1. General

1.1 Units and Conventions

1. Energy and temperature:

$$\frac{1}{40}\mathrm{eV} = 300\mathrm{K}k_B. \tag{1}$$

2. Natural units: energy is related to length by

$$1\mathrm{fm} \times 197\mathrm{MeV} = \hbar c = 1. \tag{2}$$

As a reference, $1\mathrm{GeV} = 5.08 \times 10^{15}\mathrm{m}^{-1}$.

3. For light, energy 1eV corresponds to wavelength 1.24μm.
4. Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

5. Transformations of Pauli matrices

$$\mathbf{U}^\dagger \sigma_3 \mathbf{U} = \cos 2\theta\sigma_3 + \sin 2\theta\sigma_1 \tag{4}$$
$$\mathbf{U}^\dagger \sigma_1 \mathbf{U} = -\sin 2\theta\sigma_3 + \cos 2\theta\sigma_1 \tag{5}$$
$$\mathbf{U}^\dagger \sigma_2 \mathbf{U} = \sigma_2. \tag{6}$$

1.2 Check Results

1. Are the dimensions correct?
2. Do the limits of the result make sense?
3. Does the result make sense when the complexity of the system is removed?
4. Is it the right basis to draw a conclusion?
5. Have you double checked analytical expression with other people or Mathematica?
6. Have you checked numerical results with previously calculated simple results?
7. Does unitarity hold for the result? If not is the Hamiltonian Hermitian?

2. Field Theory

2.1 Equations of Motion

1. Dirac Equation

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0 \tag{7}$$

2. Klein Gordon Equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \tag{8}$$

3. Neutrinos

3.1 Fundamental Parameters

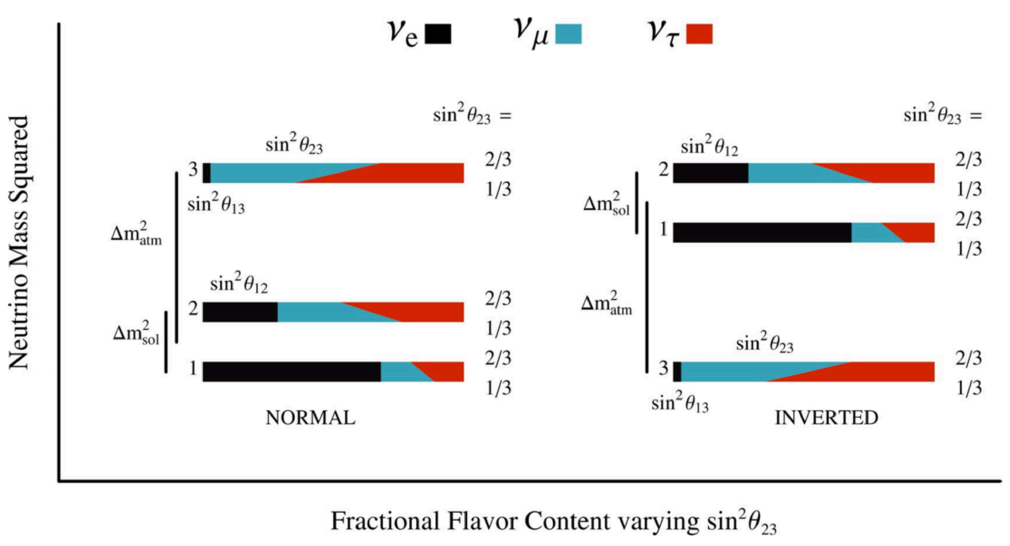

Figure 1: Neutrino mass hierarchy. Mena, O., & Parke, S. (2004). Unified graphical summary of neutrino mixing parameters. Physical Review D, 69(11), 117301.

1. Mixing angles

$$\sin^2 2\theta_{12} = 0.857 \pm 0.024 \tag{9}$$
$$\sin^2 2\theta_{23} > 0.95 \tag{10}$$
$$\sin^2 2\theta_{13} = 0.095 \pm 0.010 \tag{11}$$
$$\tag{12}$$

2. Masses (fig 1)

$$\Delta m^2_{12} = \Delta m^2_{\mathrm{sol}} = 7.53^{+0.18}_{-0.18} \times 10^{-5}\mathrm{eV}^2 \tag{13}$$
$$|\Delta m^2_{31}| = \Delta m^2_{\mathrm{atm}} = 2.44^{+0.06}_{-0.06} \times 10^{-3}\mathrm{eV}^2(\mathrm{NH}) \tag{14}$$

3. Typical oscillation frequencies

$$\omega_{v,21} = \frac{\Delta m^2_{21}}{2E} = 1.90 \times 10^{-4}\mathrm{m}^{-1}\frac{\delta m^2}{7.5 \times 10^{-5}\mathrm{eV}^2}\frac{1\mathrm{MeV}}{E} \tag{15}$$
$$\omega_{v,32} = \frac{\Delta m^2_{32}}{2E} = 6.3 \times 10^{-3}\mathrm{m}^{-1}\frac{\Delta m^2_{32}}{2.5 \times 10^{-3}\mathrm{eV}^2}\frac{1\mathrm{MeV}}{E}. \tag{16}$$

3.2 Nuclear Reactions

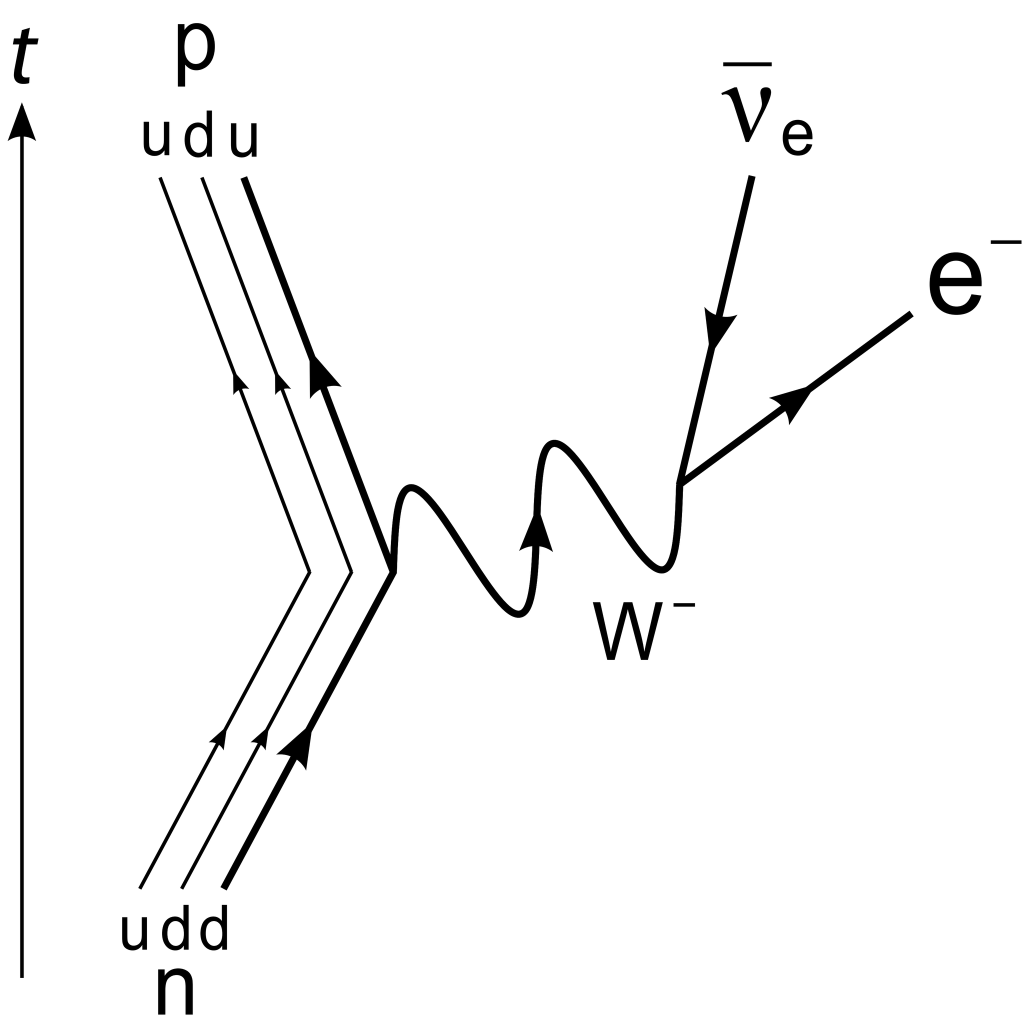

Figure 2: Feynman diagram of beta decay. The charged current weak interaction boson in this case is a W^-. Credit: Joel Holdsworth, within public domain.

Table 1: Neutrino related nuclear or leptonic reactions

Reaction	Equation	Boson
Electron emission	${}^A_ZX \to {}^A_{Z+1}X + e^- + \bar{\nu}_e$	W
Positron emission	${}^A_ZX \to {}^A_{Z-1}X + e^+ + \nu_e$	W
Electron capture	${}^A_ZX + e^- \to {}^A_{Z-1}X + \nu_e$	W
Positron capture	${}^A_ZX + e^+ \to {}^A_{Z+1}X + \bar{\nu}_e$	W
Electron annihilation	$e^- + e^+ \to \nu_e + \bar{\nu}_e$	W
Electron annihilation	$e^- + e^+ \to \nu + \bar{\nu}$	Z
Neutrino capture	${}^A_ZX + \overset{(-)}{\nu_e} \to {}^A_{Z\mp1}X + e^\pm$	W
$e^-\nu$ scattering	$e^- + \overset{(-)}{\nu_e} \to e^- + \overset{(-)}{\nu_e}$	W
$e^-\nu$ scattering	$e^\pm + \overset{(-)}{\nu_e} \to e^\pm + \overset{(-)}{\nu_e}$	Z
Neutrino scattering	${}^A_ZX + \overset{(-)}{\nu} \to {}^A_ZX + \overset{(-)}{\nu}$	Z
Bremsstrahlung	$N + N \rightleftharpoons N + N + \nu + \bar{\nu}$	
Annihilation	$e^+e^- \rightleftharpoons \nu + \bar{\nu}$	
Neutrino annihilation	$\nu + \bar{\nu} \rightleftharpoons \nu + \bar{\nu}$	

3.3 Neutrino Mixing

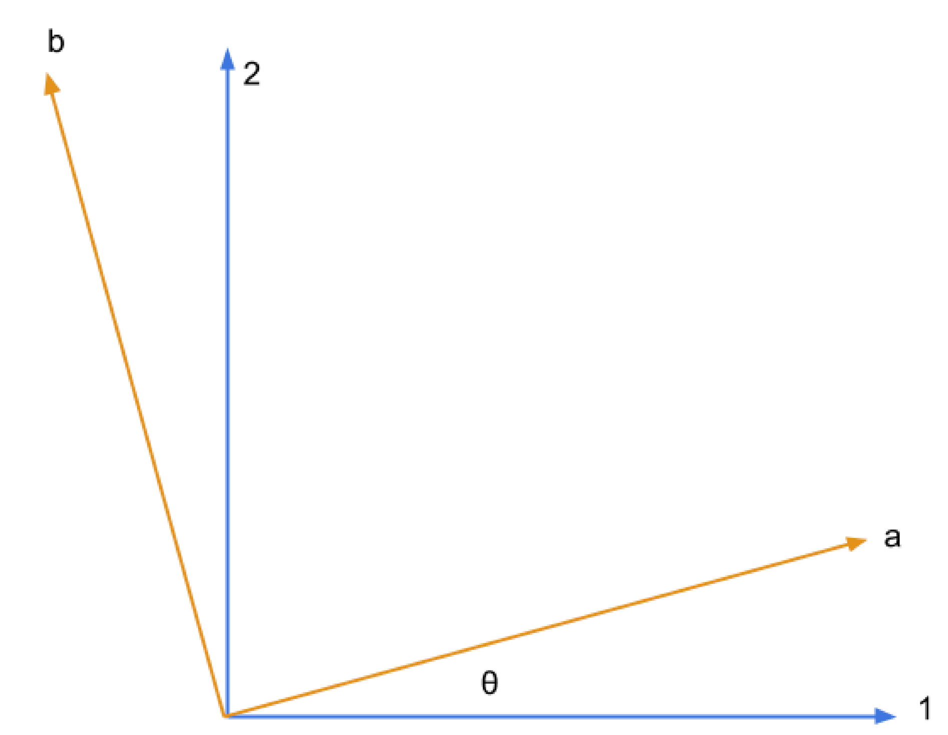

Figure 3: Neutrino mixing. Blue states ($\{1, 2\}$) are the VACUUM energy basis while the orange states ($\{a, b\}$) are the flavor basis. Blue: flavor basis; Red: propagation basis.

The two basis are related to each other through a unitary matrix $\mathbf{U}$,

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} = \mathbf{U}\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}, \tag{17}$$

where

$$\mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{18}$$

One thing to notice is that the relation of the amplitude has the same transformation

$$\begin{pmatrix} \psi_e \\ \psi_x \end{pmatrix} = \mathbf{U}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{19}$$

where the amplitude is define as the components of a state in a certain basis.

$$|\Psi\rangle = \begin{pmatrix} \psi_e & \psi_x \end{pmatrix}\begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}\begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \tag{20}$$

3.4 Hamiltonian and Basis

3.4.1 Basis

Rotation from a Hamiltonian diagonalized basis wave function Ψ_v to flavor basis wave function Ψ_f is

$$\Psi_f = R_{d2f}(\theta_x)\Psi_d, \tag{21}$$

where d can be v for vacuum eigenbasis, m for matter eigenbasis and

$$R_{d2f}(\theta_x) = \begin{pmatrix} \cos\theta_x & \sin\theta_x \\ -\sin\theta_x & \cos\theta_x \end{pmatrix}. \tag{22}$$

Matter mixing angle θ_m is determined through

$$\sin 2\theta_m = \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v}\cos 2\theta_v}}, \tag{23}$$
$$\cos 2\theta_m = \frac{\cos 2\theta_v - \lambda/\omega_v}{\sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v}\cos 2\theta_v}}, \tag{24}$$
$$\tan 2\theta_m = \frac{\sin 2\theta_m}{\cos 2\theta_m - \lambda/\omega_v}, \tag{25}$$

where $\lambda = \sqrt{2}G_F n_e$.

3.4.2 Hamiltonian

With the appearance of matter perturbation $\lambda(x) = \lambda_0 + \delta\lambda(x)$ to the system, we have the Hamiltonian in vacuum basis

$$\mathbf{H_v} = -\frac{\omega_v}{2}\sigma_3 + \frac{\lambda(x)}{2}\cos 2\theta_v\sigma_3 + \frac{\lambda(x)}{2}\sin 2\theta_v\sigma_1. \tag{26}$$

In flavor basis,

$$\mathbf{H_f} = \frac{\omega}{2}(-\cos 2\theta\sigma_3 + \sin 2\theta\sigma_1) + \frac{\lambda(x)}{2}\sigma_3. \tag{27}$$

In background matter potential λ_0 basis

$$\mathbf{H_{\tilde{m}}} = -\frac{\omega_m}{2}\sigma_3 + \frac{\delta\lambda(x)}{2}\cos 2\theta_m\sigma_3 - \frac{\delta\lambda}{2}\sin 2\theta_m\sigma_1, \tag{28}$$

where $\omega_m = \omega_v\sqrt{\left(\frac{\lambda}{\omega_v}\right)^2 + 1 - 2\frac{\lambda}{\omega_v}\cos 2\theta_v}$.

The caveat is the position/time dependence of the matter potential leads to position/time dependent transformation of the wave function which will cause an term in the Hamiltonian of the form

$$-i\mathbf{U}^\dagger(x)\frac{\partial}{\partial x}\mathbf{U}(x). \tag{29}$$

3.4.3 Physics of Neutrino Oscillation

1. Nature of neutrino oscillation
2. MSW effect
3. Collective oscillation
4. Gravitation effect on neutrino oscillations
5. Kinetic equation ($\nu\bar{\nu}$ pairing correlations?)

3.4.4 Applications of Neutrino Oscillation

1. Supernova shock revive
2. Accretion disc
3. Cosmology
4. Tomography
5. Detection of nuclear activities

3.4.5 Big Questions

1. Are neutrinos Majorana or Dirac?
2. What is the mass hierarchy?

4. Numerical

4.1 Writing Code

1. Reduce quantities to dimensionless;
2. Always calculate the corresponding value of reduced quantities in suitable units.

4.2 Limits

1. Simple cases where analytical is possible;
2. Verified numerical calculations had previously.

4.3 Zen of Python

import this

4.4 Check You Code (Neutrino)

Check the code step by step:

1. Vacuum oscillation amplitude and frequency
2. Constant matter potential oscillation amplitude and frequency
3. MSW resonance