Stimulated Neutrino Flavor Conversions and Rabi Oscillations

Lei Ma

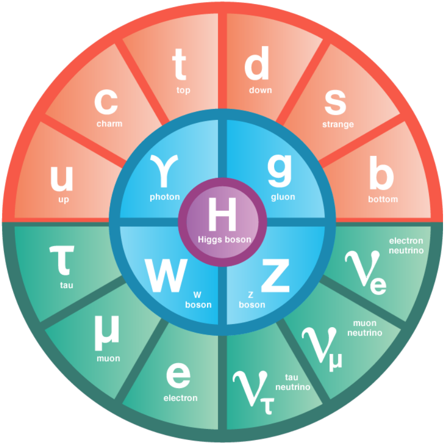
Department of Physics
UNM

January 22, 2017

OUTLINE

Overview

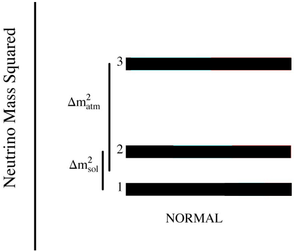
What are Neutrinos?



Elementary particles.
Source: symmetrymagazine.org

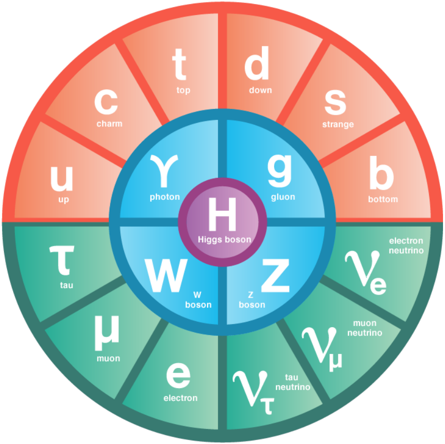
Neutrinos are

- fermions,
- electrically neutral,
- light.



Adapted from Olga Mena & Stephen Parke (2004)

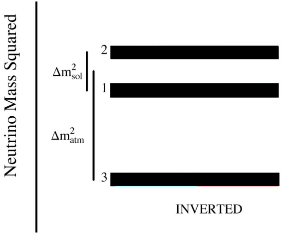
WHAT ARE NEUTRINOS?



Elementary particles.
Source: symmetrymagazine.org

Neutrinos are

- fermions,
- electrically neutral,
- light.



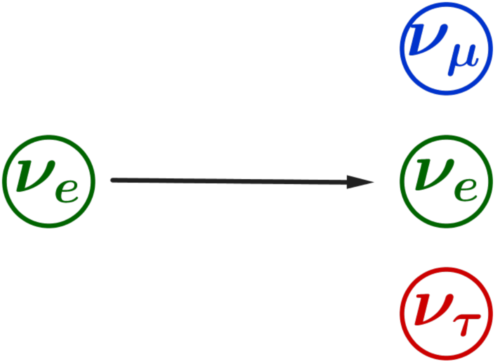
Adapted from Olga Mena & Stephen Parke (2004)

What Are Neutrino Oscillations?

Neutrino Oscillations

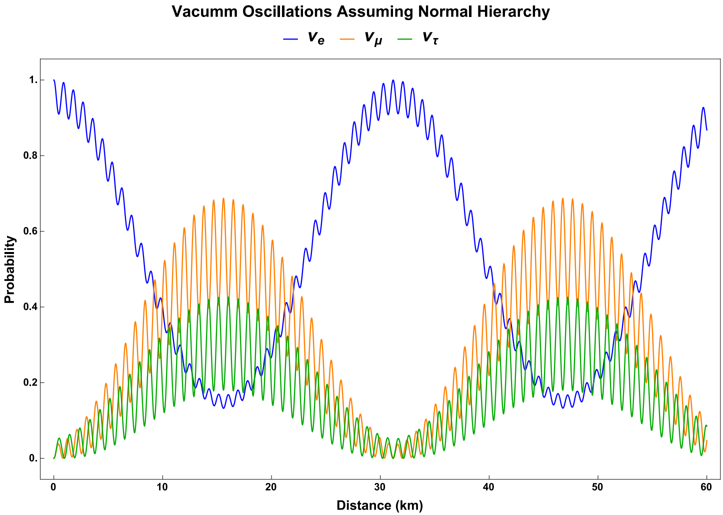
||

Neutrino Flavor Conversions



Neutrino Oscillations

What Are Neutrino Oscillations?



Probabilities of finding neutrinos to be in each flavor.

WHY DO NEUTRINOS OSCILLATE?

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\nu & \sin\theta_\nu \\ -\sin\theta_\nu & \cos\theta_\nu \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

- Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_{v}

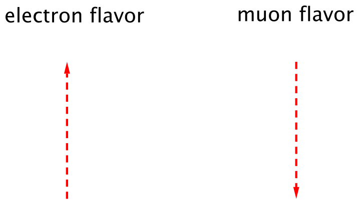
Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\boldsymbol{\sigma}}}{2} \Psi$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\boldsymbol{\sigma}}}{2} \Psi$

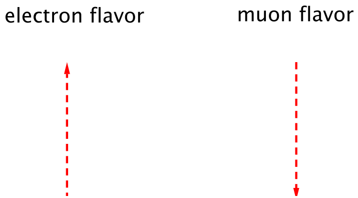
Equation of motion

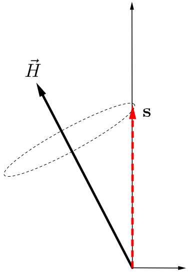
$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

Vacuum oscillation Hamiltonian

$$\frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

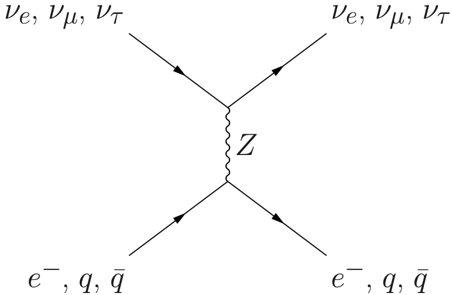
$$\rightarrow \cos 2\theta_{\mathrm{v}} \begin{pmatrix} 0 \\ 0 \\ \omega_{\mathrm{v}} \end{pmatrix} - \sin 2\theta_{\mathrm{v}} \begin{pmatrix} \omega_{\mathrm{v}} \\ 0 \\ 0 \end{pmatrix}$$



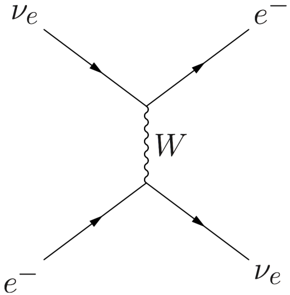


OVERVIEW

INTERACTIONS WITH MATTER



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, quarks etc.



Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)$

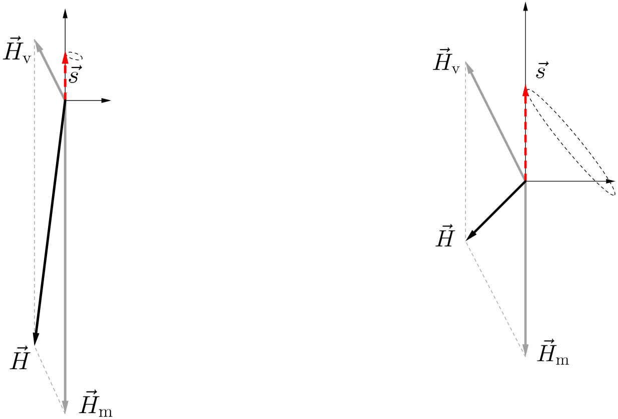
MSW EFFECT

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

$$\rightarrow \omega_{\mathrm{v}}\begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

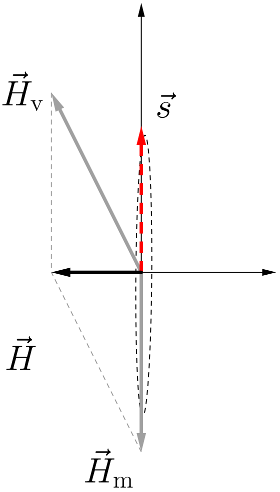
$$\rightarrow \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)$$

MSW EFFECT

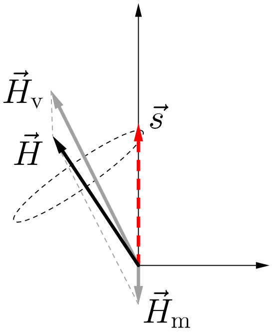


Large density

MSW EFFECT



MSW Resonance

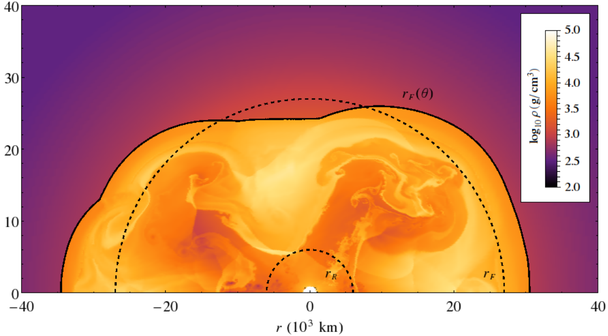


Low density

More Complicated Matter Effect

Why Do We Care

Astrophysical environments: supernovae, accretion disks etc



Supernova shock and turbulence. E. Borriello, et al (2014)

$$\Delta n_e(r) = \sum_n c_n \sin(k_n r + \phi_n)$$

Stimulated Neutrino Oscillations

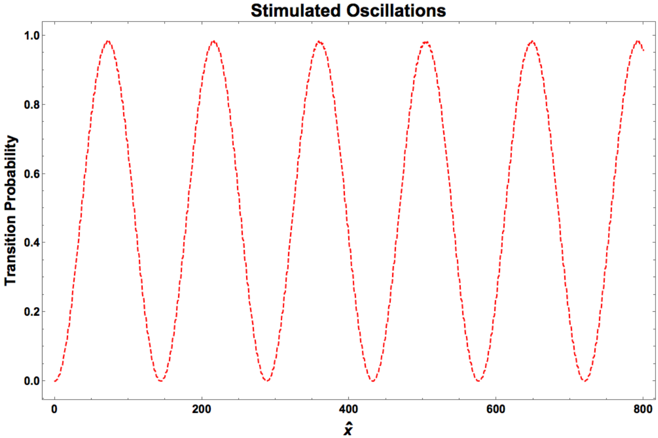
$\lambda(x) = \sqrt{2}G_F n_e$

Matter Profile

$$n_e(x) = n_0 + \delta n \sin(kx + \phi)$$
$$\Rightarrow \lambda(x) = \lambda_0 + \delta\lambda \sin(kx + \phi)$$

Stimulated Neutrino Oscillations

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013); K. Patton et al (2014);



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\sin(kx)$ with $\hat{x} = \omega_{\mathrm{m}} x$, $A = 0.1\omega_{\mathrm{m}}$, $k = 0.995\omega_{\mathrm{m}}$, $\theta_{\mathrm{m}} = \pi/6$

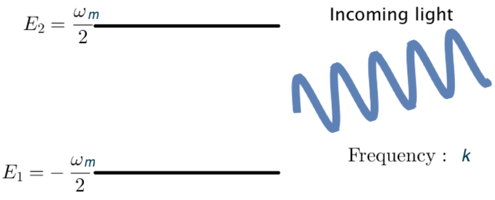
Overview

Rabi Oscillations

Rabi Oscillation

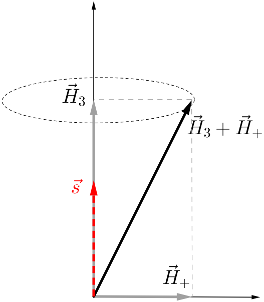
Hamiltonian

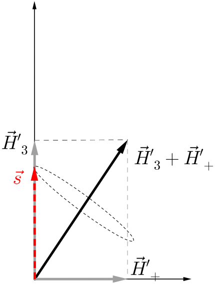
$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3-\frac{\alpha}{2}\begin{pmatrix}0 & e^{ikt}\\ e^{-ikt} & 0\end{pmatrix}$$



Rabi Oscillations

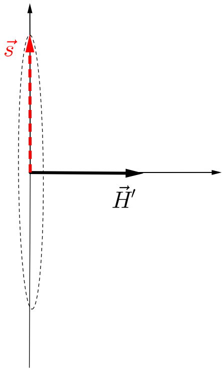
$$\vec{H}_3 = \omega_\mathrm{m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix} \qquad \vec{H}'_3 = (\omega_\mathrm{m} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$





Rabi Oscillations

$$\vec{H}_3' = (\omega_\mathrm{m} - k)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

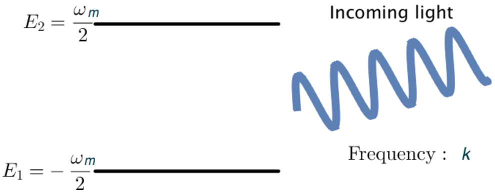


Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$



Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

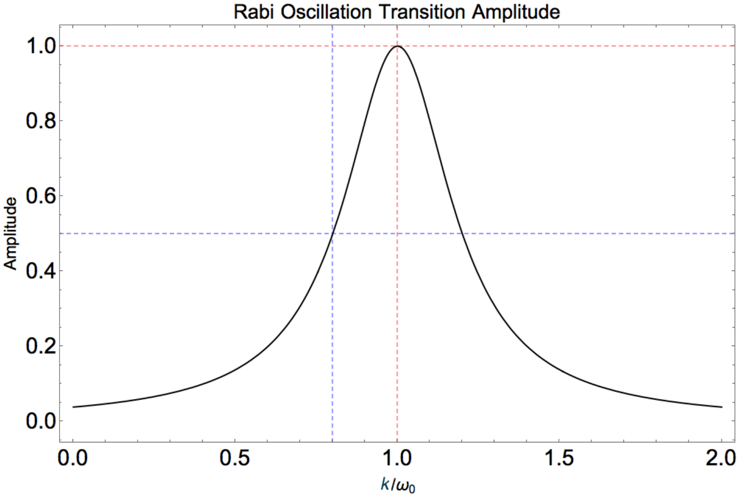
Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = \sqrt{\alpha^2 + (\omega_{\mathrm{m}} - k)^2}$$

Rabi Oscillations



Amplitude of Rabi oscillations for different driving field frequency k

Hamiltonian in Matter Basis

Matter Profile

$$\lambda(x) = \lambda_0$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}}\right)\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis

Matter Profile

$$\lambda(x) = \lambda_0 + A\sin(kx)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\sin(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\sin(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A \sin(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A \sin(kx) \sigma_1$$

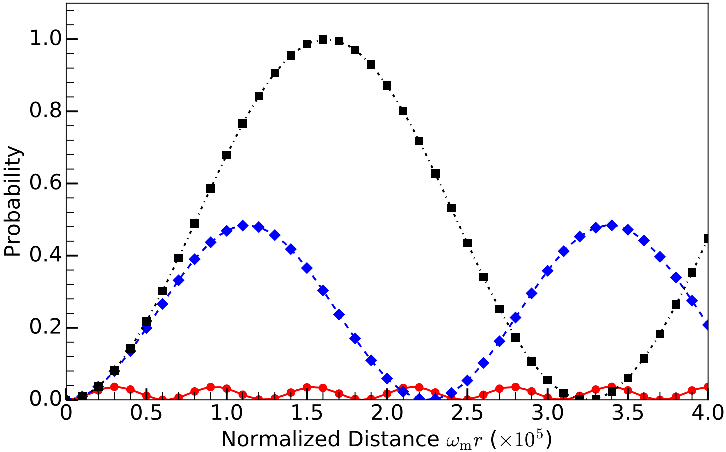
$$\rightarrow \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ -\cos(kx) \\ 0 \end{pmatrix}$$

where

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

Rabi Formula Works

$$\vec{H} \sim \omega_\mathrm{m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix}$$



Lines: Rabi formula

Dots, diamonds, triangles, and squares are for $k = \omega_\mathrm{m}$, $k = (1 - 2 \times 10^{-5})\omega_\mathrm{m}$, and $k = (1 - 10^{-4})\omega_\mathrm{m}$ respectively.

Oscillations With Multifrequencies $\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$

$$\vec{H} = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ -\cos(kx) \\ 0 \end{pmatrix}$$

Dropping off-resonance frequency: Requirement?

Oscillations With Multifrequencies

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}}$$

Corotating frame of the second frequency,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - K_2 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Oscillations With Multifrequencies

Energy gap in this frame becomes the length of the vector

$$\sqrt{(\omega_{\mathrm{m}} - k_2)^2 + \alpha_2^2} \to \omega_{\mathrm{m}} - k_2 + \frac{1}{2}\frac{\alpha_2^2}{\omega_{\mathrm{m}} - k_2}$$

Relative detuning

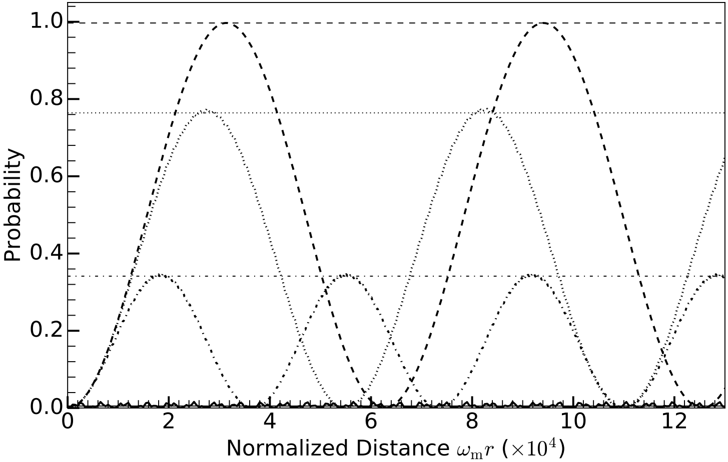
$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Oscillations With Multifrequencies

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
Destruction effect: $k_1 = \omega_{\mathrm{m}}$, $|\alpha_2| \gg \sqrt{2\omega_{\mathrm{m}}|\alpha_1(k_2 - \omega_{\mathrm{m}})|} \equiv \alpha_{2,\mathrm{C}}$

Oscillations With Multifrequencies

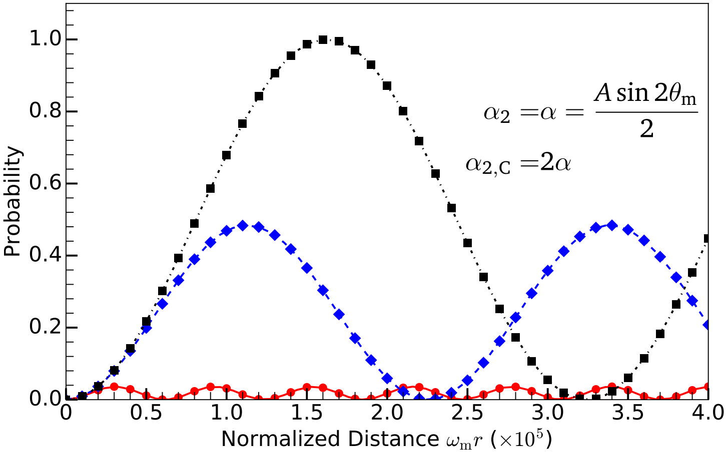


Grid lines: amplitude predicted using $1/(1+D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$	$5.0 \times 10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$5 \times 10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$

Rabi Formula Works

$$\vec{H} \sim \omega_{\rm m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix}$$



Lines: Rabi formula
Dots, diamonds, triangles, and squares are for $k = \omega_{\rm m}$, $k = (1 - 2\times 10^{-5})\omega_{\rm m}$, and $k = (1 - 10^{-4})\omega_{\rm m}$ respectively.

Overview

Hamiltonian for Single Frequency Matter Profile

We have been making approximations.

$$\begin{aligned}\mathbf{H} =& \frac{1}{2}\left(-\omega_{\mathrm{m}} \boxed{+\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 \\ \to & -\frac{\omega_m}{2}\sigma_3 - \frac{A\sin 2\theta_{\mathrm{m}}}{2}\cos(kx)\sigma_1\end{aligned}$$

We need a better basis.

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis $\{|\tilde{\nu}_{\mathrm{L}}\rangle, |\tilde{\nu}_{\mathrm{H}}\rangle\}$ is related to background matter basis $\{|\nu_{\mathrm{L}}\rangle, |\nu_{\mathrm{H}}\rangle\}$ through

$$\begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} |\tilde{\nu}_{\mathrm{L}}\rangle \\ |\tilde{\nu}_{\mathrm{H}}\rangle \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x \delta\lambda(\tau) d\tau.$$

Single Frequency Matter Profile

Matter profile

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_\mathrm{m}\begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2}e^{2i\eta(x)} \\ &= \frac{i}{4}\left[\exp\left(ikx + \boxed{i\cos 2\theta_\mathrm{m}\frac{A}{k}\cos(kx)}\right)\right. \\ &\quad \left. -\exp\left(-ikx + \boxed{i\cos 2\theta_\mathrm{m}\frac{A}{k}\cos(kx)}\right)\right] \end{aligned}$$

Single Frequency Matter Profile

Off-diagonal Term in Our System

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$\begin{aligned} h \propto \bigg[&\exp\left(ikx + \boxed{i\cos 2\theta_{\mathrm{m}} \frac{A}{k}\cos(kx)} \right) \\ &- \exp\left(-ikx + \boxed{i\cos 2\theta_{\mathrm{m}} \frac{A}{k}\cos(kx)} \right) \bigg] \end{aligned}$$

Jacobi-Anger expansion

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

Single Frequency Matter Profile

Scaled Quantities

Characteristic scale: ω_m

- $\hat{A} = A/\omega_\mathrm{m}$
- $\hat{k} = k/\omega_\mathrm{m}$
- $\hat{x} = \omega_\mathrm{m} x$
- $\hat{h} = h/\omega_\mathrm{m}$

Single Frequency Matter Profile

Rotation Wave Approximation

The off-diagonal element of Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\hat{B}_n e^{i(n\hat{k})\hat{x}} \\ \frac{1}{2}\hat{B}_n^* e^{-i(n\hat{k})\hat{x}} & 0 \end{pmatrix}$$

where $\hat{B}_n = -(-i)^n n\hat{k} \tan 2\theta_{\mathrm{m}} J_n(\hat{A}\cos 2\theta_{\mathrm{m}}/\hat{k})$.

Single Frequency Matter Profile

Transition Probability

$$P^{(n)}_{\mathrm{L}\to\mathrm{H}} = \frac{\left|\hat{B}_n\right|^2}{\left|\hat{B}_n\right|^2 + (n\hat{k}-1)^2} \sin^2\left(\frac{q^{(n)}}{2}x\right),$$

where

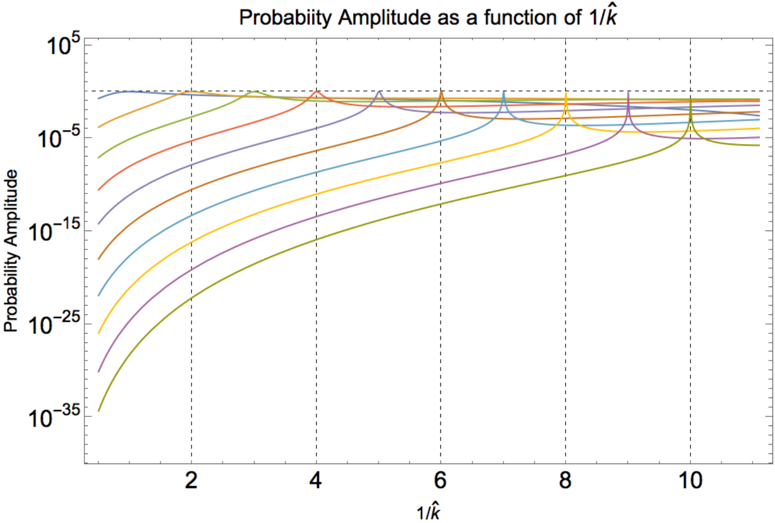
$$q^{(n)} = \sqrt{\left|\Gamma^{(n)}\right|^2 + (n\hat{k}-1)^2}, \quad \text{frequency of oscillations}$$

$$\Gamma^{(n)} = \left|\hat{B}_n\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

Resonance conditions

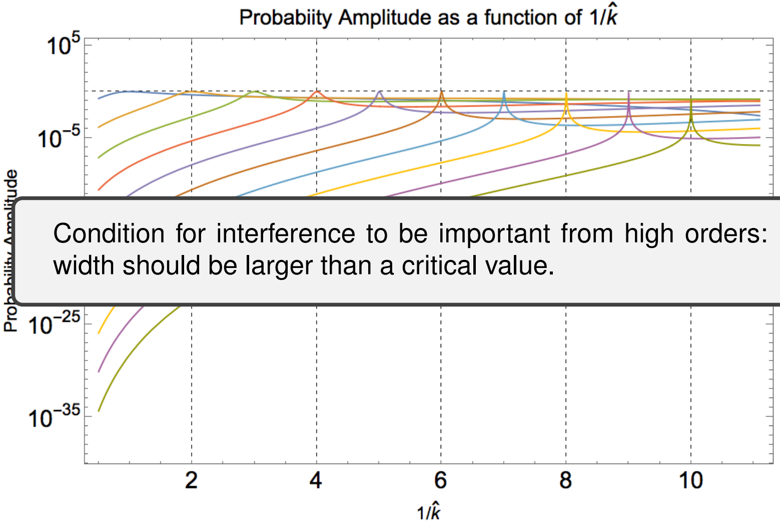
$$\hat{k} \sim \frac{1}{n}$$

SINGLE FREQUENCY MATTER PROFILE



Resonances of different $n = 1/\hat{k}$. Width becomes extremely narrow for high orders.

SINGLE FREQUENCY MATTER PROFILE



Condition for interference to be important from high orders: width should be larger than a critical value.

Resonances of different $n = 1/\hat{k}$. Width becomes extremely narrow for high orders.

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

$$k_1 = \omega_{\rm m}$$

n	D	D_1'	$2\pi\omega_{\rm m}/\Omega_n$
1	0	-	3.2×10^5
-1	10^5	4.8×10^{-6}	3.1
2	1.1×10^9	2.1×10^{-14}	6.3
-2	3.4×10^9	6.9×10^{-15}	2.1

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

$$k_1 = (1 - 2\times 10^{-5})\omega_{\mathrm{m}}$$

n	D	D'_1	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	1	-	2.2×10^5
-1	10^5	1	3.1
2	1.1×10^9	1	6.3
-2	3.4×10^9	1	2.1

Single Frequency Matter Profile Revisited

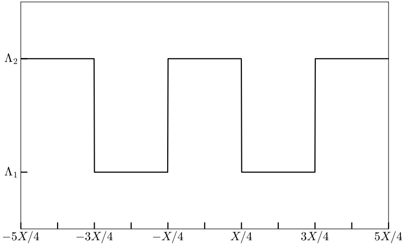
Matter profile

$$\lambda(x) = \lambda_0 + A \sin(kx),$$

$$k_1 = (1 - 10^{-4})\omega_{\mathrm{m}}$$

n	D	D'_1	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	5.2	-	6.2×10^4
-1	10^5	5.2	3.1
2	1.1×10^9	5.2	6.3
-2	3.4×10^9	5.2	2.1

Castle Wall Matter Profile



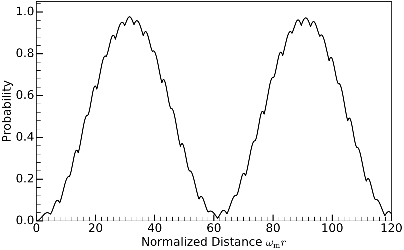
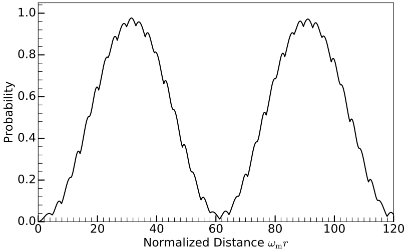


Figure: Castle wall matter profile

Castle Wall Matter Profile

Table: Relative detuning of each frequency.

$\{n_1, n_2\}$	D	$D'_{\{1,0\}}$
$\{1, 0\}$	0	-
$\{-1, 0\}$	48	1.0×10^{-2}
$\{0, 1\}$	1.5×10^{2}	1.1×10^{-3}
$\{2, 0\}$	2.4×10^{2}	2.0×10^{-4}



OVERVIEW

SUMMARY

- The fact that neutrino flavor sates are not mass states causes vacuum oscillations.
- MSW resonance happens when matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
- Even matter profile doesn't match MSW requirement, variation in matter profile can cause resonances.
- Single frequency perturbations in matter profile is a combination of many Rabi oscillations.
- Rabi oscillations with two driving fields of different frequencies: large width to destroy the resonance.

Backup Slides

BACKUP SLIDES

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \left|\Psi\right\rangle = \hat{\mathbf{H}} \left|\Psi\right\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{\left|\nu_1\right\rangle, \left|\nu_2\right\rangle\}$.
- $$\mathbf{H} = -\frac{\omega_{\mathrm{v}}}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$
- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp\left(i\omega_{\mathrm{v}}x/2\right) \\ \langle\nu_2|\Psi(0)\rangle \exp\left(-i\omega_{\mathrm{v}}x/2\right) \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_{\mathrm{e}}\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_{v}: vacuum mixing angle

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_v}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_v}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix} = \frac{\omega_v}{2}\left(-\cos 2\theta_v\boldsymbol{\sigma_3} + \sin 2\theta_v\boldsymbol{\sigma_1}\right)$$

Nature of Neutrino Oscillation

Transition Probability

$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_v)\sin^2(\omega_v x/2)$$

- $\omega_v = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- Mixing angle θ_v determines flavor oscillation amplitude.

MSW EFFECT

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_{m}

Define matter basis (eigenenergy basis) $\{|\nu_{\mathrm{L}}\rangle, |\nu_{\mathrm{H}}\rangle\}$

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & \sin\theta_{\mathrm{m}} \\ -\sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}$$

MSW Effect

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_m

Define matter basis (eigenenergy basis) $\{|\nu_\mathrm{L}\rangle, |\nu_\mathrm{H}\rangle\}$

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} |\nu_\mathrm{L}\rangle \\ |\nu_\mathrm{H}\rangle \end{pmatrix}$$

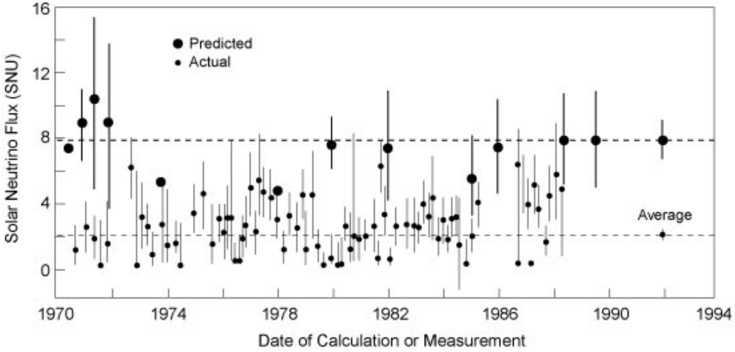
In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_\mathrm{m}}{2}\boldsymbol{\sigma_3}$$

Transition Probability

$$P(|\nu_\mathrm{e}\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_\mathrm{m})\sin^2(\omega_\mathrm{m} x)$$

SOLAR NEUTRINO PROBLEM



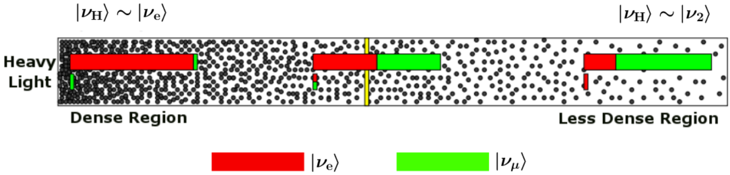
Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW Effect and Solar Neutrinos

$$\mathbf{H} = \frac{\lambda(x) - \omega_{\mathrm{v}} \cos 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_3} + \frac{\omega_{\mathrm{v}} \sin 2\theta_{\mathrm{v}}}{2} \boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_{\mathrm{L}}\rangle \\ |\nu_{\mathrm{H}}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{m}} & -\sin\theta_{\mathrm{m}} \\ \sin\theta_{\mathrm{m}} & \cos\theta_{\mathrm{m}} \end{pmatrix} \begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\mathrm{m}}}{2} \boldsymbol{\sigma_3}$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW Effect Inverted Hierarchy

Suppose $\omega_\mathrm{v} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = -\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} + \sqrt{2} G_\mathrm{F} n_\mathrm{e}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\mathbf{H} = \left(\frac{-\omega_\mathrm{v}}{2} \cos 2\theta_\mathrm{v} + \frac{\lambda(x)}{2} \right) \boldsymbol{\sigma_3} - \frac{\omega_\mathrm{v}}{2} \sin 2\theta_\mathrm{v} \boldsymbol{\sigma_1}$$

Hamiltonian

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1 .$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 .$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1 .$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 .$$

Rabi Oscillations

The coupling strength is calculated as

$$\alpha = \langle 1|\mathbf{d} \cdot \mathbf{E}|2\rangle$$

where the electric field is

$$\mathbf{E} = \mathbf{E}_0 \sin(kt).$$

and $\mathbf{d}$ is the dipole moment.

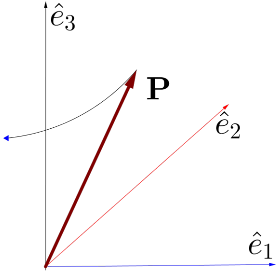
Visualizing Rabi Oscillations

$$-\frac{\omega_0}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

$$-\frac{\omega_0}{2}\sigma_3 - \frac{\alpha}{2}\cos(kt)\sigma_1 + \frac{\alpha}{2}\sin(kt)\sigma_2$$

$$= \begin{pmatrix} \alpha\cos(kt) & -\alpha\sin(kt) & \omega_0 \end{pmatrix} \boxed{\begin{pmatrix} -\sigma_1/2 \\ -\sigma_2/2 \\ -\sigma_3/2 \end{pmatrix}}$$

$$= \vec{H} \cdot (-\vec{\sigma}/2)$$



$$D = \left| \frac{\omega_0 - k}{\alpha} \right|$$

is ratio of the energy gap in corotating frame to width of resonance.

Interferences of Rabi Oscillations

$$-\frac{\omega_0}{2}\sigma_3-\frac{\alpha}{2}\begin{pmatrix}0 & e^{ikt}\\ e^{-ikt} & 0\end{pmatrix}$$

$$\begin{aligned}\mathbf{H} =& \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx) \right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1\\ \to & -\frac{\omega_m}{2}\sigma_3 - \frac{A\sin 2\theta_{\mathrm{m}}}{2}\cos(kx)\sigma_1\\ = & -\frac{\omega_m}{2}\sigma_3 - \frac{A\sin 2\theta_{\mathrm{m}}}{2}\frac{1}{2}\begin{pmatrix}0 & e^{ikx}\\ e^{-ikx} & 0\end{pmatrix} - \frac{A\sin 2\theta_{\mathrm{m}}}{2}\frac{1}{2}\begin{pmatrix}0 & e^{i(-k)x}\\ e^{-i(-k)x} & 0\end{pmatrix}\end{aligned}$$

Parameters Used for Vacuum Oscillations

$\theta_{12} = 33.36/180\pi$; $\theta_{13} = 8.66/180\pi$; $\theta_{23} = 40/180 * \pi$; $\delta_{cp} = 0$;
$m_1^2 = 0.01$; $m_2^2 = m_1^2 + 0.000079$; $E = 1\text{MeV}$

SINGLE FREQUENCY MATTER PROFILE

Why Does It Work?

$$J_n(n\,\mathrm{sech}\,\alpha) \sim \frac{e^{-n(\alpha - \tanh\alpha)}}{\sqrt{2\pi n \tanh\alpha}}, \quad \text{for large } n$$

$\Rightarrow$

$$\Gamma \propto \hat{B}_n \propto \frac{e^{-n(\alpha - \tanh\alpha)}}{\sqrt{2\pi n \tanh\alpha}}$$

Small perturbation $\Rightarrow$ Small $\hat{A}$ $\Rightarrow$ Large α $\Rightarrow$ Drops fast at large n.

Two-frequency Matter Profile

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

Two-frequency Matter Profile $\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}}$,

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1\hat{k}_1 + n_2\hat{k}_2 - 1)\hat{x}},$$

where

$$\begin{aligned}&\hat{B}_{n_1,n_2}(\hat{k}_1, \hat{k}_2)\\ =& -(-i)^{n_1+n_2}(n_1\hat{k}_1 + n_2\hat{k}_2) J_{n_1}\left(\frac{\hat{A}_1 \cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right) J_{n_2}\left(\frac{\hat{A}_2 \cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)\end{aligned}$$

Which terms are important?

Two-frequency Matter Profile

Resonance Lines

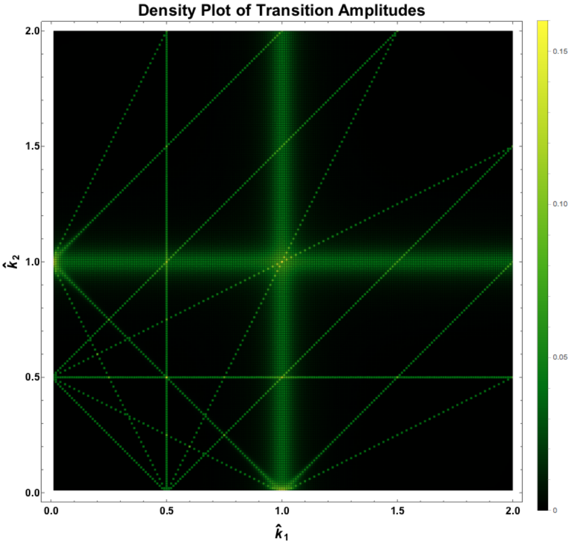
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.

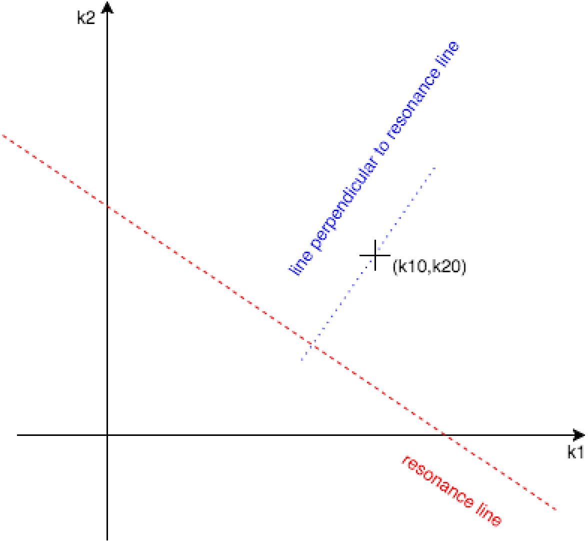
Two-frequency Matter Profile

$$\hat{h} = \sum_{n_1}\sum_{n_2} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

TWO-FREQUENCY MATTER PROFILE



Resonance line, distance to resonance, and width

Two-frequency Matter Profile

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\text{intercept}}, \hat{k}_{2,\text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

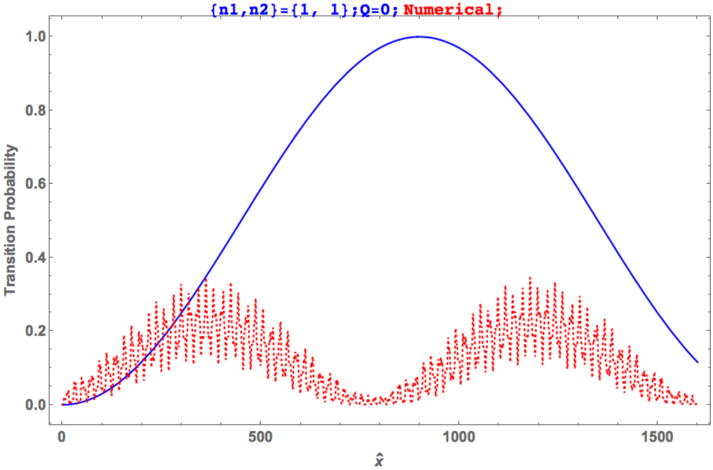
Distance to Resonance Line

$$d = \frac{|n_1\hat{k}_{10} + n_2\hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

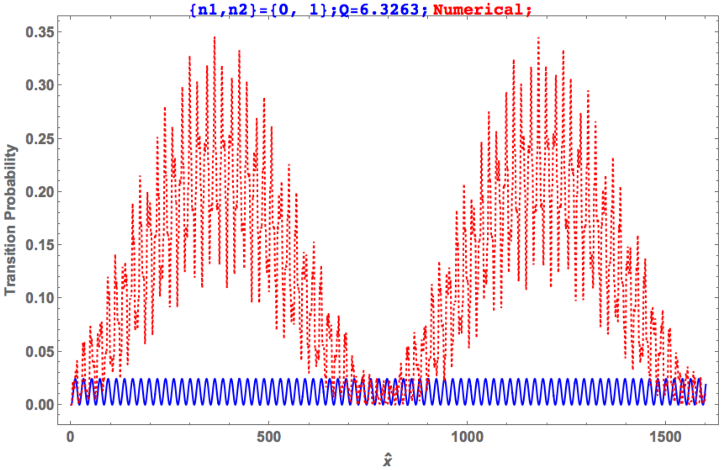
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

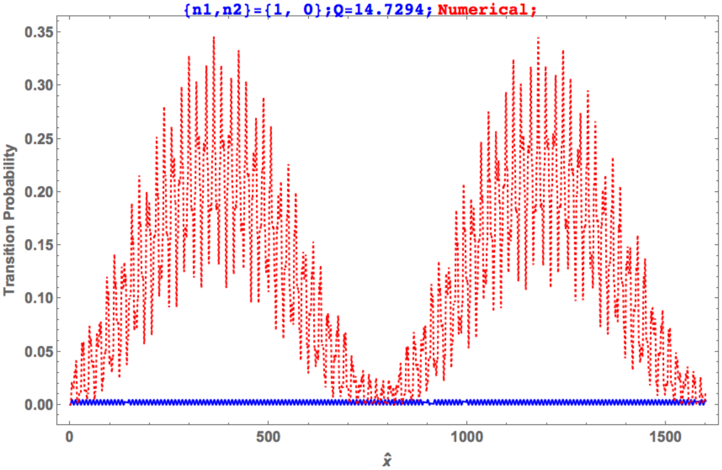
TWO-FREQUENCY MATTER PROFILE



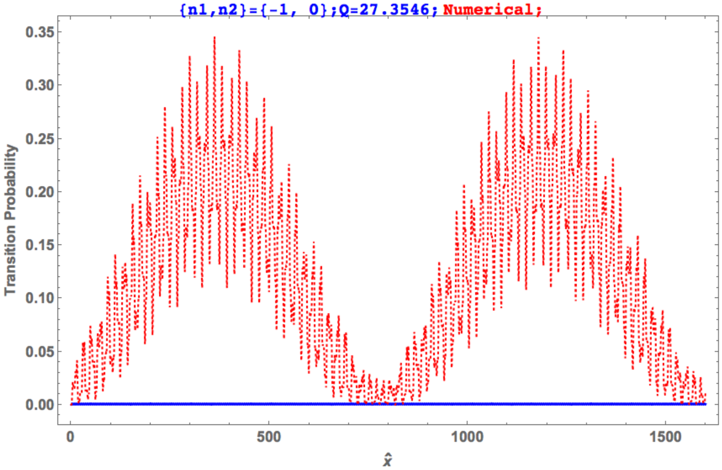
TWO-FREQUENCY MATTER PROFILE



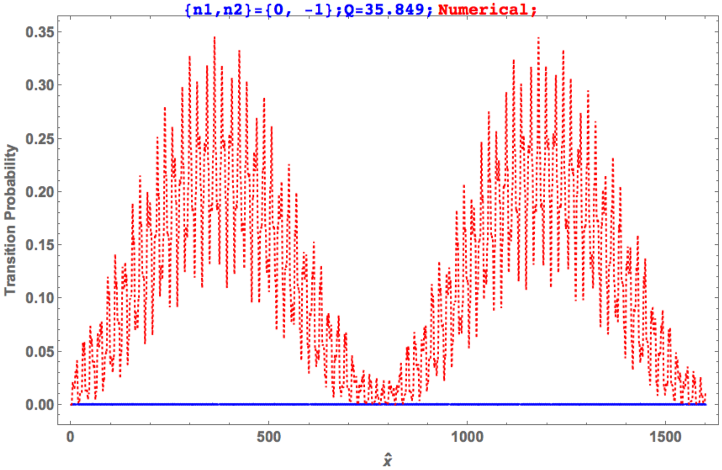
TWO-FREQUENCY MATTER PROFILE



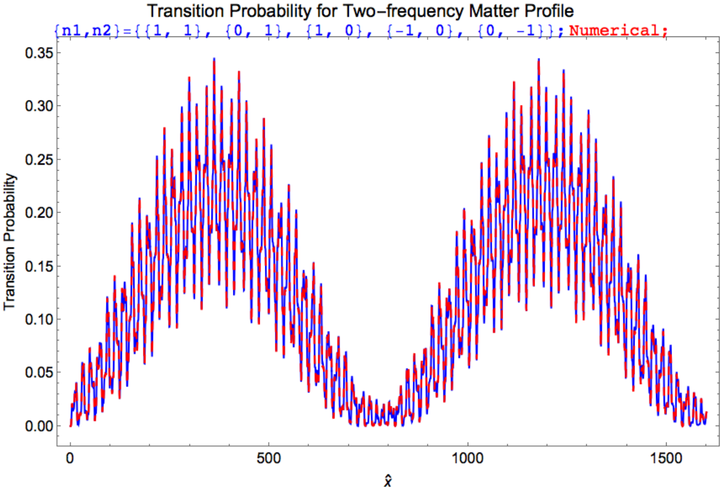
Two-frequency Matter Profile



TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



Bessel's Function

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I