

## 1. General

### 1.1 Units and Conventions

1. Energy and temperature:

$$\frac{1}{40}\text{eV} = 300\text{K}k_B. \quad (1)$$

2. Natural units: energy is related to length by

$$1\text{fm} \times 197\text{MeV} = \hbar c = 1. \quad (2)$$

3. For light, energy 1eV corresponds to wavelength  $1.24\mu\text{m}$ .

4. Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

5. Transformations of Pauli matrices

$$\mathbf{U}^\dagger \sigma_3 \mathbf{U} = \cos 2\theta \sigma_3 - \sin 2\theta \sigma_1 \quad (4)$$

$$\mathbf{U}^\dagger \sigma_1 \mathbf{U} = -\sin 2\theta \sigma_3 + \cos 2\theta \sigma_1 \quad (5)$$

$$\mathbf{U}^\dagger \sigma_2 \mathbf{U} = \sigma_2. \quad (6)$$

### 1.2 Check Results

- Are the dimensions correct?
- Does the limits of the result make sense?
- Does the result make sense when the complexity of the system is removed?
- Is it the right basis to draw a conclusion?
- Have you double checked analytical expression with other people or Mathematica?
- Have you checked numerical results with previously calculated simple results?
- Does unitarity hold for the result? If not is the Hamiltonian Hermitian?

## 2. Field Theory

### 2.1 Equations of Motion

1. Dirac Equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \quad (7)$$

2. Klein Gordon Equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \quad (8)$$

## 3. Neutrinos

### 3.1 Fundamental Parameters

1. Mixing angles

$$\sin^2 2\theta_{12} = 0.857 \pm 0.024 \quad (9)$$

$$\sin^2 2\theta_{23} > 0.95 \quad (10)$$

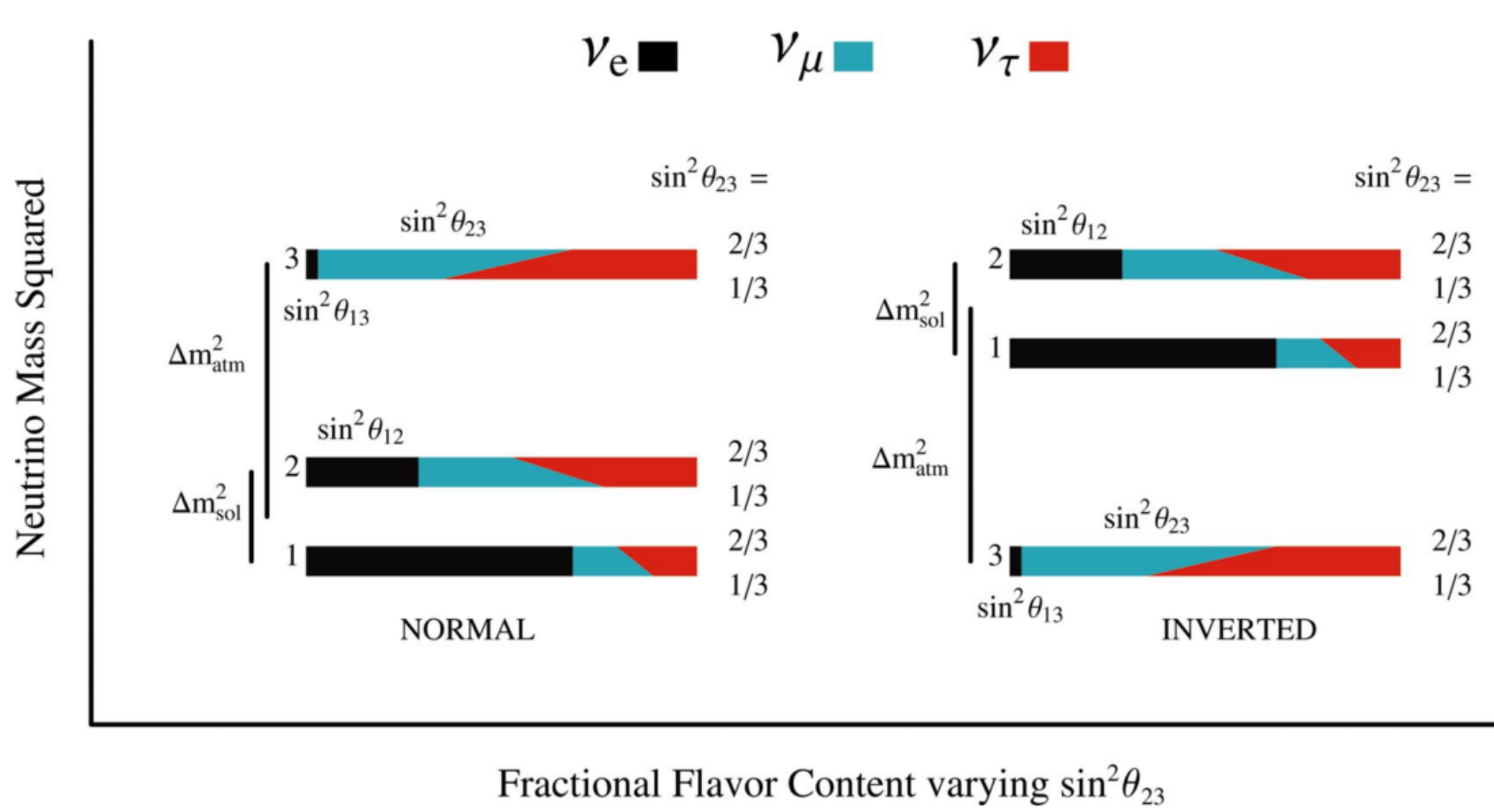
$$\sin^2 2\theta_{13} = 0.095 \pm 0.010 \quad (11)$$

$$\quad (12)$$

2. Masses (fig 1)

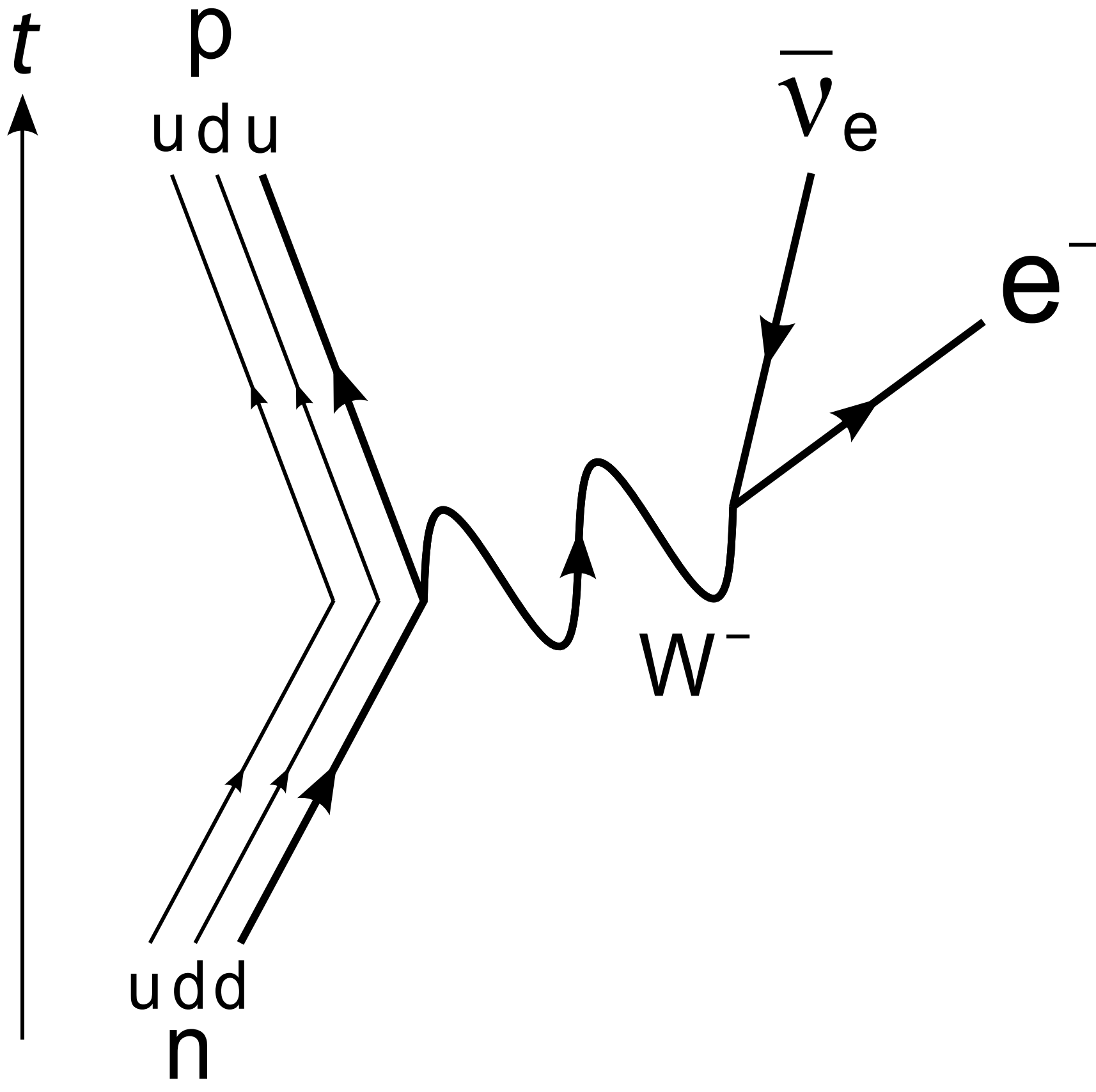
$$\Delta m_{12}^2 = \Delta m_{sol}^2 = 7.53_{-0.18}^{+0.18} \times 10^{-5}\text{eV}^2 \quad (13)$$

$$|\Delta m_{31}^2| = \Delta m_{atm}^2 = 2.44_{-0.06}^{+0.06} \times 10^{-3}\text{eV}^2 \quad (14)$$



**Figure 1:** Neutrino mass hierarchy. Mena, O., & Parke, S. (2004). Unified graphical summary of neutrino mixing parameters. Physical Review D, 69(11), 117301.

### 3.2 Nuclear Reactions

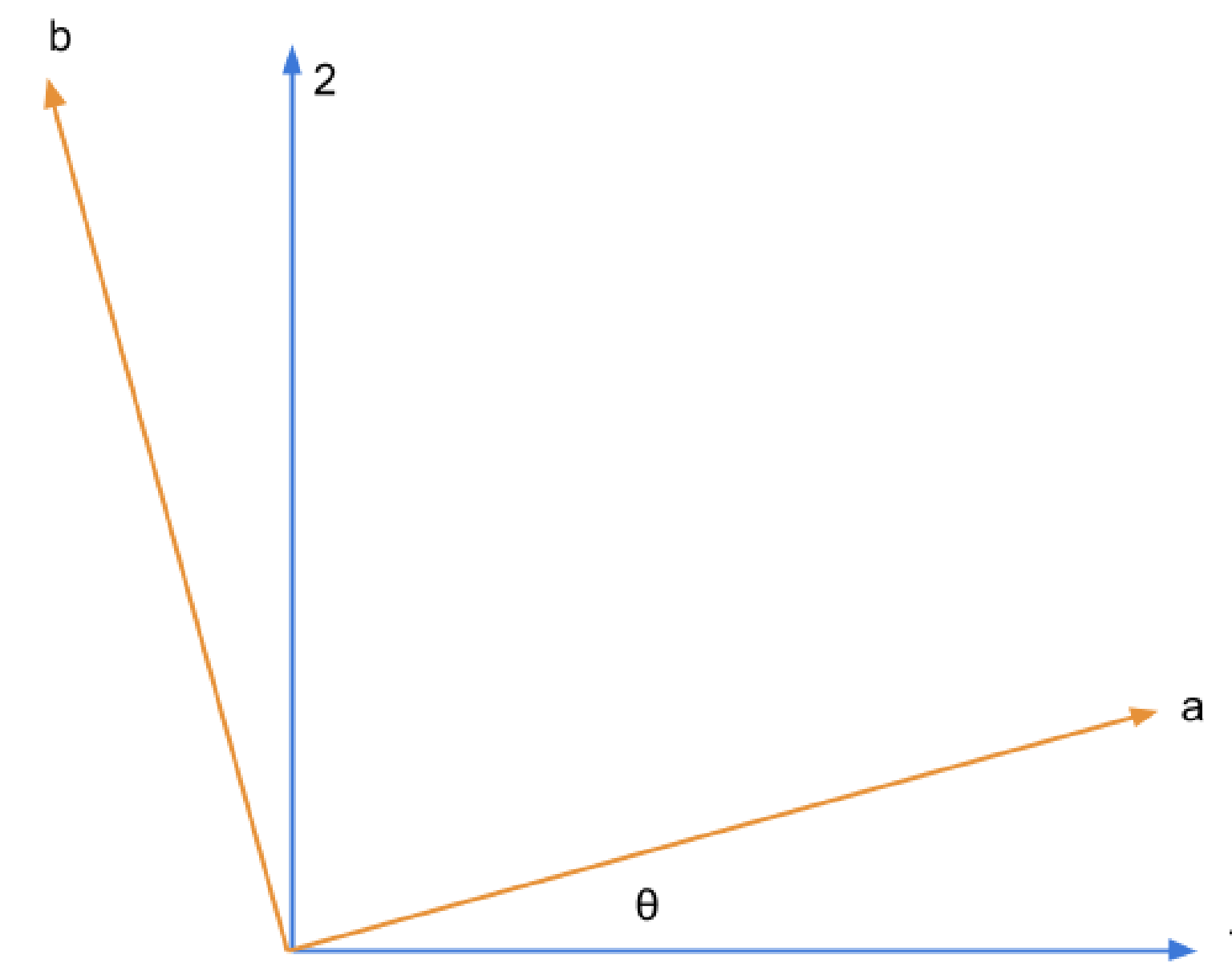


**Figure 2:** Feynman diagram of beta decay. The charged current weak interaction boson in this case is a  $W^-$ . Credit: Joel Holdsworth, within public domain.

Reaction	Equation	Boson
Electron emission	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X \rightarrow \begin{smallmatrix} A \\ Z+1 \end{smallmatrix} X + e^- + \bar{\nu}_e$	$W^-$
Positron emission	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X \rightarrow \begin{smallmatrix} A \\ Z-1 \end{smallmatrix} X + e^+ + \nu_e$	$W^+$
Electron capture	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X + e^- \rightarrow \begin{smallmatrix} A \\ Z-1 \end{smallmatrix} X + \nu_e$	$W^+$
Positron capture	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X + e^+ \rightarrow \begin{smallmatrix} A \\ Z+1 \end{smallmatrix} X + \bar{\nu}_e$	$W^-$
Electron annihilation	$e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$	$W^-$
Electron annihilation	$e^- + e^+ \rightarrow \nu + \bar{\nu}$	$Z$
Neutrino capture	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X + \begin{smallmatrix} (-) \\ \nu_e \end{smallmatrix} \rightarrow \begin{smallmatrix} A \\ Z\mp 1 \end{smallmatrix} X + e^\pm$	$W$
$e^- \nu$ scattering	$e^- + \begin{smallmatrix} (-) \\ \nu_e \end{smallmatrix} \rightarrow e^- + \begin{smallmatrix} (-) \\ \nu_e \end{smallmatrix}$	$W^-$
$e^- \nu$ scattering	$e^\pm + \begin{smallmatrix} (-) \\ \nu_e \end{smallmatrix} \rightarrow e^\pm + \begin{smallmatrix} (-) \\ \nu_e \end{smallmatrix}$	$Z$
Neutrino scattering	$\begin{smallmatrix} A \\ Z \end{smallmatrix} X + \begin{smallmatrix} (-) \\ \nu \end{smallmatrix} \rightarrow \begin{smallmatrix} A \\ Z \end{smallmatrix} X + \begin{smallmatrix} (-) \\ \nu \end{smallmatrix}$	$Z$
Bremsstrahlung	$N + N \rightleftharpoons N + N + \nu + \bar{\nu}$	
Annihilation	$e^+ e^- \rightleftharpoons \nu + \bar{\nu}$	
Neutrino annihilation	$\nu + \bar{\nu} \rightleftharpoons \nu + \bar{\nu}$	

**Table 1:** Neutrino related nuclear or leptonic reactions

### 3.3 Neutrino Mixing



**Figure 3:** Neutrino mixing. Blue states  $\{|1, 2\rangle\}$  are the VAC-UUM energy basis while the orange states  $\{|a, b\rangle\}$  are the flavor basis. Blue: flavor basis; Red: propagation basis.

The two basis are related to each other through a unitary matrix  $\mathbf{U}$ ,

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} = \mathbf{U} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}, \quad (15)$$

where

$$\mathbf{U} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (16)$$

One thing to notice is that the relation of the amplitude has the same transformation

$$\begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix} = \mathbf{U} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (17)$$

where the amplitude is define as the components of a state in a certain basis.

$$|\Psi\rangle = (\psi_e \ \psi_x) \begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} = (\psi_1 \ \psi_2) \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \quad (18)$$

### 3.4 Hamiltonian and Basis

#### 3.4.1 Basis

Rotation from a Hamiltonian diagonalized basis wave function  $\Psi_v$  to flavor basis wave function  $\Psi_f$  is

$$\Psi_f = R_{d2f}(\theta_x)\Psi_d, \quad (19)$$

where  $d$  can be  $v$  for vacuum eigenbasis,  $m$  for matter eigenbasis and

$$R_{d2f}(\theta_x) = \begin{pmatrix} \cos \theta_x & \sin \theta_x \\ -\sin \theta_x & \cos \theta_x \end{pmatrix}. \quad (20)$$

Matter mixing angle  $\theta_m$  is determined through

$$\sin 2\theta_m = \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v} \cos 2\theta_v}}, \quad (21)$$

$$\cos 2\theta_m = \frac{\cos 2\theta_v - \lambda/\omega_v}{\sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v} \cos 2\theta_v}}, \quad (22)$$

$$\tan 2\theta_m = \frac{\sin 2\theta_m}{\cos 2\theta_m - \lambda/\omega_v}, \quad (23)$$

where  $\lambda = \sqrt{2}G_F n_e$ .

#### 3.4.2 Hamiltonian

With the appearance of matter perturbation  $\lambda(x) = \lambda_0 + \delta\lambda(x)$  to the system, we have the Hamiltonian in vacuum basis

$$\mathbf{H}_v = -\frac{\omega_v}{2}\sigma_3 + \frac{\lambda(x)}{2}\cos 2\theta_v\sigma_3 + \frac{\lambda(x)}{2}\sin 2\theta_v\sigma_1. \quad (24)$$

In flavor basis,

$$\mathbf{H}_f = \frac{\omega}{2}(-\cos 2\theta\sigma_3 + \sin 2\theta\sigma_1) + \frac{\lambda(x)}{2}\sigma_3. \quad (25)$$

In background matter potential  $\lambda_0$  basis

$$\mathbf{H}_{\bar{m}} = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda(x)}{2}\cos 2\theta_m\sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m\sigma_1, \quad (26)$$

where  $\omega_m = \omega_v \sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v} \cos 2\theta_v}$ .

The caveat is the position/time dependence of the matter potential leads to position/time dependent transformation of the wave function which will cause an term in the Hamiltonian of the form

$$-i\mathbf{U}^\dagger(x)\frac{\partial}{\partial x}\mathbf{U}(x). \quad (27)$$

#### 3.4.3 Physics of Neutrino Oscillation

- Nature of neutrino oscillation
- MSW effect
- Collective oscillation
- Gravitation effect on neutrino oscillations
- Kinetic equation ( $\nu\bar{\nu}$  pairing correlations?)

#### 3.4.4 Applications of Neutrino Oscillation

- Supernova shock revive
- Accretion disc
- Cosmology
- Tomography
- Detection of nuclear activities

#### 3.4.5 Big Questions

- Are neutrinos Majorana or Dirac?
- What is the mass hierarchy?

## 4. Numerical

### 4.1 Writing Code

- Reduce quantities to dimensionless;
- Always calculate the corresponding value of reduced quantities in suitable units.

### 4.2 Limits

- Simple cases where analytical is possible;
- Verified numerical calculations had previously.

### 4.3 Zen of Python

import this

### 4.4 Check You Code (Neutrino)

Check the code step by step:

- Vacuum oscillation amplitude and frequency
- Constant matter potential oscillation amplitude and frequency
- MSW resonance