Neutrino Flavor Conversions in Dense Media

by

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Abstract

One of the interesting and important problems in astrophysics is the mechanism of core-collapse supernova explosions. Many numerical simulations have shown that the explosion shock would stall. Different proposals have been made to explain the core-collapse supernovae, among which the neutrino mechanism is promising and most researched one. To explore the mechanism, prediction of the neutrino flavors in core-collapse supernovae is crucial. Neutrino flavor conversions are altered by the matter, neutrinos themselves, as well as other factors such as the geometries of the neutrino emissions. The complexity of the problem requires breaking it down into investigations of each simple yet specific situation.

Neutrinos propagating through a matter background experience a potential which changes the flavor conversions. One of the important mechanisms is the Mikheyev–Smirnov–Wolfenstein effect. However, much more complicated density profiles of matter, such as periodic density profiles, may lead to large flavor conversion, which is dubbed as stimulated oscillations by J. Kneller et al. Mathematics of such large conversion has been established but without clear pictures. For the two-flavor scenario, neutrino oscillations is a two-level quantum system, and it reminds us of many two-level quantum problems that have been solved in the past. We draw analogies between neutrinos passing through matter and Rabi oscillations in optics, which allows us to calculate resonance conditions and flavor survival probability easily.

As for neutrinos flavors with high number densities, nonlinear interactions come into play since neutrino forward scattering provides another potential that is related to the flavor of the neutrinos themselves. Nonlinearity makes the flavor conversion hard to predict by intuition. The treatment is linearizing the equation of motion and identifying instabilities. One of the tricks in the realm is to utilize the dispersion relation. In principle, dispersion relations tell us how waves propagate for different wave numbers and frequencies. However, the neutrino problem is much more complicated. Situations that are inconsistent with the dispersion relation approach are identified.

Finally, forward scattering of supernova neutrinos are not the only thing that happens. During propagation around a supernova, neutrinos may be scattered in every direction, which forms a neutrino halo. The halo couples the neutrinos nonlocally, which then becomes a nonlocal boundary value problem. One of the solutions is the relaxation method. Starting from some state of neutrinos and relaxing the system into equilibrium has proven to be a working algorithm. A numerical algorithm is developed and neutrino line model with back scattering is investigated.

Chapter 1

Introduction

1.1 The Little Neutral One

The neutrino has been one of the most interesting particles that has ever been discovered. Its fanscinating history started with the observation of beta decay, i.e., the emission of electrons in nuclear decays, such as

$$^{14}_{6}\mathrm{C} \to ^{17}_{7}\mathrm{N} + \mathrm{e}^{-} + \bar{\nu}_{\mathrm{e}}.$$

The fact that the electron energy spectrum in the beta decay process is continuous indicates the existence of a third product other than $^{17}_{7}\mathrm{N}$ and e^{-}. [Some History here!!!!!!!!!!!!!!!!!!!!!!!!!!!] It was then proven to be an anti-neutrino. In nuclear beta decays, the charged current weak interaction converts a down quark in the neutron to an up quark while releasing an electron and an anti-electron neutrino,

$$n \to p + e^{-} + \bar{\nu}_{e}. \tag{1.1}$$

More generally, the positron/electron emission and capture processes are all neutrino-related nuclear reactions which are listed in Table 1.1. There are three different flavors of neutrinos, namely the electron flavor, the muon flavor, and the tau flavor as shown in Table 1.2. The first direct detection of neutrinos was done by Clyde Cowan and Frederick Reines in 1956 [1] who used nuclear reactor neutrinos as the source of the experiment.

Reaction Type	Process	Mediator(s)
Electron emission	${}^A_ZX \to {}^{A}_{Z+1}X' + e^- + \bar{\nu}_e$	$W^\pm$
Positron emission	${}^A_ZX \to {}^{A}_{Z-1}X' + e^+ + \nu_e$	$W^\pm$
Electron capture	${}^A_ZX + e^- \to {}^{A}_{Z-1}X' + \nu_e$	$W^\pm$
Positron capture	${}^A_ZX + e^+ \to {}^{A}_{Z+1}X' + \bar{\nu}_e$	$W^\pm$
$e^\pm$ annihilation	$e^- + e^+ \to \nu + \bar{\nu}$	$W^\pm$, Z
Bremsstrahlung	$X + X' \to X + X' + \nu + \bar{\nu}$	Z
$\nu(\bar{\nu})$ capture	${}^A_ZX + \overset{(-)}{\nu}_e \to {}^{A}_{Z\mp1}X' + e^\pm$	$W^\pm$
$e^\pm\nu$ scattering	$e^- + \overset{(-)}{\nu} \to e^- + \overset{(-)}{\nu}$	$W^\pm$, Z
Nucleon scattering	${}^A_ZX + \overset{(-)}{\nu} \to {}^A_ZX + \overset{(-)}{\nu}$	Z

Table 1.1: Neutrino related nuclear and leptonic reactions.

[0.5ex] Electric Charge	0
Spin	1/2
Mass	< 2 eV
Interactions	Weak, Gravitation
Flavors	ν_e, ν_μ, ν_τ
Chirality	Left
Hypercharge	-1

Table 1.2: The physical properties of the neutrino [9].

1.2 Stellar Neutrinos

Besides man-made sources, neutrinos are also produced in many astrophysical environments.For example, numerous nuclear reactions occur in stellar cores which pro-

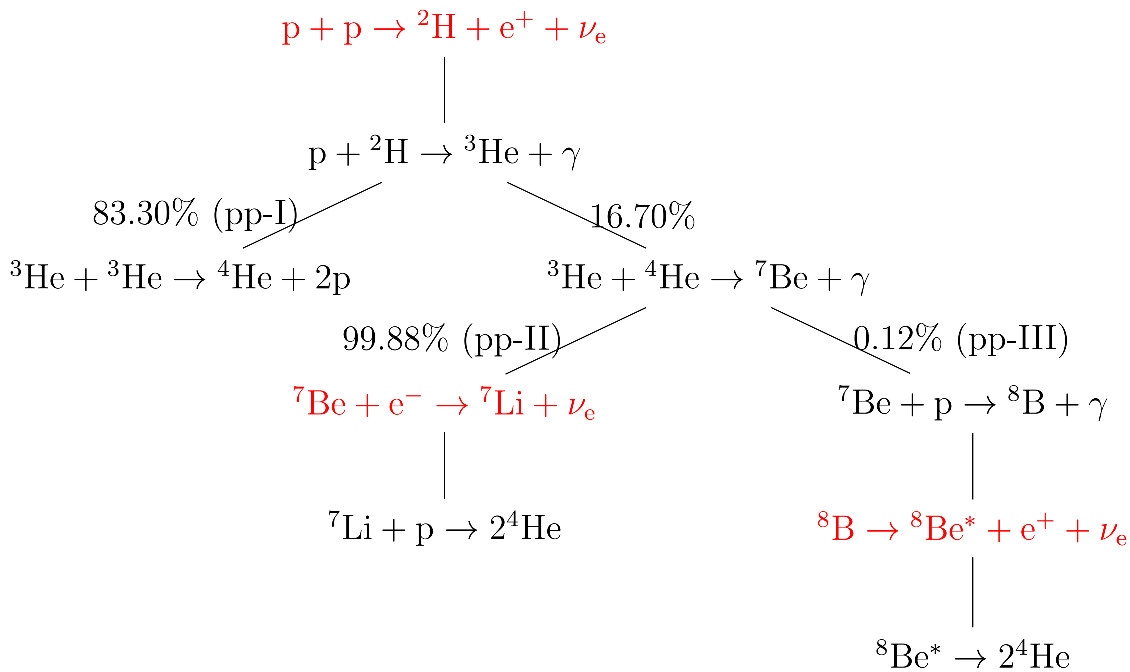


Figure 1.1: The pp chain reactions with the corresponding branching ratios. The branching ratios are taken from Ref. [5].

duce luminous neutrino fluxes. The most important nuclear reactions in the Sun are the pp chain reactions which are shown in Fig. 1.1. In order to calculate the neutrino spectrum we need the neutrino production rate in each reaction and the branching ratios. Solar neutrinos are mostly produced in the pp reaction, Be electron capture and B decay which are labeled in red in Fig 1.1:

$$\begin{aligned}
&\mathrm{p} + \mathrm{p} \to {}^2\mathrm{H} + \mathrm{e}^+ + \nu_\mathrm{e} && \leq 0.422\mathrm{MeV}, \\
&{}^7\mathrm{Be} + \mathrm{e}^- \to {}^7\mathrm{Li} + \nu_\mathrm{e} && 0.862\mathrm{MeV} \text{ for } 90\%, \\
& && 0.384\mathrm{MeV} \text{ for } 10\%, \\
&{}^8\mathrm{B} \to {}^8\mathrm{Be}^* + \mathrm{e}^+ + \nu_\mathrm{e} && \leq 15\mathrm{MeV}.
\end{aligned}$$

Even without the knowledge of the detailed reactions, the conservation of the electric charge and the electron lepton number will lead to the overall neutrino production formula

$$4\mathrm{p} + 2\mathrm{e}^- \rightarrow {}^4\mathrm{He} + 2\nu_\mathrm{e}. \tag{1.2}$$

It is important to notice that two neutrinos are emitted for each α particle, i.e., ^{4}He, produced in the Sun. Using this simple relation, we can estimate the neutrino number flux emitted by the Sun. The energy released during the production of each α particle is the difference between the initial and final rest masses of the particles,

$$Q = 4m_p + 2m_e - m_\alpha = 26.7\mathrm{MeV}, \tag{1.3}$$

where the mass of the neutrinos are neglected. On average, each neutrino carries away an energy of 0.2MeV and the rest of the energy is in the form of thermal energy $Q_\gamma = 26.3\mathrm{MeV}$ [7]. Since two neutrinos are emitted for the production of thermal energy Q_γ, the number flux of the solar neutrinos near the Earth is approximately

$$\Phi_\nu = \frac{2S_0}{Q_\gamma} \approx 6 \times 10^{10}\mathrm{cm}^{-2}\mathrm{s}^{-1}, \tag{1.4}$$

where the solar constant S_0 is the energy flux of solar photons on the top of the Eath atmosphere.

As the detection of neutrinos became feasible, Ray Davis and John Bahcall et al worked out the solar neutrino flux and led the Homestake experiment to measure the solar neutrinos. The results revealed that the neutrino flux detected was less than what was predicted by the standard solar model [3]. This is the solar neutrino problem. It is now known that the solution to the problem is related to the neutrino. The electron neutrinos produced in the solar core transform to other flavors while they travel to the Earth. This phonomenon is referred known as the flavor transformation of the neutrino, or neutrino oscillations. The theory of neutrino oscillations was first proposed by Pontecorvo in 1968 [2]. The field of neutrino oscillations has grown significantly into a broad field in physics since then.

1.3 Supernova Neutrinos

Another astronomical source of neutrinos is the core-collapse supernova explosion. Massive stars with masses larger than 6—8 solar masses are very bright. However, violent delights have violent ends. When the core of a massive star runs out of nuclear fuel, it collapses under it's own gravity. During the collapse, the inner core is compressed to almost nuclear density, which has a stiff equation of state. The materials falling onto the highly compressed inner core are bounced outward which generates a shock wave and may lead to an explosion. However, supernova simulations to date show that the shock wave itself is not always energetic enough to produce the explosion [8]. In most cases, it stalls and becomes a standing accretion shock wave. To revive the shock, more energy has to be deposited behind the shock. A possible solution is to introduce reheating of the shock by neutrinos [8]. In fact 99% percent the energy released in a core-collapse supernova is carried away by neutrinos. In order to implement the neutrino-driven mechanism in computer simulations of supernovae, the flux and flavor content of the neutrinos have to be known everywhere behind the shock. Thus neutrino oscillations in dense matter become a key to the supernova explosion problem.

The average energy of the neutrinos $\langle E \rangle$ emitted during a supernova explosion is of the order of 10MeV [10], and the neutrino luminosity at the early epoch is approximately $10^{52}\text{ergs} \cdot \text{s}^{-1}$[CHANGE CITATIONSSSSSSSSS!!!!!!!]. Therefore, the number density of the neutrinos at the radius R is

$$n \sim 10^{18}\text{cm}^{-3} \left(\frac{100\text{km}}{R} \right)^2 \left(\frac{10\text{MeV}}{\langle E \rangle} \right).$$

It turns out that the ambient dense neutrino medium has a significant impact on neutrino oscillations, which has been intensely investigated in the last decade [CITATION NEEDED].

Observation-wise, the neutrino signals from a galactic supernova can reveal a great amount of information about the physical conditions inside the supernova. In fact, the detection of supernova neutrinos is on the task list of the Deep Underground Neutrino Experiment (DUNE) [11].

1.4 Organization of the Disseration

The rest of the disseration is organized as follows. In Chapter 2, I will review neutrino oscillations in vacuum and in environments with smooth matter density profiles. In Chapter **??**, I will discuss my work on neutrino oscillations in oscillatory matter profiles, which can be decomposed into Fourier modes and interpreted as a superposition of Rabi oscillations. In Chapter **??**, I will first review how neutrino self-interactions can cause a dense neutrino medium to oscillate collectively. Then I will discuss my study on the dispersion relations of the collective modes of neutrino oscillations. I will also discuss a premilinary work on neutrino oscillations when both forward and backward neutrino fluxes are present. In Chapter **??**, I will summarize my work and discuss possible future directions of the field.

Chapter 2

General Principles of Neutrino Oscillations

Because the flavor eigenstates of the neutrino are not the same as its propagation eigenstates, it can change flavor while it propagates. In this chapter, I will use the two-flavor scheme to explain neutrino oscillations in some simple scenarios[1]. I will first discuss neutrino oscillations in vacuum. After explaining the general principles of neutrino oscillations in matter, I will show how the solar neutrino problem can be explained by neutrino oscillations. Finally, I will demonstrate the flavor isospin picture which can be used to visualize neutrino oscillations.

[1]In most physical problems, the two-flavor scheme is a good approximation to the phenomena of neutrino oscillations. The mass splits between the three mass eigenstates are so different that the corresponding oscillations occur on very different length scales. On the right length scale, the two-flavor scheme captures the prominent features of the neutrino oscillations of the corresponding mass split.

2.1 Vacuum Oscillations

Before working out the math, I can estimate the frequency of the oscillations of the neutrino between its flavors. In the natural units, frequency has the same dimension as energy (see Appendix **??**). Consider an electron neutrino with momentum p which is a superposition of the two mass eigenstates $|\nu_i\rangle$ $(i = 1, 2)$ with masses m_i, respectively. Since the neutrino masses are small, I can Taylor expand the energy of each mass eigenstate in terms of the corresponding mass:

$$\begin{aligned} E_i &= \sqrt{m_i^2 + p^2} \\ &= p\sqrt{\frac{m_i^2}{p^2} + 1} \\ &\approx p + \frac{1}{2}\frac{m_i^2}{p}. \end{aligned} \tag{2.1}$$

The first term in the above equation produces a global phase to the flavor wave function of the neutrino which does not affect neutrino flavor oscillations. The characteristic energy scale in the problem is the difference between the energies of the two mass eigenstates,

$$\omega_{\mathrm{v}} = \frac{m_2^2 - m_1^2}{2E} = \frac{\delta m^2}{2E}, \tag{2.2}$$

which turns out to be the vacuum oscillation frequency. Here $E = p$ is approximately the energy of the neutrino.

To work out the exact solution, I will utilize the Schrödinger equation. The wave function in flavor basis $\Psi^{(\mathrm{f})}$ is related to the wave function in mass basis $\Psi^{(\mathrm{v})}$ through a unitary mixing matrix U,

$$\Psi^{(\mathrm{f})} = \mathsf{U}\Psi^{(\mathrm{v})}, \tag{2.3}$$

where the upper indices $^{(\mathrm{v})}$ and $^{(\mathrm{f})}$ are used to denote the corresponding bases. The mixing matrix can be expressed using the vacuum mixing angle θ_{v},

$$\mathsf{U} = \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix}. \tag{2.4}$$

In vacuum mass basis, the neutrino has a free propagation Hamiltonian

$$\mathsf{H}^{(\mathrm{v})} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{2.5}$$

To the first order, the Hamiltonian becomes

$$\begin{aligned} \mathsf{H}^{(\mathrm{v})} &\approx \frac{1}{2E}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} + E\mathsf{I} \\ &= \frac{1}{4E}\begin{pmatrix} -\delta m^2 & 0 \\ 0 & \delta m^2 \end{pmatrix} + \left(\frac{m_2^2 + m_1^2}{4E} + E\right)\mathsf{I}. \end{aligned} \tag{2.6}$$

Because a multiple of the identity matrix I only gives an global phase to the neutrino flavor wave function, I will neglect it from now on, and the vacuum Hamiltonian simplifies to

$$\mathsf{H}^{(\mathrm{v})} = \frac{\delta m^2}{4E}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{\delta m^2}{4E}\sigma_3 = -\frac{\omega_{\mathrm{v}}}{2}\sigma_3, \tag{2.7}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

are the three Pauli matrices. The Schrödinger equation has the following simple solution in mass basis:

$$\Psi^{(\mathrm{v})}(t) = \begin{pmatrix} c_1(0)e^{i\omega_{\mathrm{v}}t/2} \\ c_2(0)e^{-i\omega_{\mathrm{v}}t/2} \end{pmatrix}. \tag{2.9}$$

Using Eqn. 2.3, I obtain the wave function in flavor basis,

$$\Psi^{(\mathrm{f})}(t) = \mathsf{U}\Psi^{(\mathrm{v})}(t) \tag{2.10}$$

$$= \begin{pmatrix} \cos\theta_{\mathrm{v}} & \sin\theta_{\mathrm{v}} \\ -\sin\theta_{\mathrm{v}} & \cos\theta_{\mathrm{v}} \end{pmatrix}\begin{pmatrix} c_1(0)e^{i\omega_{\mathrm{v}}t/2} \\ c_2(0)e^{-i\omega_{\mathrm{v}}t/2} \end{pmatrix}. \tag{2.11}$$

Alternatively, I can also determine the Hamiltonian in flavor basis first, which is

$$\mathsf{H}^{(\mathrm{f})} = \mathsf{U}\mathsf{H}^{(\mathrm{v})}\mathsf{U}^{\dagger} = -\frac{\omega_{\mathrm{v}}}{2}\cos 2\theta_{\mathrm{v}}\sigma_3 + \frac{\omega_{\mathrm{v}}}{2}\sin 2\theta_{\mathrm{v}}\sigma_1. \tag{2.12}$$

By solving the Schrödinger equation in flavor basis, I will obtain the same wave function as in Eqn. 2.11.

The probability for a neutrino emitted in the electron flavor at time $t = 0$ to be detected as the electron flavor at a later time t is

$$P(t) = 1 - \sin^2(2\theta_{\mathrm{v}})\sin^2\left(\frac{\omega_{\mathrm{v}} t}{2}\right). \tag{2.13}$$

Since the neutrino travels with approximately the speed of light, the electron neutrino survival probability at a distance r from the source is

$$P(r) = 1 - \sin^2(2\theta_{\mathrm{v}})\sin^2\left(\frac{\omega_{\mathrm{v}}}{2} r\right). \tag{2.14}$$

I plot the above result in Fig. 2.1 which clearly shows the oscillatory behavior. The oscillation length is determined by the characteristic energy scale ω_{v}, which confirms our qualitative method result in Eqn. 2.2. The oscillation amplitude is determined by $\sin^2(2\theta_{\mathrm{v}})$.

In nature, there are three neutrino flavors and, correspondingly, three neutrino mass eigenstates, which are shown in Fig. 2.2. Because there are two different characteristic energy scales, $\omega_{\mathrm{v},21} = \delta m_{21}^2/2E$ and $\omega_{\mathrm{v},32} = \delta m_{31}^2/2E$, two oscillation periods should occur, as shown in Fig. 2.3. The fast oscillations are determined by the larger energy scale, $\omega_{\mathrm{v},32}$, while the slow oscillations are determined by the smaller one $\omega_{\mathrm{v},21}$. For the inverted neutrino mass hierarchy (with $m_3 < m_1 < m_2$), the oscillation frequencies are the same as in the normal mass hierarchy (with $m_3 > m_2 > m_1$) since they have the same characteristic energy scales. However, they will develop different phases during oscillations.

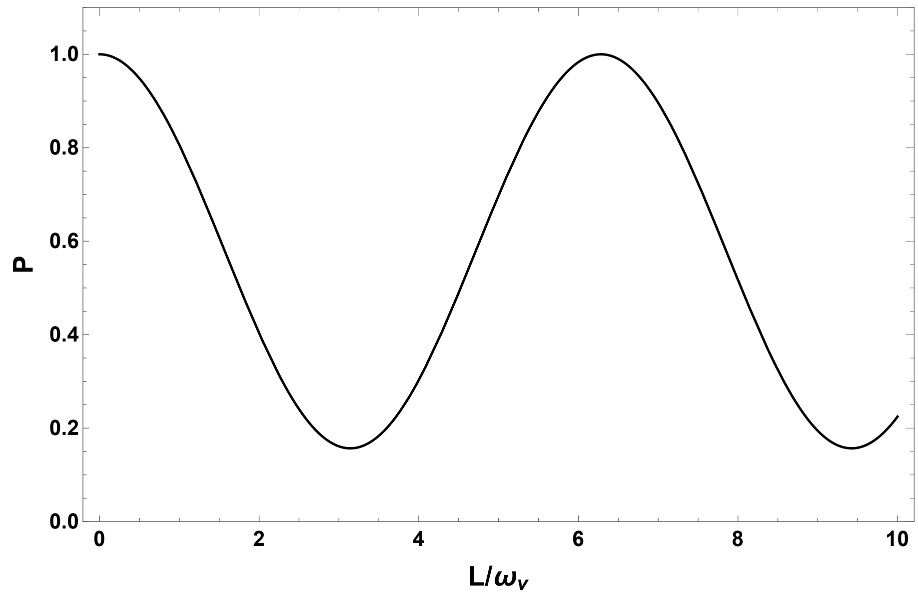
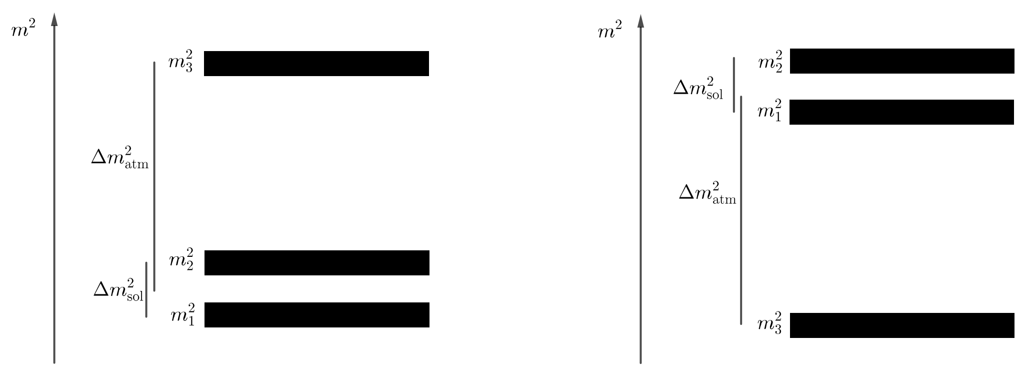


Figure 2.1: The electron flavor neutrino survival probability in vacuum oscillations as a function of distance r which is measured in terms of vacuum oscillation frequency ω_{v}. The mixing angle θ_{v} is given by $\sin^2\theta_{\mathrm{v}} = 0.30 \approx \sin^2\theta_{12}$.



(a) Normal hierarchy: the third mass is heavier than the first two.

(b) Inverted hierarchy: the third mass is smaller than the first two.

Figure 2.2: The order of the three neutrino masses. The difference between the first two masses is responsible for solar neutrino oscillations, and the difference between the third mass and the first two is responsible for atmospheric neutrino oscillations.

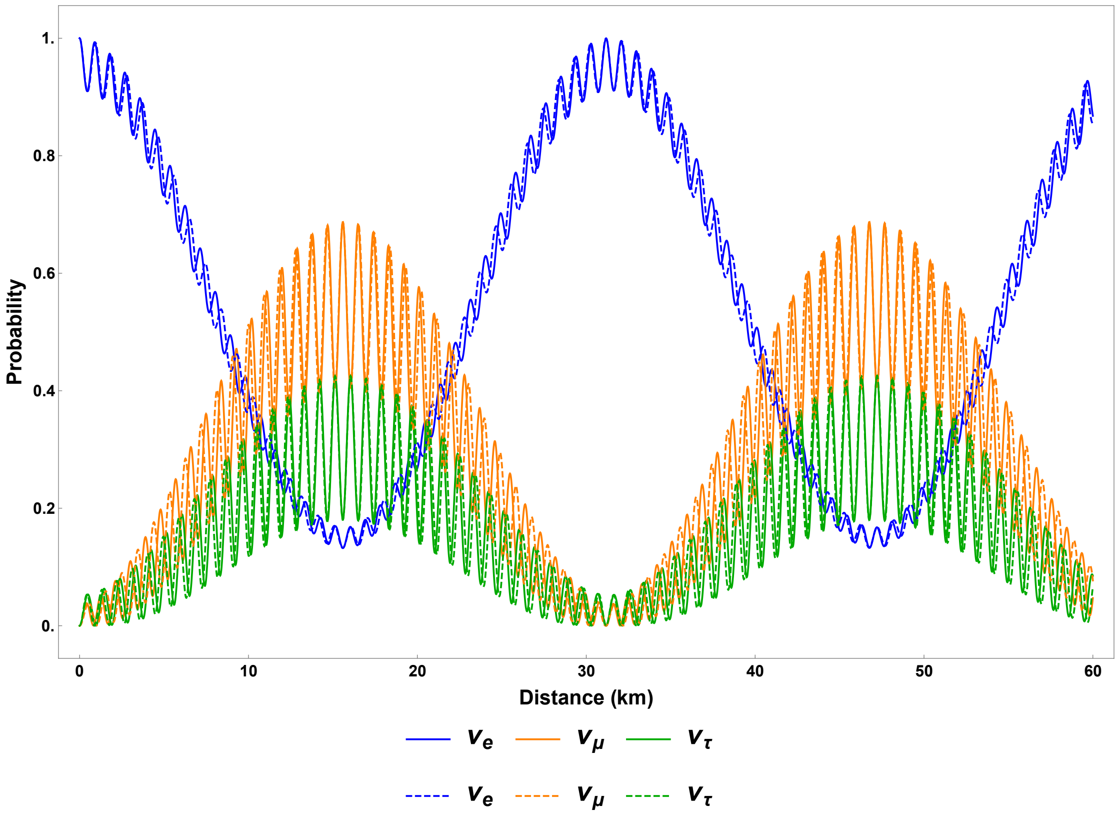


Figure 2.3: The probabilities for a 1MeV neutrino, which is in the electron flavor initially, in different flavors as functions of the distance in vacuum. The solid lines represent the normal hierarchy and the dashed lines represent the inverted hierarchy. The mixing angles are $\sin^2\theta_{12} = 0.30$, $\sin^2\theta_{13} = 0.023$, and $\sin^2\theta_{23} = 0.41$, respectively, and the mass differences are $\delta m^2_{21} = 7.9 \times 10^{-5}\text{eV}^2$ and $\delta m^2_{23} = 2.7 \times 10^{-3}\text{eV}^2$.

2.2 Neutrino Oscillations in Matter

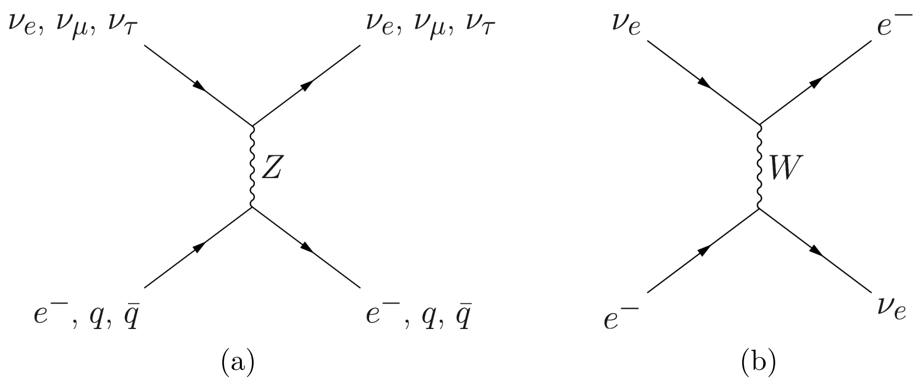


Figure 2.4: (a) The neutral current weak interaction does not distinguish between neutrino flavors and has no impact on neutrino oscillations. (b) The electron flavor neutrino acquires a unique refractive index contribution from the charged current weak interaction with ambient electrons.

In many astrophysical environments, such as stars and core-collapse supernovae, neutrinos are mostly produced at the center of the object and propagate through the dense matter envelope. Although this matter envelope is essentially transparent to neutrinos, the refractive indices of the neutrinos in matter are different than those in vacuum.[2] Because the neutral current weak interaction does not distinguish between neutrino flavors (see Fig. 2.4a), it has no impact on neutrino oscillations, and I will ignore it from now on. Meanwhile, the electrons and positrons in matter will cause electron flavor neutrinos to have refractive indices different than other neutrino flavors through the charged current weak interaction (see Fig. 2.4b). This leads to an effective potential

$$\mathsf{V}^{(\mathrm{f})} = \frac{\sqrt{2} G_{\mathrm{F}} n_{\mathrm{e}}}{2} \sigma_3, \tag{2.15}$$

where G_{F} is Fermi constant and n_{e} is net number density of the electron. As usual,

[2]The word "matter" in this disseration refers to ordinary matter composed of electrons, positrons, nucleons and nuclei. We assume that the temperature and density of the environment are not high enough to produce muons and tau particles. We will discuss the effect of dense neutrino medium in Chapter **??**.

I have ignored the trace terms in the above equation.

The Hamiltonian with the matter effect is the combination of Eqn. 2.12 and Eqn. 2.15:

$$\mathsf{H}^{(\mathrm{f})} = \left(\frac{\lambda}{2} - \frac{\omega_{\mathrm{v}}}{2}\cos 2\theta_{\mathrm{v}}\right)\sigma_3 + \frac{\omega_{\mathrm{v}}}{2}\sin 2\theta_{\mathrm{v}}\sigma_1, \tag{2.16}$$

where

$$\lambda = \sqrt{2}G_{\mathrm{F}}n_{\mathrm{e}}. \tag{2.17}$$

Due to the off-diagonal terms in $\mathsf{H}^{(\mathrm{f})}$, the neutrino will experience oscillations in flavor. A resonance with the maximum flavor mixing occurs when the diagonal terms of $\mathsf{H}^{(\mathrm{f})}$ vanish,

$$\frac{\lambda}{2} - \frac{\omega}{2}\cos 2\theta_{\mathrm{v}} = 0, \tag{2.18}$$

which gives the Mikheyev–Smirnov–Wolfenstein (MSW) resonance condition.

2.3 Neutrino Oscillations in the Sun

The neutrinos produced in the solar core experience decreasing matter density as they travel outward through the Sun. The neutrino propagation eigenstates are different from the flavor states in general [4]. Because the density change inside the Sun is not dramatic, the flavor quantum states of the neutrinos will evolve adiabatically inside the Sun.

The values of the instantaneous eigenstates of the Hamiltonian, known as the heavy and light states, are

$$\varepsilon_{\mathrm{H}} = \frac{\omega_{\mathrm{v}}}{2}\sqrt{\hat{\lambda} + 1 - 2\hat{\lambda}\cos 2\theta_{\mathrm{v}}}, \tag{2.19}$$

$$\varepsilon_{\mathrm{L}} = -\frac{\omega_{\mathrm{v}}}{2}\sqrt{\hat{\lambda} + 1 - 2\hat{\lambda}\cos 2\theta_{\mathrm{v}}}, \tag{2.20}$$

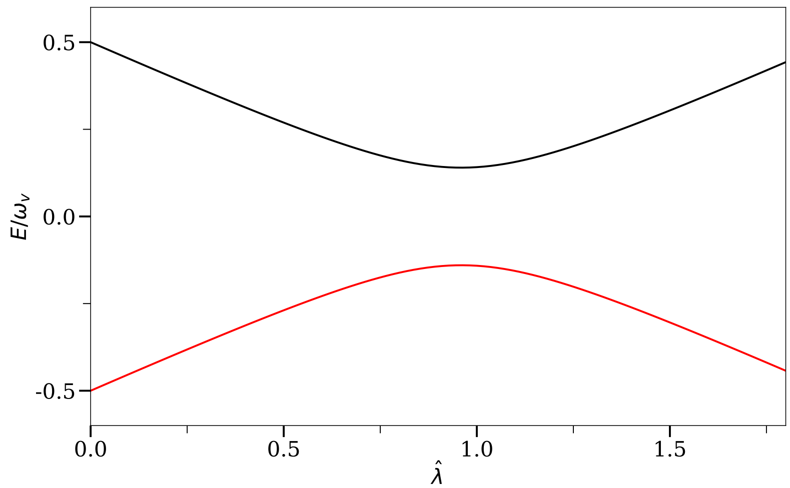


Figure 2.5: The two eigenvalues of the neutrino Hamiltonian as functions of matter potential $\hat{\lambda}$. I have used $\sin^2\theta_{\rm v} = 0.02 \approx \sin^2\theta_{13}$.

where

$$\hat{\lambda} = \frac{\lambda}{\omega_{\rm v}}. \tag{2.21}$$

In Fig. 2.5, I show the two eigenvalues of the neutrino Hamiltonian as functions of the matter potential $\hat{\lambda}$.

For a very high matter density, the heavy state of the neutrino $|\nu_{\rm H}\rangle$ is almost the same as $|\nu_{\rm e}\rangle$. As the matter density decreases, $|\nu_{\rm H}\rangle$ becomes a mixture of different neutrino flavors. As the neutrino reaches the surface of the Sun, where the matter density is approximately zero, $|\nu_{\rm H}\rangle$ is about the same as vacuum mass eigenstate $|\nu_2\rangle$. As a result, the electron flavor neutrinos produced at the solar core are partially converted to other flavors as they reach the surface of the Sun. This explains the solar neutrino problem.

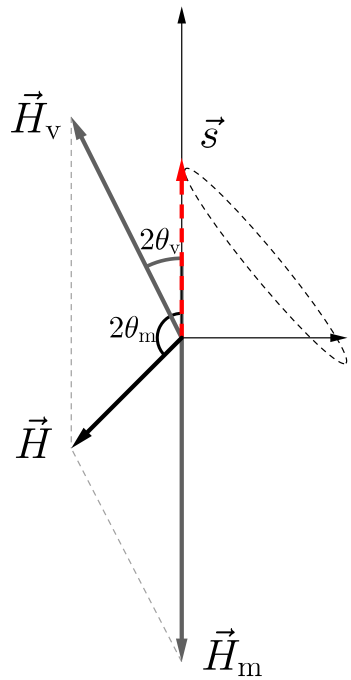


Figure 2.6: Neutrino oscillations in flavor isospin picture, with the presence of matter potential. The flavor isospin is denoted as red dashed arrow. It starts from electron flavor. The two gray vectors stand for the Hamiltonians of vacuum $\vec{H}_\mathrm{v}$ and matter $\vec{H}_\mathrm{m}$.

2.4 Flavor Isospin Formalism

The oscillations in the two flavor scenario are consequences of the Hamiltonian in this two-level quantum system. It is known that two-level quantum systems can be visualized using the Bloch sphere. In the realm of neutrino physics, the neutrino flavor isospin was introduced for such purpose [6]. The Hamiltonian for neutrino oscillations can be reformulated into a vector form.

Every two-by-two Hermitian matrix can be expanded in the quaternian basis.

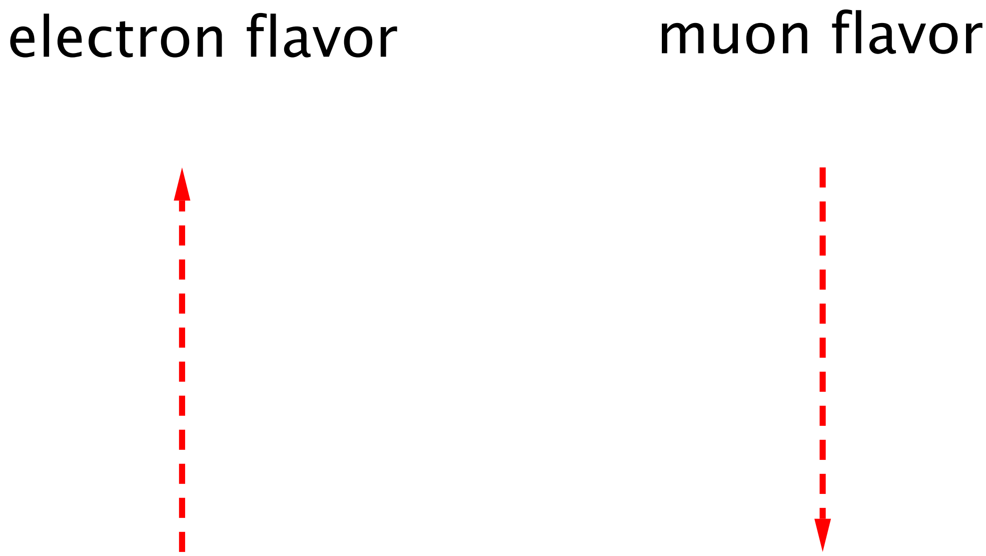


Figure 2.7: In the flavor isospin picture, a flavor isospin pointing upward, i.e., along the third axis in flavor space, indicates that the neutrino is in the electron flavor, while the downward direction indicates the other flavor, such as the muon flavor.

For example, the Hamiltonian for neutrino oscillations in vacuum can be written as

$$\mathsf{H}^{(\mathrm{f})} = -\frac{\vec{\sigma}}{2} \cdot \vec{H}^{(\mathrm{f})}. \tag{2.22}$$

I will use $\vec{\ }$ to denote vectors in flavor space. Meanwhile, the flavor quantum state of the neutrino is represented by the flavor isospin, which is defined as

$$\vec{s}^{(\mathrm{f})} = \Psi^{(\mathrm{f})\dagger}\frac{\vec{\sigma}}{2}\Psi^{(\mathrm{f})}. \tag{2.23}$$

As shown in Fig. 2.7, the directions of the flavor isospin in flavor space tell us the flavor content of the neutrino. A flavor isospin pointing upward in flavor space, i.e., along the direction of the third axis, denotes the electron flavor by definition. In the flavor isospin formalism, the electron flavor survival probability is

$$P = \frac{1}{2} + s_3^{(\mathrm{f})},$$

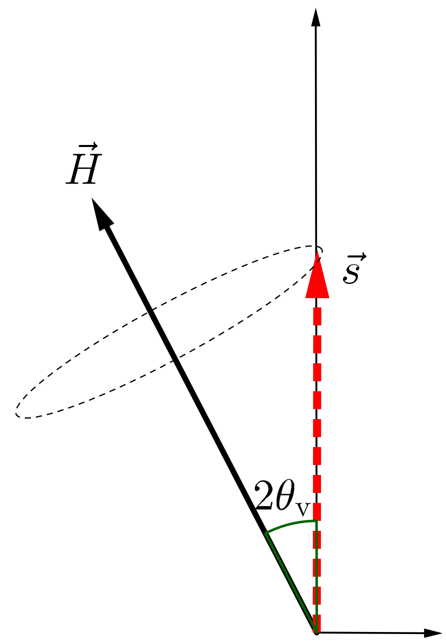


Figure 2.8: Vacuum oscillations in the flavor isospin picture. The flavor isospin of a neutrino starting with the electron flavor will precess around the static "Hamiltonian vector" $\vec{H}$, which gives periodic flavor oscillations according to Eqn. 2.4.

where $s_3^{(\mathrm{f})}$ is the third component of the flavor isospin. Correspondingly, the equation of motion for the flavor isospin describes its precession around the vector $\vec{H}^{(\mathrm{f})}$,

$$\dot{\vec{s}}^{(\mathrm{f})} = \vec{s}^{(\mathrm{f})} \times \vec{H}^{(\mathrm{f})}. \tag{2.24}$$

This precession corresponds to periodic oscillations between the two neutrino flavors. For example, in vacuum oscillations, the Hamiltonian becomes

$$\mathsf{H}^{(f)} \rightarrow \frac{\omega_{\mathrm{v}}}{2} \left(-\cos 2\theta_{\mathrm{v}}\sigma_3 + \sin 2\theta_{\mathrm{v}}\sigma_1\right) \rightarrow \omega_{\mathrm{v}} \begin{pmatrix} \sin\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix},$$

which is a vector of length ω_{v} and tilted away from the third axis by the angle $2\theta_{\mathrm{v}}$.

Eqn. 2.24 depicts the precession of the flavor isospin for a neutrino which starts with the electron flavor and propagates in vacuum. The oscillation frequency is trivially read out from Eqn. 2.24,

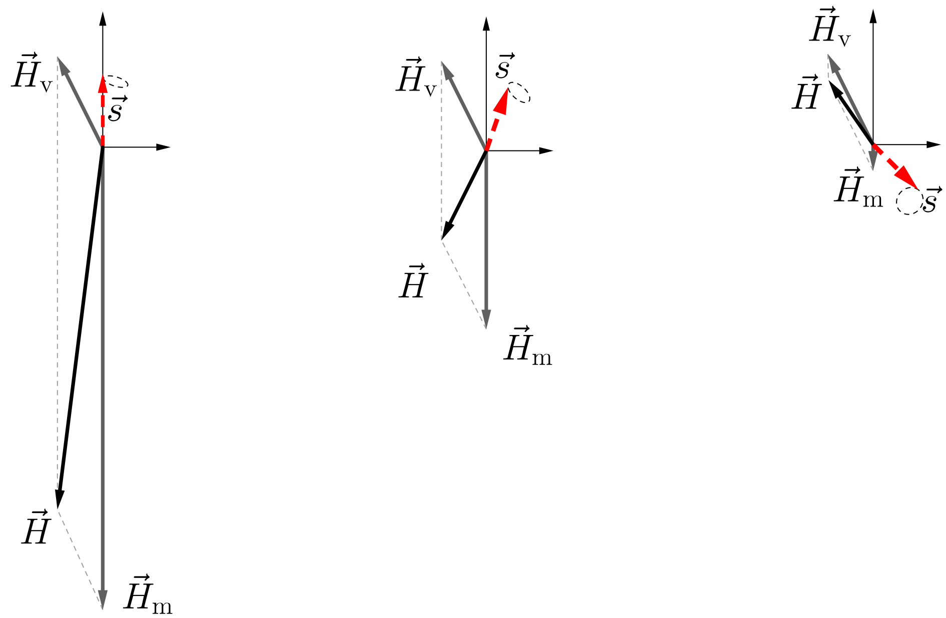
$$\omega_{\mathrm{v}} = |\vec{H}^{(\mathrm{f})}| \tag{2.25}$$

MSW effect is also easily explained using flavor isospin picture. The Hamiltonian in flavor isospin picture

$$\begin{aligned}
\mathsf{H} =& \frac{\omega_{\mathrm{v}}}{2}\left(-\cos 2\theta_{\mathrm{v}}\sigma_3 + \sin 2\theta_{\mathrm{v}}\sigma_1\right) + \frac{\lambda(x)}{2}\sigma_3 \\
\rightarrow& \omega_{\mathrm{v}}\begin{pmatrix} -\sin 2\theta_{\mathrm{v}} \\ 0 \\ \cos 2\theta_{\mathrm{v}} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\lambda(x) \end{pmatrix} \\
=& \vec{H}_{\mathrm{v}} + \vec{H}_{\mathrm{m}}(x),
\end{aligned}$$

where $\vec{H}_{\mathrm{v}}$ is vacuum contribution and $\vec{H}_{\mathrm{m}}(x)$ is the matter potential contribution. The two vectors are visualized in Fig. 2.6. We discussed in Sec. **??** the adiabatic transitions of neutrino states in varying matter density. Fig. 2.9 shows the adiabatic evolution of neutrino flavor isospin. For region of high density matter background, which provides large matter potential, the total Hamiltonian is almost pointing downward. We observe almost no flavor oscillations since flavor isospin precession is tiny. As the neutrinos moving into smaller matter density regions, the flavor isospin is approximately following the evolution of Hamiltonian. Flavor conversion happens because of the evolution of Hamiltonian, even though flavor oscillations are still tiny. In the end, neutrinos reach the region with almost no matter, where they are almost converted to one of the mass eigenstates. In fact, those neutrinos won't oscillate that much in vacuum following this initial condition.

Neutrinos might experience a critical matter density, when the overall Hamiltonian is perpendicular to the upright axis. Assuming we have electron neutrinos going through such regions, they will experience maximum flavor oscillations, c.f. Fig. 2.10.



(a) High matter density (b) Medium matter density (c) Low matter density

Figure 2.9: Flavor isospin picture of neutrino oscillations in matter. $\vec{H}_{\rm v}$ is the vacuum contribution to Hamiltonian, and $\vec{H}_{\rm m}$ corresponds to the matter potential.

2.5 Summary

Vacuum neutrino oscillations can be easily explained and calculated. However, it conveys the message of the nature of neutrino oscillations. Neutrinos are usually produced in flavor states through weak interactions. The neutrino does not remain in the same flavor state during its propagation because the flavor states are not the eigenstates of the propagation Hamiltonian. An extrapolation of this idea is that neutrinos might also oscillate in a uniform linear potential, the Hamiltonian of which would be similar to vacuum Hamiltonian but with different values. One of such situations is that neutrinos propagate through a region with a constant matter

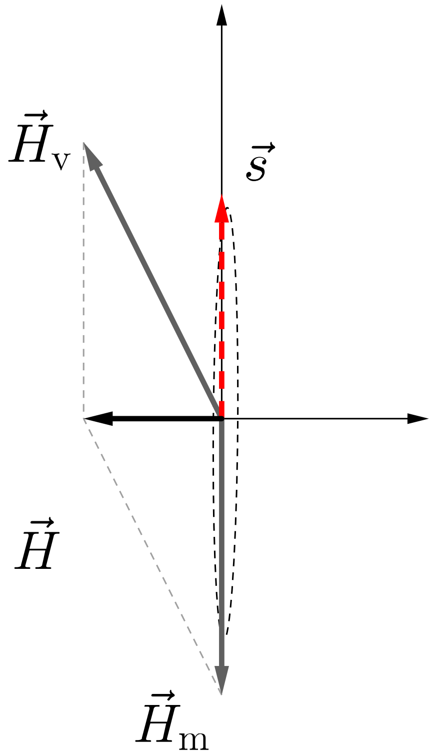


Figure 2.10: MSW resonance happens when electron neutrinos go through a critical matter density.

density, which I will explain in the next chapter.

Bibliography

[1] C L Cowan et al. "Detection of the Free Neutrino: a Confirmation." In: *Science (New York, N.Y.)* 124.3212 (July 1956), pp. 103–4.

[2] B Pontecorvo. "Neutrino experiments and the problem of conservation of leptonic charge". In: *Sov. Phys. JETP* 26.5 (1968), pp. 984–988.

[3] J.N. Bahcall. "The solar neutrino problem". In: *Nuclear Instruments and Methods* 110 (July 1973), pp. 381–384.

[4] L. Wolfenstein. "Neutrino oscillations in matter". In: *Physical Review D* 17.9 (May 1978), pp. 2369–2374.

[5] Michael F Altmann, Rudolf L Mößbauer, and Lothar J N Oberauer. "Solar neutrinos". In: *Reports on Progress in Physics* 64.1 (Jan. 2001), pp. 97–146.

[6] Huaiyu Duan, George M. Fuller, and Yong-Zhong Qian. "Analysis of collective neutrino flavor transformation in supernovae". In: *Physical Review D* 74.12 (Dec. 2006), p. 123004. arXiv: `0703776 [astro-ph]`.

[7] Eg Adelberger and a García. "Solar fusion cross sections. II. The pp chain and CNO cycles". In: *Reviews of Modern ...* 83.March (2011). arXiv: `arXiv:1004.2318v3`.

[8] Hans-Thomas Janka, Tobias Melson, and Alexander Summa. "Physics of Core-Collapse Supernovae in Three Dimensions: A Sneak Preview". In: *Annual Review of Nuclear and Particle Science* 66.1 (2016), pp. 341–375. arXiv: `1602.05576`.

[9] C. Patrignani et al. "Review of Particle Physics". In: *Chin. Phys.* C40.10 (2016), p. 100001.

[10] Hans-Thomas Janka. *Neutrino Emission from Supernovae*. Ed. by Athem W. Alsabti and Paul Murdin. Cham: Springer International Publishing, 2017, pp. 1575–1604. ISBN: 978-3-319-21846-5.

[11] E. Kemp. "The Deep Underground Neutrino Experiment: The precision era of neutrino physics". In: *Astronomische Nachrichten* 338.9-10 (2017), pp. 993–999.