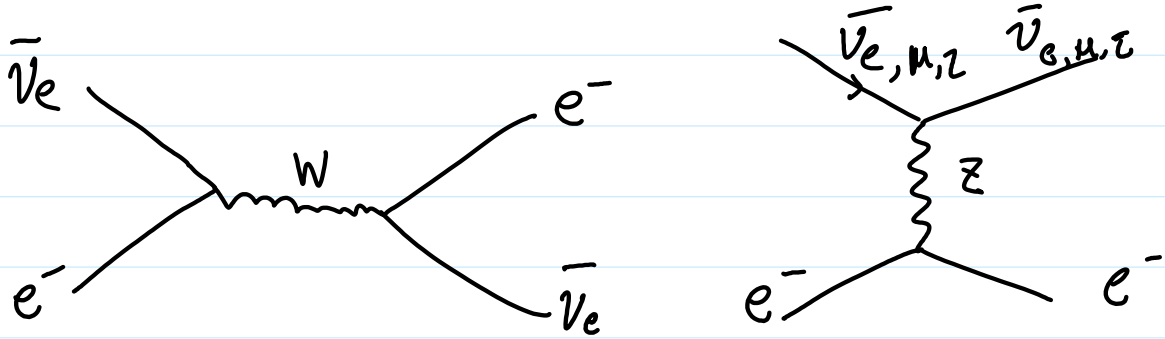
$\square$ why parity symmetry breaking

☐ What exactly is the interaction in math & physics

⊙ Neutrino - electron scattering

$$\nu_e + e^- \longrightarrow \nu_e + e^-$$





Effective Lagrangian for elastic scattering in low energy (negleting propagators since they are heavy)

$$L_{eff}\left(\overset{(-)}{\nu}_e e^- \to \overset{(-)}{\nu}_e e^-\right)$$

$$= -\frac{1}{\sqrt{2}} G_F \Big\{ \overbrace{\left[\bar{\nu}_e \gamma^\rho (1-\gamma^5) e\right]\left[\bar{e} \gamma_\rho (1-\gamma^5) \nu_e\right]}^{\text{charged current}}$$

$$+ \underbrace{[\bar{\nu}_e \gamma^\rho (1-\gamma^5) \nu_e][\bar{e} \gamma_\rho (g_V^L - g_A^L \gamma^5) e]}_{\text{neutral current}} \}$$

Use Fierz transformation to transform charged current (should read why in chap 2)

A Fierz transformation

$$\mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4) = (\bar{\psi}_1 \gamma^\mu \underline{(1-\gamma^5)} \psi_2)(\bar{\psi}_3 \gamma_\mu \underline{(1-\gamma^5)} \psi_4)$$

violates parity symmetry

pseudo vector

Recall that $\bar{\psi}\gamma^5\gamma^\mu\psi$ (pseudo vector) $\bar{\psi}\gamma^5\psi$ (pseudo scalar)

It can be proved that we exchange $\psi_2 \leftrightarrow \psi_4$ the Lagrangian is the same.

$$\mathcal{L}(\psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}(\psi_1, \psi_4, \psi_3, \psi_2)$$

$$\mathcal{L}_{eff}(\overset{(-)}{\nu}_e e^- - \overset{(-)}{\nu} e^-) = -\frac{1}{\sqrt{2}} G_F [\bar{\nu}_e \gamma^\rho (1-\gamma^5) \nu_e][\bar{e}\gamma_\rho((1+g_V^L) - (1+g_A^L)\gamma^5) e]$$

$$\mathcal{L}_{eff}(\overset{(-)}{\nu}_\alpha e^- - \overset{(-)}{\nu}_\alpha e^-) = -\frac{1}{\sqrt{2}} G_F [\bar{\nu}_\alpha \gamma^\rho (1-\gamma^5) \nu_\alpha][\bar{e}\gamma_\rho (g_V^L - g_A^L \gamma^5) e]$$

for $\alpha = \mu, \tau$

back to Stodolsky's paper

neutrino in rest frame of sea, ρ

also electron $\vec{v}$

So if we switch to electron rest frame, we using velocity transformation in SR

$$u' = \frac{u - v}{}$$

velocity in S frame

$$u' = \frac{u - v}{\sqrt{1 - v^2}}$$

velocity in S frame

velocity in S' frame which is moving with velocity v with respect to S

$$\vec{J} = 2\rho\vec{v}/\sqrt{1 - v^2}$$

By definition, $\vec{J} = -\frac{i\hbar}{2m}(\phi^* \nabla\phi - \phi\nabla\phi^*)$

$$\boxed{\text{Recall that energy of magnetic moment in magnetic field } \Delta H = -\vec{\mu}\cdot\vec{B}}$$

The energy related to spin is

$$2\frac{G}{\sqrt{2}}\vec{\sigma}\cdot\vec{J}$$

Plug in the current, we have the energy difference between 2 spin states

$$\Delta E = 2\sqrt{2}\, G_F\, \rho\, \vec{v}/\sqrt{1 - v^2}$$

* what is the density ρ?
number density can be calculated using Fermi distribution
in phase space $\propto p_F^3$

$$\rho \propto p_F^3 \qquad (\rho = (6\pi^2)^{-1} p_F^2$$

* Plug in the numbers

$$\Delta E \sim 0.6 \times 10^{-4} \left(\frac{p_F}{eV}\right)^3 v/(1 - v^2)^{\frac{1}{2}} \quad eV$$

* So what is p_F?

β decay (Weinberg) . $P_f \leq 60\,eV$

cosmic-ray transmission : $P_f \leq 1\,eV$

cosmological : $P_f \leq 0.75 \times 10^{-2}\,eV$

So $\delta E \sim 10^{-24} \; v \left(\frac{P_f}{eV}\right)^3$

$$\begin{cases} \leq 10^{-19} \; \frac{v}{c} \\ \leq 10^{-14} \; \frac{v}{c} \\ \leq 10^{-30} \; \frac{v}{c} \end{cases}$$

⊙ Apply to what kind of system

① Polarized beams :

equally polarized electron beam

$\pm$ helicity equally distributed

The have different frequency

phase difference of $\pm$ helicity states

$$\varphi \simeq \underbrace{2\sqrt{2G}\,\rho\, z}_{\sim \delta E t}$$

↑ in rest frame of electron

z : distance travelled by electron in the CMB background.

#'s : $\frac{\varphi}{z} = 3 \times 10^{-20}\,\pi^3$ rad/cm

: 1 ly → π^3 rad

How do we do the experiment?

* Conceal relativistic electrons in a box for a long time

* Using the fact the earth is moving around galactic center ($280\ km \cdot s^{-1}$)

$\frac{v}{c} \sim 10^{-3}$

$\frac{\varphi}{2} = 0.05\ n^3\ sec/yr$

$1\ sec \sim 4.8 \times 10^{-6}\ rad$

* A big chunk of Ferromagnetic material

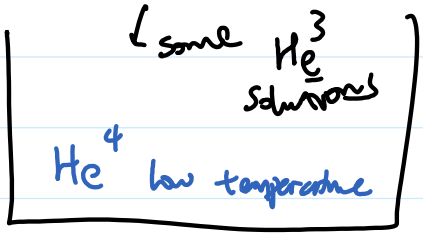
1 ton ($\sim 10^{27}$ aligned electron spins)

So there will be a torque due to ν_B

$n^3 eV$

but external B field can interfere the data

⊙ Notice that the neutral current only for ν_μ, ν_τ ~~can~~ also has a similar L_{eff}, thus these ~~de~~estimations also ~~Acc~~ ~~antion~~ works for this NC



Can retain nuclear polarization for a long time.

Mixing

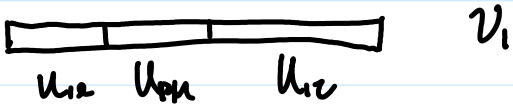
Monday, February 8, 2016 10:11 AM

* How to visualize mixing matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \qquad \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U_{1e} & U_{1\mu} & U_{1\tau} \\ U_{2e} & U_{2\mu} & U_{2\tau} \\ U_{3e} & U_{3\mu} & U_{3\tau} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$\nu_e = U_{e1}\nu_1 + U_{e2}\nu_2 + U_{e3}\nu_3$$

$$\nu_1 = U_{1e}\nu_e + \cdots$$



Thursday, February 11, 2016 11:33 AM

A matter picture with ~~[illegible]~~ V-V interaction

Bessel functions

Thursday, March 3, 2016 9:48 AM

$$J_n(z) = \frac{\left(\frac{1}{2}z\right)^n}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{izt}\left(1-t^2\right)^{n-\frac{1}{2}} dt$$

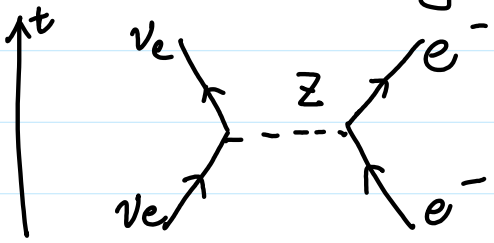
$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y)\, J_{n-m}(z)$$

$$J_n(y+z) = \frac{1}{2\pi i} \int^{(0+)} t^{-n-1} e^{\frac{1}{2}(y+z)(t-1/t)} dt$$

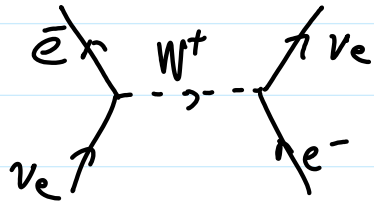
Neutrino Feynman diagrams

Tuesday, March 22, 2016 1:03 PM

Elastic scattering

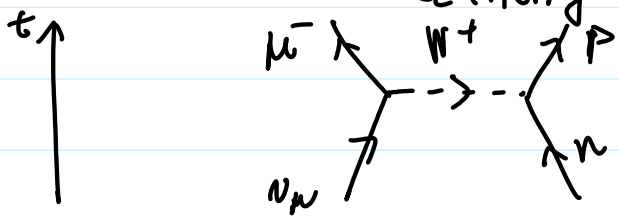


NC

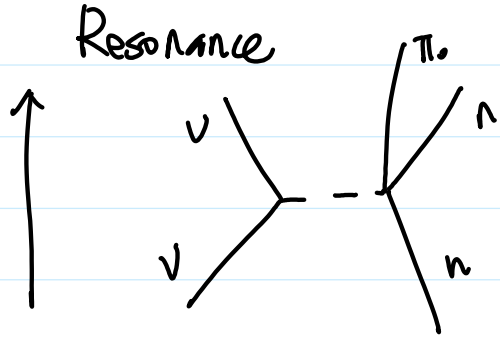


CC

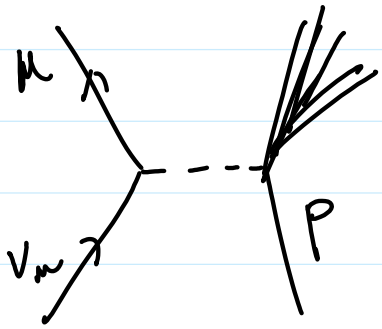
Quasi Elastic Scattering



Resonance

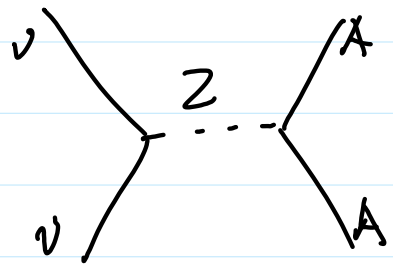


Deep inelastic scattering



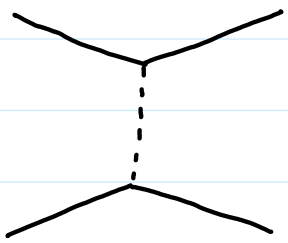
Energy higher

Coherent NC neutrino-nucleus elastic scattering

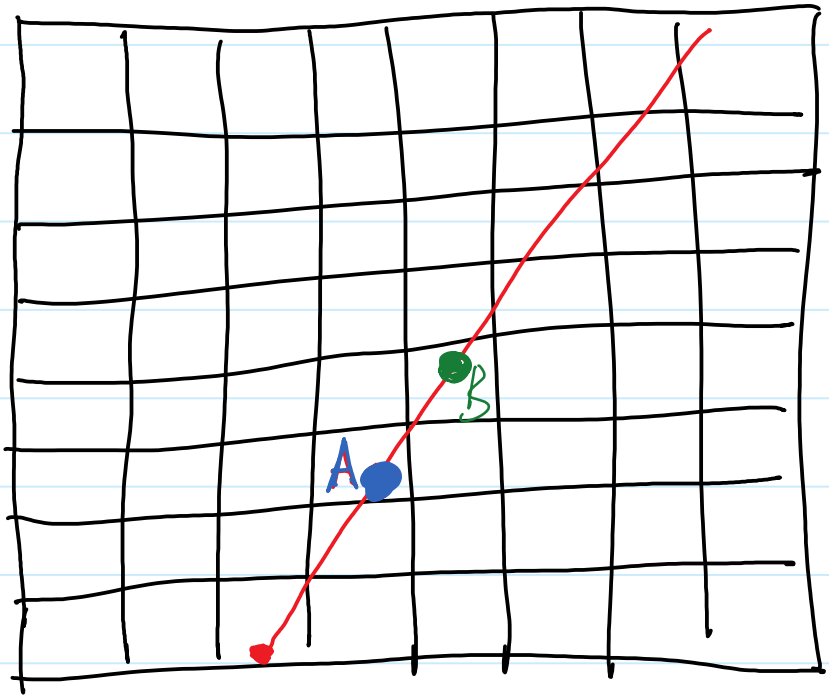


If energy is higher than 50 MeV, the neutrino will scatter with the actual neutrons or protons.

To summarize :



Friday, September 9, 2016 2:33 PM



Questions

1. Mass of electron flavor?
   Hard to say. But we can always define an effective mass which is composed of ν_1 & ν_2
   using $\nu_e = \cos\theta\, \nu_1 + \sin\theta\, \nu_2$

2. Difference in hierarchy will change

   Vacuum Hamiltonian

Normal Hierarchy	Inverted
$\Delta m^2_{21} > 0$	$\Delta m^2_{21} < 0$

   thus changing ω

   meaning of ω: angular frequency of oscillation

3. Resonance factor

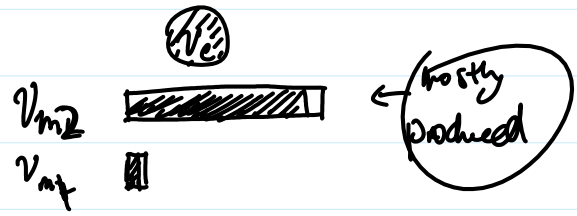
$$R = \left(\frac{\lambda}{\omega_v} - \cos^2 2\theta_v\right)^2 + \sin^2 2\theta_v$$

   $R \to 1$ at $\lambda \to 0$

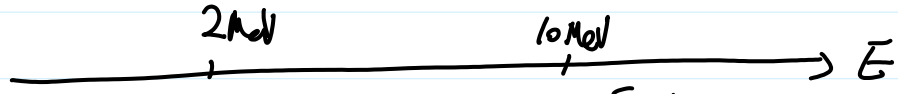
4. How is solar neutrino missing solved

exactly?

LMA: ν_e produced in core



adiabatic transition



Vacuum osc important

Small vacuum oscillation + adiabatic conversion

5 Why is matter interaction

$$\sqrt{2}\, G_F n_e(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Why is it ok to remove an identity ?

$$\frac{\sqrt{2}\, G_F n_e(x)}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sqrt{2}\, G_F n_e(x)}{2} I$$

$$i\partial_x \psi = \left(H(x) + z(x) I\right)\psi$$

$$i\partial_x \psi = z(x) I \cdot \psi$$

$$i\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = z(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = z(x) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

6. $H = z\,\sigma_1$

$$i\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = z \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

$$i\partial_x \psi_1 = z\psi_2 \Big|_{\partial} \quad i\partial_x^2 \psi_1 = z\,\partial_x \psi_2$$

$$\left.\begin{aligned} i\partial_x \psi_1 &= z\psi_2 \\ i\partial_x \psi_2 &= z\psi_1 \end{aligned}\right\} \Rightarrow \quad i\partial_x^2 \psi_1 = z\,\partial_x \psi_2$$

$$= -z\,i\,(i\partial_x \psi_2)$$

$$= -i\,z \cdot z\psi_1$$

$$i\partial_x^2 \psi_1 = -i\,z^2 \psi_1$$

$$\partial_x^2 \psi_1 = z^2 \psi_1$$

$$\psi_1 = e$$

$$\boxed{\begin{aligned} &H = z\sigma_3 + z\sigma_1 \\ &z \to 0 \quad \text{no conversion} \\ &z \uparrow \quad \text{max conversion} \end{aligned}}$$

Problems of slides

Thursday, April 7, 2016 11:32 AM

To Contents

Neutrino oscillations
Conclusions

P7 Large flow

add in simple explanation for oscillation
P7

* Dirac equation → schrödinger
Choose a proper basis → sth else

* P10 Remove diagram

P11 weak interaction states

P14 Orange words

$\nu_\mu \leftrightarrow \nu_x$

P15 $\omega_\nu(x) = \omega_\nu \sqrt{\left(\frac{\Delta}{\omega_\nu} - \cos\theta\right)^2}$

NO 2

$P_{1b} \geq \sin\theta_v$

X

$P_{1b} \quad \omega_{vs} > \omega_{eg} \sin\theta_v$

~~Use~~ Define the relation of Dirac's

$\langle V_e|\psi\rangle$

$\langle V_{M,1}|\psi\rangle$

$-\frac{1}{2}\begin{pmatrix} 0 & \delta\lambda e^{2in} \\ \delta\lambda e^{-in} & 0 \end{pmatrix}$ $\quad \sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$

$\frac{1}{2}\begin{pmatrix} & \delta\lambda e^{2in} \end{pmatrix}$

$= \sin(kx)\, e^{2in}$

$\delta = \begin{pmatrix} e & e^{i(-kx+2\eta)} \end{pmatrix}$

$\frac{i}{4}\begin{pmatrix} & e^{i} \frac{1}{2i} & & i(kx+2\eta) \\ & & 0 & \end{pmatrix}$

① Research done about matter

② Mention some examples

③ Other

$$i\partial_x\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix} -\frac{w}{2} & we^{ikx} \\ we^{-ikx} & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

$$\begin{cases} i\partial_x\psi_1 = -\frac{w}{2}\psi_1 \\ i\partial_x\psi_2 = \psi_1 + \frac{w}{2}\psi_2 \end{cases}$$

$\frac{w}{2}$ $\quad + we^{ikx}\psi_2$

$w^* e^{-ikx}$

② Solve ψ_2

$\cancel{i\partial_x\psi_2 =}$

$w^* \quad = \quad \frac{w}{2}\psi_2 \quad \psi_1$

$$\frac{ie^{ikx}}{w^*}\partial_x\psi_2 - e^{-ikx}\frac{1}{w^*}e^{ikx}\psi_2 =$$

$$\frac{i}{w^*}\partial_x(e^{i} \quad - \frac{w}{2}\frac{1}{w^*}\frac{w}{2}\partial_x(e^{ikx}\psi_2) = \partial_x\psi_1$$

$$\Rightarrow \frac{i}{w^*}(e^{ikx} \quad {}^{kx}(\partial_x\psi_2)\, e^{ikx}\partial_x\psi_2) = \partial_x\psi_1$$

$ik\partial_x\psi_2$

$\partial_x\psi_1 = i\frac{w}{2}\psi_1 \quad we^{ikx}\psi_2$

ie^{ikx}

$$\partial_x\psi_1 \quad -iwe^{ikx}\psi_2 + i\frac{w}{2}\quad \partial_x\psi_2 - \frac{w}{2}\frac{1}{w^*}e^{ikx}\psi_2) =$$

Combine?

$w^* = w$ ⇐ Assumption

$\frac{1}{w}(ik\partial_x\psi_2$

$$= -W\psi_2 + \frac{W}{2}\left(\frac{i}{W}\partial_x\psi_2 - \frac{W}{2}\frac{1}{W}\psi_2\right)$$

$$\partial_x^2\psi_2 +$$

$$= -W^2\psi_2 + \frac{W}{2}\left(i\partial_x\psi_2 - \frac{W}{2}\psi_2\right)$$

$ik\partial_x\psi$

$$\cancel{\partial_x^2\psi_2 + ik\partial_x\psi_2 - \frac{W}{2}\partial_x\psi_2 + i\frac{W}{2}\partial_x\psi_2}$$

$$\cancel{-\frac{W}{2}ik\psi_2 + iW^2\psi_2 + i\frac{W^2}{4}\psi_2 = 0}$$

$$\cancel{\partial_x^2\psi_2 + \left(ik - \frac{W}{2} - i\frac{W}{2}\right)\partial_x\psi_2 + \left(iW^2 - i\frac{W}{2}k + i\frac{W^2}{4}\right)\psi_2 = 0}$$

Rabi

Sunday, April 10, 2016 3:14 PM

Hamiltonian

$$H = \begin{pmatrix} E_1 & h \\ h^* & E_2 \end{pmatrix}$$

where h is periodic

$$h = e^{i\omega x}$$

$$i\frac{\partial}{\partial x}\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} E_1 & e^{i\omega x} \\ e^{-i\omega x} & E_2 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

① Try Mathematica. 9

$$i\frac{\partial}{\partial x}\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = ($$

②

$$i\partial_x \psi_1 = E_1\psi_1 + e^{i\omega x}\psi_2$$

E_1, E_2 constant

$$i\partial_x \psi_2 = E_2\psi_2 + e^{-i\omega x}\psi_1$$

$$i e^{-i\omega x}\partial_x \psi_1 = E_1 e^{-i\omega x}\psi_1 + \psi_2$$

$$\partial_x \psi_2 = -iE_2\psi_2 - i e^{-i\omega x}\psi_1$$

$$\partial_x \psi_2 = -iE_2\left(i e^{-i\omega x}\partial_x\psi_1 - E_1 e^{-i\omega x}\psi_1\right) - i e^{-i\omega x}\psi_1$$

$$i(-i\omega)e^{-i\omega x}\partial_x\psi_1 + i e^{-i\omega x}\partial_x^2\psi_1 =$$

$$-i\omega E_1 e^{-i\omega x}\psi_1 + E_1 e^{-i\omega x}\partial_x\psi_1 - iE_2\left(i e^{-i\omega x}\partial_x\psi_1 - E_1 e^{-i\omega x}\psi_1\right) - i e^{-i\omega x}\psi_1$$

$$\omega\partial_x\psi_1 + i\partial_x^2\psi_1 = -i\omega E_1\psi_1 + E_1\partial_x\psi_1 + E_2\partial_x\psi_1 + iE_1E_2\psi_1 - i\psi_1$$

We can always remove an identity term from H

$$H = \frac{\omega_0}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + A\begin{pmatrix} 0 & e^{i\omega x} \\ e^{-i\omega x} & 0 \end{pmatrix}$$

We assume the wave function alway has the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} C_1(x)\, e^{-iE_1 x} \\ C_2(x)\, e^{-iE_2 x} \end{pmatrix}$$

initial condition $\begin{pmatrix} \psi_1(0) \\ \psi_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$i\partial\begin{pmatrix} C_1(x)\, e^{-iE_1 x} \\ \end{pmatrix} = \frac{\omega_0}{2}\begin{pmatrix} -1 & 0 \\ \end{pmatrix}\begin{pmatrix} C_1(x)e^{-iE_1 x} \\ \end{pmatrix} + A\begin{pmatrix} 0 & e^{i\omega x} \\ \end{pmatrix}\begin{pmatrix} C_1(x)e^{-iE_1 x} \\ \end{pmatrix}$$

$$i\partial_x \begin{pmatrix} C_1(x)\, e^{-iE_L x} \\ C_2(x)\, e^{-iE_h x} \end{pmatrix} = \frac{\omega_0}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_1(x) e^{-iE_L x} \\ C_2(x) e^{-iE_h x} \end{pmatrix} + A\begin{pmatrix} 0 & e^{i\omega x} \\ e^{-i\omega x} & 0 \end{pmatrix}\begin{pmatrix} C_1(x) e^{-iE_L x} \\ C_2(x) e^{-iE_h x} \end{pmatrix}$$

$$i\begin{pmatrix} \partial_x C_1 - iE_L C_1 \\ \partial_x C_2 - iE_h C_2 \end{pmatrix} = \begin{pmatrix} -\frac{\omega_0}{2} C_1 + A e^{i\omega x} C_2 e^{-i(E_h - E_L)x} \\ \frac{\omega_0}{2} C_2 + A e^{-i\omega x} C_1 e^{-i(E_L - E_h)x} \end{pmatrix}$$

Define $\Delta E = E_h - E_L$

$$\begin{cases} i\partial_x C_1 + E_L C_1 = -\frac{\omega_0}{2} C_1 + A e^{i\omega x} C_2 e^{-i\Delta E x} \\ i\partial_x C_2 + E_h C_2 = \frac{\omega_0}{2} C_2 + A e^{-i\omega x} C_1 e^{i\Delta E x} \end{cases}$$

Solve C_2:

$$\frac{1}{A} e^{i(\omega - \Delta E)x}\left(i\partial_x C_2 + E_h C_2 - \frac{\omega_0}{2} C_2\right) = C_1$$

$$\frac{1}{A} i(\omega - \Delta E) e^{i(\omega - \Delta E)x}\left(i\partial_x C_2 + \left(E_h - \frac{\omega_0}{2}\right) C_2\right)$$
$$+ \frac{1}{A} e^{i(\omega - \Delta E)x}\left(i\partial_x^2 C_2 + \left(E_h - \frac{\omega_0}{2}\right)\partial_x C_2\right) = \partial_x C_1$$

Meanwhile,

$$\frac{1}{A} i(-\omega + \Delta E) e^{i(\Delta E - \omega)x}\left(i\partial_x C_1 + \left(E_L + \frac{\omega_0}{2}\right) C_1\right)$$
$$+ \frac{1}{A} e^{-i(\omega - \Delta E)x}\left(i\partial_x^2 C_1 + \left(E_L + \frac{\omega_0}{2}\right)\partial_x C_1\right) = \partial_x C_2$$

$$\Rightarrow \quad i\partial_x C_1 + \left(E_L + \frac{\omega_0}{2}\right) C_1 = A e^{i(\omega - \Delta E)x} C_2$$

$$i\partial_x\left(\frac{1}{A} e^{i(\omega - \Delta E)x}\left(i\partial_x C_2 + E_h C_2 - \frac{\omega_0}{2} C_2\right)\right)$$
$$+ \left(E_L + \frac{\omega_0}{2}\right)\left(\frac{1}{A} e^{i(\omega - \Delta E)x}\left(i\partial_x C_2 + E_h C_2 - \frac{\omega_0}{2} C_2\right)\right)$$
$$= A e^{i(\omega - \Delta E)x} C_2$$

$$-\frac{1}{A}(\omega - \Delta E)\cancel{e^{i(\omega - \Delta E)x}}\left(i\partial_x C_2 + \left(E_h - \frac{\omega_0}{2}\right) C_2\right) + \frac{i}{\cancel{A}} \cancel{e^{i(\omega - \Delta E)x}}\left(i\partial_x^2 C_2 + \left(E_h - \frac{\omega_0}{2}\right)\partial_x C_2\right)$$
$$+ \frac{E_L + \frac{\omega_0}{2}}{\cancel{A}} \cancel{e^{i(\omega - \Delta E)x}}\left(i\partial_x C_2 + \left(E_h - \frac{\omega_0}{2}\right) C_2\right) = A^2 \cancel{e^{i(\omega - \Delta E)x}} C_2$$

Collect terms

$$-(\omega - \Delta E)\left(i\partial_x C_2 + \left(E_h - \frac{\omega_0}{2}\right) C_2\right) - \partial_x^2 C_2 + i\left(E_h - \frac{\omega_0}{2}\right)\partial_x C_2$$
$$+ \left(E_L + \frac{\omega_0}{2}\right)\left(i\partial_x C_2 + \left(E_h - \frac{\omega_0}{2}\right) C_2\right) = A^2 C_2$$

$$\partial_x^2 C_2 + A^2 C_2 = -(\omega - \Delta E)\left(E_h - \frac{\omega_0}{2}\right) C_2 + \left(E_L + \frac{\omega_0}{2}\right)\left(E_h - \frac{\omega_0}{2}\right) C_2$$

$$-i(\omega-\Delta E)\partial_x C_2 + i\left(E_h-\frac{\omega_0}{2}\right)\partial_x C_2 + i\left(E_L+\frac{\omega_0}{2}\right)\partial_x C_2$$

$$\partial_x^2 C_2 + \left(A^2 + (\omega-\Delta E)\left(E_h-\frac{\omega_0}{2}\right) - \left(E_L+\frac{\omega_0}{2}\right)\left(E_h-\frac{\omega_0}{2}\right)\right)C_2$$
$$+ i\Big(\underbrace{(\omega-\Delta E) - E_h + \cancel{\frac{\omega_0}{2}} - E_L - \cancel{\frac{\omega_0}{2}}}_{\omega - E_h + \cancel{E_L} - E_h - \cancel{E_L}}\Big)\partial_x C_2 = 0$$

wrong ?

Mathematica $\Rightarrow$ $m_1 = -m_2$

$$C_2 = e^{\frac{1}{2}i(E_h-\omega+i\sqrt{})x} - e^{\frac{1}{2}i(2E_h-\omega-i\sqrt{})x}$$

A small

I guess it should be $-4A^2 - \omega^2 + 2\omega\omega_0 - \omega_0^2$

$$C_2 = e^{\frac{1}{2}i(E_h-\omega)}\left(e^{-\frac{1}{2}\sqrt{}\,x} - e^{\frac{1}{2}\sqrt{}\,x}\right)$$

problem.

$$I = \frac{2\hat{F}}{n_0} = \frac{2}{n_0}\left|k\tan 2\theta_m\, n_0 J_{n_0}(z_k)\right|$$
$$= \left|2k\tan 2\theta_m\, J_{n_0}(z_k)\right|$$

$$B = \frac{A\sin 2\theta_m}{\cancel{4}\,2}\,\cancel{(-(-i)^n)}\,\frac{2n}{\frac{A}{k}\cos 2\theta_m}\,J_n(z_n)$$
$$= \frac{1}{2}k\tan 2\theta_m\, n\, J_n$$

$$\frac{1}{2}\left|k\tan 2\theta_m\, n_0 J_{n_0}\right|^2 = \cancel{\frac{1}{2}\left|\qquad\right|} + \frac{1}{2}(n_0 k-1)^2$$
$$\left|k\tan 2\theta_m n_0 J_{n_0}\right| = (n_0 k-1)$$

$$n_0 \hat{k}\left(1 - \tan 2\theta_m J_{m0}\right) = 1$$

$$\hat{k} = \frac{1}{n_0\left(1 - \tan 2\theta_m Z_{k}\right)}$$

$$\hat{k} \tan 2\theta_m \, n_0 \, J_{m0}(z_{k_0})$$

$$\frac{F^2}{F^2 + g^2} = \frac{1}{2}$$

$$\frac{1}{2} F^2 = \frac{1}{2} g^2 \qquad g = F$$

$$n\hat{k} = 1 \pm |F|$$

$$\hat{k}$$

Rabi system

$$H = \begin{pmatrix} -\frac{\omega}{2} & \alpha e^{ikx} \\ \alpha e^{-ikx} & \frac{\omega}{2} \end{pmatrix}$$

init: $\begin{cases} \psi_1(0) = 0 \\ \psi_2(0) = 0 \end{cases}$

$$i\psi_1'(0) = -\frac{\omega}{2}$$

$$i\psi_2'(0) = \alpha e^{ik}$$

$$\psi_2'(0) = -i\alpha$$

$$\begin{cases} i\frac{d}{dx}\psi_1 = -\frac{\omega}{2}\psi_1 + \alpha e^{ikx}\psi_2 \\ i\frac{d}{dx}\psi_2 = \alpha e^{-ikx}\psi_1 + \frac{\omega}{2}\psi_2 \end{cases}$$

Solve ψ_2

$$i\frac{d}{dx}\psi_2 - \frac{\omega}{2}\psi_2 = \alpha e^{-ikx}\psi_1$$

$$i\frac{1}{\alpha} e^{+ikx}\partial_x\psi_2 - \frac{1}{\alpha} e^{ikx}\frac{\omega}{2}\psi_2 = \psi_1$$

$$i\frac{1}{\alpha}\, ik e^{ikx}\partial_x\psi_2 + i\frac{1}{\alpha} e^{ikx}\partial_x^2\psi_2$$

$$-\frac{1}{\alpha}\frac{\omega}{2}\, ik e^{ikx}\psi_2 - \frac{1}{\alpha}\frac{\omega}{2} e^{ikx}\partial_x\psi_2 = \partial_x\psi_1$$

$$\parallel$$

$$i\frac{\omega}{2}\psi_1 - i\alpha e^{ikx}\psi_2$$

$$i\frac{\omega}{2}\left(i\frac{1}{\alpha}e^{ikx}\partial_x\psi_2 - \frac{1}{\alpha}\,e^{ikx}\frac{\omega}{2}\psi_2\right) - i\alpha e^{ikx}\psi_2$$

$$i\frac{1}{\alpha}\,ik\partial_x\psi_2 + i\frac{1}{\alpha}\partial_x^2\psi_2 - \frac{1}{\alpha}\frac{\omega}{2}ik\psi_2 - \frac{1}{\alpha}\frac{\omega}{2}\partial_x\psi_2$$

$$= i\frac{\omega}{2}\left(i\frac{1}{\alpha}\partial_x\psi_2 - \frac{1}{\alpha}\frac{\omega}{2}\psi_2\right) - i\alpha\psi_2$$

$$-k\partial_x\psi_2 + i\partial_x^2\psi_2 - \frac{\omega}{2}ik\psi_2 - \frac{\omega}{2}\partial_x\psi_2$$

$$= -\frac{\omega}{2}\partial_x\psi_2 - i\left(\frac{\omega}{2}\right)^2\psi_2 - i\alpha^2\psi_2$$

$$\partial_x^2\psi_2 + ik\partial_x\psi_2 - \frac{\omega}{2}k\psi_2 + \cancel{i\frac{\omega}{2}\partial_x\psi_2}$$

$$= \cancel{i\frac{\omega}{2}\partial_x\psi_2} - \left(\frac{\omega}{2}\right)^2\psi_2 - \alpha^2\psi_2$$

$$\partial_x^2\psi_2 + ik\partial_x\psi_2 + \left[\left(\frac{\omega}{2}\right)^2 + \alpha^2 - \frac{\omega}{2}k\right]\psi_2 = 0$$

Solution to ψ_2:

$$\frac{i\alpha\left(e^{\frac{1}{2}\left(-ik - \sqrt{-(\omega-k)^2-4\alpha^2}\right)x} - e^{\frac{1}{2}\left(-ik + \sqrt{-(\omega-k)^2-4\alpha^2}\right)x}\right)}{\sqrt{-(\omega-k)^2-4\alpha^2}}$$

$$= \frac{\alpha\left(e^{-\frac{i}{2}\left(+k+\sqrt{}\right)x} - e^{-\frac{i}{2}\left(k-\sqrt{}\right)x}\right)}{\sqrt{(\omega-k)^2+4\alpha^2}}$$

$$\frac{2\alpha^2 e^{-\frac{i}{2}kx}}{\sqrt{(\omega-k)^2+4\alpha^2}}\left(\frac{e^{-\frac{i}{2}\sqrt{(\omega-k)^2+4\alpha^2}\,x} - e^{\frac{i}{2}\sqrt{(\omega-k)^2+4\alpha^2}\,x}}{2}\right)$$

$$\shortparallel$$

$$-i\sin\left(\frac{1}{2}\sqrt{(\omega-k)^2+4\alpha^2}\,x\right)$$

$\cancel{\cos\frac{1}{2}\sqrt{}}$ $\quad 2i\sin\frac{1}{2}\sqrt{}$ $\quad \cancel{\omega}$

Transition ~~Ox~~ amplitude

$$P \propto \frac{\alpha^2}{\frac{1}{4}(\omega-k)^2+\alpha^2} \sim \frac{1}{\frac{1}{4}\left(\frac{\omega}{\alpha}-\frac{k}{\alpha}\right)^2+1}$$

1

$$\sim \frac{1}{\frac{1}{4}\left(1-\frac{k}{\omega}\right)^2\left(\frac{\omega}{\alpha}\right)^2+1}$$

choose $\omega = \alpha$

$$p = \frac{1}{\frac{1}{4}\left(1-\frac{k}{\omega}\right)^2\left(\frac{\omega}{\alpha}\right)^2+1}\sin^2\left(\frac{1}{2}\sqrt{\left(\omega-\frac{k}{\omega}\right)^2\omega^2+4\alpha^2}\,x\right)$$

$$= \frac{1}{\frac{1}{4}\left(1-\frac{k}{\omega}\right)^2\left(\frac{\omega}{\alpha}\right)^2+1}\sin^2\left(\frac{1}{2}\sqrt{\left(1-\frac{k}{\omega}\right)^2\omega^2+4\left(\frac{\alpha}{\omega}\right)^2}\,\omega x\right)$$

$$\alpha = \frac{1}{10}\omega$$

Thursday, June 16, 2016 3:36 PM

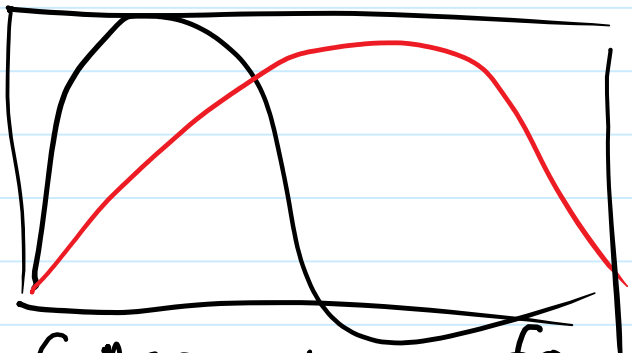
Stuart's paper

Paper#1

Friday, July 29, 2016 9:31 AM

1. Write about neutrino oscillations in matter to present that the resonance condition ~~of~~ in both Kneller & Smirnov are $\omega_m \sim nk$ (cite papers)

Present plots:



$k = 1\ \omega_m$

$k = 0.5\ \omega_m$

(parameters from Kneller's paper)

2. Rabi oscillations

How to make connections?

$$H^{(m)} = -\frac{\omega_m}{2}\sigma_3 + 0\,\sigma_3 - 0\,\sigma_1$$

6

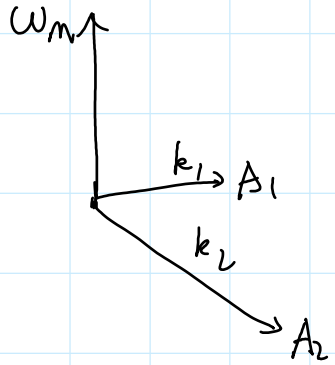
$$0\sigma_1 + 0\sigma_2$$

$$\frac{e^{+ikx} + e^{-ikx}}{2}\sigma_1 + \frac{e^{ikx} - e^{-ikx}}{2i}\sigma_2$$

Paper #1 Project Transform

Wednesday, October 12, 2016 10:29 AM

1. Derive the condition of interference

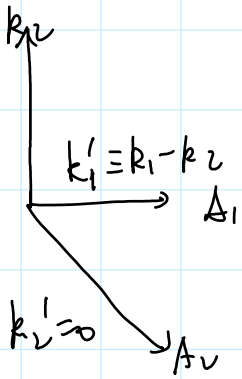
① 

$k_1 \sim \omega_m$

regardless of k_2

To the frame that k_2 mode is stationary

$\omega_m' = \omega_m - k_2$



shift of energy gap is

$$\bar{\omega} = \sqrt{\omega_m'^2 + A_2^2}$$

$$\approx |\omega_m'| \sqrt{1 + \frac{A_2^2}{\omega_m'^2}}$$

$$= |\omega_m'| \left(1 + \frac{A_2^2}{2|\omega_m'|^2}\right)$$

$$= |\omega_m'| + \frac{1}{2} \frac{A_2^2}{|\omega_m'|}$$

$$= |\omega_m - k_2| + \frac{1}{2} \frac{A_2^2}{|\omega_m - k_2|}$$

In fact the direction should also be important

In the case $|A_2| \ll |\omega_m - k_2|$ we can do the expansion, but if ω_m' is pointing down,

I need k_1' also change direction to have resonance.

So we define

$$\bar{\omega} = (\omega_m - k_2)\sqrt{1 + \frac{A_2^2}{(\omega_m - k_2)^2}}$$

$$\approx (\omega_m - k_2) + \frac{1}{2}\frac{A_2^2}{\omega_m - k_2} \quad \text{if } |A_2| \ll |\omega_m - k_2|$$

② Next step is to calculate relative detuning of the k_1 frequency in this new corotating frame.

$$D = \frac{|\bar{\omega} - k_1'|}{A_1} = \frac{|(\omega_m - k_2)\sqrt{1 + A_2^2/(\omega_m - k_2)^2} - (k_1 - k_2)|}{A_1}$$

If $|A_1| \ll |\omega_m - k_2|$, we have

$$D \approx \frac{|\omega_m - k_2 + \frac{1}{2}\frac{A_2^2}{\omega_m - k_2} - k_1 + k_2|}{A_1}$$

$$\approx \frac{|\omega_m - k_1 + \frac{1}{2}\frac{A_2^2}{\omega_m - k_2}|}{A_1}$$

2. What needs to be verified.

① A_2 is super large and becomes even comparable with $|\omega_m - k_2|$?

② $A_2 \ll |\omega_m - k_2|$ use the approximation

How to verify?
Using two frequencies.

Is there a consistancy with D of each mode?

So I choose a a three-frequency neutrino oscillation

The different modes are

$$e^{+ik_1 r} \qquad e^{i k_2 r} \qquad e^{-ik_2 r}$$

$$e^{i k_1 r} \qquad e^{i(k_1 + 2k_2)r} \qquad e^{i(k_1 - 2k_2)r}$$

$$e^{i(k_1 + k_2)r} \qquad e^{i(k_1 - k_2)r}$$

$$\frac{A_2}{2A_1 \left|\frac{(\omega_m - k_2)}{A_2}\right|} = \frac{A_2}{2A_1 \Delta_2}$$

$$S_k = S_i \wedge S_j$$

$$\sum_{i,j} A_{ij}(1 + S_i S_j) + \sum_{i,j,k} B_{ijk}(\cdots\cdots)$$

$$S_k = S_i \vee S_j$$

$$+ \sum_{ijk} C_{ijk} (\cdots\cdots)$$

$$+ \sum_{ij} D_{ij} (1 - S_i S_j)$$

How a Supernova explodes

Friday, October 7, 2016 2:56 PM

[By Hans Bethe & Gerald Brown.]

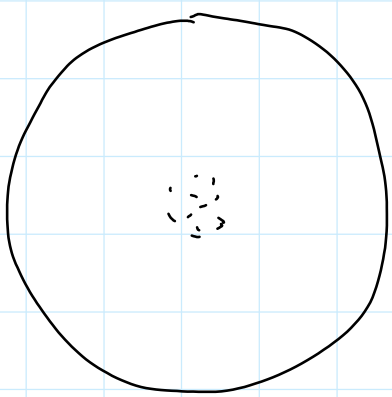
* Energy scales : total

Kinetic energy : $K \sim 10\ EM$

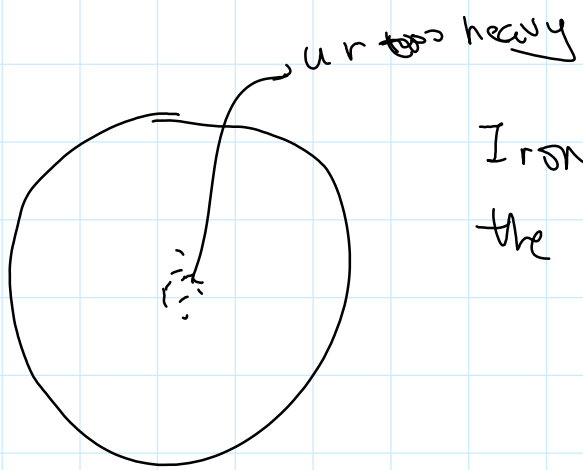
Electromagnetic energy : EM

Neutrinos : $\nu \sim 100\ K$

I need to be able to make graphs

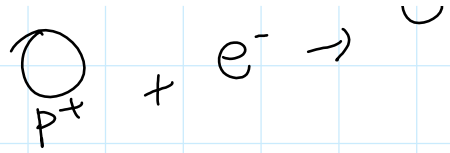


A star burning ~~to~~ and reaches a iron core (through silicon burning)

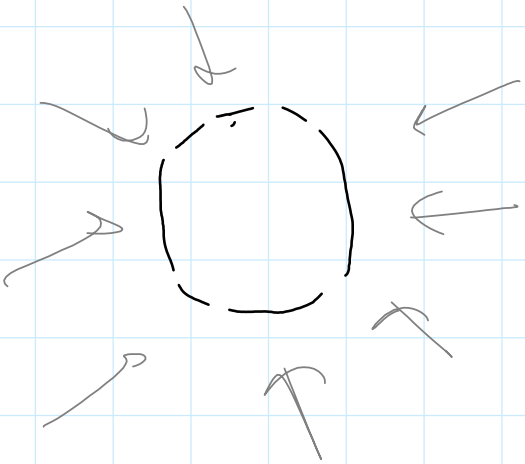


Iron core can not support the gravitational attraction.

$O + e^- \rightarrow n_O + \nu_e$ electron capture

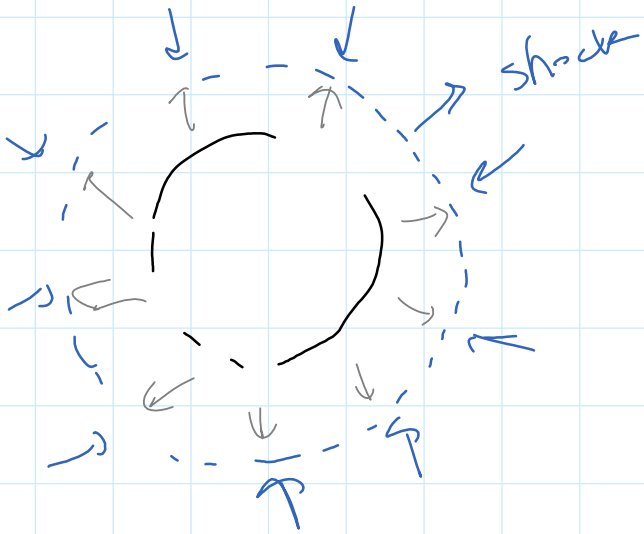


neutron degenerate pressure
a very stiff/hard core



In falling matter hit the surface

BOUNCE



forming a shock

but the shock lose energy because of the confrontation,

It stalls!!

(no explosion)

How to revive the shock?

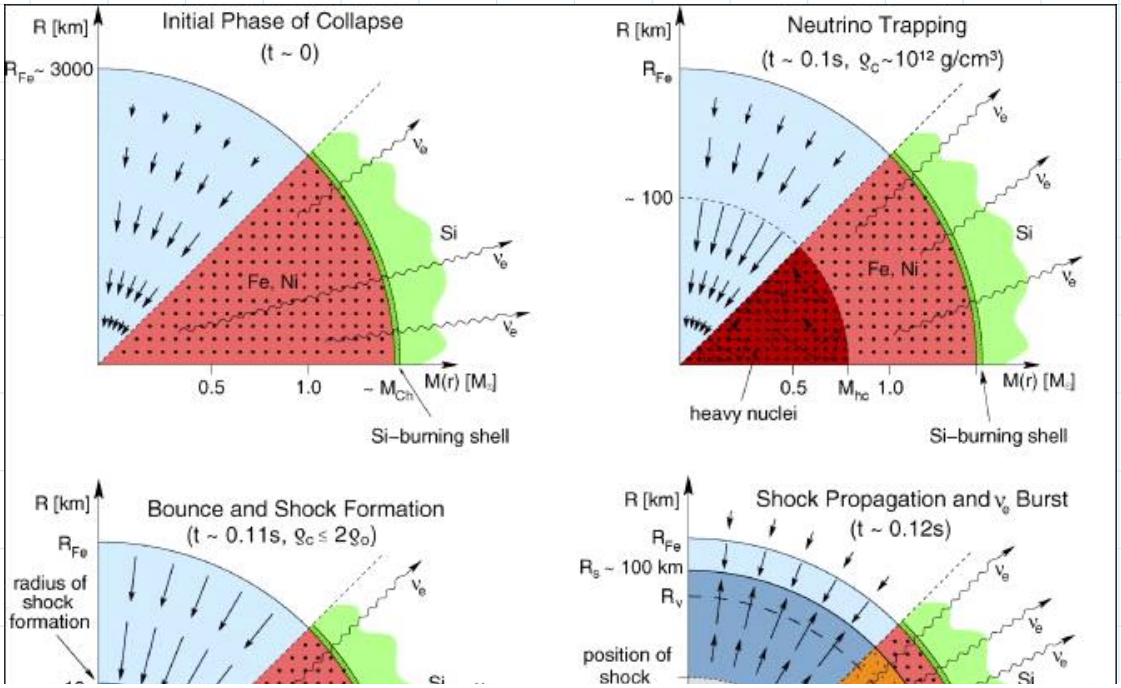
① Neutrino : deposit energy

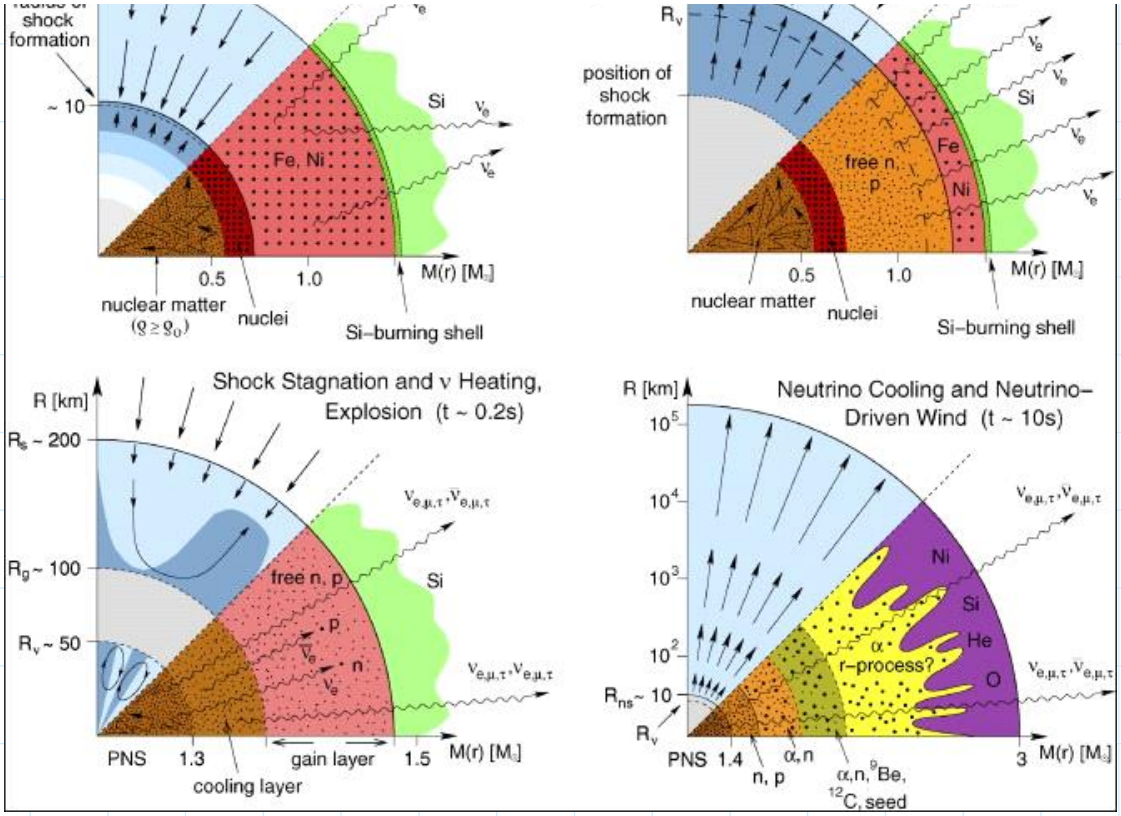
In the explosion

ν process → Ni, Co

About 10 days after the explosion

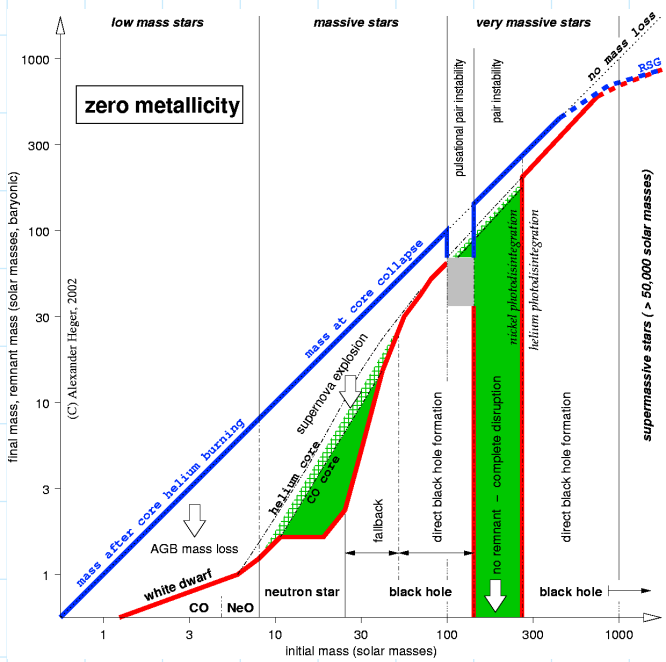
optically thin → EM release





from http://arxiv.org/abs/astro-ph/0612072

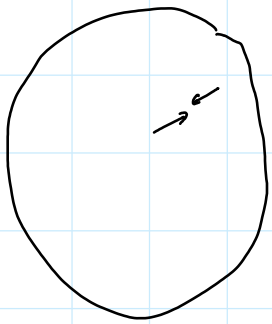
From <http://physics.stackexchange.com/questions/63558/why-does-a-supernova-explode>



from http://rmp.aps.org/abstract/RMP/v74/i4/p1015_1

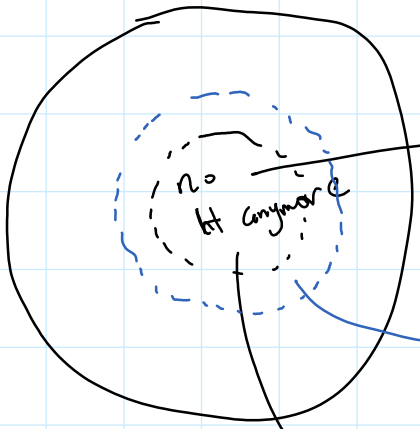
From <http://physics.stackexchange.com/questions/63558/why-does-a-supernova-explode>

Before all these explosions, we also explain how can a star be stable.



$4p \rightarrow He$

pp chain



core contracts due to lessened fusion (~~energy~~ pressure support)

The near core region heated

pp → He begins in the surrounding

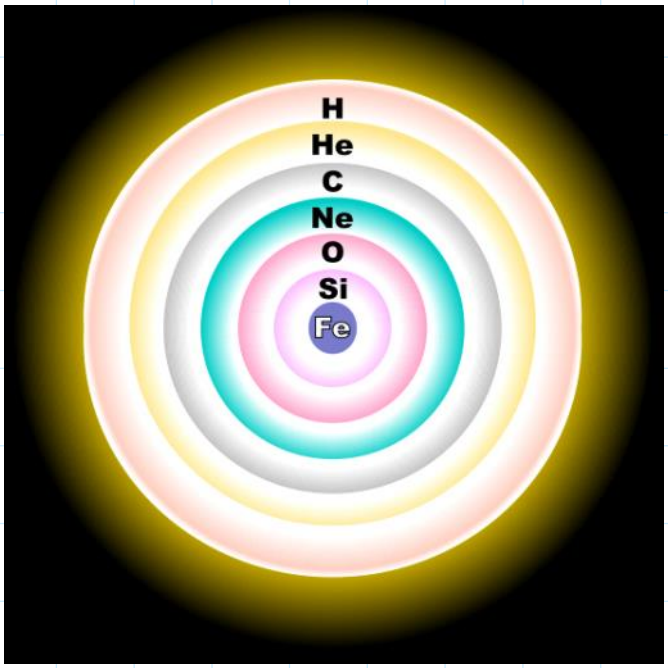
heated and new reactions

$He \rightarrow C$

$\leftrightarrow$ Neon, O, Si

Finally $Si \rightarrow Fe$ (^{56}Fe)

(spontaneous fusion ends here)



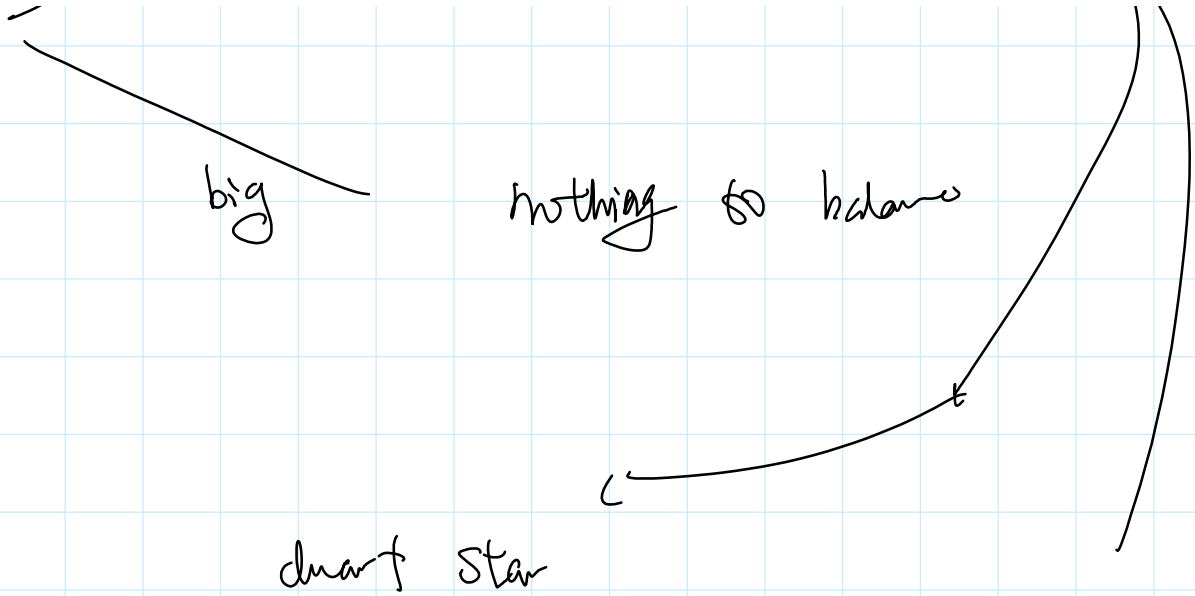
onion-like structure

* How to estimate the life time of a star

☆ How do the reaction rate depend on temperature!!
How is temperature depend on mass of star

small (how small?)

something to balance the star



* Chandrasekhar limit

○ Increasing temperature $\xrightarrow{?}$ increase pressure?

not necessarily true:

Iron core $\xrightarrow{T \uparrow}$ dissociation (which uses energy)

$\downarrow$

energy comes from e^-

$\downarrow$

reduces pressure of e^-

$\downarrow$

which is a main source of pressure

main source of pressure

* Why is e^- pressure more important?

* Why is then entropy decreasing

Basics

Thursday, November 3, 2016 5:09 PM

I. The equations

1. Hamiltonian

$$H = H_v + H_m + H_{\nu\nu}$$

$$H_v = -\frac{1}{2}\eta\omega_v \sigma_3 \qquad \eta = +1 \quad NH$$
$$\eta = -1 \quad IH$$

$$H_m = \frac{1}{2}(\sqrt{2}G_F N_e)\,\sigma_3 = \frac{1}{2}\lambda\sigma_3$$

$$H_{\nu\nu} = \sqrt{2}\,G_F \int d\omega\, d\Omega_{\hat{v}'}\, \rho_\alpha\,(1-\hat{v}\cdot\hat{v}')$$

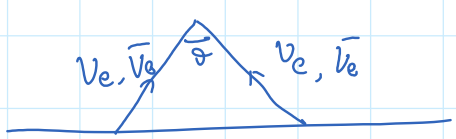
Notice that the $H_{\nu\nu}$ term should include all the possible contributions

① For example we use two beams

For the beam of ν_e

$$H_{\nu\nu}^{(\nu_e)} = \sqrt{2}\,G_F \int d\omega_{\bar{\nu}_e}\, d\Omega_{\hat{v}'}\, \rho_{\bar{\nu}_e}\,(1-\cos\theta)$$

② For more complicated beams



$$H_{\nu\nu}^{(\nu_e,\,\text{Left})} = \sqrt{2}\,G_F \int d\omega_{\nu_e,R}\, d\Omega_{\hat{v}'}\, \rho_{\nu_e,R}\,(1-\cos\theta)$$
$$+\sqrt{2}\,G_F \int d\omega_{\bar{\nu}_e,R}\, d\Omega_{\hat{v}'}\, \rho_{\bar{\nu}_e,R}\,(1-\cos\theta)$$

because the ν_e beam on Left feels the contribution from the <u>beams</u> on the Right, both ν_e and $\bar{\nu}_e$!

2. Equation of motion

$$i\partial_t \rho = [H, \rho]$$

For linear stability analysis, we use density matrix

$$\rho_{\nu_e} = \begin{pmatrix} 1 & \delta \\ \delta^* & 0 \end{pmatrix}$$

$$\rho_{\bar{\nu}_e} = \begin{pmatrix} 1 & \bar{\delta} \\ \bar{\delta}^* & 0 \end{pmatrix}$$

The equation of motion becomes

$$i\frac{d}{dt}\begin{pmatrix} 1 & \delta \\ \delta^* & 0 \end{pmatrix} = \begin{pmatrix} h_0 & h \\ h^* & -h_0 \end{pmatrix}\begin{pmatrix} 1 & \delta \\ \delta^* & 0 \end{pmatrix} - \begin{pmatrix} 1 & \delta \\ \delta^* & 0 \end{pmatrix}\begin{pmatrix} h_0 & h \\ h^* & -h_0 \end{pmatrix}$$

$$\text{RHS} = \cancel{\begin{pmatrix} h_0 & 0 \\ 0 & -h_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + \begin{pmatrix} h_0 & 0 \\ 0 & -h_0 \end{pmatrix}\begin{pmatrix} 0 & \delta \\ \delta^* & 0 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 0 & -h_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \end{pmatrix}} + \begin{pmatrix} 0 & \\ & \end{pmatrix}\begin{pmatrix} \delta^* & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}\begin{pmatrix} 0 & \delta \\ \delta^* & 0 \end{pmatrix}$$

$$- \cancel{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h_0 & 0 \\ 0 & -h_0 \end{pmatrix}} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & \delta \\ \delta^* & 0 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & -h_0 \end{pmatrix} - \begin{pmatrix} 0 & \delta \\ \delta^* & 0 \end{pmatrix}\begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & h_0\delta \\ -h_0\delta^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ h^* & 0 \end{pmatrix} + \begin{pmatrix} h\delta^* & 0 \\ 0 & h^*\delta \end{pmatrix}$$

$$- \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -h_0\delta \\ \delta^* h_0 & 0 \end{pmatrix} - \begin{pmatrix} \delta h^* & 0 \\ 0 & \delta^* h \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2h_0\delta \\ -2h_0\delta^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -h \\ h^* & 0 \end{pmatrix} + \begin{pmatrix} h\delta^* - h^*\delta & 0 \\ 0 & h^*\delta - \delta^* h \end{pmatrix}$$

?