

Stimulated Neutrino Flavor Conversions and Rabi Oscillations

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OUTLINE

1. Background
 - What are Neutrinos
 - Neutrino Oscillations
 - Why Do Neutrinos Oscillate
2. Matter Effect
 - Interactions with Matter
 - MSW Effect
3. Stimulated Neutrino Flavor Conversions
 - Rabi Oscillations
 - Single Frequency Matter Profile and Rabi Oscillations
4. Single Frequency Matter Potential Decomposed
 - Basis and Formalism
 - Rabi Oscillations With Multiple Potentials
 - Multiple Frequencies in Matter Potential
5. Summary

OVERVIEW

Background

- What are Neutrinos

- Neutrino Oscillations

- Why Do Neutrinos Oscillate

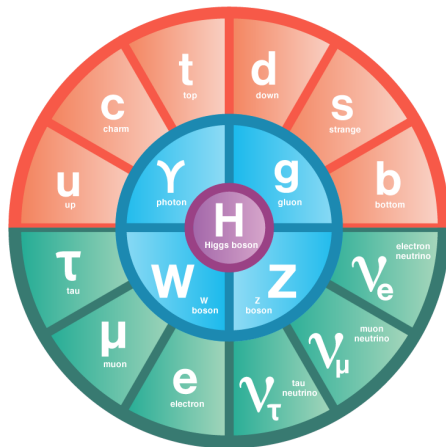
Matter Effect

Stimulated Neutrino Flavor Conversions

Single Frequency Matter Potential Decomposed

Summary

WHAT ARE NEUTRINOS?

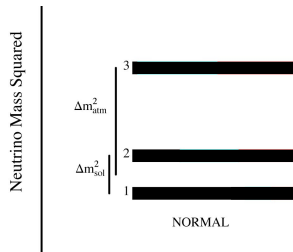


Elementary particles.

Source: symmetrymagazine.org

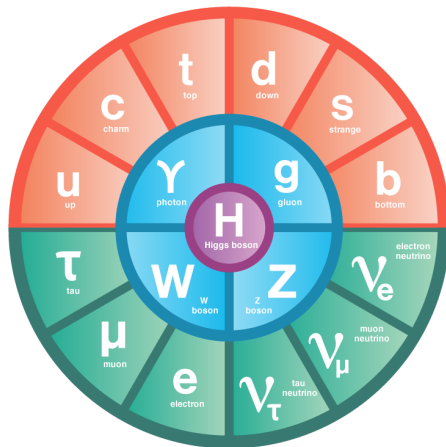
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- light.



Adapted from Olga Mena & Stephen Parke (2004)

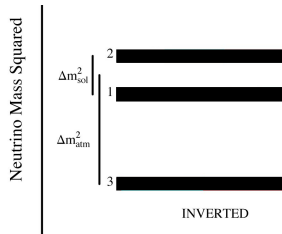
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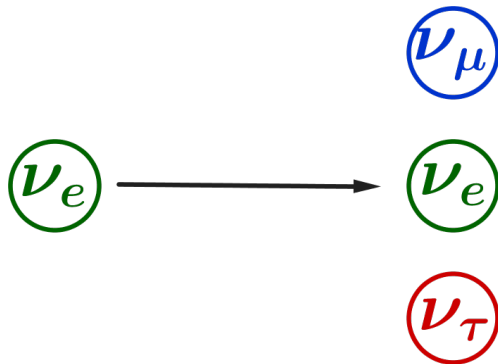
Adapted from Olga Mena &
Stephen Parke (2004)

WHAT ARE NEUTRINO OSCILLATIONS?

Neutrino Oscillations

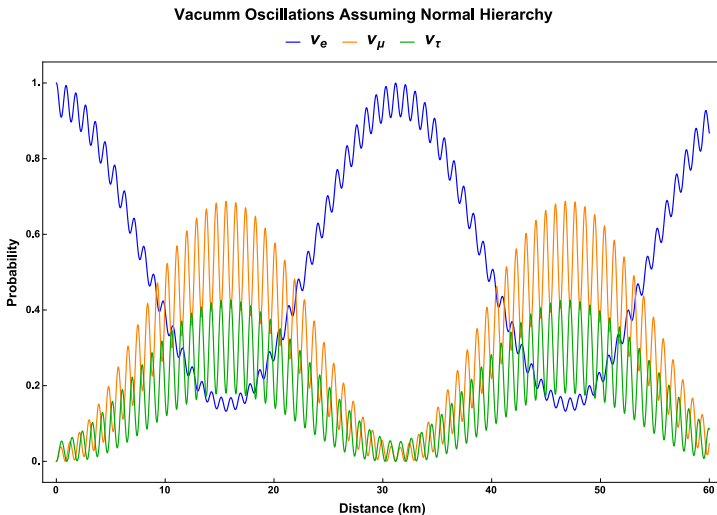
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Neutrino Flavor Conversions



Neutrino Oscillations

WHAT ARE NEUTRINO OSCILLATIONS?



Probabilities of finding neutrinos to be in each flavor.

WHY DO NEUTRINOS OSCILLATE?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_v}{2} (-\cos 2\theta_v \boldsymbol{\sigma}_3 + \sin 2\theta_v \boldsymbol{\sigma}_1)$$

- Oscillation frequency:

$$\omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_v

FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\sigma}}{2} \Psi$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

electron flavor



muon flavor



FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}$

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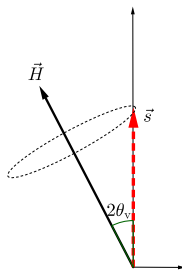
Vacuum oscillation Hamiltonian

$$\frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1)$$

$$\rightarrow \cos 2\theta_v \begin{pmatrix} 0 \\ 0 \\ \omega_v \end{pmatrix} - \sin 2\theta_v \begin{pmatrix} \omega_v \\ 0 \\ 0 \end{pmatrix}$$

electron flavor

muon flavor



OVERVIEW

Background

Matter Effect

Interactions with Matter

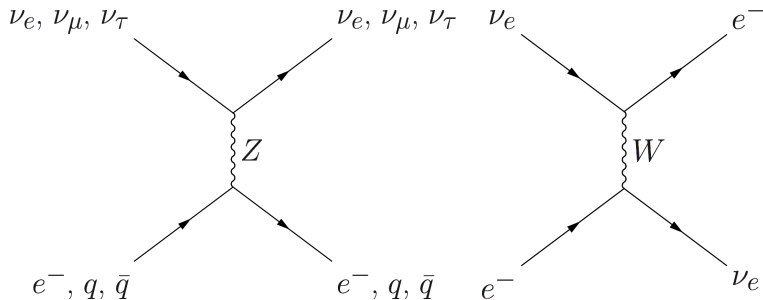
MSW Effect

Stimulated Neutrino Flavor Conversions

Single Frequency Matter Potential Decomposed

Summary

INTERACTIONS WITH MATTER



Neutral current interaction between ν_e, ν_μ, ν_τ , and e^- , quarks etc.

Charged current interaction between ν_e and e^-

MATTER INTERACTION

Hamiltonian with matter interaction in flavor basis ($\omega_v = \delta m^2/2E$):

$$\mathbf{H} = \frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1) + \frac{\lambda(x)}{2} \sigma_3$$

► Vacuum Hamiltonian

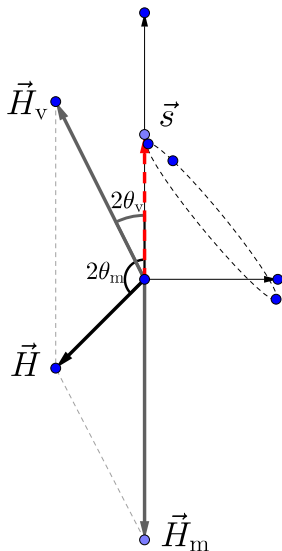
► Matter interaction

► $\lambda(x) = \sqrt{2}G_F n_e(x)$

MSW EFFECT

$$\begin{aligned}\mathbf{H} &= \frac{\omega_v}{2} (-\cos 2\theta_v \boldsymbol{\sigma}_3 + \sin 2\theta_v \boldsymbol{\sigma}_1) + \frac{\lambda(x)}{2} \boldsymbol{\sigma}_3 \\ &\rightarrow \omega_v \begin{pmatrix} -\sin 2\theta_v & \\ & 0 \\ & & \cos 2\theta_v \end{pmatrix} + \begin{pmatrix} 0 & \\ & 0 \\ & & -\lambda(x) \end{pmatrix} \\ &= \vec{H}_v + \vec{H}_m(x)\end{aligned}$$

MSW EFFECT



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

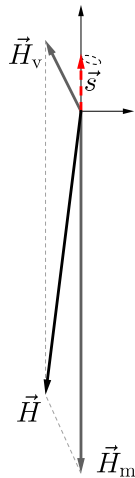
$$\omega_v = |\vec{H}_v|$$

Oscillation frequency in **matter**:

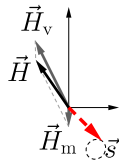
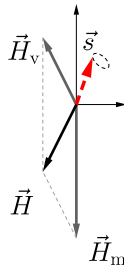
$$\omega_m = |\vec{H}|$$

MSW EFFECT

Adiabatic matter density change

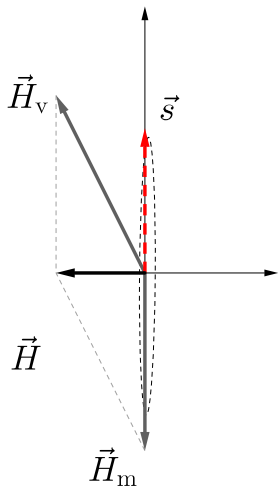


Large density



Low density

MSW EFFECT



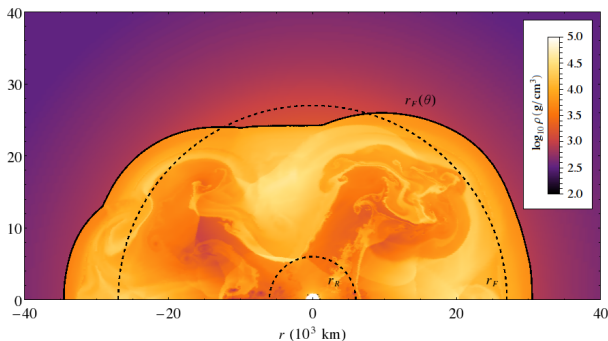
- ▶ Maximum possible flavor transition probability amplitude
- ▶ MSW Resonance
- ▶ A specific matter density

$$\sqrt{2}G_F n_e \equiv \omega_v \cos 2\theta_v$$

MORE COMPLICATED MATTER EFFECT

Why Do We Care

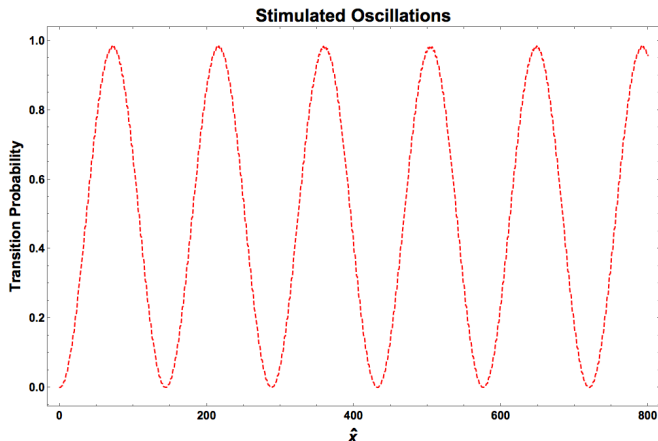
Astrophysical environments: supernovae etc



Turbulence in supernova. E. Borriello, et al (2014)

STIMULATED NEUTRINO FLAVOR CONVERSIONS

$$\lambda(x) = \lambda_0 + A \cos(kx)$$



P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);

OVERVIEW

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Stimulated Neutrino Flavor Conversions

Rabi Oscillations

Single Frequency Matter Profile and Rabi Oscillations

Single Frequency Matter Potential Decomposed

Summary

RABI OSCILLATIONS

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2} \begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

$$E_2 = \frac{\omega_m}{2}$$

Periodic Driving Potential



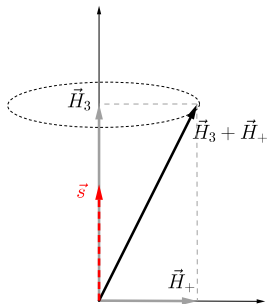
$$E_1 = -\frac{\omega_m}{2}$$

Frequency : k

RABI OSCILLATIONS

Static Frame

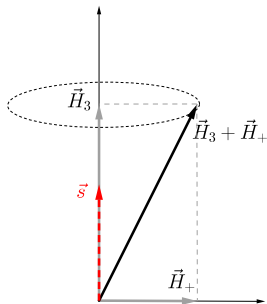
$$\vec{H}_3 = \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



RABI OSCILLATIONS

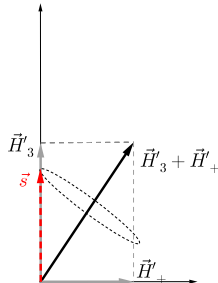
Static Frame

$$\vec{H}_3 = \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Corotating Frame

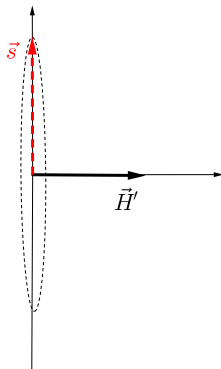
$$\vec{H}'_3 = (\omega_m - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



RABI OSCILLATIONS

Corotating Frame

$$\vec{H}'_3 = (\omega_m - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow k = \omega_m$$



RABI OSCILLATIONS

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2} \begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi formula

$$P_{1 \rightarrow 2} = \frac{1}{1 + D^2} \sin^2 \left(\frac{\Omega_R}{2} t \right).$$

Relative detuning

$$D = \left| \frac{\omega_m - k}{\alpha} \right|.$$

Rabi frequency

$$\Omega_R = |\alpha| \sqrt{1 + D^2}$$

$$E_2 = \frac{\omega_m}{2}$$

$$E_1 = -\frac{\omega_m}{2}$$

Periodic Driving Potential



Frequency : k

HAMILTONIAN IN MATTER BASIS $\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2} (-\omega_m) \sigma_3$$

HAMILTONIAN IN MATTER BASIS $\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0 + A \cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2} (-\omega_m + A \cos(kx) \cos 2\theta_m) \sigma_3 - \frac{A \cos(kx)}{2} \sin 2\theta_m \sigma_1$$

HAMILTONIAN IN MATTER BASIS

$$\alpha = \frac{\sin 2\theta_m}{2} A$$

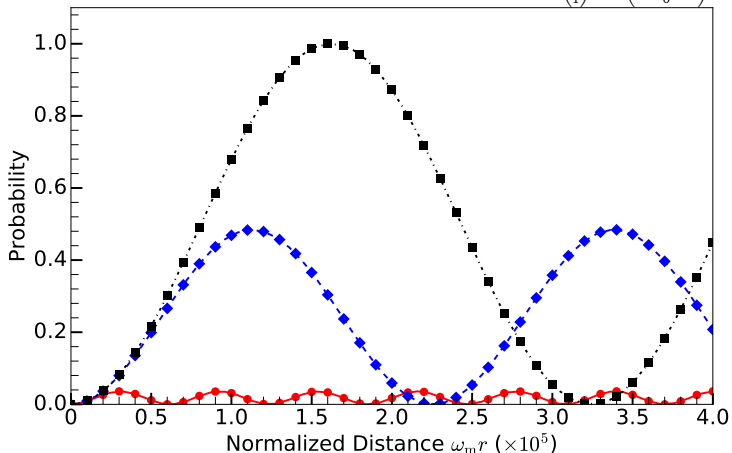
Matter potential frequency

$$k \sim \omega_m$$

$$\begin{aligned} \mathbf{H} &= \frac{1}{2} \left(-\omega_m + \cos 2\theta_m A \cos(kx) \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1 \\ &\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix} \end{aligned}$$

RABI FORMULA WORKS

$$\vec{H} \sim \omega_m \begin{pmatrix} 0 & \\ 0 & \\ 1 & \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) & \\ -\sin(kx) & \\ 0 & \end{pmatrix}$$



Lines: Rabi formula

Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_m$, $k = (1 - 2 \times 10^{-5})\omega_m$, and $k = (1 - 10^{-4})\omega_m$ respectively.

OVERVIEW

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Single Frequency Matter Potential Decomposed
Basis and Formalism
Rabi Oscillations With Multiple Potentials
Multiple Frequencies in Matter Potential

Summary

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

We have been making approximations.

$$\mathbf{H} = \frac{1}{2} \left(-\omega_m + \cos 2\theta_m A \cos(kx) \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1$$
$$\rightarrow \omega_m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

RABI BASIS

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2} (-\omega_m + A \cos(kx) \cos 2\theta_m) \sigma_3 - \frac{A \cos(kx)}{2} \sin \theta_m \sigma_1.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)} \end{pmatrix} \begin{pmatrix} \tilde{\psi}_L \\ \tilde{\psi}_H \end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_m}{2} \int_0^x A \cos(k\tau) d\tau.$$

SINGLE FREQUENCY MATTER POTENTIAL

$$\lambda(x) = \lambda_0 + A \sin(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n n k \tan 2\theta_m J_n(A \cos 2\theta_m/k)$.

SINGLE FREQUENCY MATTER POTENTIAL

$$\lambda(x) = \lambda_0 + A \sin(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n n k \tan 2\theta_m J_n(A \cos 2\theta_m/k)$.

Multiple potentials with different frequencies!

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

Relative detuning

$$D' = \left| \frac{\omega_m - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

Consider $k_1 = \omega_m$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_m - k_2)} \right|$$

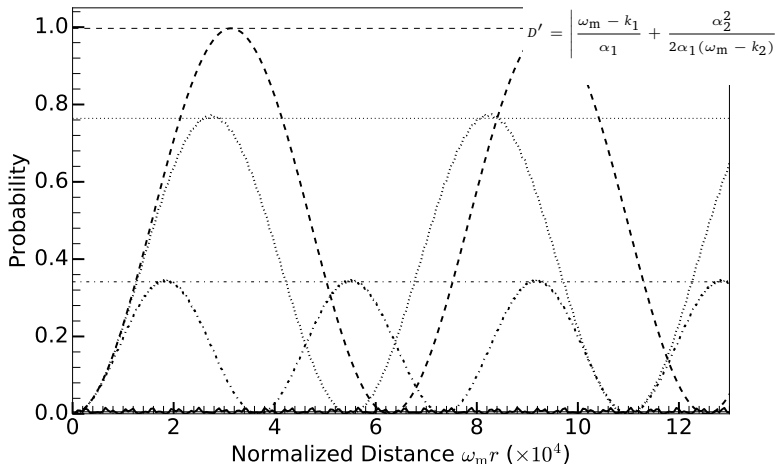
Amplitude reduces from 1 to 1/2 if

$$D' = 1 \Rightarrow \alpha_{2,c} \equiv \sqrt{2|\alpha_1(k_2 - \omega_m)|}.$$

Two driving frequencies k_1 , and k_2 , with amplitude α_1 , and α_2
For $k_1 = \omega_m$, survival of resonance requires

$$|\alpha_2| \ll \alpha_{2,c} \equiv \sqrt{2|\alpha_1(k_2 - \omega_m)|}$$

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS



Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_m, 10\omega_m$	$10^{-2}\omega_m, 10^{-1}\omega_m$	$5.0 \times 10^{-2}\omega_m, 10\omega_m$	$5 \times 10^{-2}\omega_m, 10^{-1}\omega_m$

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

Matter potential

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

Consider the resonance condition ($k = \omega_m$)

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2}\sigma_3 + \sum_{n=-\infty}^{\infty} \begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

Matter potential

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

Consider the resonance condition ($k = \omega_m$)

$$\tilde{\mathbf{H}} \sim -\frac{\omega_m}{2}\sigma_3 + \frac{1}{2} \begin{pmatrix} 0 & \alpha_1 e^{ikx} \\ \alpha_1^* e^{-ikx} & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \alpha_n e^{inkx} \\ \alpha_n^* e^{-inkx} & 0 \end{pmatrix}$$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_n(\omega_m - nk)} \right|$$

$k = \omega_m$	
n	D'
-1	4.8×10^{-6}
2	2.1×10^{-14}
-2	6.9×10^{-15}

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

Width

$$\alpha_n = -(-i)^n n k \tan 2\theta_m J_n(A \cos 2\theta_m / k)$$

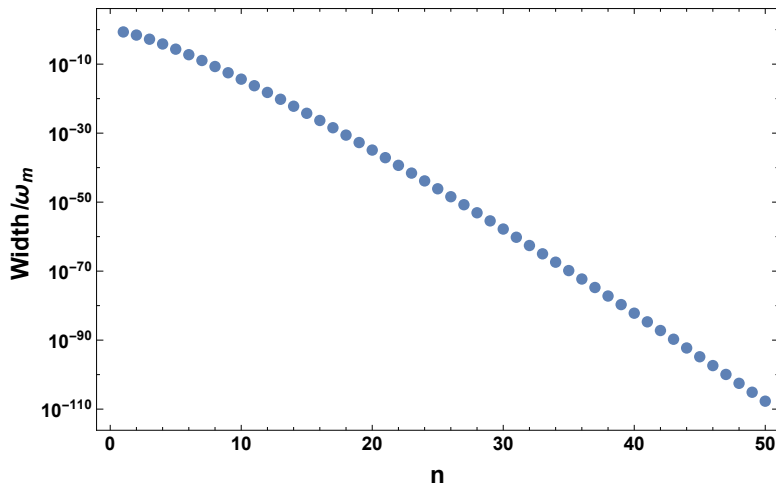
$$|\alpha_n| \propto \sqrt{\frac{n}{2\pi}} \left(\frac{eA \cos 2\theta_m}{2nk} \right)^n, \quad \text{for large } n$$

Width drops fast at large n .

But the critical value for each mode becomes larger for large n 's

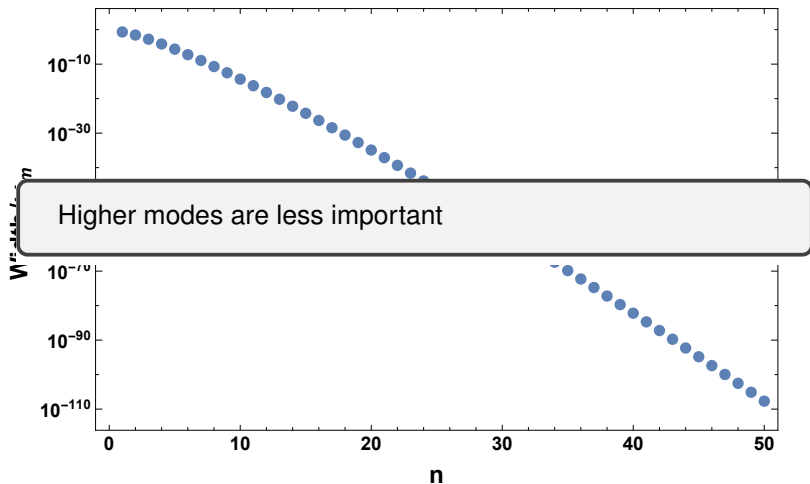
$$\alpha_{n,C} \equiv \sqrt{2|\alpha_1(nk - \omega_m)|}$$

SINGLE FREQUENCY MATTER POTENTIAL



Width of different modes given value of matter potential frequency k

SINGLE FREQUENCY MATTER POTENTIAL



Width of different modes given value of matter potential frequency k

MULTIPLE FREQUENCIES IN MATTER POTENTIAL

$$\lambda(x) = \lambda_0 + \sum_{a=1}^N A_a \sin(k_a x)$$

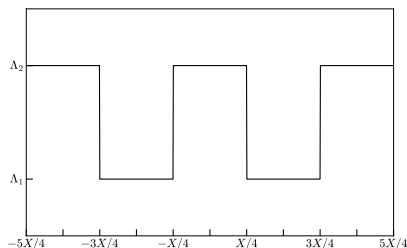
Hamiltonian in Rabi Basis

$$\tilde{H} = -\frac{\omega_m}{2}\sigma_3 + \frac{1}{2} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \begin{pmatrix} 0 & B_{\{n_a\}} e^{i \sum_a n_a k_a x} \\ B_{\{n_a\}}^* e^{-i \sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a \right) \left(\prod_a J_{n_a} \left(\frac{A_a}{k_a} \cos 2\theta_m \right) \right)$$

CASTLE WALL MATTER POTENTIAL



Castle wall matter profile:

$$\Lambda_2 = 0.35\omega_v \cos 2\theta_v,$$

$$\Lambda_1 = 0.15\omega_v \cos 2\theta_v \text{ and period}$$

$$X = 2\pi/\omega_m$$

$$\lambda(x) = \lambda_0 + \sum_1^{\infty} \lambda_n \cos(k_n x)$$

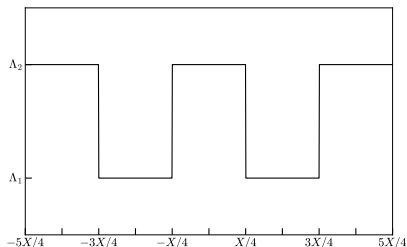
where

$$\lambda_0 = (\Lambda_1 + \Lambda_2)/2$$

$$\lambda_n = 2(-1)^n (\Lambda_1 - \Lambda_2) / (2n\pi - \pi)$$

$$k_n = 2\pi(2n - 1)/X$$

CASTLE WALL MATTER POTENTIAL

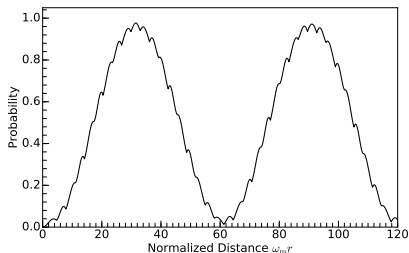


Castle wall matter profile:

$$\Lambda_2 = 0.35\omega_v \cos 2\theta_v,$$

$$\Lambda_1 = 0.15\omega_v \cos 2\theta_v \text{ and period}$$

$$X = 2\pi/\omega_m$$



Transition probability is a Rabi resonance with small variations due to higher orders.

OVERVIEW

Background

Matter Effect

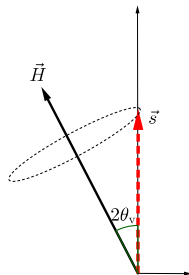
Stimulated Neutrino Flavor Conversions

Single Frequency Matter Potential Decomposed

Summary

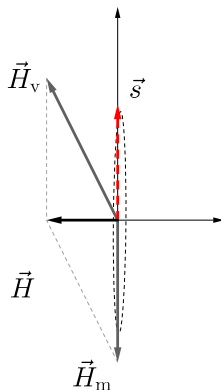
SUMMARY

1. Vacuum oscillations: flavor states are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. Oscillations with two driving fields of different frequencies: large potential to destroy the resonance.



SUMMARY

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SUMMARY

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For matter potential

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

Resonance condition

$$nk = \omega_m$$

SUMMARY

1. Vacuum oscillations: flavor states are not mass states.
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$$|\alpha_2| \gg \alpha_{2,c} \equiv \sqrt{2|\alpha_1(k_2 - \omega_m)|}$$

BACKUP SLIDES

BACKUP SLIDES

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x |\Psi\rangle = \hat{\mathbf{H}} |\Psi\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{|\nu_1\rangle, |\nu_2\rangle\}$.

►

$$\mathbf{H} = -\frac{\omega_v}{2}\sigma_3, \quad \text{where } \omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$

- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp(i\omega_v x/2) \\ \langle\nu_2|\Psi(0)\rangle \exp(-i\omega_v x/2) \end{pmatrix}$$

WHY DO NEUTRINOS OSCILLATE?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v : vacuum mixing angle

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θ_v : vacuum mixing angle

Hamiltonian H

Mass basis

$$\begin{aligned} & \frac{\omega_v}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -\frac{\omega_v}{2} \sigma_3 \end{aligned}$$

Flavor basis

$$\begin{aligned} & \frac{\omega_v}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix} \\ &= \frac{\omega_v}{2} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1) \end{aligned}$$

NATURE OF NEUTRINO OSCILLATION

Transition Probability

$$P(|\nu_e\rangle \rightarrow |\nu_\mu\rangle) = \sin^2(2\theta_v) \sin^2(\omega_v x/2)$$

- ▶ $\omega_v = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- ▶ Mixing angle θ_v determines flavor oscillation amplitude.

MSW EFFECT

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_m

Define matter basis (eigenenergy basis) $\{|\nu_L\rangle, |\nu_H\rangle\}$

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m \\ -\sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2} \sigma_3$$

MSW EFFECT

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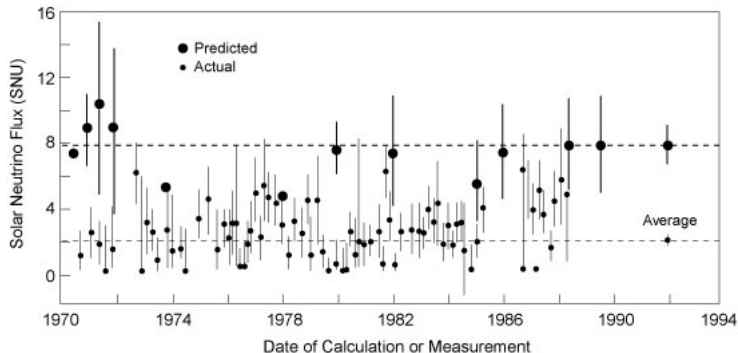
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Transition Probability

$$P(|\nu_e\rangle \rightarrow |\nu_\mu\rangle) = \sin^2(2\theta_m) \sin^2(\omega_m x)$$

SOLAR NEUTRINO PROBLEM



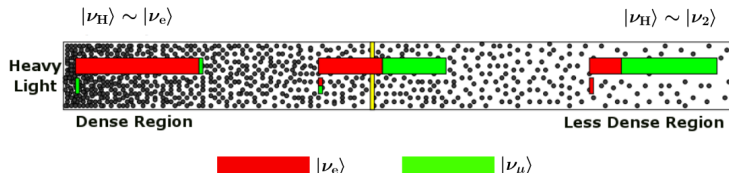
Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW EFFECT AND SOLAR NEUTRINOS

$$\mathbf{H} = \frac{\lambda(x) - \omega_\nu \cos 2\theta_\nu}{2} \sigma_3 + \frac{\omega_\nu \sin 2\theta_\nu}{2} \sigma_1$$

$$\begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix} = \begin{pmatrix} \cos \theta_m & -\sin \theta_m \\ \sin \theta_m & \cos \theta_m \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2} \sigma_3$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW EFFECT INVERTED HIERARCHY

Suppose $\omega_v = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = -\frac{\omega_v}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix} + \sqrt{2}G_F n_e(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



$$\mathbf{H} = \left(\frac{-\omega_v}{2} \cos 2\theta_v + \frac{\lambda(x)}{2} \right) \sigma_3 - \frac{\omega_v}{2} \sin 2\theta_v \sigma_1$$

HAMILTONIAN

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0 ,

$$H_{\text{background}} = -\frac{\omega_m}{2} \sigma_3.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2} (-\omega_m + \delta\lambda(x) \cos 2\theta_m) \sigma_3 - \frac{\delta\lambda(x)}{2} \sin \theta_m \sigma_1.$$

HAMILTONIAN

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2} (-\omega_m + \delta\lambda(x) \cos 2\theta_m) \sigma_3 - \frac{\delta\lambda(x)}{2} \sin 2\theta_m \sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$$\mathbf{H} = \frac{1}{2} (-\omega_m + \cos 2\theta_m A \cos(kx)) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1.$$

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Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$$\mathbf{H} = \frac{1}{2} \left(-\omega_m + \cos 2\theta_m A \cos(kx) \right) \sigma_3 - \frac{\sin 2\theta_m}{2} A \cos(kx) \sigma_1.$$

RABI OSCILLATIONS

The coupling strength is calculated as

$$\alpha = \langle 1 | \mathbf{d} \cdot \mathbf{E} | 2 \rangle$$

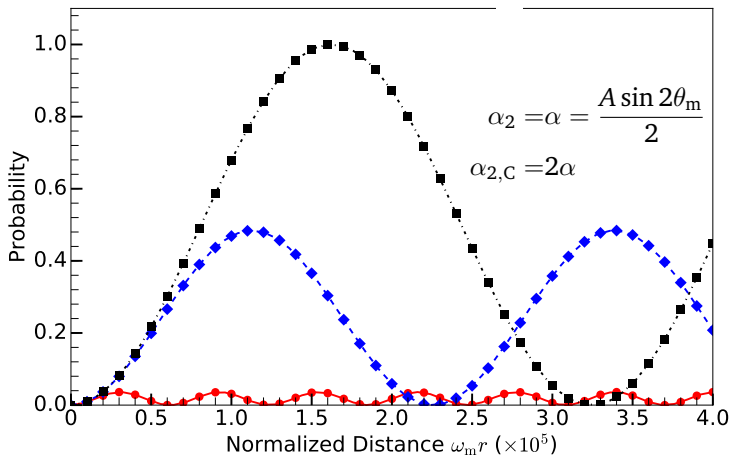
where the electric field is

$$\mathbf{E} = \mathbf{E}_0 \sin(kt).$$

and \mathbf{d} is the dipole moment.

RABI FORMULA WORKS

$$\vec{H} \sim \omega_m \begin{pmatrix} 0 & \\ & 0 \\ & & 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) & \\ \cos(kx) & \\ & & 0 \end{pmatrix}$$



Lines: Rabi formula

Dots, diamonds, triangles, and squares are for $k = \omega_m$,
 $k = (1 - 2 \times 10^{-5})\omega_m$, and $k = (1 - 10^{-4})\omega_m$ respectively.

PARAMETERS USED FOR VACUUM OSCILLATIONS

$$\theta_{12} = 33.36/180\pi; \theta_{13} = 8.66/180\pi; \theta_{23} = 40/180 * \pi; \delta_{cp} = 0;$$
$$m_1^2 = 0.01; m_2^2 = m_1^2 + 0.000079; E = 1\text{MeV}$$

SINGLE FREQUENCY MATTER POTENTIAL

Matter potential

$$\lambda(x) = \lambda_0 + A \sin(kx),$$

Hamiltonian in new basis

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2}\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_m \begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = -\frac{\omega_m}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2}e^{2i\eta(x)} \\ &= \frac{i}{4} \left[\exp \left(ikx + i \cos 2\theta_m \frac{A}{k} \cos(kx) \right) \right. \\ &\quad \left. - \exp \left(-ikx + i \cos 2\theta_m \frac{A}{k} \cos(kx) \right) \right] \end{aligned}$$

SINGLE FREQUENCY MATTER POTENTIAL

Off-diagonal Term in Our System

$$\tilde{\mathbf{H}} = -\frac{\omega_m}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$h \propto \left[\exp \left(ikx + i \cos 2\theta_m \frac{A}{k} \cos(kx) \right) - \exp \left(-ikx + i \cos 2\theta_m \frac{A}{k} \cos(kx) \right) \right]$$

Jacobi-Anger expansion (Kneller et al, 2013)

$$e^{i\beta \cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

SINGLE FREQUENCY MATTER PROFILE

Transition Probability

$$P_{L \rightarrow H}^{(n)} = \frac{|\hat{B}_n|^2}{|\hat{B}_n|^2 + (n\hat{k} - 1)^2} \sin^2 \left(\frac{q^{(n)}}{2} x \right),$$

where

$$q^{(n)} = \sqrt{|\Gamma^{(n)}|^2 + (n\hat{k} - 1)^2}, \quad \text{frequency of oscillations}$$

$$\Gamma^{(n)} = |\hat{B}_n|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

Resonance conditions

$$\hat{k} \sim \frac{1}{n}$$

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

Width

$$\alpha_n = -(-i)^n n k \tan 2\theta_m J_n(A \cos 2\theta_m/k)$$

$$J_n(n \operatorname{sech} \beta) \sim \frac{e^{-n(\beta - \tanh \beta)}}{\sqrt{2\pi n \tanh \beta}}, \quad \text{for large } n$$

\Rightarrow

$$|\alpha_n| \propto \frac{e^{-n(\beta - \tanh \beta)}}{\sqrt{2\pi n \tanh \beta}}, \quad \text{for large } n$$

where $\operatorname{sech} \beta = A \cos 2\theta_m / \omega_m$.

$\beta - \tanh \beta > 0 \Rightarrow$ **Width** drops fast at large n .

TWO-FREQUENCY MATTER PROFILE

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

TWO-FREQUENCY MATTER P

$$\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2} \hat{B}_n e^{i(n\hat{k}-1)\hat{x}},$$

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$

where

$$\begin{aligned} & \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) \\ &= -(-i)^{n_1+n_2} (n_1 \hat{k}_1 + n_2 \hat{k}_2) J_{n_1} \left(\frac{\hat{A}_1 \cos 2\theta_m}{\hat{k}_1} \right) J_{n_2} \left(\frac{\hat{A}_2 \cos 2\theta_m}{\hat{k}_2} \right) \end{aligned}$$

Which terms are important?

SINGLE FREQUENCY MATTER PROFILE REVISITED

Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$k_1 = \omega_m$			
n	D	D'_1	$2\pi\omega_m/\Omega_n$
1	0	-	3.2×10^5
-1	10^5	4.8×10^{-6}	3.1
2	1.1×10^9	2.1×10^{-14}	6.3
-2	3.4×10^9	6.9×10^{-15}	2.1

SINGLE FREQUENCY MATTER PROFILE REVISITED

Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$k_1 = (1 - 2 \times 10^{-5})\omega_m$			
n	D	D'_1	$2\pi\omega_m/\Omega_n$
1	1	-	2.2×10^5
-1	10^5	1	3.1
2	1.1×10^9	1	6.3
-2	3.4×10^9	1	2.1

SINGLE FREQUENCY MATTER PROFILE REVISITED

Matter profile

$$\lambda(x) = \lambda_0 + A \cos(kx),$$

$$k_1 = (1 - 10^{-4})\omega_m$$

n	D	D'_1	$2\pi\omega_m/\Omega_n$
1	5.2	-	6.2×10^4
-1	10^5	5.2	3.1
2	1.1×10^9	5.2	6.3
-2	3.4×10^9	5.2	2.1

CASTLE WALL MATTER PROFILE

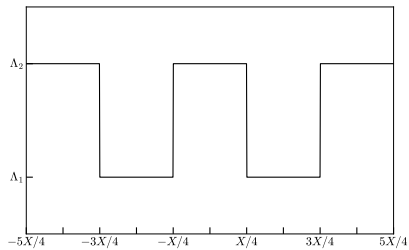
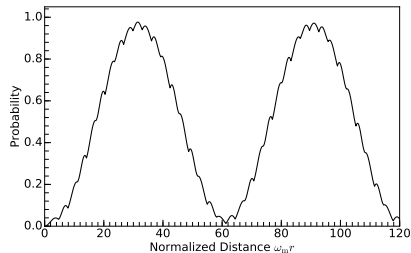


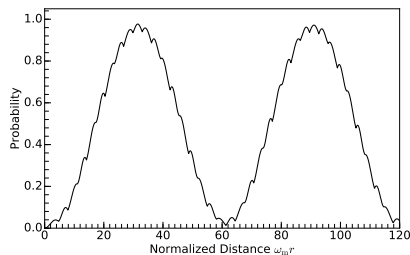
Figure: Castle wall matter profile



CASTLE WALL MATTER PROFILE

Table: Relative detuning of each frequency.

$\{n_1, n_2\}$	D	$D'_{\{1,0\}}$
$\{1, 0\}$	0	-
$\{-1, 0\}$	48	1.0×10^{-2}
$\{0, 1\}$	1.5×10^2	1.1×10^{-3}
$\{2, 0\}$	2.4×10^2	2.0×10^{-4}



TWO-FREQUENCY MATTER PROFILE

Resonance Lines

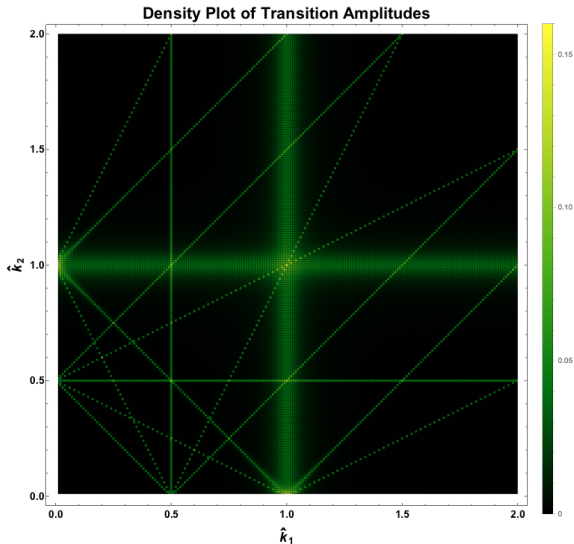
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. \Rightarrow Resonance width for each point on resonance lines.

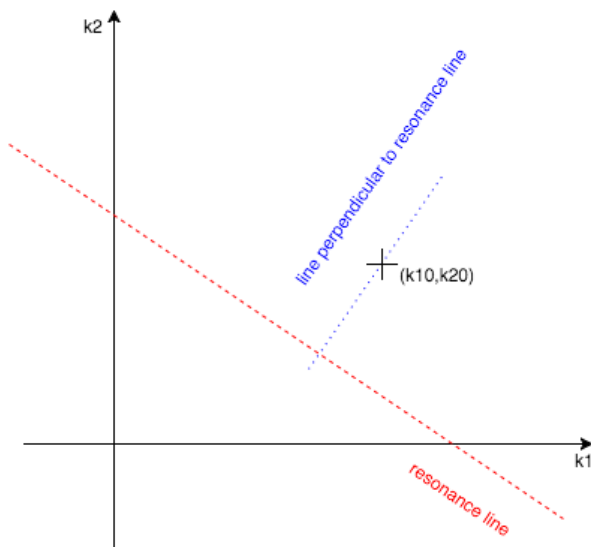
TWO-FREQUENCY MATTER Pf

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

TWO-FREQUENCY MATTER PROFILE



Resonance line, distance to resonance, and width

TWO-FREQUENCY MATTER PROFILE

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1, n_2}(\hat{k}_{1, \text{intercept}}, \hat{k}_{2, \text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

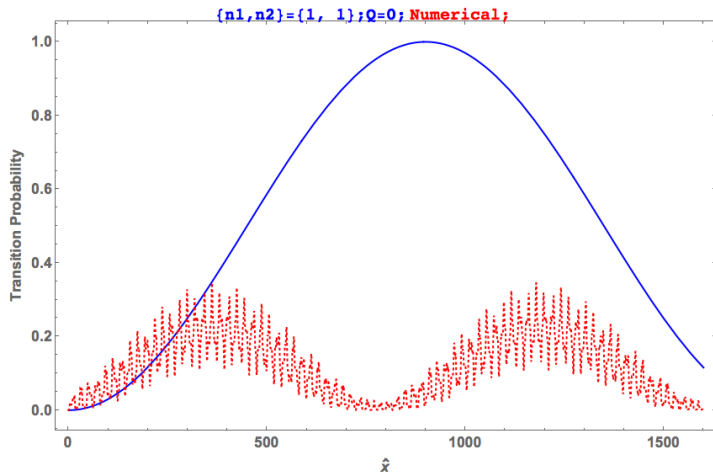
Distance to Resonance Line

$$d = \frac{|n_1 \hat{k}_{10} + n_2 \hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

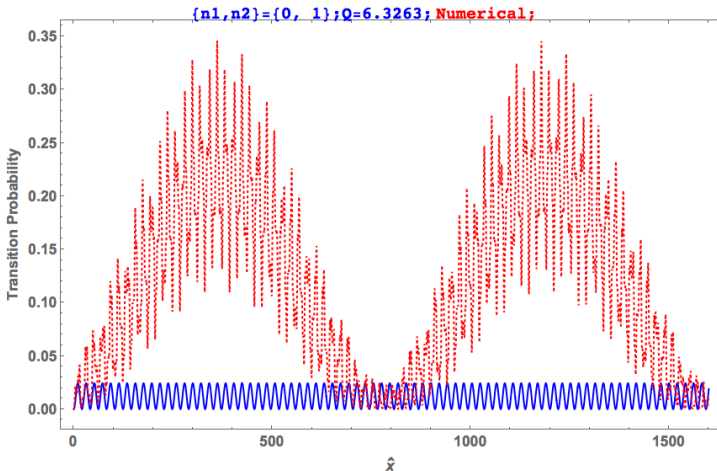
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

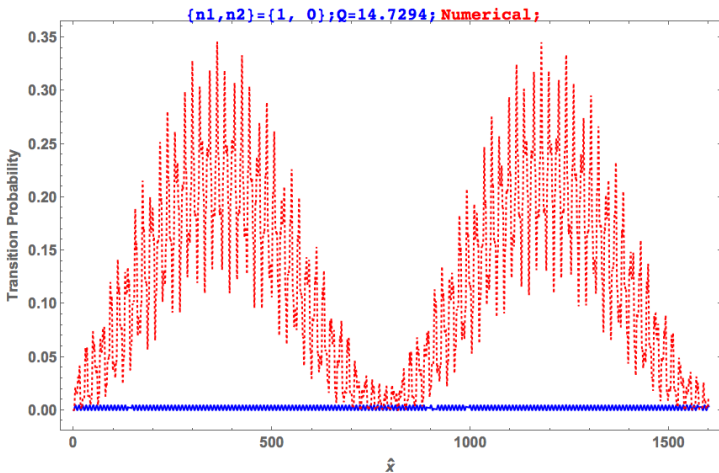
TWO-FREQUENCY MATTER PROFILE



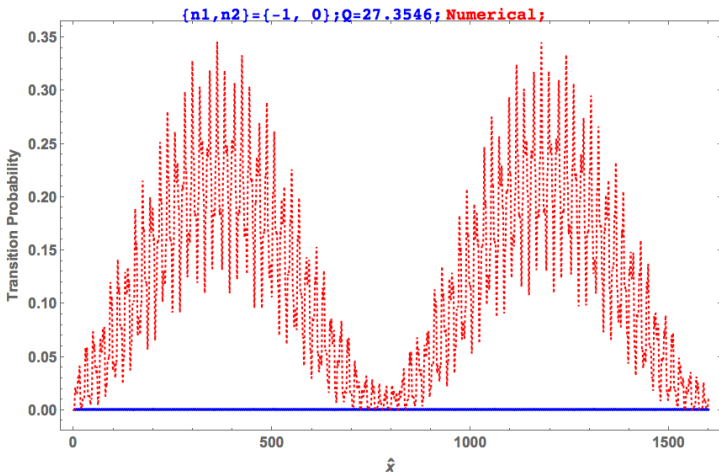
TWO-FREQUENCY MATTER PROFILE



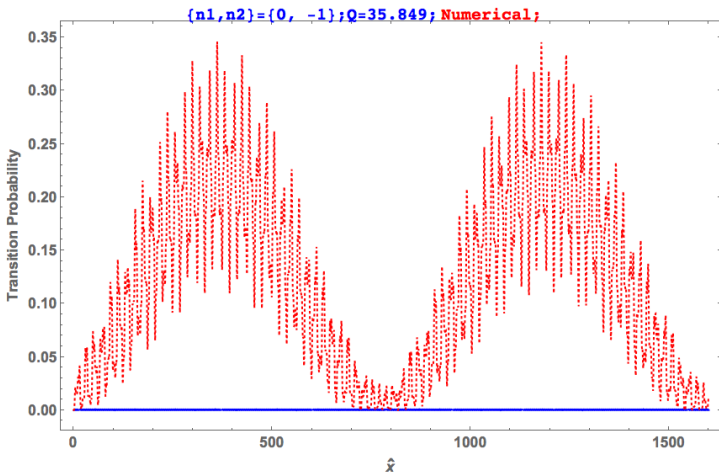
TWO-FREQUENCY MATTER PROFILE



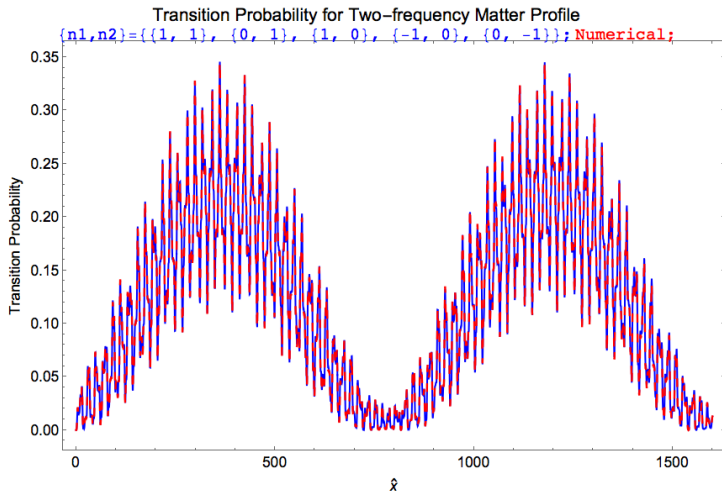
TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



BESSEL'S FUNCTION

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I