Stimulated Neutrino Flavor Conversions and Rabi Oscillations

Lei Ma

Department of Physics
UNM

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OUTLINE

OVERVIEW

WHAT ARE NEUTRINOS?

Elementary particles.
Source: symmetrymagazine.org

Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- light.

Adapted from Olga Mena &
Stephen Parke (2004)

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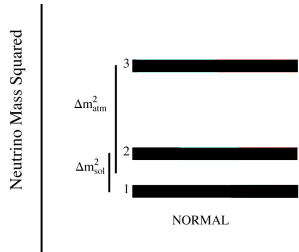
Neutrino Oscillations

WHAT ARE NEUTRINO OSCILLATIONS?
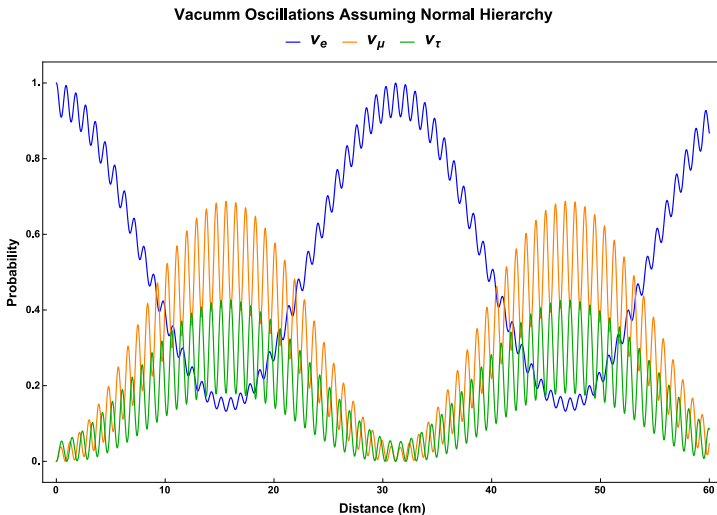
Probabilities of finding neutrinos to be in each flavor.

WHY DO NEUTRINOS OSCILLATE?

Flavor states are different from mass states (eigen energy states).

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

► Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

► Mixing angle θ_{v}

FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\dfrac{\vec{\sigma}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{s} = \Psi^{\dagger} \dfrac{\vec{\sigma}}{2} \Psi$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

electron flavor   muon flavor

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Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

Vacuum oscillation Hamiltonian

$$\frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

$$\rightarrow \cos 2\theta_{\mathrm{v}} \begin{pmatrix} 0 \\ 0 \\ \omega_{\mathrm{v}} \end{pmatrix} - \sin 2\theta_{\mathrm{v}} \begin{pmatrix} \omega_{\mathrm{v}} \\ 0 \\ 0 \end{pmatrix}$$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

electron flavor　　　　muon flavor
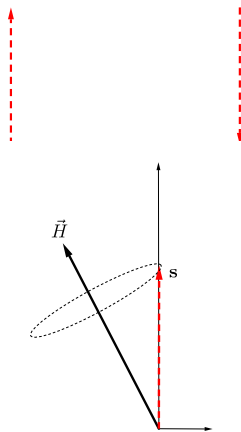
OVERVIEW

INTERACTIONS WITH MATTER
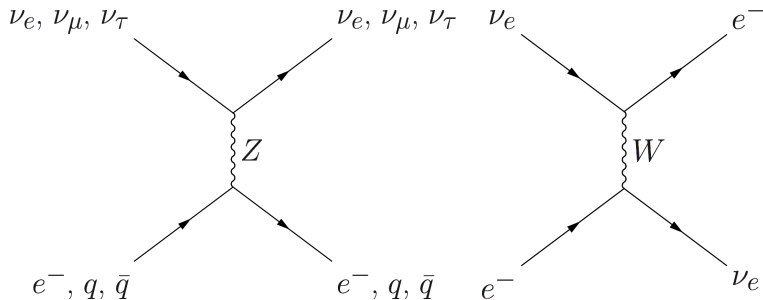
Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, quarks etc.

Charged current interaction between ν_e and e^-

MATTER INTERACTION

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)$

MSW EFFECT

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma_1} \right) + \frac{\lambda(x)}{2} \boldsymbol{\sigma_3}$$

$$\rightarrow \omega_{\mathrm{v}} \begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

$$= \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)$$
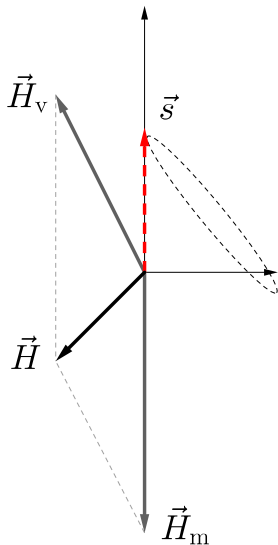
MSW EFFECT

Energy gap:

$$|\vec{H}|$$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

MSW EFFECT

Adiabatic matter density change
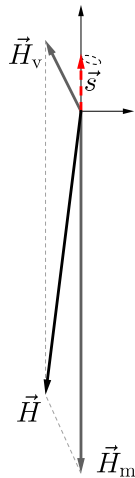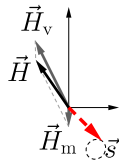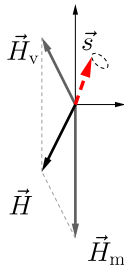
Large density   Low density

MSW EFFECT
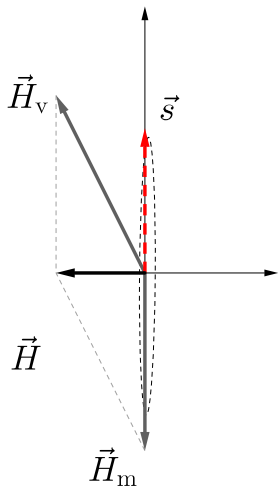
- ▶ MSW Resonance
- ▶ Maximum possible flavor transition probability amplitude
- ▶ A specific matter density

MORE COMPLICATED MATTER EFFECT

Why Do We Care

Astrophysical environments: supernovae etc
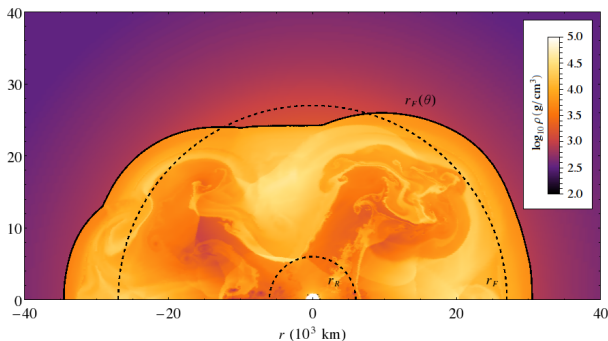
Supernova shock and turbulence. E. Borriello, et al (2014)

STIMULATED NEUTRINO FLAVOR CONVERSIONS

$$\lambda(x) = \lambda_0 + A\cos(kx)$$
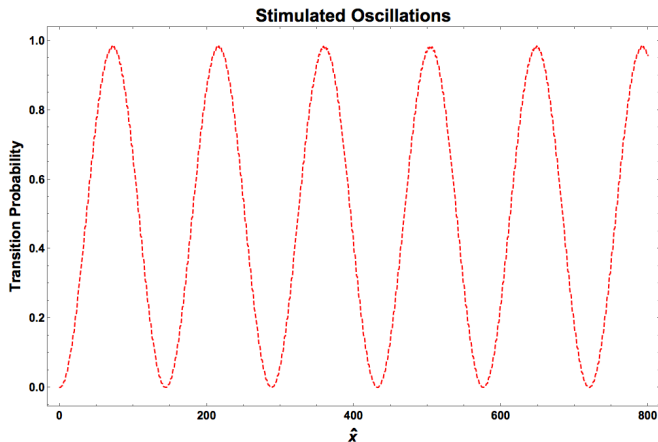
Stimulated oscillations. $\lambda(x) = \lambda_0 + A\cos(kx)$

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

$E_2 = \frac{\omega_m}{2}$ ——————

Periodic Driving Potential

Frequency : k

$E_1 = -\frac{\omega_m}{2}$ ——————

RABI OSCILLATIONS

Static Frame

$$\vec{H}_3 = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$
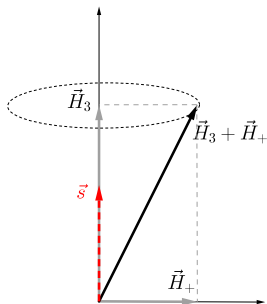
Corotating Frame

$$\vec{H}'_3 = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
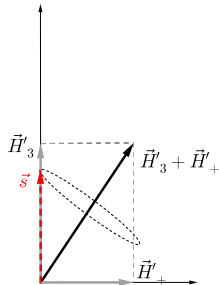
RABI OSCILLATIONS

Corotating Frame

$$\vec{H}_3' = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow k = \omega_{\mathrm{m}}$$
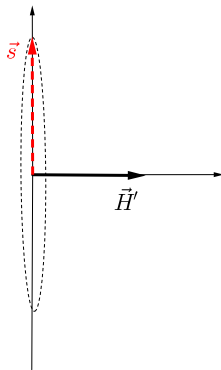
RABI OSCILLATIONS

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

$E_2 = \frac{\omega_m}{2}$ ———————

Periodic Driving Potential

Frequency : k

$E_1 = -\frac{\omega_m}{2}$ ———————

Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = \sqrt{\alpha^2 + (\omega_{\mathrm{m}} - k)^2}$$
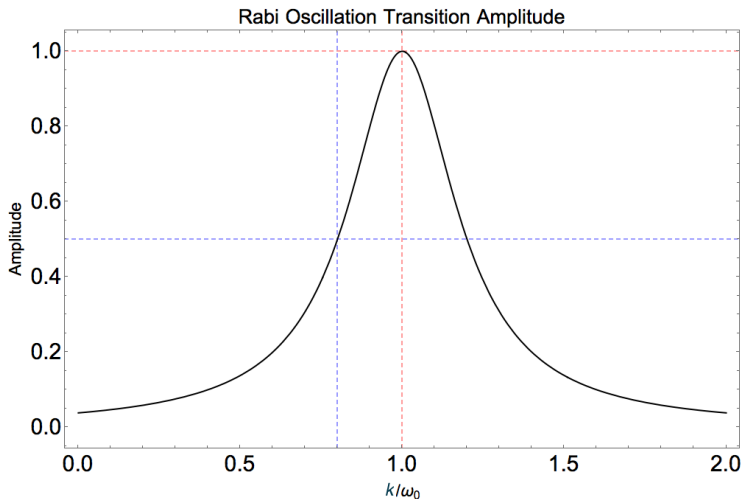
Amplitude of Rabi oscillations for different driving field frequency k

$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

Background matter basis (eigen energy basis):

$$\mathbf{H} = \frac{1}{2}\left(-\omega_m\right)\boldsymbol{\sigma_3}$$

HAMILTONIAN IN MATTER BASIS

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis (eigen energy basis):

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

HAMILTONIAN IN MATTER BASIS

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Matter potential frequency

$$k \sim \omega_{\mathrm{m}}$$

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{2} \left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A \cos(kx) \right) \sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A \cos(kx) \sigma_1 \\
&\to \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}
\end{aligned}
$$

RABI FORMULA WORKS
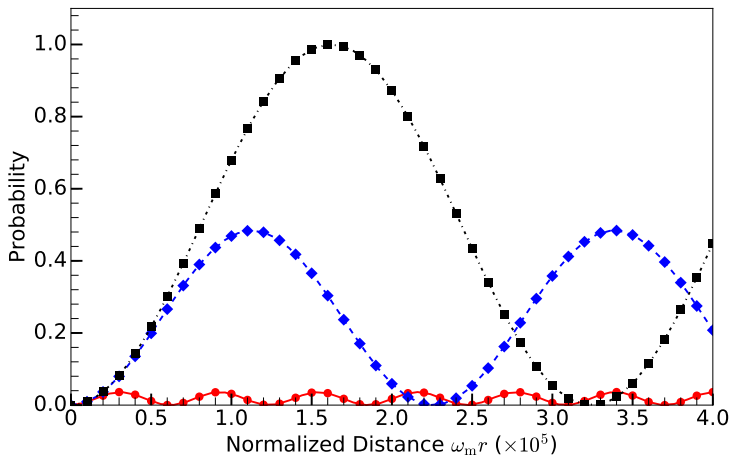
$$\vec{H} \sim \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix}$$

Lines: Rabi formula
Dots, diamonds, triangles, and squares are for $k = \omega_{\mathrm{m}}$,
$k = (1 - 2 \times 10^{-5})\omega_{\mathrm{m}}$, and $k = (1 - 10^{-4})\omega_{\mathrm{m}}$ respectively.

OVERVIEW

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_\mathrm{m} + \cos 2\theta_\mathrm{m} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_\mathrm{m}}{2}A\cos(kx)\sigma_1$$

$$\rightarrow \omega_\mathrm{m}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha\begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha\begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

RABI BASIS

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2} \left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}} \right) \boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{\mathrm{L}} \\ \tilde{\psi}_{\mathrm{H}} \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2} \int_0^x A\cos(k\tau)d\tau.$$

SINGLE FREQUENCY MATTER POTENTIAL

Hamiltonian in Rabi Basis

The off-diagonal element of Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}B_n e^{i(nk)x} \\ \frac{1}{2}B_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $B_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)/\omega_{\mathrm{m}}$.

SINGLE FREQUENCY MATTER POTENTIAL

Hamiltonian in Rabi Basis

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where $B_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)/\omega_{\mathrm{m}}$.

Multiple potentials with different frequencies!

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

Energy gap in this frame becomes the length of the vector

$$\sqrt{(\omega_m - k_2)^2 + \alpha_2^2} \to \omega_m - k_2 + \frac{1}{2}\frac{\alpha_2^2}{\omega_m - k_2}$$

Relative detuning

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

Consider $k_1 = \omega_{\mathrm{m}}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Amplitude reduces from 1 to 1/2 if

$$D' = 1.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
Destruction effect: $k_1 = \omega_{\mathrm{m}}$,

$$|\alpha_2| \gg \sqrt{2\omega_{\mathrm{m}}|\alpha_1(k_2 - \omega_{\mathrm{m}})|} \equiv \alpha_{2,\mathrm{C}}$$

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$
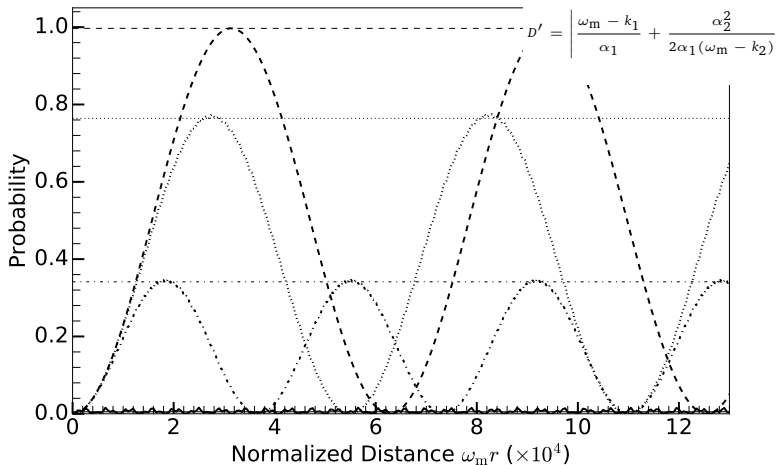
Grid lines: amplitude predicted using $1/(1 + D'^2)$

		α_2, k_1 values		
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$	$5.0 \times 10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$5 \times 10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$

Matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

		$k = \omega_{\mathrm{m}}$
n	D	D'
1	0	-
-1	10^5	4.8×10^{-6}
2	1.1×10^9	2.1×10^{-14}
-2	3.4×10^9	6.9×10^{-15}

OVERVIEW

1. **Vacuum oscillations: flavor sates are not mass states.**

2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.

3. Stimulated oscillations: variation in matter profile can cause resonances.

4. Oscillations with two driving fields of different frequencies: large potential to destroy the resonance.

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. Oscillations with two driving fields of different frequencies: large potential to destroy the resonance.

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. Oscillations with two driving fields of different frequencies: large potential to destroy the resonance.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
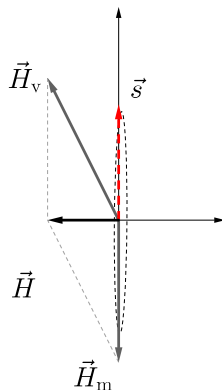2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
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$$|\alpha_2| \gg \sqrt{2\omega_{\mathrm{m}}|\alpha_1(k_2 - \omega_{\mathrm{m}})|} \equiv \alpha_2$$

BACKUP SLIDES

BACKUP SLIDES

WHY DO NEUTRINOS OSCILLATE?

> ### Equation of Motion
>
> $$i\partial_x |\Psi\rangle = \hat{\mathbf{H}} |\Psi\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{|\nu_1\rangle, |\nu_2\rangle\}$.
-
  $$\mathbf{H} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$

- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

  $$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp\left(i\omega_v x/2\right) \\ \langle\nu_2|\Psi(0)\rangle \exp\left(-i\omega_v x/2\right) \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

WHY DO NEUTRINOS OSCILLATE?

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θ_{v}: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_{\mathrm{v}}}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= -\frac{\omega_{\mathrm{v}}}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_{\mathrm{v}}}{2} \begin{pmatrix} -\cos 2\theta_{\mathrm{v}} & \sin 2\theta_{\mathrm{v}} \\ \sin 2\theta_{\mathrm{v}} & \cos 2\theta_{\mathrm{v}} \end{pmatrix}$$
$$= \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)$$

NATURE OF NEUTRINO OSCILLATION

Transition Probability

$$P(|\nu_{\mathrm{e}}\rangle \to |\nu_{\mu}\rangle) = \sin^2(2\theta_{\mathrm{v}}) \sin^2(\omega_{\mathrm{v}} x/2)$$

- $\omega_{\mathrm{v}} = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- Mixing angle θ_{v} determines flavor oscillation amplitude.

MSW EFFECT

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_m

Define matter basis (eigenenergy basis) $\{|\nu_L\rangle, |\nu_H\rangle\}$

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2}\boldsymbol{\sigma_3}$$

MSW EFFECT

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

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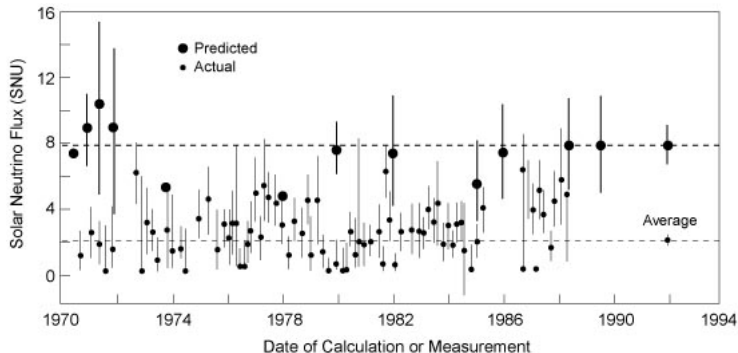

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2}\boldsymbol{\sigma_3}$$

Transition Probability

$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_m)\sin^2(\omega_m x)$$
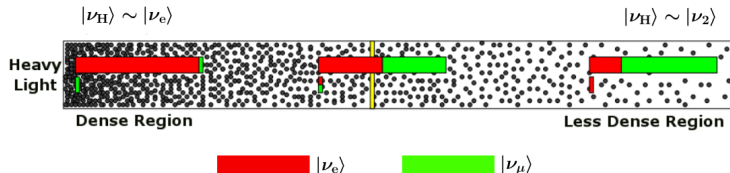

SOLAR NEUTRINO PROBLEM

Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW EFFECT AND SOLAR NEUTRINOS

$$\mathbf{H} = \frac{\lambda(x) - \omega_{\mathrm{v}}\cos 2\theta_{\mathrm{v}}}{2}\boldsymbol{\sigma_3} + \frac{\omega_{\mathrm{v}}\sin 2\theta_{\mathrm{v}}}{2}\boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & -\sin\theta_{\mathrm{m}} \\ \sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}$$

Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW EFFECT INVERTED HIERARCHY

Suppose $\omega_{\mathrm{v}} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = -\frac{\omega_{\mathrm{v}}}{2} \begin{pmatrix} -\cos 2\theta_{\mathrm{v}} & \sin 2\theta_{\mathrm{v}} \\ \sin 2\theta_{\mathrm{v}} & \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \sqrt{2} G_{\mathrm{F}} n_{\mathrm{e}}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\mathbf{H} = \left(\frac{-\omega_{\mathrm{v}}}{2} \cos 2\theta_{\mathrm{v}} + \frac{\lambda(x)}{2} \right) \boldsymbol{\sigma_3} - \frac{\omega_{\mathrm{v}}}{2} \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma_1}$$

HAMILTONIAN

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\text{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

HAMILTONIAN

> **Hamiltonian in Background Matter Basis**
>
> $$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}}A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2}A\cos(kx)\sigma_1.$$

HAMILTONIAN

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2}A\cos(kx)\sigma_1.$$

RABI OSCILLATIONS

The coupling strength is calculated as

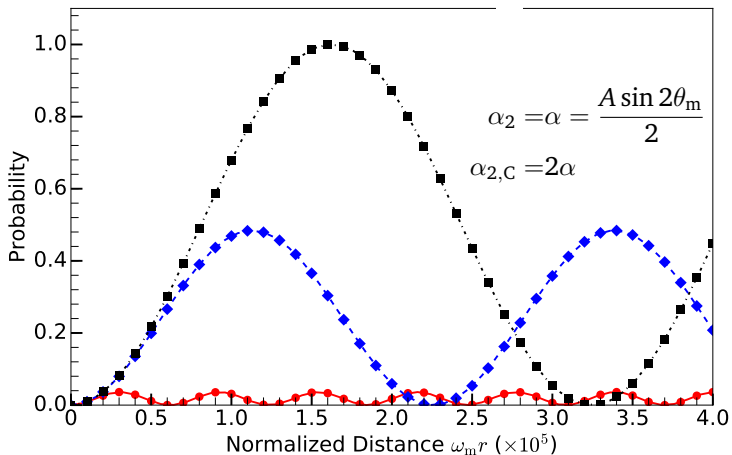$$\alpha = \langle 1 | \mathbf{d} \cdot \mathbf{E} | 2 \rangle$$

where the electric field is

$$\mathbf{E} = \mathbf{E}_0 \sin(kt).$$

and $\mathbf{d}$ is the dipole moment.

RABI FORMULA WORKS

$$\vec{H} \sim \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix}$$

$$\alpha_2 = \alpha = \frac{A \sin 2\theta_{\mathrm{m}}}{2}$$

$$\alpha_{2,\mathrm{C}} = 2\alpha$$

Lines: Rabi formula
Dots, diamonds, triangles, and squares are for $k = \omega_{\mathrm{m}}$,
$k = (1 - 2 \times 10^{-5})\omega_{\mathrm{m}}$, and $k = (1 - 10^{-4})\omega_{\mathrm{m}}$ respectively.

PARAMETERS USED FOR VACUUM OSCILLATIONS

$\theta_{12} = 33.36/180\pi; \theta_{13} = 8.66/180\pi; \theta_{23} = 40/180 * \pi; \delta_{cp} = 0;$
$m_1^2 = 0.01; m_2^2 = m_1^2 + 0.000079; E = 1\text{MeV}$

SINGLE FREQUENCY MATTER POTENTIAL

Matter potential

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

$$h \equiv -\frac{\delta\lambda(x)}{2}e^{2i\eta(x)}$$

$$= \frac{i}{4}\left[\exp\left(ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right.$$

$$\left. - \exp\left(-ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right]$$

SINGLE FREQUENCY MATTER POTENTIAL

Off-diagonal Term in Our System

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$h \propto \left[\exp\left(ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx) \right) \right.$$
$$\left. - \exp\left(-ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx) \right) \right]$$

Jacobi-Anger expansion (Kneller et al, 2013)

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta)e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

SINGLE FREQUENCY MATTER PROFILE

Why Does It Work?

$$J_n(n \operatorname{sech} \alpha) \sim \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}, \quad \text{for large } n$$

$\Rightarrow$

$$\Gamma \propto \hat{B}_n \propto \frac{e^{-n(\alpha - \tanh \alpha)}}{\sqrt{2\pi n \tanh \alpha}}$$

Small perturbation $\Rightarrow$ Small $\hat{A}$ $\Rightarrow$ Large α $\Rightarrow$ Drops fast at large n.

SINGLE FREQUENCY MATTER PROFILE

Transition Probability

$$P_{\text{L}\to\text{H}}^{(n)} = \frac{\left|\hat{B}_n\right|^2}{\left|\hat{B}_n\right|^2 + (n\hat{k}-1)^2} \sin^2\left(\frac{q^{(n)}}{2}x\right),$$

where

$$q^{(n)} = \sqrt{\left|\Gamma^{(n)}\right|^2 + (n\hat{k}-1)^2}, \quad \text{frequency of oscillations}$$

$$\Gamma^{(n)} = \left|\hat{B}_n\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

Resonance conditions

$$\hat{k} \sim \frac{1}{n}$$

TWO-FREQUENCY MATTER PROFILE

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

$$\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2} \hat{B}_n e^{i(n\hat{k}-1)\hat{x}},$$

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2} \hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1\hat{k}_1 + n_2\hat{k}_2 - 1)\hat{x}},$$

where

$$\hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2)$$
$$= -(-i)^{n_1+n_2}(n_1\hat{k}_1 + n_2\hat{k}_2) J_{n_1}\left(\frac{\hat{A}_1 \cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right) J_{n_2}\left(\frac{\hat{A}_2 \cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)$$

Which terms are important?

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

		$k_1 = \omega_{\mathrm{m}}$	
n	D	D'_1	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	0	-	3.2×10^5
-1	10^5	4.8×10^{-6}	3.1
2	1.1×10^9	2.1×10^{-14}	6.3
-2	3.4×10^9	6.9×10^{-15}	2.1

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
$k_1 = (1 - 2 \times 10^{-5})\omega_{\mathrm{m}}$
1	1	-	2.2×10^5
-1	10^5	1	3.1
2	1.1×10^9	1	6.3
-2	3.4×10^9	1	2.1

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = (1 - 10^{-4})\omega_{\mathrm{m}}$$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	5.2	-	6.2×10^4
-1	10^5	5.2	3.1
2	1.1×10^9	5.2	6.3
-2	3.4×10^9	5.2	2.1

CASTLE WALL MATTER PROFILE
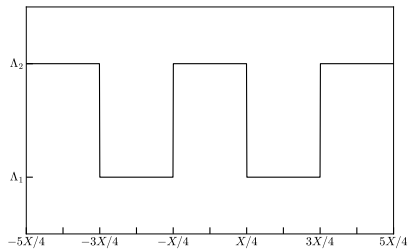
Figure: Castle wall matter profile
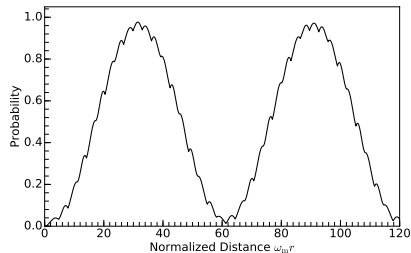
CASTLE WALL MATTER PROFILE

Table: Relative detuning of each frequency.

$\{n_1, n_2\}$	D	$D'_{\{1,0\}}$
$\{1, 0\}$	0	-
$\{-1, 0\}$	48	1.0×10^{-2}
$\{0, 1\}$	1.5×10^2	1.1×10^{-3}
$\{2, 0\}$	2.4×10^2	2.0×10^{-4}
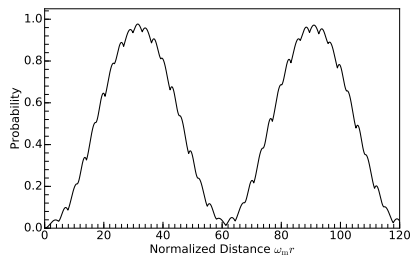
TWO-FREQUENCY MATTER PROFILE

> ### Resonance Lines
>
> There are still resonances, i.e., (almost) zero phases, on lines
>
> $$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$
>
> in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.
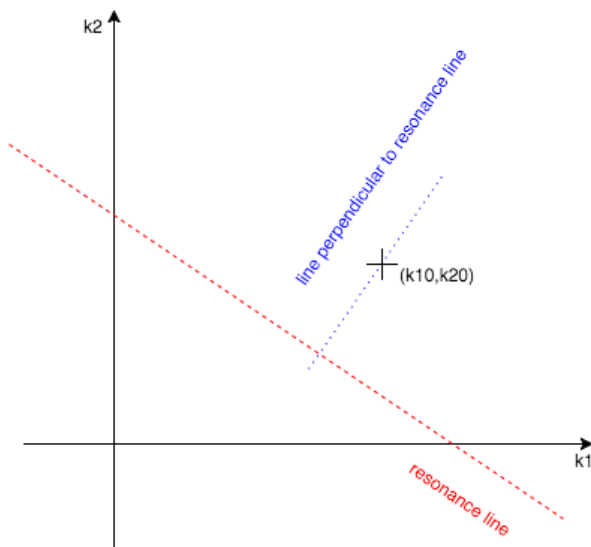
TWO-FREQUENCY MATTER PE

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$
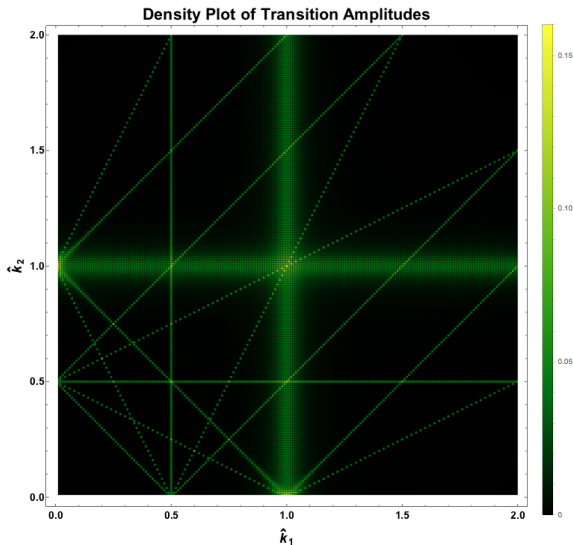
Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

Resonance line, distance to resonance, and width

TWO-FREQUENCY MATTER PROFILE

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\text{intercept}}, \hat{k}_{2,\text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$
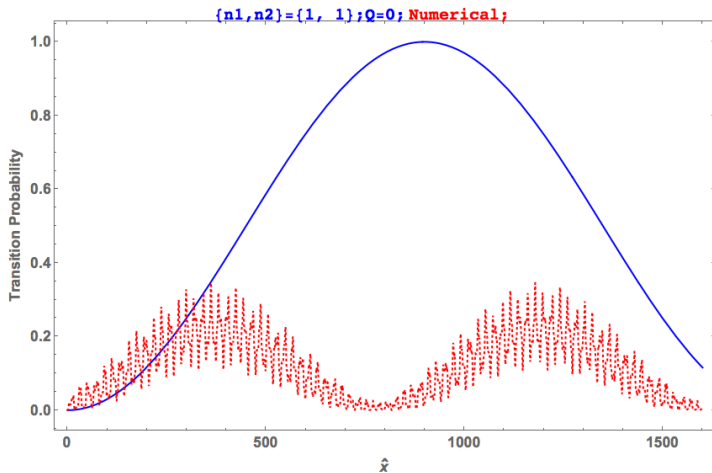
Distance to Resonance Line

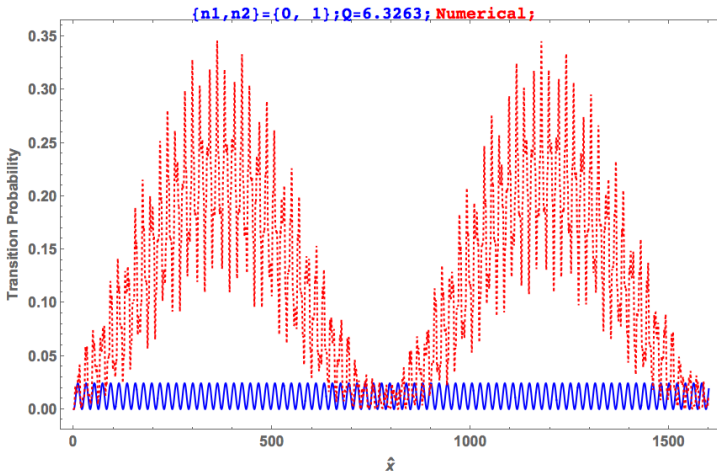$$d = \frac{|n_1 \hat{k}_{10} + n_2 \hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

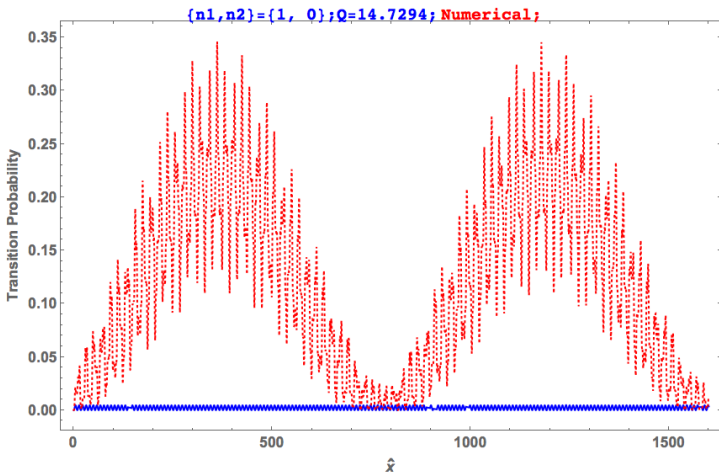### Distance to Resonance Width Ratio

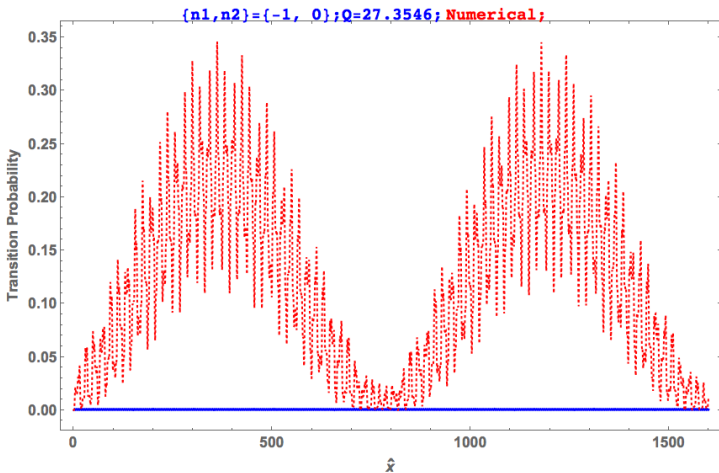$$Q_2 = \frac{d}{\Gamma_2}.$$

TWO-FREQUENCY MATTER PROFILE

{n1,n2}={1, 1};Q=0; Numerical;

TWO-FREQUENCY MATTER PROFILE

$\{n1,n2\}=\{0,\ 1\};Q=6.3263;$ Numerical;

TWO-FREQUENCY MATTER PROFILE

{n1,n2}={1, 0};Q=14.7294; Numerical;

TWO-FREQUENCY MATTER PROFILE

{n1,n2}={-1, 0};Q=27.3546; Numerical;

TWO-FREQUENCY MATTER PROFILE

{n1,n2}={0, -1};Q=35.849; Numerical;

Two-frequency Matter Profile

Transition Probability for Two−frequency Matter Profile

{n1,n2}={{1, 1}, {0, 1}, {1, 0}, {−1, 0}, {0, −1}}; Numerical;
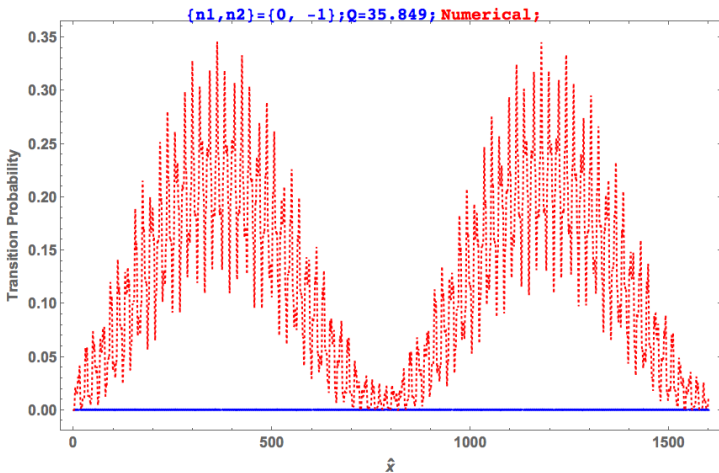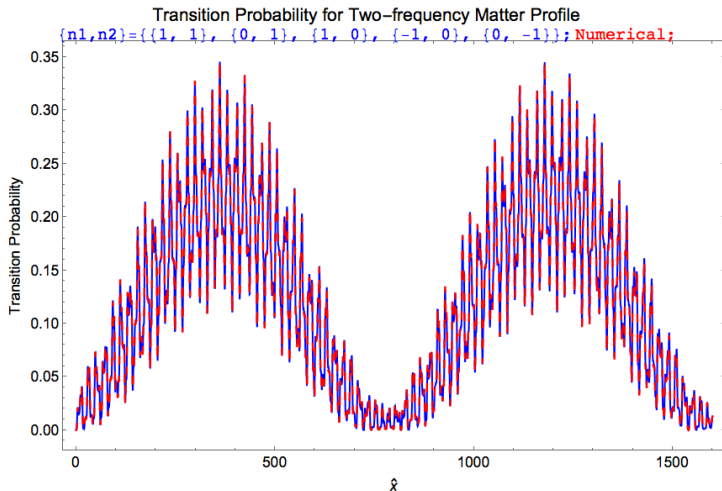
BESSEL'S FUNCTION

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I