Stimulated Neutrino Flavor Conversions and Rabi Oscillations

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OUTLINE

OVERVIEW

Background
- What are Neutrinos
- Neutrino Oscillations
- Why Do Neutrinos Oscillate

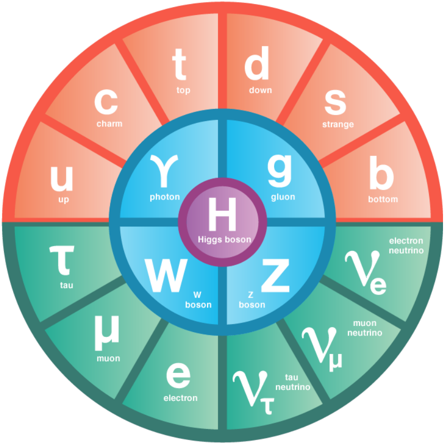
Matter Effect

Stimulated Neutrino Flavor Conversions

Single Frequency Matter Potential Decomposed

Summary

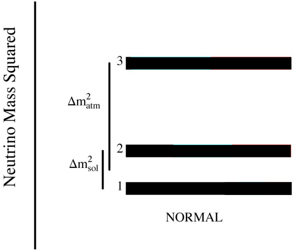
WHAT ARE NEUTRINOS?



Elementary particles.
Source: symmetrymagazine.org

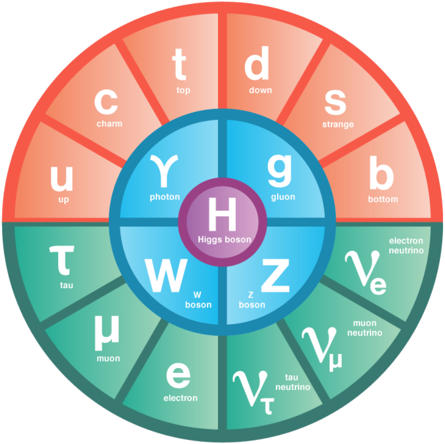
Neutrinos are

- fermions,
- electrically neutral,
- three flavors,
- light.



Adapted from Olga Mena & Stephen Parke (2004)

What are Neutrinos?



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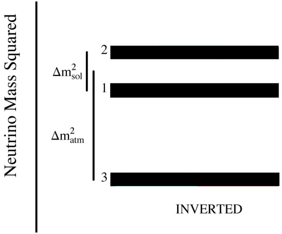

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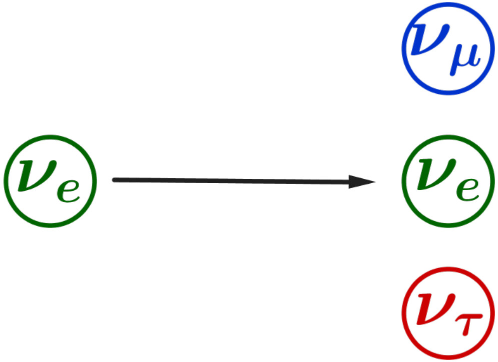
What Are Neutrino Oscillations?

Neutrino Oscillations

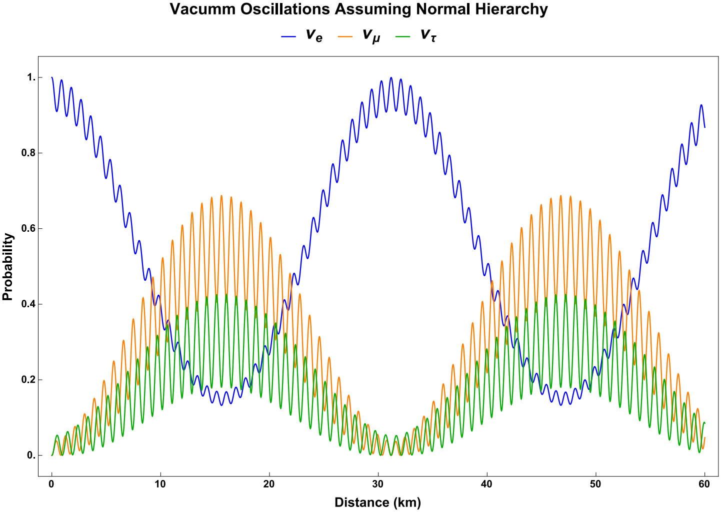
||

Neutrino Flavor Conversions



Neutrino Oscillations

WHAT ARE NEUTRINO OSCILLATIONS?



Probabilities of finding neutrinos to be in each flavor.

Why Do Neutrinos Oscillate?

Flavor states are different from mass states (eigen energy states).

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Equation of Motion

$$i\partial_x \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \mathbf{H} \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix}$$

$$\mathbf{H} = \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_3 + \sin 2\theta_{\mathrm{v}} \boldsymbol{\sigma}_1 \right)$$

- Oscillation frequency:

$$\omega_{\mathrm{v}} = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}$$

- Mixing angle θ_{v}

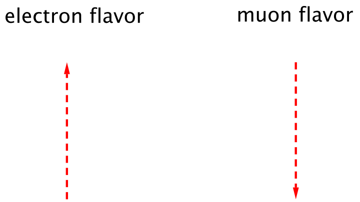
Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

Flavor isospin: $\vec{s} = \Psi^\dagger \frac{\vec{\boldsymbol{\sigma}}}{2} \Psi$

Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$



Flavor Isospin

Hamiltonian: $\mathbf{H} = -\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{H}$

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Equation of motion

$$\dot{\vec{s}} = \vec{s} \times \vec{H}$$

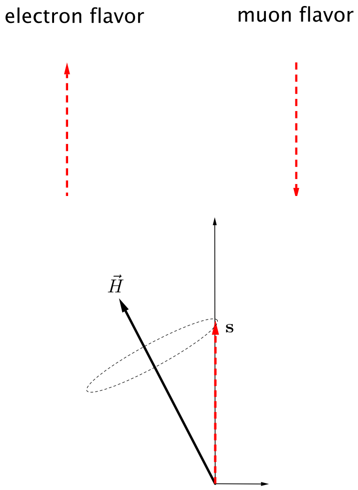
Vacuum oscillation Hamiltonian

$$\frac{\omega_{\rm v}}{2} \left(-\cos 2\theta_{\rm v} \boldsymbol{\sigma}_3 + \sin 2\theta_{\rm v} \boldsymbol{\sigma}_1 \right)$$

$$\to \cos 2\theta_{\rm v} \begin{pmatrix} 0 \\ 0 \\ \omega_{\rm v} \end{pmatrix} - \sin 2\theta_{\rm v} \begin{pmatrix} \omega_{\rm v} \\ 0 \\ 0 \end{pmatrix}$$

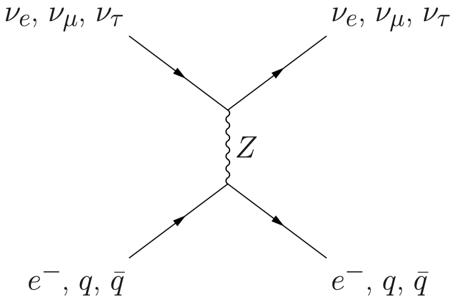
Electron flavor survival probability

$$P = \frac{1}{2} + s_3$$

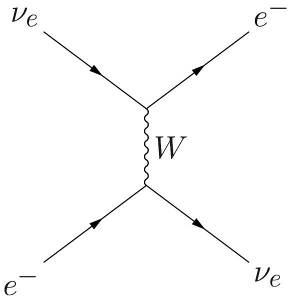


OVERVIEW

Interactions with Matter



Neutral current interaction between ν_e, ν_μ, ν_τ, and e^-, quarks etc.



Charged current interaction between ν_e and e^-

Matter Interaction

Hamiltonian with matter interaction in flavor basis ($\omega_{\rm v} = \delta m^2/2E$):

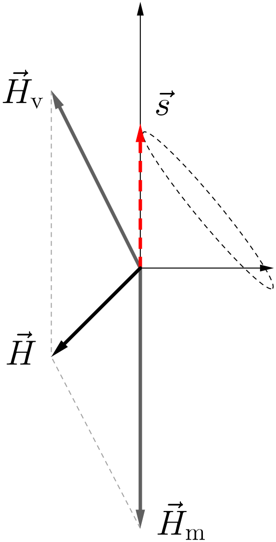
$$\mathbf{H} = \frac{\omega_{\rm v}}{2}\left(-\cos 2\theta_{\rm v}\boldsymbol{\sigma_3} + \sin 2\theta_{\rm v}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3}$$

- Vacuum Hamiltonian
- Matter interaction
- $\lambda(x) = \sqrt{2}G_{\rm F}n_{\rm e}(x)$

MSW Effect

$$
\begin{aligned}
\mathbf{H} &= \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\boldsymbol{\sigma_3} + \sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}\right) + \frac{\lambda(x)}{2}\boldsymbol{\sigma_3} \\
&\to \omega_{\mathrm{v}}\begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix} \\
&= \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x)
\end{aligned}
$$

MSW EFFECT



Electron flavor survival probability

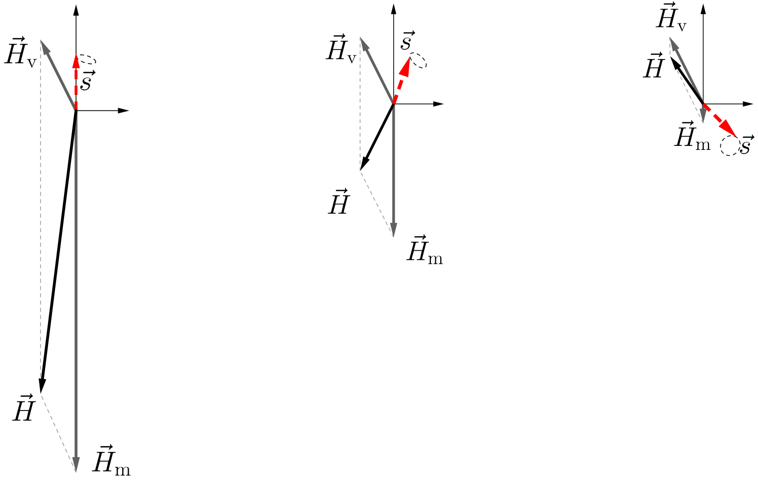
$$P = \frac{1}{2} + s_3$$

Energy gap:

$$|\vec{H}|$$

MSW EFFECT

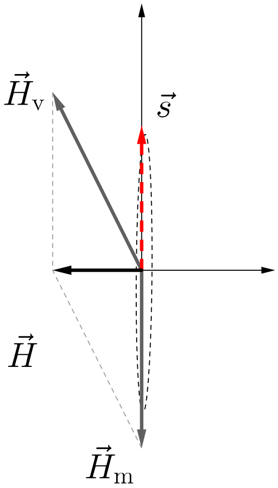
Adiabatic matter density change



Large density

Low density

MSW EFFECT

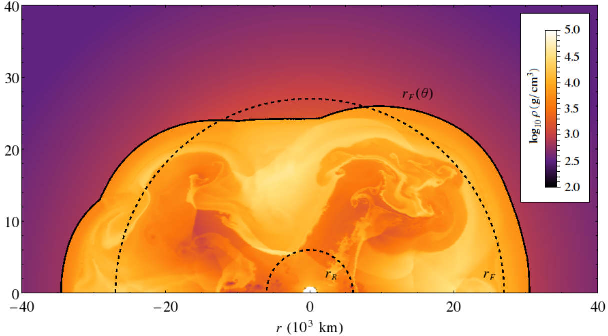


- MSW Resonance
- Maximum possible flavor transition probability amplitude
- A specific matter density

More Complicated Matter Effect

Why Do We Care

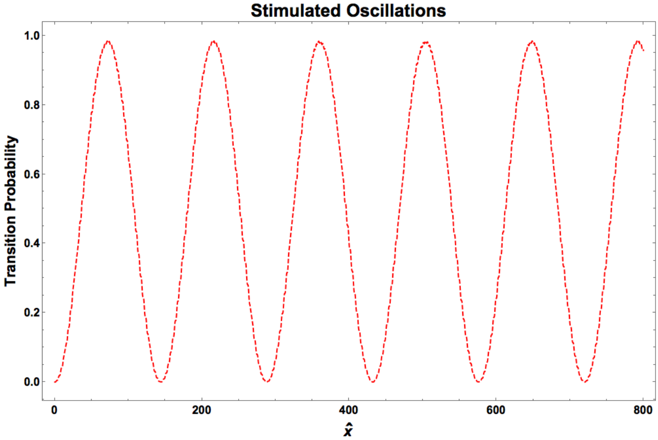
Astrophysical environments: supernovae etc



Supernova shock and turbulence. E. Borriello, et al (2014)

Stimulated Neutrino Flavor Conversions

$$\lambda(x) = \lambda_0 + A\cos(kx)$$



Stimulated oscillations. $\lambda(x) = \lambda_0 + A\cos(kx)$

P. Krastev and A. Smirnov (1989); J. Kneller et al (2013); K. Patton et al (2014);

OVERVIEW

Rabi Oscillations

Rabi Oscillation

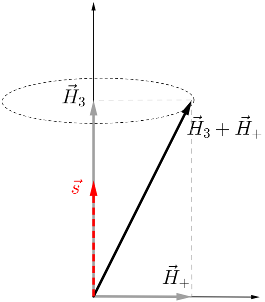
Hamiltonian

$$-\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$

Rabi Oscillations

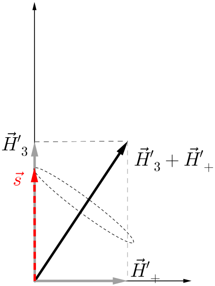
Static Frame

$$\vec{H}_3 = \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+ = \alpha \begin{pmatrix} \cos(kt) \\ -\sin(kt) \\ 0 \end{pmatrix}$$



Corotating Frame

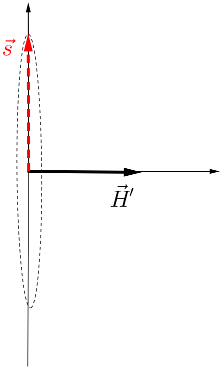
$$\vec{H}_3' = (\omega_{\mathrm{m}} - k) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \vec{H}_+' = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$



Rabi Oscillations

Corotating Frame

$$\vec{H}'_3 = (\omega_\mathrm{m} - k)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \Rightarrow k = \omega_\mathrm{m}$$

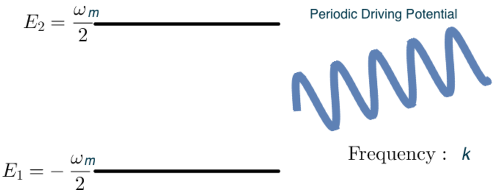


Rabi Oscillations

Rabi Oscillation

Hamiltonian

$$-\frac{\omega_m}{2}\sigma_3 - \frac{\alpha}{2}\begin{pmatrix} 0 & e^{ikt} \\ e^{-ikt} & 0 \end{pmatrix}$$



Rabi formula

$$P_{1\to 2} = \frac{1}{1+D^2}\sin^2\left(\frac{\Omega_R}{2}t\right).$$

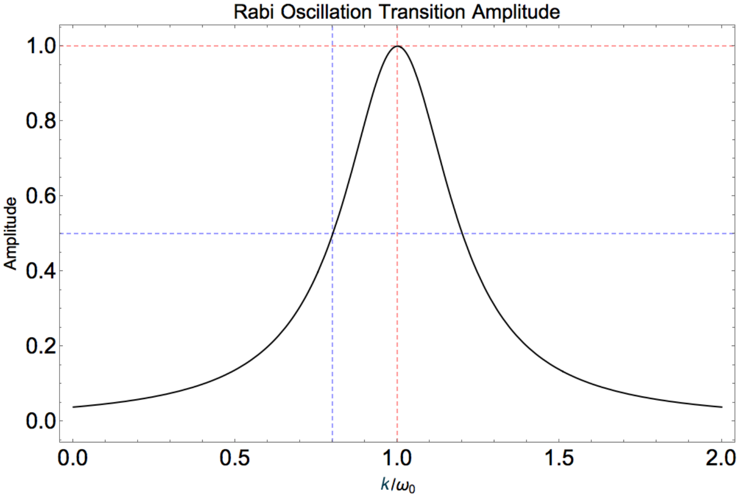
Relative detuning

$$D = \left|\frac{\omega_m - k}{\alpha}\right|.$$

Rabi frequency

$$\Omega_R = \sqrt{\alpha^2 + (\omega_m - k)^2}$$

Rabi Oscillations



Amplitude of Rabi oscillations for different driving field frequency k

Hamiltonian in Matter Basis $\begin{pmatrix}\psi_e \\ \psi_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m\end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_H\end{pmatrix}$

Matter Potential

$$\lambda(x) = \lambda_0$$

Basis

Background matter basis (eigen energy basis):

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}}\right)\boldsymbol{\sigma_3}$$

HAMILTONIAN IN MATTER BASIS

$$\begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_H \end{pmatrix}$$

Matter Potential

$$\lambda(x) = \lambda_0 + A\cos(kx)$$

Basis

Background matter basis (eigen energy basis):

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin 2\theta_{\mathrm{m}}\boldsymbol{\sigma_1}$$

Hamiltonian in Matter Basis

$$\alpha = \frac{\sin 2\theta_{\mathrm{m}}}{2} A$$

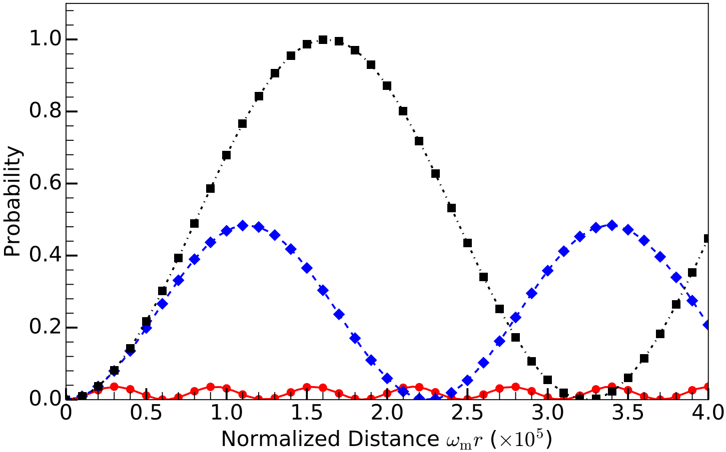
Matter potential frequency

$$k \sim \omega_{\mathrm{m}}$$

$$\begin{aligned}\mathbf{H} =& \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A \cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A \cos(kx) \sigma_1 \\ \rightarrow& \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}\end{aligned}$$

Rabi Formula Works

$$\vec{H} \sim \omega_{\rm m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix}$$



Lines: Rabi formula
Dots, diamonds, triangles, and squares are for $k = \omega_{\rm m}$, $k = (1 - 2 \times 10^{-5})\omega_{\rm m}$, and $k = (1 - 10^{-4})\omega_{\rm m}$ respectively.

OVERVIEW

Single Frequency Matter Potential Revisited

We have been making approximations.

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1$$

$$\rightarrow \omega_{\mathrm{m}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \cos(kx) \\ -\sin(kx) \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} \cos(-kx) \\ -\sin(-kx) \\ 0 \end{pmatrix}$$

RABI BASIS

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + A\cos(kx)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{A\cos(kx)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

A Better Basis

Define Rabi basis in which the wave function is related to wave function in background matter basis through

$$\begin{pmatrix}\psi_{\mathrm{L}}\\ \psi_{\mathrm{H}}\end{pmatrix} = \begin{pmatrix} e^{-i\eta(x)} & 0 \\ 0 & e^{i\eta(x)}\end{pmatrix}\begin{pmatrix}\tilde{\psi}_{\mathrm{L}}\\ \tilde{\psi}_{\mathrm{H}}\end{pmatrix},$$

where

$$\eta(x) - \eta(0) = \frac{\cos 2\theta_{\mathrm{m}}}{2}\int_0^x A\cos(k\tau)d\tau.$$

Single Frequency Matter Potential

$$\delta\lambda_N = \sum_{a=1}^{N} A_a \sin(k_a x + \phi_a)$$

Hamiltonian in Rabi Basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_\mathrm{m}}{2}\sigma_3 + \frac{1}{2}\sum_{n_1=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\begin{pmatrix} 0 & B_{\{n_i\}}e^{i\sum_a n_a k_a x} \\ B^*_{\{n_i\}}e^{-i\sum_a n_a k_a x} & 0 \end{pmatrix}$$

where

$$B_{\{n_i\}} = -(-i)^{\sum_a n_a} \tan 2\theta_m \left(\sum_a n_a k_a\right)\left(\prod_a J_{n_a}\left(\frac{A_a}{k_a}\cos 2\theta_m\right)\right),$$

$$\Phi_{\{n_i\}} = e^{i\left(\sum_a n_a \phi_a\right)}.$$

Single Frequency Matter Potential

$$\delta\lambda_N = \sum_{a=1}^{N} A_a \sin(k_a x + \phi_a)$$

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Multiple potentials with different frequencies!

RABI OSCILLATIONS WITH MULTIPLE POTENTIALS

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 x) \\ -\sin(k_1 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} \cos(k_2 x) \\ -\sin(k_2 x) \\ 0 \end{pmatrix}}$$

Corotating frame of the second potential,

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \alpha_1 \begin{pmatrix} \cos(k_1 - k_2 x) \\ -\sin(k_1 - k_2 x) \\ 0 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Energy gap in this frame becomes the length of the vector

$$\begin{pmatrix} 0 \\ 0 \\ \omega_m - k_2 \end{pmatrix} + \boxed{\alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

Rabi Oscillations With Multiple Potentials

Energy gap in this frame becomes the length of the vector

$$\sqrt{(\omega_{\mathrm{m}} - k_2)^2 + \alpha_2^2} \to \omega_{\mathrm{m}} - k_2 + \frac{1}{2}\frac{\alpha_2^2}{\omega_{\mathrm{m}} - k_2}$$

Relative detuning

$$D' = \left|\frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)}\right|$$

Rabi Oscillations With Multiple Potentials

Consider $k_1 = \omega_{\mathrm{m}}$

$$D' = \left| \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

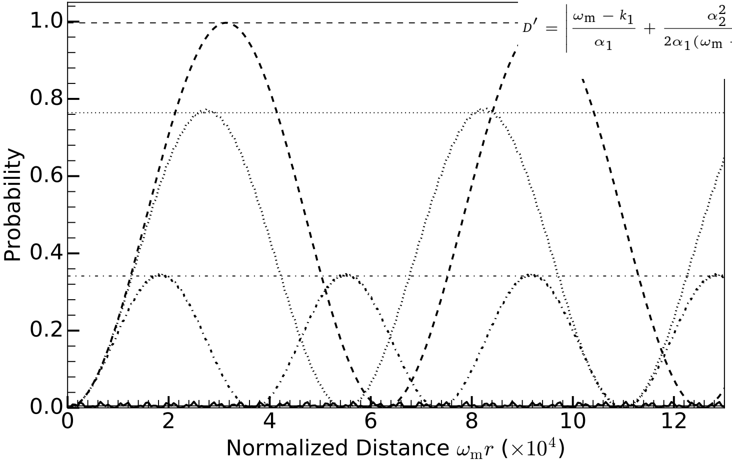
Amplitude reduces from 1 to 1/2 if

$$D' = 1.$$

Two driving frequencies k_1, and k_2, with amplitude α_1, and α_2
Destruction effect: $k_1 = \omega_{\mathrm{m}}$,

$$|\alpha_2| \gg \sqrt{2\omega_{\mathrm{m}}|\alpha_1(k_2 - \omega_{\mathrm{m}})|} \equiv \alpha_{2,\mathrm{C}}$$

Rabi Oscillations With Multiple Potentials



$$D' = \left| \frac{\omega_{\mathrm{m}} - k_1}{\alpha_1} + \frac{\alpha_2^2}{2\alpha_1(\omega_{\mathrm{m}} - k_2)} \right|$$

Grid lines: amplitude predicted using $1/(1 + D'^2)$

α_2, k_1 values

Dashed	dotted	dash-dotted	solid
$10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$	$5.0 \times 10^{-2}\omega_{\mathrm{m}}, 10\omega_{\mathrm{m}}$	$5 \times 10^{-2}\omega_{\mathrm{m}}, 10^{-1}\omega_{\mathrm{m}}$

Single Frequency Matter Potential Revisited

Matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Consider the resonance mode ($k = \omega_{\mathrm{m}}$) and one other mode

$$\widetilde{\mathbf{H}} \sim -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\left(\begin{pmatrix} 0 & B_1 e^{ikx} \\ B_1^* e^{-ikx} & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_n e^{inkx} \\ B_n^* e^{-inkx} & 0 \end{pmatrix}\right)$$

Single Frequency Matter Potential Revisited

Matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Consider the resonance mode ($k = \omega_{\mathrm{m}}$) and one other mode

$$\widetilde{\mathbf{H}} \sim -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \frac{1}{2}\left(\begin{pmatrix} 0 & B_1 e^{ikx} \\ B_1^* e^{-ikx} & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_n e^{inkx} \\ B_n^* e^{-inkx} & 0 \end{pmatrix}\right)$$

$$D' = \left|\frac{B_2^2}{2B_n(\omega_{\mathrm{m}} - nk)}\right|$$

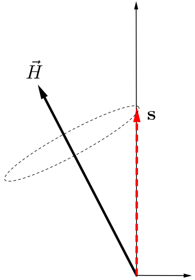
$k = \omega_{\mathrm{m}}$

n	D'
1	-
-1	4.8×10^{-6}
2	2.1×10^{-14}
-2	6.9×10^{-15}

Overview

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
3. Stimulated oscillations: variation in matter profile can cause resonances.
4. Oscillations with two driving fields of different frequencies: large potential to destroy the resonance.



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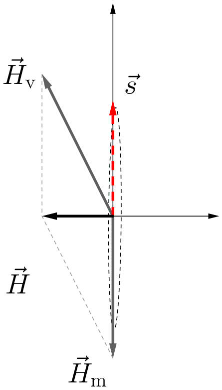

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For matter potential

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

Resonance condition

$$nk = \omega_{\mathrm{m}}$$

SUMMARY

1. Vacuum oscillations: flavor sates are not mass states.
2. MSW resonance: matter potential cancels out the vacuum diagonal elements of the Hamiltonian.
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$$|\alpha_2| \gg \sqrt{2\omega_{\rm m}|\alpha_1(k_2-\omega_{\rm m})|} \equiv \alpha_2$$

BACKUP SLIDES

BACKUP SLIDES

WHY DO NEUTRINOS OSCILLATE?

Equation of Motion

$$i\partial_x |\Psi\rangle = \hat{\mathbf{H}} |\Psi\rangle$$

- Basis: Hamiltonian diagonalized basis/mass basis/propagation basis, $\{|\nu_1\rangle, |\nu_2\rangle\}$.
- $$\mathbf{H} = -\frac{\omega_v}{2}\boldsymbol{\sigma_3}, \qquad \text{where } \omega_v = \frac{\delta m^2}{2E} = \frac{m_2^2 - m_1^2}{2E}.$$
- The system can be solved given initial condition of the amplitudes of the two eigenstates $(\langle\nu_1|\Psi(0)\rangle, \langle\nu_2|\Psi(0)\rangle)^T$,

$$\begin{pmatrix} \langle\nu_1|\Psi(x)\rangle \\ \langle\nu_2|\Psi(x)\rangle \end{pmatrix} = \begin{pmatrix} \langle\nu_1|\Psi(0)\rangle \exp(i\omega_v x/2) \\ \langle\nu_2|\Psi(0)\rangle \exp(-i\omega_v x/2) \end{pmatrix}$$

Why Do Neutrinos Oscillate?

Flavor basis

Neutrino wave function in flavor basis $\{|\nu_\mathrm{e}\rangle, |\nu_\mu\rangle\}$ is related to state in energy basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ through

$$\begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{v} & \sin\theta_\mathrm{v} \\ -\sin\theta_\mathrm{v} & \cos\theta_\mathrm{v} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

θ_v: vacuum mixing angle

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θ_v: vacuum mixing angle

Hamiltonian $\mathbf{H}$

Mass basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\omega_\mathrm{v}}{2}\boldsymbol{\sigma_3}$$

Flavor basis

$$\frac{\omega_\mathrm{v}}{2}\begin{pmatrix} -\cos 2\theta_\mathrm{v} & \sin 2\theta_\mathrm{v} \\ \sin 2\theta_\mathrm{v} & \cos 2\theta_\mathrm{v} \end{pmatrix} = \frac{\omega_\mathrm{v}}{2}\left(-\cos 2\theta_\mathrm{v}\boldsymbol{\sigma_3} + \sin 2\theta_\mathrm{v}\boldsymbol{\sigma_1}\right)$$

NATURE OF NEUTRINO OSCILLATION

Transition Probability

$$P(|\nu_e\rangle \to |\nu_\mu\rangle) = \sin^2(2\theta_v)\sin^2(\omega_v x/2)$$

- $\omega_v = (m_2^2 - m_1^2)/2E$ determines oscillation wavelength.
- Mixing angle θ_v determines flavor oscillation amplitude.

MSW EFFECT

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an example,

Significance of θ_m

Define matter basis (eigenenergy basis) $\{|\nu_L\rangle, |\nu_H\rangle\}$

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} |\nu_L\rangle \\ |\nu_H\rangle \end{pmatrix}$$

In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_m}{2}\boldsymbol{\sigma_3}$$

MSW Effect

$$\begin{pmatrix} |\nu_{\mathrm{e}}\rangle \\ |\nu_{\mu}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}$$

Constant matter profile λ_0 as an examp ,

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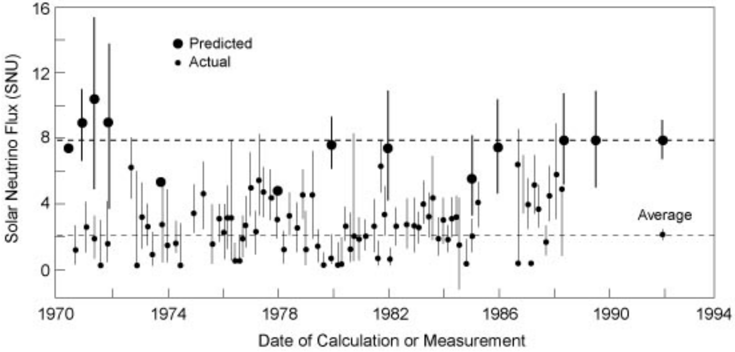
In matter basis

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}$$

Transition Probability

$$P(|\nu_{\mathrm{e}}\rangle \to |\nu_{\mu}\rangle) = \sin^2(2\theta_{\mathrm{m}})\sin^2(\omega_{\mathrm{m}} x)$$

Solar Neutrino Problem



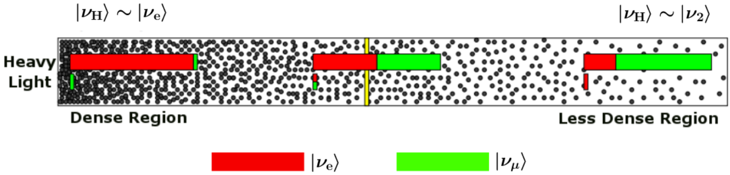
Chlorine detector (Homestake experiment) results and theory predictions.
SNU: 1 event for 10^{36} target atoms per second. Kenneth R. Lang (2010)

MSW Effect and Solar Neutrinos

$$\mathbf{H} = \frac{\lambda(x) - \omega_\mathrm{v}\cos 2\theta_\mathrm{v}}{2}\boldsymbol{\sigma_3} + \frac{\omega_\mathrm{v}\sin 2\theta_\mathrm{v}}{2}\boldsymbol{\sigma_1}$$

$$\begin{pmatrix} |\nu_\mathrm{L}\rangle \\ |\nu_\mathrm{H}\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{m} & -\sin\theta_\mathrm{m} \\ \sin\theta_\mathrm{m} & \cos\theta_\mathrm{m} \end{pmatrix} \begin{pmatrix} |\nu_\mathrm{e}\rangle \\ |\nu_\mu\rangle \end{pmatrix}$$

$$\mathbf{H}_{\text{matter-basis}} = -\frac{\omega_\mathrm{m}}{2}\boldsymbol{\sigma_3}$$



Yellow bar is the resonance point. Red: $|\nu_e\rangle$. Green: $|\nu_\mu\rangle$. Adapted from Smirnov, 2003.

MSW Effect Inverted Hierarchy

Suppose $\omega_{\mathrm{v}} = (m_2^2 - m_1^2)/2E < 0$,

$$\mathbf{H} = -\frac{\omega_{\mathrm{v}}}{2}\begin{pmatrix} -\cos 2\theta_{\mathrm{v}} & \sin 2\theta_{\mathrm{v}} \\ \sin 2\theta_{\mathrm{v}} & \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\downarrow$$

$$\mathbf{H} = \left(\frac{-\omega_{\mathrm{v}}}{2}\cos 2\theta_{\mathrm{v}} + \frac{\lambda(x)}{2}\right)\boldsymbol{\sigma_3} - \frac{\omega_{\mathrm{v}}}{2}\sin 2\theta_{\mathrm{v}}\boldsymbol{\sigma_1}$$

HAMILTONIAN

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x)$$

Basis

Background matter basis (eigen energy basis): Hamiltonian is diagonalized with only background matter profile λ_0,

$$\mathrm{H}_{\mathrm{background}} = -\frac{\omega_{\mathrm{m}}}{2}\boldsymbol{\sigma_3}.$$

Hamiltonian

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\boldsymbol{\sigma_3} - \frac{\delta\lambda(x)}{2}\sin\theta_{\mathrm{m}}\boldsymbol{\sigma_1}.$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1 .$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 .$$

Hamiltonian

Hamiltonian in Background Matter Basis

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \delta\lambda(x)\cos 2\theta_{\mathrm{m}}\right)\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\sigma_1 .$$

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$\mathbf{H} = \frac{1}{2}\left(-\omega_{\mathrm{m}} + \cos 2\theta_{\mathrm{m}} A\cos(kx)\right)\sigma_3 - \frac{\sin 2\theta_{\mathrm{m}}}{2} A\cos(kx)\sigma_1 .$$

Rabi Oscillations

The coupling strength is calculated as

$$\alpha = \langle 1|\mathbf{d} \cdot \mathbf{E}|2\rangle$$

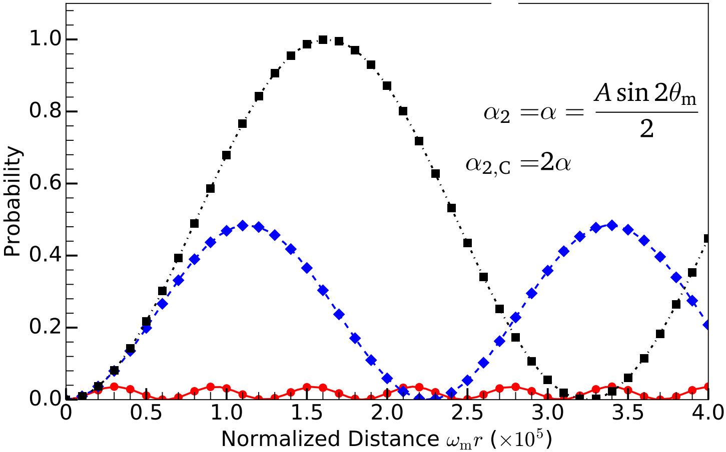
where the electric field is

$$\mathbf{E} = \mathbf{E}_0 \sin(kt).$$

and $\mathbf{d}$ is the dipole moment.

Rabi Formula Works

$$\vec{H} \sim \omega_{\rm m} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \sin(kx) \\ \cos(kx) \\ 0 \end{pmatrix}$$



Lines: Rabi formula

Dots, diamonds, triangles, and squares are for $k = \omega_{\rm m}$, $k = (1 - 2 \times 10^{-5})\omega_{\rm m}$, and $k = (1 - 10^{-4})\omega_{\rm m}$ respectively.

Parameters Used for Vacuum Oscillations

$\theta_{12} = 33.36/180\pi$; $\theta_{13} = 8.66/180\pi$; $\theta_{23} = 40/180 * \pi$; $\delta_{cp} = 0$;
$m_1^2 = 0.01$; $m_2^2 = m_1^2 + 0.000079$; $E = 1$MeV

Single Frequency Matter Potential

Matter potential

$$\lambda(x) = \lambda_0 + A\sin(kx),$$

Hamiltonian in new basis

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 - \frac{\delta\lambda(x)}{2}\sin 2\theta_{\mathrm{m}}\begin{pmatrix} 0 & e^{2i\eta(x)} \\ e^{-2i\eta(x)} & 0 \end{pmatrix} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

Hamiltonian in New Basis

$$\begin{aligned} h &\equiv -\frac{\delta\lambda(x)}{2}e^{2i\eta(x)} \\ &= \frac{i}{4}\left[\exp\left(ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right. \\ &\quad \left. - \exp\left(-ikx + i\cos 2\theta_{\mathrm{m}}\frac{A}{k}\cos(kx)\right)\right] \end{aligned}$$

Single Frequency Matter Potential

Off-diagonal Term in Our System

$$\widetilde{\mathbf{H}} = -\frac{\omega_{\mathrm{m}}}{2}\sigma_3 + \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$$

$$\begin{aligned} h \propto \Bigg[&\exp\left(ikx + \boxed{i\cos 2\theta_{\mathrm{m}} \frac{A}{k}\cos(kx)} \right) \\ &- \exp\left(-ikx + \boxed{i\cos 2\theta_{\mathrm{m}} \frac{A}{k}\cos(kx)} \right) \Bigg] \end{aligned}$$

Jacobi-Anger expansion (Kneller et al, 2013)

$$e^{i\beta\cos(kx)} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta) e^{inkx},$$

where $J_n(\beta)$ are Bessel's functions of the first kind.

Single Frequency Matter Profile

Why Does It Work?

$$J_n(n\,\mathrm{sech}\,\alpha) \sim \frac{e^{-n(\alpha - \tanh\alpha)}}{\sqrt{2\pi n \tanh\alpha}}, \quad \text{for large } n$$

$\Rightarrow$

$$\Gamma \propto \hat{B}_n \propto \frac{e^{-n(\alpha - \tanh\alpha)}}{\sqrt{2\pi n \tanh\alpha}}$$

Small perturbation $\Rightarrow$ Small $\hat{A}$ $\Rightarrow$ Large α $\Rightarrow$ Drops fast at large n.

Single Frequency Matter Profile

Transition Probability

$$P^{(n)}_{\mathrm{L}\to\mathrm{H}} = \frac{\left|\hat{B}_n\right|^2}{\left|\hat{B}_n\right|^2 + (n\hat{k}-1)^2} \sin^2\left(\frac{q^{(n)}}{2}x\right),$$

where

$$q^{(n)} = \sqrt{\left|\Gamma^{(n)}\right|^2 + (n\hat{k}-1)^2}, \quad \text{frequency of oscillations}$$

$$\Gamma^{(n)} = \left|\hat{B}_n\right|, \quad \text{width of resonance } (n\hat{k} \text{ as parameter})$$

Resonance conditions

$$\hat{k} \sim \frac{1}{n}$$

TWO-FREQUENCY MATTER PROFILE

Matter Profile

$$\lambda(x) = \lambda_0 + \delta\lambda(x), \quad \delta\lambda(x) = A_1 \sin(k_1 x) + A_2 \sin(k_2 x).$$

TWO-FREQUENCY MATTER P

$$\hat{h} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\hat{B}_n e^{i(n\hat{k}-1)\hat{x}},$$

Hamiltonian Off-diagonal Element

Apply Jacobi-Anger expansion,

$$\hat{h} = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \frac{1}{2}\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)e^{i(n_1\hat{k}_1+n_2\hat{k}_2-1)\hat{x}},$$

where

$$\begin{aligned}&\hat{B}_{n_1,n_2}(\hat{k}_1,\hat{k}_2)\\ =&-(-i)^{n_1+n_2}(n_1\hat{k}_1+n_2\hat{k}_2)J_{n_1}\left(\frac{\hat{A}_1\cos 2\theta_{\mathrm{m}}}{\hat{k}_1}\right)J_{n_2}\left(\frac{\hat{A}_2\cos 2\theta_{\mathrm{m}}}{\hat{k}_2}\right)\end{aligned}$$

Which terms are important?

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = \omega_{\mathrm{m}}$$

n	D	D'_1	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	0	-	3.2×10^5
-1	10^5	4.8×10^{-6}	3.1
2	1.1×10^9	2.1×10^{-14}	6.3
-2	3.4×10^9	6.9×10^{-15}	2.1

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$$k_1 = (1 - 2\times 10^{-5})\omega_{\mathrm{m}}$$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	1	-	2.2×10^5
-1	10^5	1	3.1
2	1.1×10^9	1	6.3
-2	3.4×10^9	1	2.1

Single Frequency Matter Profile Revisited

Matter profile

$$\lambda(x) = \lambda_0 + A\cos(kx),$$

$k_1 = (1 - 10^{-4})\omega_{\mathrm{m}}$

n	D	D_1'	$2\pi\omega_{\mathrm{m}}/\Omega_n$
1	5.2	-	6.2×10^4
-1	10^5	5.2	3.1
2	1.1×10^9	5.2	6.3
-2	3.4×10^9	5.2	2.1

Castle Wall Matter Profile

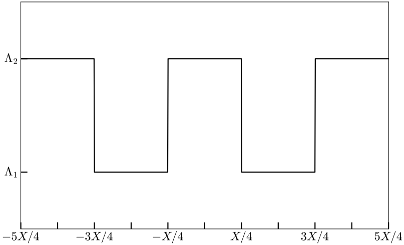
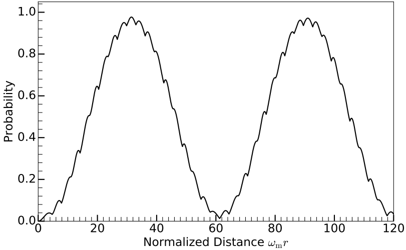


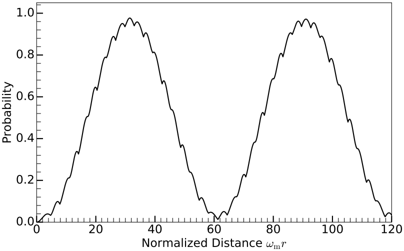
Figure: Castle wall matter profile



Castle Wall Matter Profile

Table: Relative detuning of each frequency.

$\{n_1, n_2\}$	D	$D'_{\{1,0\}}$
$\{1, 0\}$	0	-
$\{-1, 0\}$	48	1.0×10^{-2}
$\{0, 1\}$	1.5×10^{2}	1.1×10^{-3}
$\{2, 0\}$	2.4×10^{2}	2.0×10^{-4}



Two-frequency Matter Profile

Resonance Lines

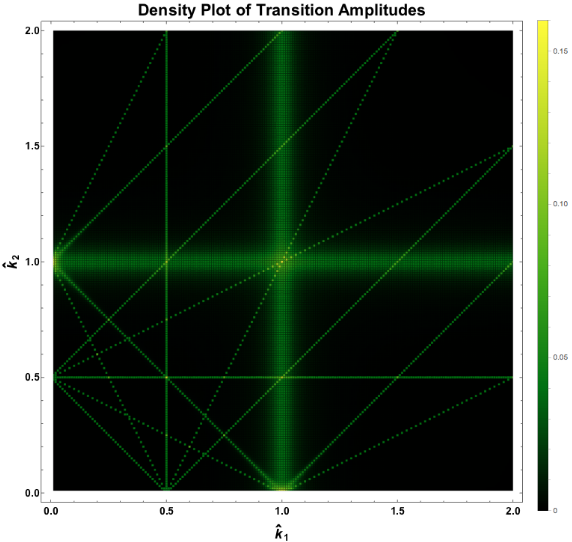
There are still resonances, i.e., (almost) zero phases, on lines

$$n_{1,0}\hat{k}_1 + n_{2,0}\hat{k}_2 - 1 = 0$$

in $\{\hat{k}_1, \hat{k}_2\}$ plane. $\Rightarrow$ Resonance width for each point on resonance lines.

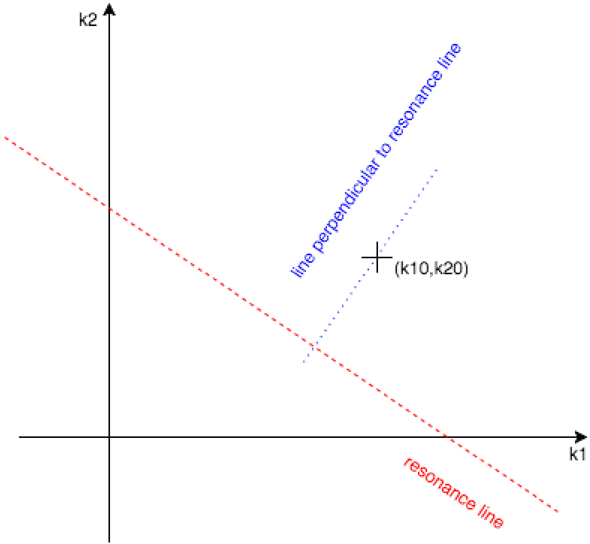
TWO-FREQUENCY MATTER PF

$$\hat{h} = \sum_{n_1} \sum_{n_2} \frac{1}{2} \hat{B}_{n_1, n_2}(\hat{k}_1, \hat{k}_2) e^{i(n_1 \hat{k}_1 + n_2 \hat{k}_2 - 1)\hat{x}},$$



Density plot of transition amplitudes calculated using only one term out of the whole summation in Hamiltonian. $n_1, n_2 \in [-2, 2]$

Two-frequency Matter Profile



Resonance line, distance to resonance, and width

Two-frequency Matter Profile

Width

$$\Gamma_2 = \frac{\hat{B}_{n_1,n_2}(\hat{k}_{1,\text{intercept}}, \hat{k}_{2,\text{intercept}})}{\sqrt{n_1^2 + n_2^2}}.$$

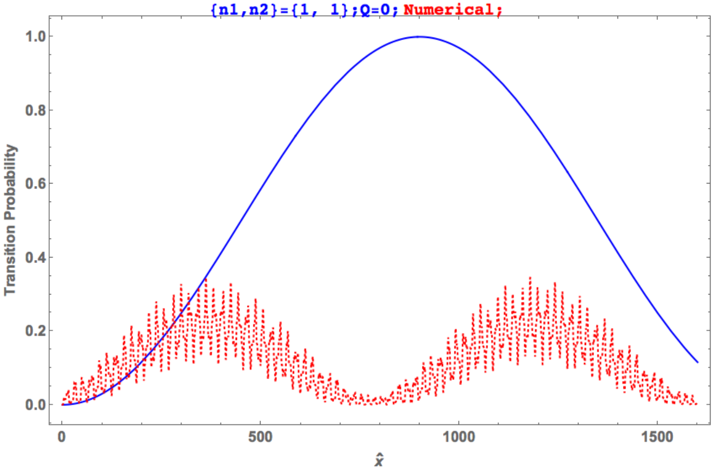
Distance to Resonance Line

$$d = \frac{|n_1\hat{k}_{10} + n_2\hat{k}_{20} - 1|}{\sqrt{n_1^2 + n_2^2}}.$$

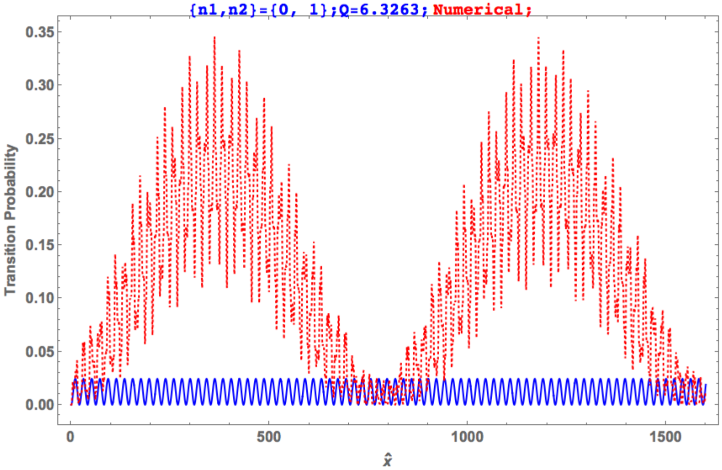
Distance to Resonance Width Ratio

$$Q_2 = \frac{d}{\Gamma_2}.$$

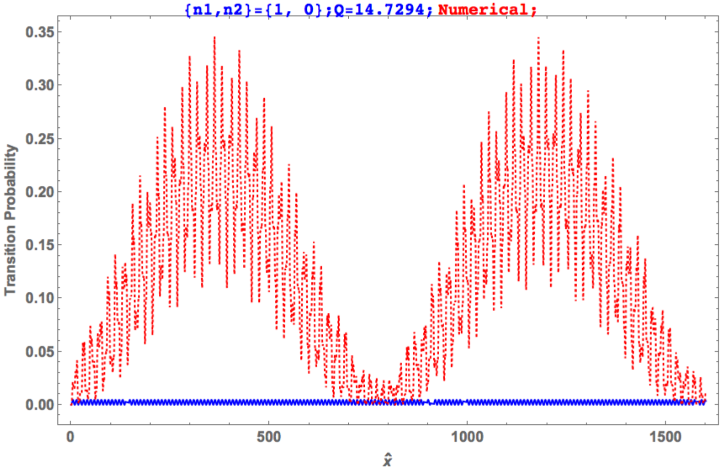
Two-frequency Matter Profile



TWO-FREQUENCY MATTER PROFILE



TWO-FREQUENCY MATTER PROFILE



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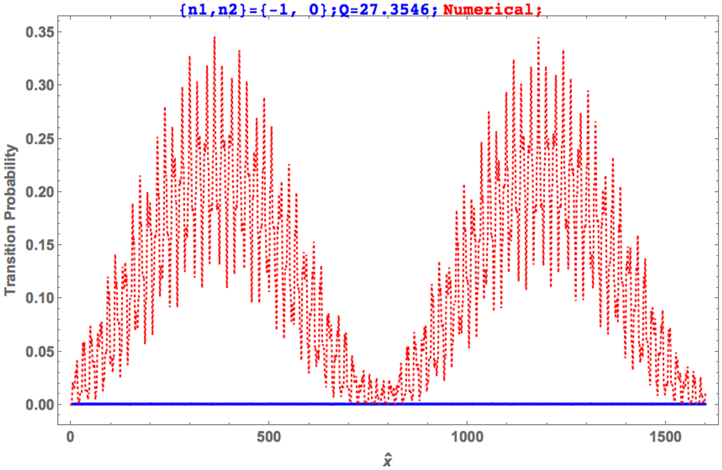
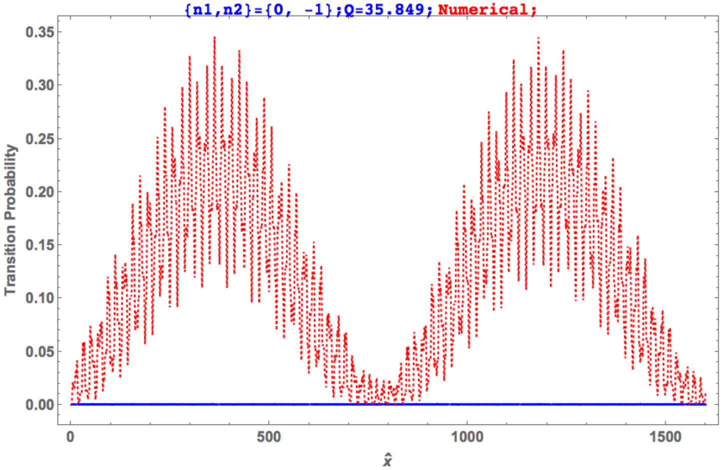
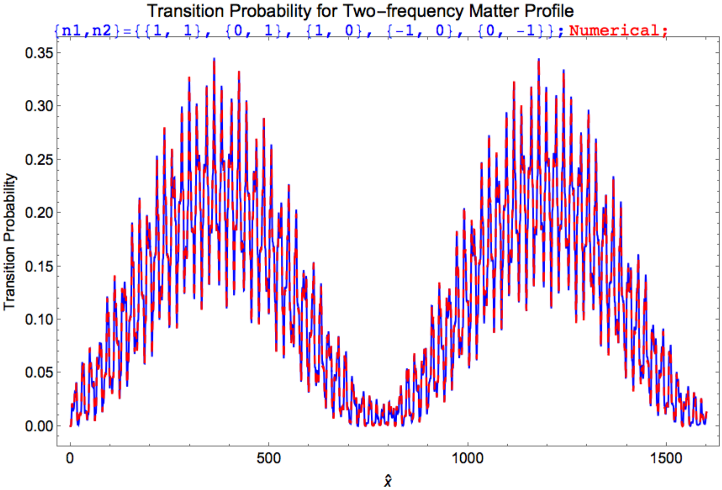


Two-frequency Matter Profile



Two-frequency Matter Profile



Bessel's Function

$$J_n(\beta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{\beta}{2}\right)^{2m+n}$$

REFERENCES I