Stimulated Neutrino Flavor Conversions and Rabi Oscillations

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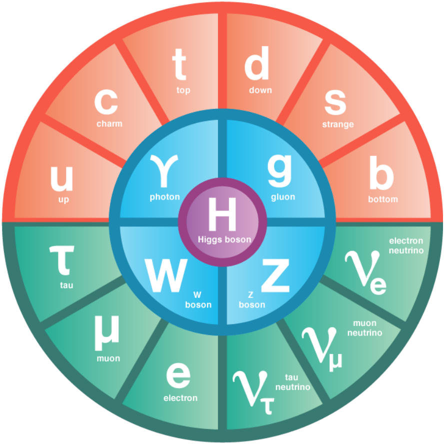
OUTLINE

1. Background
   - What are Neutrinos
   - Neutrino Oscillations
   - Why Do Neutrinos Oscillate
2. Matter Effect
   - Interactions with Matter
   - MSW Effect
3. Stimulated Neutrino Flavor Conversions
   - Rabi Oscillations
   - Single Frequency Matter Profile and Rabi Oscillations
4. Single Frequency Matter Potential Decomposed
   - Basis and Formalism
   - Rabi Oscillations With Multiple Driving Frequencies
   - Multiple Frequencies in Matter Potential
5. Summary

Overview

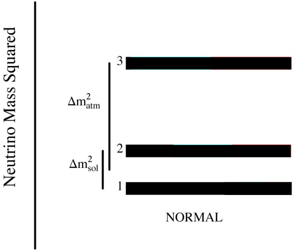
WHAT ARE NEUTRINOS?



Elementary particles.
Source: symmetrymagazine.org

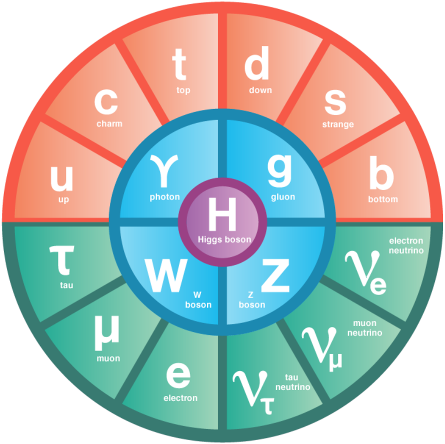
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- none vanishing mass.



Adapted from Olga Mena & Stephen Parke (2004)

What are Neutrinos?



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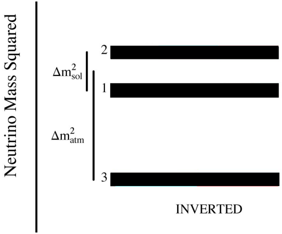

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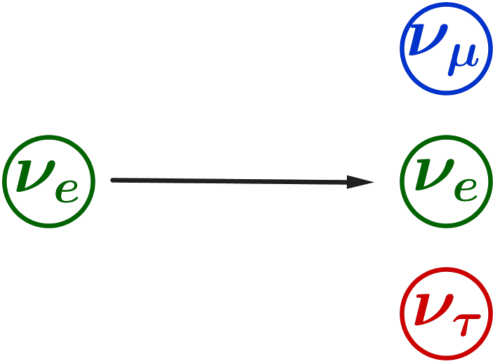
What Are Neutrino Oscillations?

Neutrino Oscillations

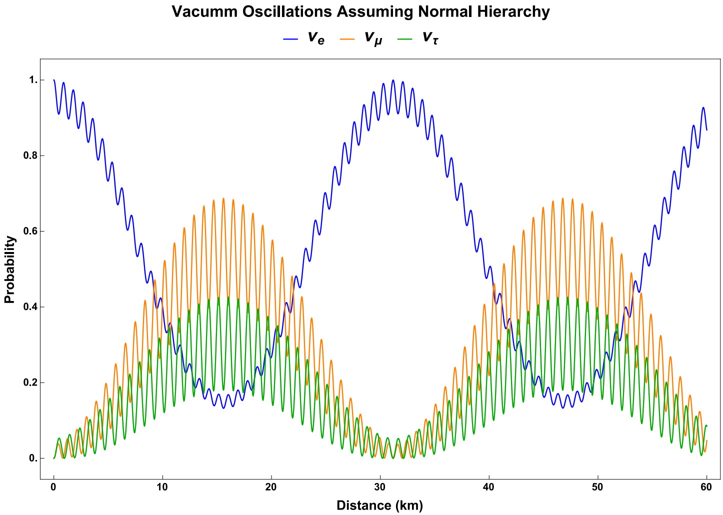
||

Neutrino Flavor Conversions



Neutrino Oscillations

What Are Neutrino Oscillations?



Probabilities of finding neutrinos to be in each flavor.

WHY DO NEUTRINOS OSCILLATE?

Flavor states are different from mass states.

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

- Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_{v}

FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

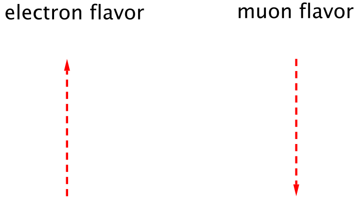
Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\boldsymbol{\sigma}}}{2} \Psi$

Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



FLAVOR ISOSPIN

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

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Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

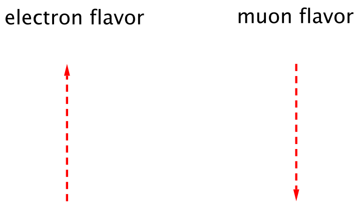
Equation of motion

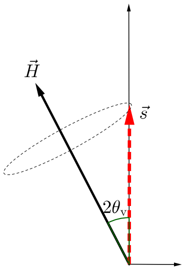
$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

Vacuum oscillation Hamiltonian

$$\frac{\omega_\mathrm{v}}{2} \left(-\cos 2\theta_\mathrm{v} \boldsymbol{\sigma}_3 + \sin 2\theta_\mathrm{v} \boldsymbol{\sigma}_1 \right)$$

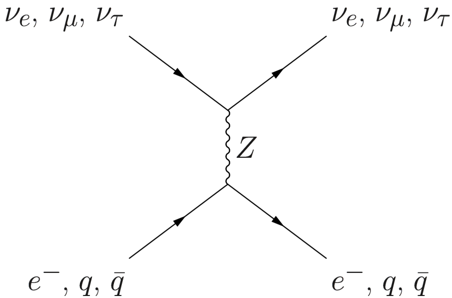
$$\rightarrow \cos 2\theta_\mathrm{v} \begin{pmatrix} 0 \\ 0 \\ \omega_\mathrm{v} \end{pmatrix} - \sin 2\theta_\mathrm{v} \begin{pmatrix} \omega_\mathrm{v} \\ 0 \\ 0 \end{pmatrix}$$



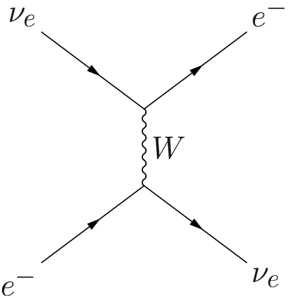


Overview

INTERACTIONS WITH MATTER



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, quarks etc.



Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\mathrm{v}} = \delta m^2/2E$):

$$\mathbf{H} = \boxed{\frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right)} \boxed{+\frac{\lambda(x)}{2}\boldsymbol{\sigma_3}}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)$

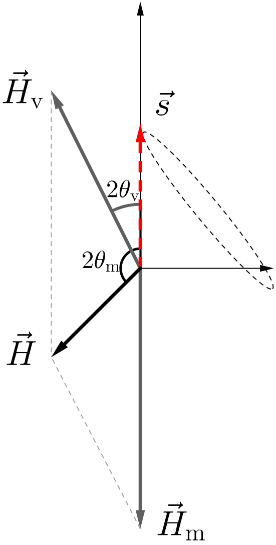
MSW Effect

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

$$\to \omega_{\mathrm{v}}\begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix}$$

$$= \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)$$

MSW EFFECT



Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

Oscillation frequency in **vacuum**:

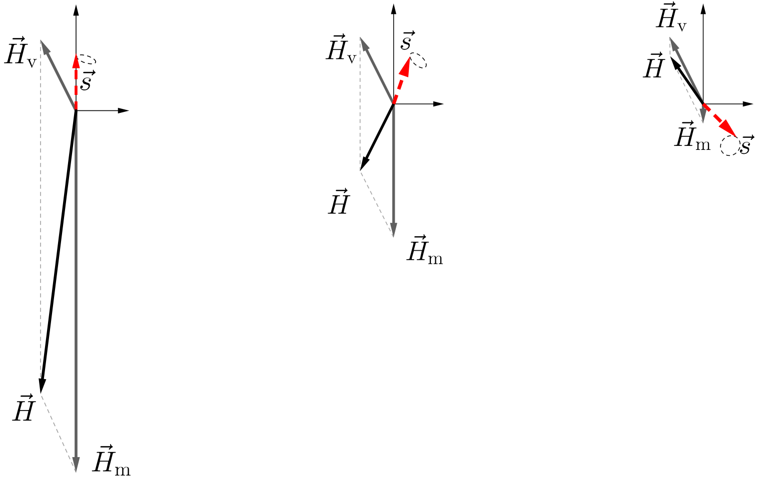
$$\omega_\mathrm{v} = |\vec{H}_\mathrm{v}|$$

Oscillation frequency in **matter**:

$$\omega_\mathrm{m} = |\vec{H}|$$

MSW EFFECT

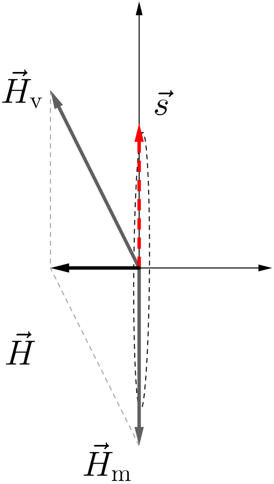
Adiabatic matter density change



Large density

Low density

MSW Effect



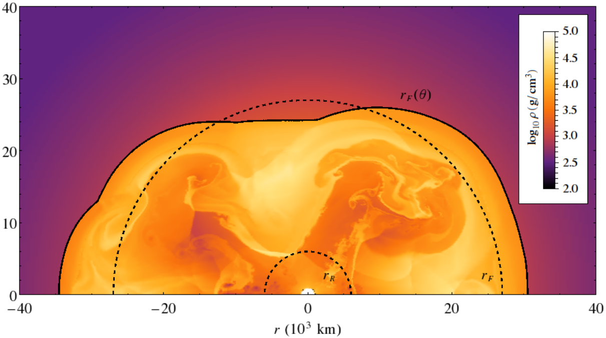
- Maximum possible flavor transition probability amplitude
- MSW Resonance
- A specific matter density

$$\sqrt{2}G_{\rm F}n_{\rm e} \equiv \omega_{\rm v}\cos 2\theta_{\rm v}$$

More Complicated Matter Effect

Why Do We Care

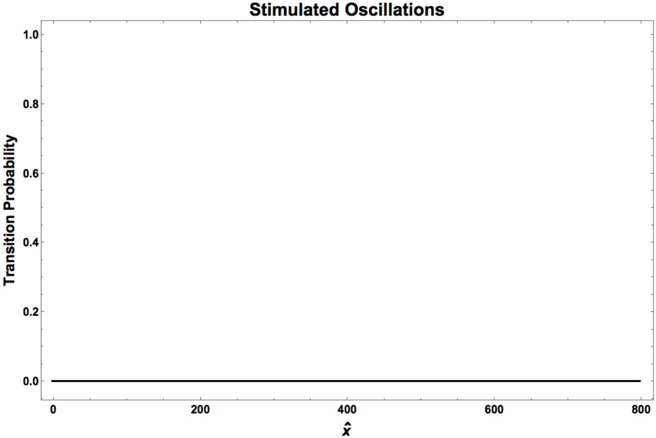
Astrophysical environments: supernovae etc



Turbulence in supernova. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

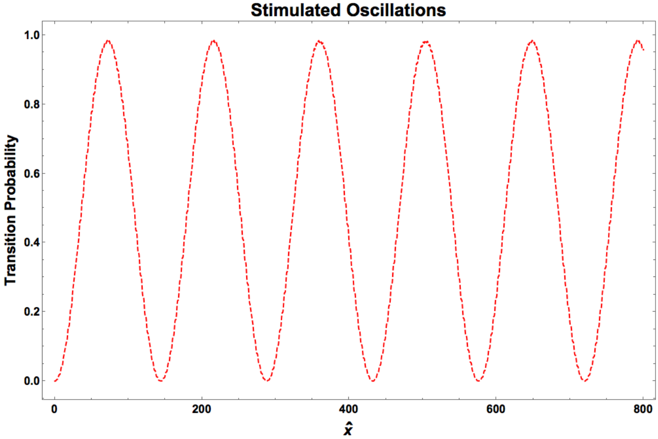
$$\lambda(x) = \lambda_0$$



Transition probabilities between mass states in matter.

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



P. Krastev and A. Smirnov (1989); J. Kneller et al (2013);
K. Patton et al (2014);

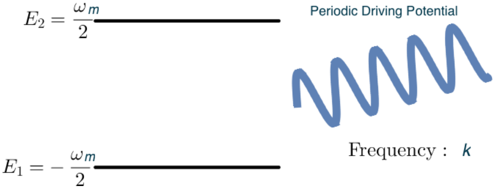
Overview

Rabi Oscillations

Rabi Oscillation

Hamiltonian

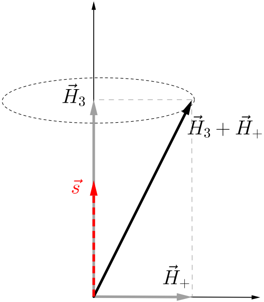
$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$



Rabi Oscillations

Static Frame

$$\vec{H}_3 = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$

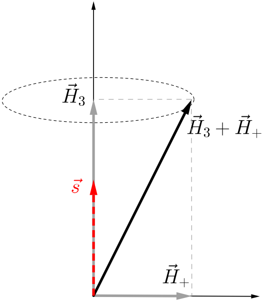


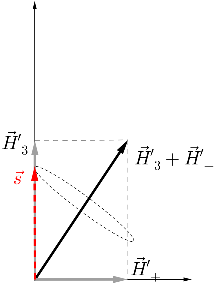
Rabi Oscillations

Static Frame

Corotating Frame

$$\vec{H}_3 = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix} \qquad \vec{H}'_3 = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}'_+ = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

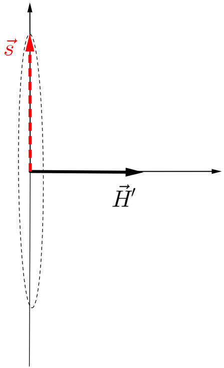




Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow k = \omega_{\mathrm{m}}$$

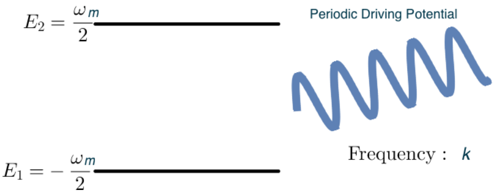


Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$



Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_{\mathrm{R}}}{2}t\right).$$

Relative detuning

$$D = \left|\frac{\omega_{\mathrm{m}} - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_{\mathrm{R}} = |\alpha|\sqrt{1+D^2}$$

Hamiltonian in Matter Basis $\begin{pmatrix}\psi_e\\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m\\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_H\end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}}\right)\boldsymbol{\sigma_3}$$

Hamiltonian in Matter Basis $\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis:

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

Matter potential frequency

$$k \sim \omega_{\rm m}$$

$$\begin{aligned}\mathbf{H} &= \frac{1}{2}\left(-\omega_{\rm m} + \cancel{\cos 2\theta_{\rm m} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\rm m}}{2} A\cos(kx)\sigma_1 \\ &\to \omega_{\rm m}\begin{pmatrix}0\\0\\1\end{pmatrix} + \alpha\begin{pmatrix}\cos(kx)\\ -\sin(kx)\\ 0\end{pmatrix} + \alpha\begin{pmatrix}\cos(-kx)\\ -\sin(-kx)\\ 0\end{pmatrix}\end{aligned}$$

$$\alpha = \frac{\sin 2\theta_{\rm m}}{2} A$$

Hamiltonian in Matter Basis

Matter potential frequency

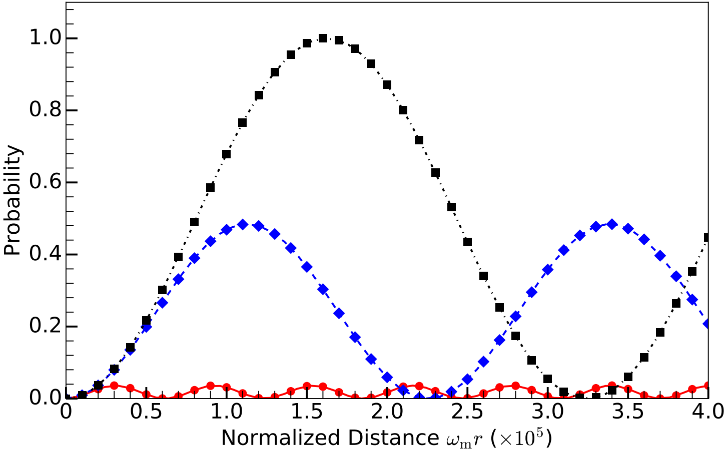
$$k \sim \omega_{\mathrm{m}}$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A \cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A \cos(kx) \sigma_1$$

$$\rightarrow \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \cancel{\alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}}$$

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

RABI FORMULA WORKS



Transition between two mass states in background matter potential λ_0

Lines: Rabi formula

Dots, diamonds, triangles, and squares are **full solutions without approximations** for $k = \omega_\mathrm{m}$, $k = (1 - 2 \times 10^{-5})\omega_\mathrm{m}$, and $k = (1 - 10^{-4})\omega_\mathrm{m}$ respectively.

Overview

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cancel{\cos 2\theta_{\mathrm{m}} A\cos(kx)}\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \cancel{\alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}}$$

Rabi Basis

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix}\psi_{\mathrm{L}}\\ \psi_{\mathrm{H}}\end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\tilde{\psi}_{\mathrm{L}}\\ \tilde{\psi}_{\mathrm{H}}\end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty}\begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk\tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Single Frequency Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Hamiltonian in Rabi Basis

The Hamiltonian

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \sum_{n=-\infty}^{\infty}\begin{pmatrix} 0 & \frac{1}{2}\alpha_n e^{i(nk)x} \\ \frac{1}{2}\alpha_n^* e^{-i(nk)x} & 0 \end{pmatrix}$$

where $\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A\cos 2\theta_{\mathrm{m}}/k)$.

Map neutrino oscillations in single frequency matter potential to Rabi oscillations with many driving potentials.

Resonance condition for each mode: $nk = \omega_{\mathrm{m}}$

Rabi Oscillations With Multiple Driving Frequencies

Consider Rabi oscillation with two driving frequencies $k_1 = n_1 k$, $k_2 = n_2 k$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

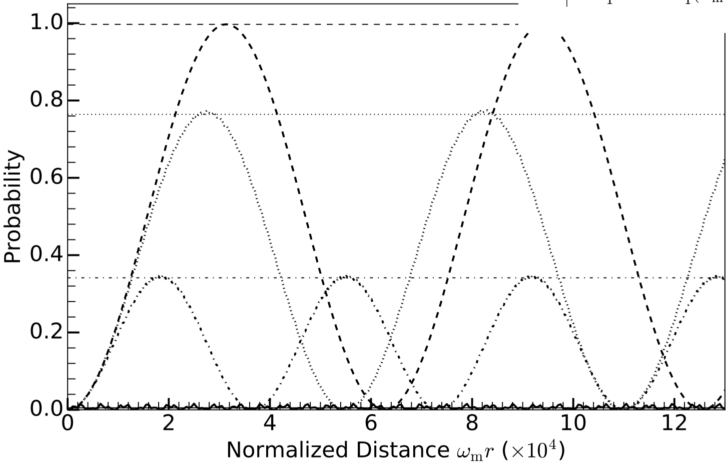
Rabi Oscillations With Multiple Driving Frequencies

Relative detuning

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Rabi Oscillations With Multiple Driving Frequencies

$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$



Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values			
Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$	$5.0 \times 10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$5 \times 10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$

Rabi Oscillations With Multiple Driving Frequencies

Consider $k_1 = \omega_\mathrm{m}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_\mathrm{m} - k_2)} \right|$$

Amplitude reduces from 1 to 1/2 if

$$D' = 1 \Rightarrow \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
For $k_1 = \omega_\mathrm{m}$, survival of resonance requires

$$|\alpha_2| \ll \alpha_{2,\mathrm{C}} \equiv \sqrt{2|\alpha_1(k_2 - \omega_\mathrm{m})|}$$

SINGLE FREQUENCY MATTER POTENTIAL REVISITED

Matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Consider the resonance condition ($k = \omega_{\mathrm{m}}$)

$$\widetilde{\mathbf{H}} \sim -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\begin{pmatrix} 0 & \alpha_1 e^{ikx} \\ \alpha_1^* e^{-ikx} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha_n e^{inkx} \\ \alpha_n^* e^{-inkx} & 0 \end{pmatrix}$$

Single Frequency Matter Potential Revisited

Matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Consider the resonance condition ($k = \omega_\mathrm{m}$)

$$\widetilde{\mathbf{H}} \sim -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \frac{1}{2}\begin{pmatrix} 0 & \alpha_1 e^{ikx} \\ \alpha_1^* e^{-ikx} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha_n e^{inkx} \\ \alpha_n^* e^{-inkx} & 0 \end{pmatrix}$$

$$D' = \left|\frac{\alpha_2^2}{2\alpha_n(\omega_\mathrm{m} - nk)}\right|$$

$k = \omega_\mathrm{m}$

n	D'	Amplitude
1	0	1
1 & -1	4.8×10^{-6}	$1 - 2.3 \times 10^{-11}$
1 & 2	2.1×10^{-14}	$1 - 4.4 \times 10^{-28}$
1 & -2	6.9×10^{-15}	$1 - 4.8 \times 10^{-29}$

Single Frequency Matter Potential Revisited

Width

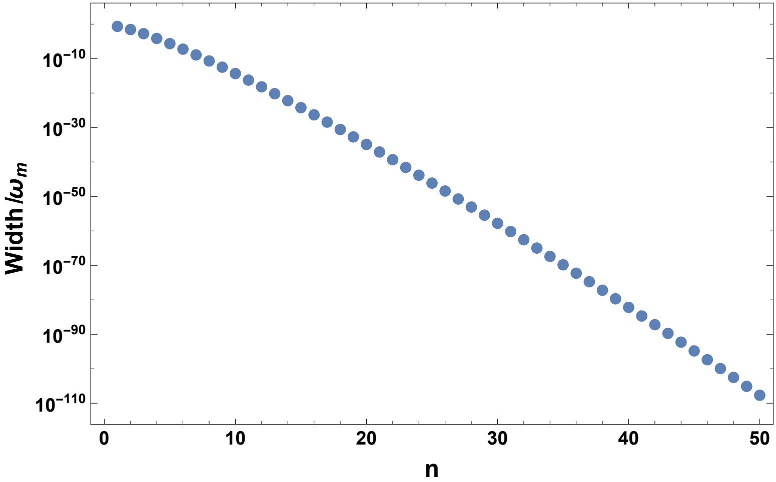
$$\alpha_n = -(-i)^n nk \tan 2\theta_{\mathrm{m}} J_n(A \cos 2\theta_{\mathrm{m}}/k)$$

$$|\alpha_n| \propto \sqrt{\frac{n}{2\pi}} \left(\frac{eA \cos 2\theta_{\mathrm{m}}}{2nk} \right)^n, \quad \text{for large } n$$

Width drops fast at large n.
But the critical value for each mode becomes larger for large n's

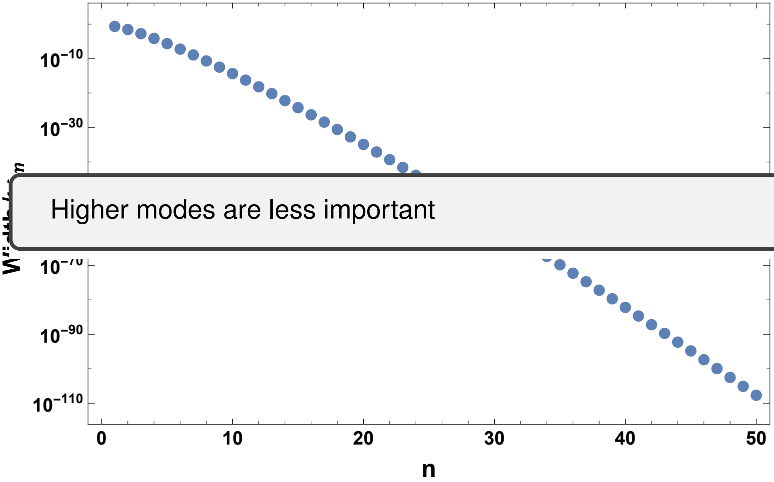
$$\alpha_{n,\mathrm{C}} \equiv \sqrt{2|\alpha_1(nk - \omega_{\mathrm{m}})|}$$

Single Frequency Matter Potential



Width of different modes given value of matter potential frequency k

Single Frequency Matter Potential



Higher modes are less important

Width of different modes given value of matter potential frequency k

Multiple Frequencies in Matter Potential

$$\lambda(x) = \lambda_0 + \sum_{a=1}^{N} A_a \sin(k_a x)$$

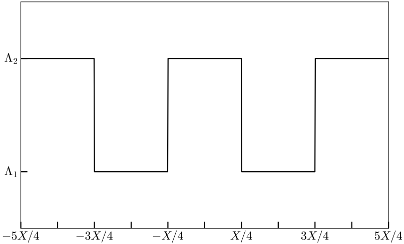
Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\begin{pmatrix} 0 & B_{\{n_a\}} e^{i\sum_a n_a k_a x} \\ B^*_{\{n_a\}} e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_a\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a\right)\left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right)$$

CASTLE WALL MATTER POTENTIAL



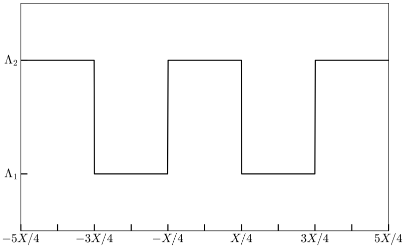
Castle wall matter profile:
$\Lambda_2 = 0.35\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$,
$\Lambda_1 = 0.15\omega_\mathrm{v}\cos 2\theta_\mathrm{v}$ and period
$X = 2\pi/\omega_\mathrm{m}$

$$\lambda(x) = \lambda_0 + \sum_1^\infty \lambda_n \cos(k_n x)$$

where

$$\begin{aligned}
\lambda_0 =& (\Lambda_1 + \Lambda_2)/2 \\
\lambda_n =& 2(-1)^n(\Lambda_1 - \Lambda_2)/(2n\pi - \pi) \\
k_n =& 2\pi(2n-1)/X
\end{aligned}$$

Castle Wall Matter Potential



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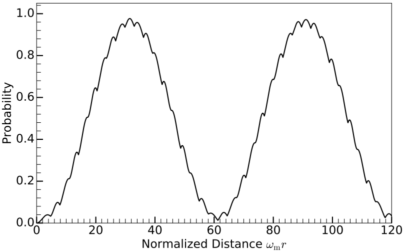
$X = 2\pi/\omega_{\rm m}$

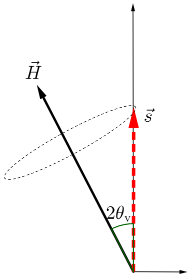


Transition probability is a Rabi resonance with small variations due to higher orders.

Overview

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.



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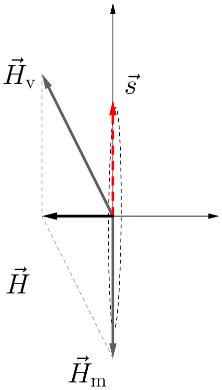

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. In many cases neutrino oscillations in multi-frequency matter potential can be viewed as Rabi oscillations with few driving frequencies.

Backup Slides

BACKUP SLIDES

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x \left|\Psi\right\rangle = \hat{\mathbf{H}} \left|\Psi\right\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{\left|\nu_1\right\rangle, \left|\nu_2\right\rangle\}$.
- $$\mathbf{H} = -\frac{\omega_{\mathrm{v}}}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$
- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp\left(i\omega_{\mathrm{v}} x/2\right) \\ \langle\nu_2|\Psi(0)\rangle \exp\left(-i\omega_{\mathrm{v}} x/2\right) \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_\mathrm{e}\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

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θ_v: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_v}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_v}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix} = \frac{\omega_v}{2}\left(-\cos 2\theta_v \boldsymbol{\sigma_3} + \sin 2\theta_v \boldsymbol{\sigma_1}\right)$$

NATURE OF NEUTRINO OSCILLATION

Transition Probability

$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_v)\sin^2(\omega_v x/2)$$

- $\omega_v = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- Mixing angle θ_v determines flavor oscillation amplitude.

MSW Effect

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_m

Define matter basis (eigenenergy basis) $\{|\nu_\mathrm{L}\rangle, |\nu_\mathrm{H}\rangle\}$

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & \sin\theta_\mathrm{m} \\ -\sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} |\nu_\mathrm{L}\rangle \\ |\nu_\mathrm{H}\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_\mathrm{m}}{2}\boldsymbol{\sigma_3}$$

MSW EFFECT

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

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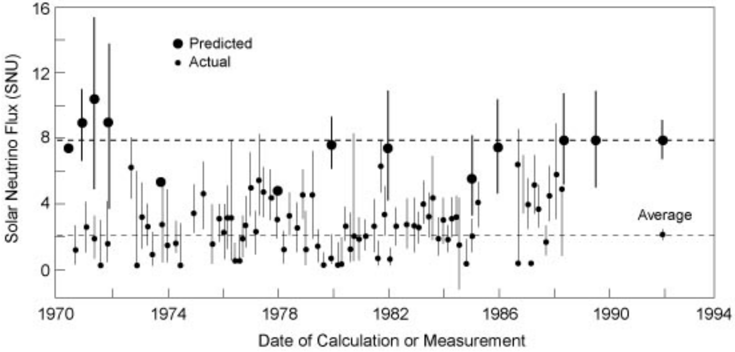
In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2}\boldsymbol{\sigma_3}$$

Transition Probability

$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_m)\sin^2(\omega_m x)$$

Solar Neutrino Problem



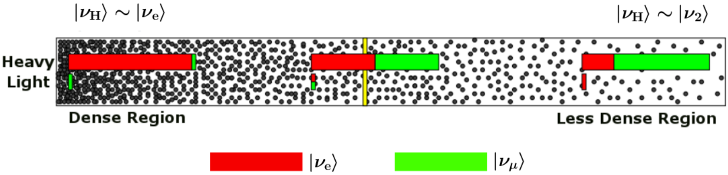
Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW Effect and Solar Neutrinos

$$\mathbf{H} = \frac{\lambda(x) - \omega_v \cos 2\theta_v}{2}\boldsymbol{\sigma_3} + \frac{\omega_v \sin 2\theta_v}{2}\boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_m & -\sin\theta_m \\ \sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2}\boldsymbol{\sigma_3}$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW Effect Inverted Hierarchy

Suppose $\omega_\mathrm{v} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = \boxed{-\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix}} \boxed{+\sqrt{2} G_\mathrm{F} n_\mathrm{e}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$\downarrow$$

$$\mathbf{H} = \left(\frac{-\omega_\mathrm{v}}{2} \cos 2\theta_\mathrm{v} + \frac{\lambda(x)}{2} \right) \boldsymbol{\sigma_3} - \frac{\omega_\mathrm{v}}{2} \sin 2\theta_\mathrm{v} \boldsymbol{\sigma_1}$$

Hamiltonian

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1 .$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 .$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\rm m} + \delta\lambda(x)\cos 2\theta_{\rm m}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\rm m}\sigma_1.$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\rm m} + \cos 2\theta_{\rm m} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\rm m}}{2} A\cos(kx)\sigma_1.$$

Rabi Oscillations

The coupling strength is calculated as

$$\alpha = \langle 1|\mathbf{d} \cdot \mathbf{E}|2\rangle$$

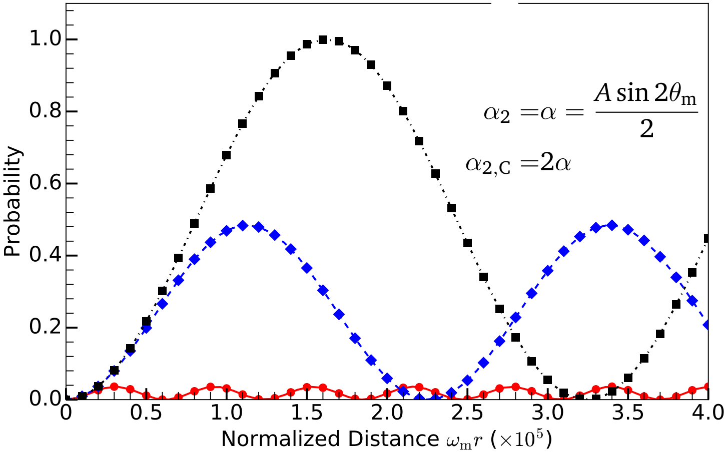
where the electric field is

$$\mathbf{E} = \mathbf{E}_0 \sin(kt).$$

and $\mathbf{d}$ is the dipole moment.

Rabi Formula Works

$\vec{H} \sim \omega_{\rm m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix}$



Lines: Rabi formula
Dots, diamonds, triangles, and squares are for $k = \omega_{\rm m}$, $k = (1 - 2 \times 10^{-5})\omega_{\rm m}$, and $k = (1 - 10^{-4})\omega_{\rm m}$ respectively.

Parameters Used for Vacuum Oscillations

$\theta_{12} = 33.36/180\pi$; $\theta_{13} = 8.66/180\pi$; $\theta_{23} = 40/180 * \pi$; $\delta_{cp} = 0$;
$m_1^2 = 0.01$; $m_2^2 = m_1^2 + 0.000079$; $E = 1$MeV

Single Frequency Matter Potential

Matter potential

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2}e^{2i\eta(x)} \\ &= \frac{i}{4}\left[\exp\left(ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right. \\ &\qquad \left. - \exp\left(-ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right] \end{aligned}$$

Single Frequency Matter Potential

Off-diagonal Term in Our System

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$h \propto \left[\exp\left(ikx + \boxed{i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)}\right) - \exp\left(-ikx + \boxed{i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)}\right)\right]$$

Jacobi-Anger expansion (Kneller et al, 2013)

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

Single Frequency Matter Profile

Transition Probability

$$P^{(n)}_{\mathrm{L}\to\mathrm{H}} = \frac{\left|\hat{B}_n\right|^2}{\left|\hat{B}_n\right|^2 + (n\hat{k}-1)^2} \sin^2\left(\frac{q^{(n)}}{2}x\right),$$

where

$$q^{(n)} = \sqrt{\left|\Gamma^{(n)}\right|^2 + (n\hat{k}-1)^2}, \quad \text{frequency of oscillations}$$
$$\Gamma^{(n)} = \left|\hat{B}_n\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

Resonance conditions

$$\hat{k} \sim \frac{1}{n}$$

Single Frequency Matter Potential Revisited

Width

$$\alpha_n = -(-i)^n nk \tan 2\theta_\mathrm{m} J_n(A \cos 2\theta_\mathrm{m}/k)$$

$$J_n(n \operatorname{sech} \beta) \sim \frac{e^{-n(\beta - \tanh \beta)}}{\sqrt{2\pi n \tanh \beta}}, \quad \text{for large } n$$

$\Rightarrow$

$$|\alpha_n| \propto \frac{e^{-n(\beta - \tanh \beta)}}{\sqrt{2\pi n \tanh \beta}}, \quad \text{for large } n$$

where $\operatorname{sech} \beta = A \cos 2\theta_\mathrm{m}/\omega_\mathrm{m}$.

$\beta - \tanh \beta > 0 \Rightarrow$ **Width** drops fast at large n.

Two-frequency Matter Profile

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

TWO-FREQUENCY MATTER P

$$\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}},$$

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}},$$

where

$$\begin{aligned}&\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)\\ =& -(-i)^{n_1+n_2}(n_1\hat{k}_1+n_2\hat{k}_2)J_{n_1}\left(\frac{\hat{A}_1\cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right)J_{n_2}\left(\frac{\hat{A}_2\cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)\end{aligned}$$

Which terms are important?

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = \omega_{\mathrm{m}}$$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	0	-	3.2×10^5
-1	10^5	4.8×10^{-6}	3.1
2	1.1×10^9	2.1×10^{-14}	6.3
-2	3.4×10^9	6.9×10^{-15}	2.1

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = (1 - 2 \times 10^{-5})\omega_{\rm m}$$

n	D	D_1'	$2\pi\omega_{\rm m}/\Omega_n$
1	1	-	2.2×10^5
-1	10^5	1	3.1
2	1.1×10^9	1	6.3
-2	3.4×10^9	1	2.1

Single Frequency Matter Profile Revisited

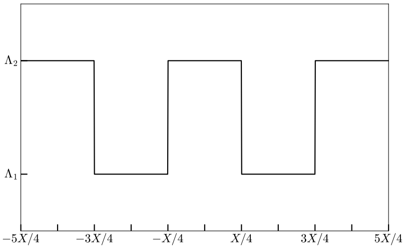
Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = (1 - 10^{-4})\omega_{\mathrm{m}}$$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	5.2	-	6.2×10^4
-1	10^5	5.2	3.1
2	1.1×10^9	5.2	6.3
-2	3.4×10^9	5.2	2.1

Castle Wall Matter Profile



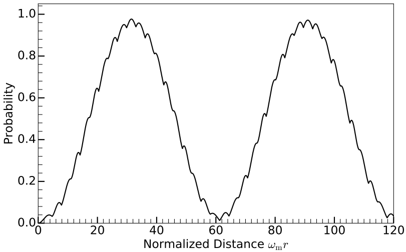
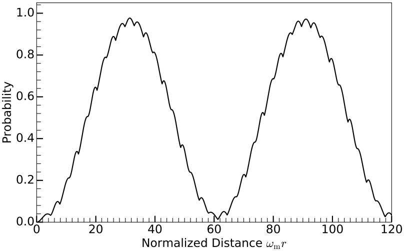


Figure: Castle wall matter profile

Castle Wall Matter Profile

Table: Relative detuning of each frequency.

$\{n_1, n_2\}$	D	$D'_{\{1,0\}}$
$\{1, 0\}$	0	-
$\{-1, 0\}$	48	1.0×10^{-2}
$\{0, 1\}$	1.5×10^{2}	1.1×10^{-3}
$\{2, 0\}$	2.4×10^{2}	2.0×10^{-4}



TWO-FREQUENCY MATTER PROFILE

Resonance Lines

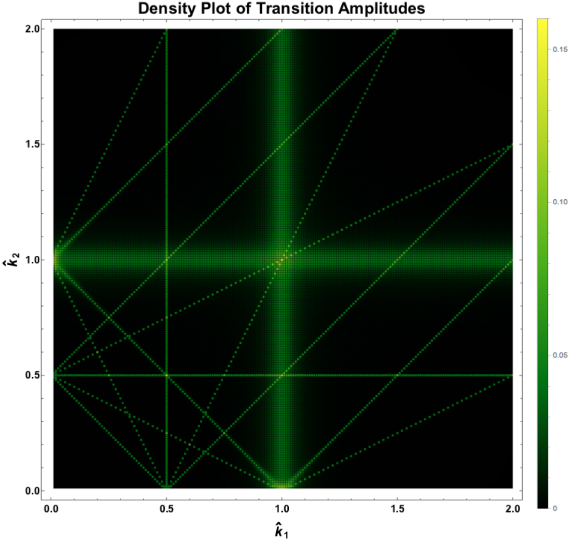
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.

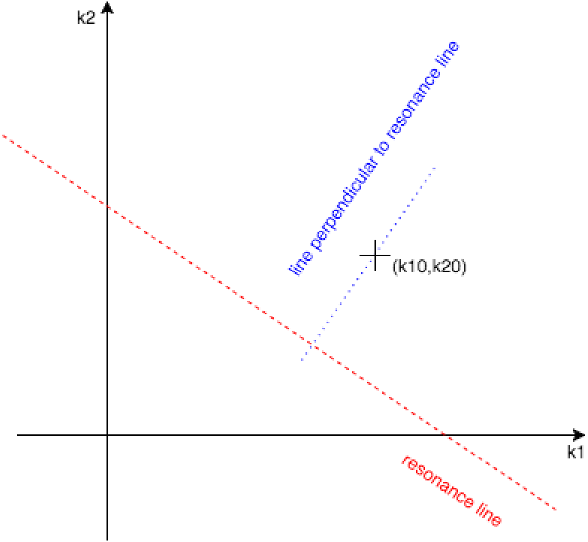
Two-frequency Matter Pr

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

Two-frequency Matter Profile



Resonance line, distance to resonance, and width

Two-frequency Matter Profile

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\mathrm{intercept}}, \hat{k}_{2,\mathrm{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

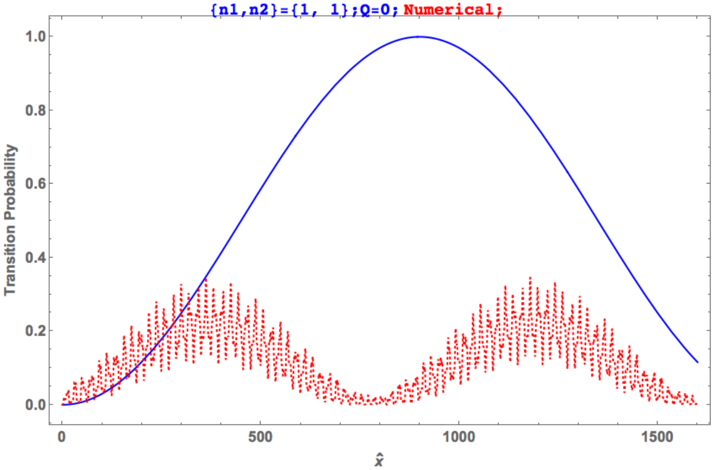
Distance to Resonance Line

$$d = \frac{|n_1 \hat{k}_{10} + n_2 \hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

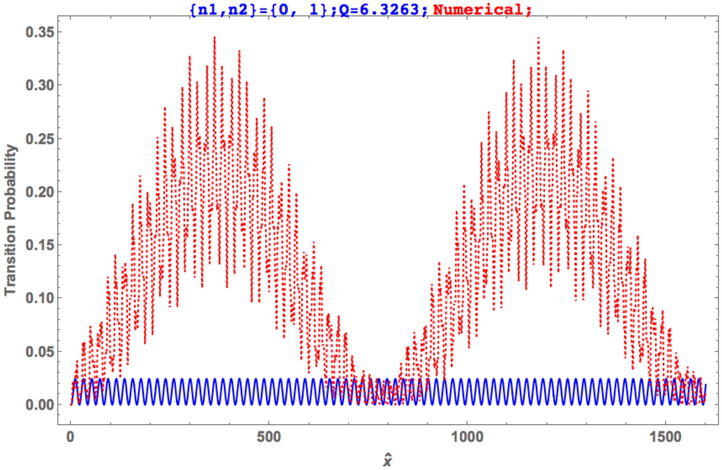
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



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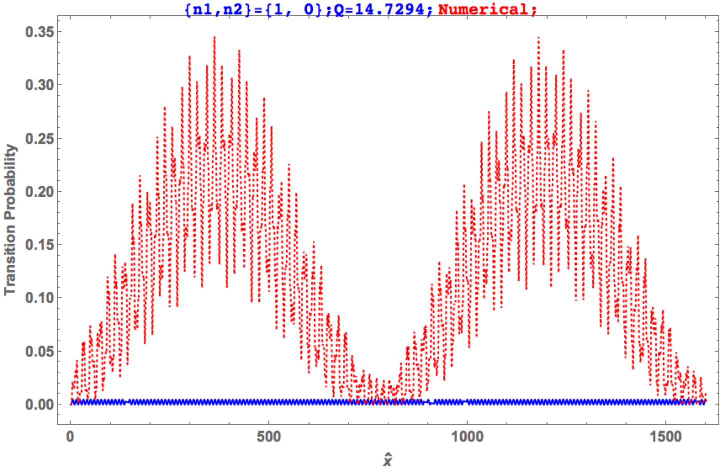
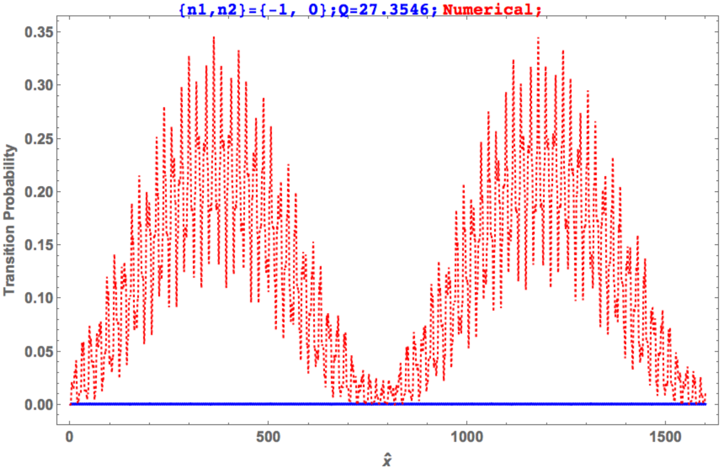
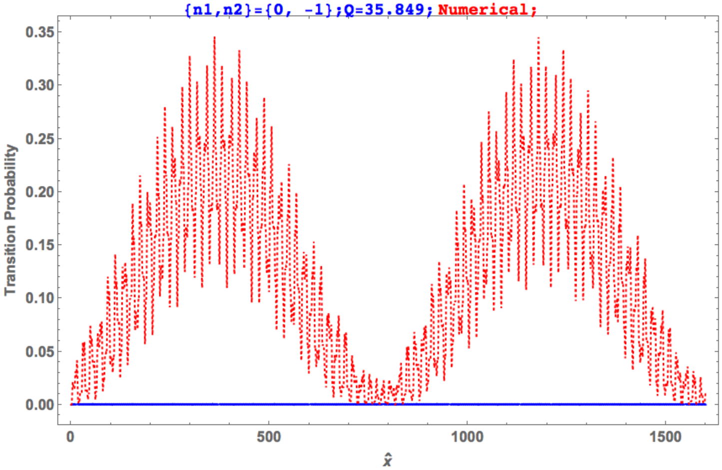


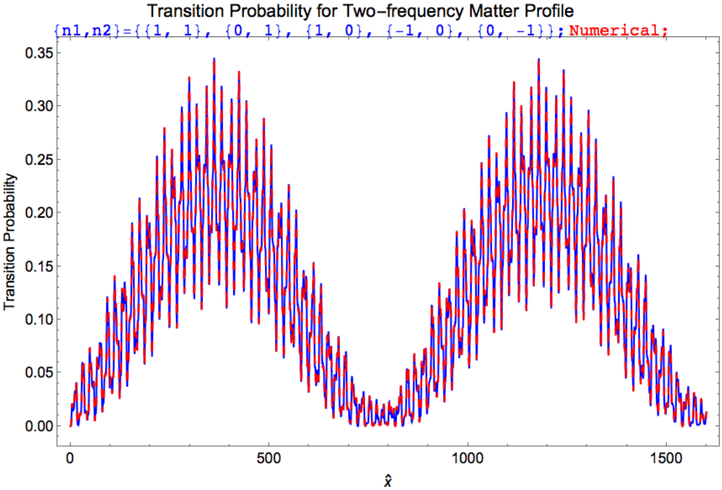
TWO-FREQUENCY MATTER PROFILE



Two-frequency Matter Profile



Two-frequency Matter Profile



Bessel's Function

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I