Optimized thinning example problem: Appendix A

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1 Introduction

This is an example of how to use optimization programming to thin forest plots to match multiple output distributions. This document is a companion to the Rmd file and contains mathematical notation, tables, and more in-depth descriptions of the decision variables, constraints, constraint matrix, and objective function of our problem.

2 Assignment matrices

This section visualizes several components of the density and basal area assignment matrices. Each assignment matrix has six rows (one row per plot) and three columns (one column per distribution bin).

2.1 Blank target density assignment matrix

This blank density assignment matrix (Table 1) will be filled out as the optimization solver finds the ideal solution to our problem. If a particular plot is assigned to a particular bin, a "1" is put in the corresponding cell of the assignment matrix. If a particular plot is not assigned to a particular bin, a "0" is put in the corresponding cell of the assignment matrix. Plots assigned to density bin 1 will have one to three trees after thinning; plots

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assigned to density bin 2 will have four to six trees after thinning; plots assigned to density bin 3 will have seven to nine trees after thinning.

	Density bin 1	**Density bin 2**	**Density bin 3**
Plot 1			
Plot 2			
Plot 3			
Plot 4			
Plot 5			
Plot 6			

Table 1: Blank density assignment matrix

2.2 Target density assignment matrix (filled example)

Table 2 is one example of what the filled density assignment matrix may look like once the problem is solved. Each plot can only be assigned to one bin, and each bin must have one to three plots assigned to it. Looking at the rows of the table individually, we can see that there is only a "1" in one cell of each row— this means that each plot was only assigned to one bin. Looking down the columns of the table, we can see that density bin 1 contains two plots (plot 2 and plot 6), density bin 2 contains three plots (plot 1, plot 3, and plot 5), and density bin 3 contains one plot (plot 4).

	Density bin 1	**Density bin 2**	**Density bin 3**
Plot 1	0	1	0
Plot 2	1	0	0
Plot 3	0	1	0
Plot 4	0	0	1
Plot 5	0	1	0
Plot 6	1	0	0

Table 2: Filled density assignment matrix

Note: the numbers in Table 2 are arbitrarily chosen for the sake of example and do not necessarily reflect the actual outcome of the solved example problem. This is just one way the filled matrix could look after solving the problem.

2.3 Target density assignment matrix with row and column sums

As discussed in section 2.1, each plot can only be assigned to one bin of each assignment matrix. This means that each row must sum to one. Similarly, each bin may only contain one to three plots after thinning. This means that the sum of each column must be between one and three (inclusive). These row and column sums are depicted in Table 3. These sums are important to keep in mind as we construct the constraints of the optimization problem.

	Density bin 1	**Density bin 2**	**Density bin 3**	**Row sums**
Plot 1				$\sum = 1$
Plot 2				$\sum = 1$
Plot 3				$\sum = 1$
Plot 4				$\sum = 1$
Plot 5				$\sum = 1$
Plot 6				$\sum = 1$
Column sums	$\sum$ = 1-3	$\sum$ = 1-3	$\sum$ = 1-3	

Table 3: Density assignment matrix with row and column sums

2.4 Blank target basal area assignment matrix

This blank basal area assignment matrix (Table 4) will be filled out as the optimization solver finds the ideal solution to our problem. If a particular plot is assigned to a particular bin, a "1" is put in the corresponding cell of the assignment matrix. If a particular plot is not assigned to a particular bin, a "0" is put in the corresponding cell of the assignment matrix. After thinning, plots in basal area bin 1 will have 0.2 to 5 square meters of total basal area (that is, the sum of the basal area of all the remaining trees in the plot will be between 0.2 and 5). Plots in basal area bin 2 will have 5.1 to 10 square meters of total basal area after thinning. Plots in basal area bin 3 will have 10.1 to 15 square meters of total basal area after thinning.

	Basal area bin 1	Basal area bin 2	Basal area bin 3
Plot 1			
Plot 2			
Plot 3			
Plot 4			
Plot 5			
Plot 6			

Table 4: Blank basal area assignment matrix

2.5 Target basal area assignment matrix (filled example)

Table 5 is one example of what the filled basal area assignment matrix may look like once the problem is solved. Each plot can only be assigned to one bin, and each bin must have one to three plots assigned to it. Looking at the rows of the table individually, we can see that there is only a "1" in one cell of each row— this means that each plot was only assigned to one bin. Looking down the columns of the table, we can see that basal area bin 1 contains three plots (plot 2, plot 3, and plot 5), basal area bin 2 contains one plot (plot 6), and basal area bin 3 contains two plots (plot 1 and plot 4).

The filled density assignment matrix and basal area assignment matrix don't have to be identical to each other. For example, not every plot in density bin 1 necessarily has to belong to basal area bin 1. Perhaps there is a plot in density bin 1 with humongous trees, and that plot belongs to basal area bin 2. While density and basal area are certainly related (that is, as the number of trees in a plot increases, the total basal area of the plot will also increase), the assignment matrices don't have to look the same after the problem is solved. The most important thing is that, in both assignment matrices, all of the rows sum to one and all of the columns sum to one, two, or three.

	Basal area bin 1	Basal area bin 2	Basal area bin 3
Plot 1	0	0	1
Plot 2	1	0	0
Plot 3	1	0	0
Plot 4	0	0	1
Plot 5	1	0	0
Plot 6	0	1	0

Table 5: Filled basal area assignment matrix

Note: the numbers in Table 5 are arbitrarily chosen and do not necessarily reflect the outcome of the detailed example problem once we apply a solver. This is just one way the filled matrix could look after solving the problem.

2.6 Target basal area assignment matrix with row and column sums

As discussed in section 2.4, each plot can only be assigned to one bin of the basal area assignment matrix. This means that the sum of each row must sum to one. Similarly, each bin may only contain one to three plots after thinning. This means that the sum of each column must be between one and three (inclusive). These row and column sums are depicted in the table below.

	Basal area bin 1	Basal area bin 2	Basal area bin 3	Row sums
Plot 1				$\sum = 1$
Plot 2				$\sum = 1$
Plot 3				$\sum = 1$
Plot 4				$\sum = 1$
Plot 5				$\sum = 1$
Plot 6				$\sum = 1$
Column sums	$\sum =$ 1-3	$\sum =$ 1-3	$\sum =$ 1-3	

Table 6: Basal area assignment matrix with row and column sums

3 Decision variables

There are two classes of decision variables in this problem: those that indicate which plots get assigned to which bins of the distribution, and those that indicate which trees get kept vs. cut down. All of these decision variables are **binary**. If the solver assigns a decision variable a value of 1, the situation that variable represents is TRUE. For example, if a decision variable from the first class (plot/bin assignment) is 1, that particular plot has been assigned to that particular bin of the density or basal area assignment matrix. If the solver assigns a decision variable a value of 0, the situation that variable represents is FALSE. For example, if a decision variable from the second class (trees kept vs. removed) is 0, that particular tree has been cut down.

Each decision variable is represented as x_n. The order of decision variables in this problem (that is, x_1, x_2, x_3...) is arbitrary in that we get to choose what order to put them in. However, the order we decide upon is very important to keep track of. Knowing which decision variable is which will allow us to fill out the constraint matrix and interpret the output of the optimization solver. The first 18 decision variables in the vector correspond to the density assignment matrix (a variable in each cell of the 6 x 3 assignment matrix). The next 18 decision variables correspond to the basal area assignment matrix (a variable in each cell of the 6 x 3 assignment matrix). The last N_TREES decision variables in the vector correspond to the individual trees. The following subsections go through the decision variables in order.

3.1 Density assignment matrix decision variables

The first 18 decision variables (x_1 through x_{18}) represent the density assignment matrix. Table 7 depicts the decision variables associated with each cell of the density assignment matrix.

	Density bin 1	**Density bin 2**	**Density bin 3**
Plot 1	x_1	x_2	x_3
Plot 2	x_4	x_5	x_6
Plot 3	x_7	x_8	x_9
Plot 4	x_{10}	x_{11}	x_{12}
Plot 5	x_{13}	x_{14}	x_{15}
Plot 6	x_{16}	x_{17}	x_{18}

Table 7: Decision variables associated with the density assignment matrix

So, for example, x_1 refers to plot 1 being assigned to bin 1 of the density assignment matrix, which means that plot 1 contains one to three total trees after thinning. x_{18} refers to plot 3 being assigned to bin 3 of the density assignment matrix, which means that plot 3 contains seven to nine total trees after thinning.

3.2 Basal area assignment matrix decision variables

The next 18 decision variables (x_{19} through x_{36}) represent the basal area assignment matrix. Table 8 depicts the decision variables associated with each cell of the basal area assignment matrix.

	Basal area bin 1	**Basal area bin 2**	**Basal area bin 3**
Plot 1	x_{19}	x_{20}	x_{21}
Plot 2	x_{22}	x_{23}	x_{24}
Plot 3	x_{25}	x_{26}	x_{27}
Plot 4	x_{28}	x_{29}	x_{30}
Plot 5	x_{31}	x_{32}	x_{33}
Plot 6	x_{34}	x_{35}	x_{36}

Table 8: Decision variables associated with the basal area assignment matrix

So, for example, x_{19} refers to plot 1 being assigned to bin 1 of the basal area assignment matrix, which means that plot 1 contains 0.2-5.0 m^2 total basal area after thinning. x_{36} refers to plot 3 being assigned to bin 3 of the basal area assignment matrix, which means that plot 3 contains 10.1-15.0 m^2 total basal area after thinning.

3.3 Individual tree assignment matrix decision variables

The last N_TREES decision variables (x_{37} through x_{64}) correspond to the individual trees in the plots that are eligible to be removed. If you check out the "removable_trees" data.frame (generated in line 243 of the Rmd), you can see that we have 28 removable trees that need to be represented by decision variables. The order of decision variables 37-64 are the same order as the trees in this data.frame.

Table 9 replicates the "removable_trees" data.frame with the decision variable number for each tree in the last column.

basal area	**plot**	**cost_coefficient**	**eligible_for_removal**	**decision_variable_#**
0.755306	1	0.379721	TRUE	x_{37}
1.369058	1	0.187526	TRUE	x_{38}
1.29647	1	0.075288	TRUE	x_{39}
0.346621	1	0.336291	TRUE	x_{40}
1.930411	1	0.401934	TRUE	x_{41}
1.878314	1	0.57024	TRUE	x_{42}
0.368072	1	0.980045	TRUE	x_{43}
1.04935	2	0.857089	TRUE	x_{44}
1.112922	2	0.112927	TRUE	x_{45}
1.166098	2	0.166455	TRUE	x_{46}
0.736422	2	0.329131	TRUE	x_{47}
0.884034	2	0.080892	TRUE	x_{48}
0.705924	2	0.753684	TRUE	x_{49}
1.798609	3	0.845185	TRUE	x_{50}
1.002511	3	0.659487	TRUE	x_{51}
0.621592	3	0.537216	TRUE	x_{52}
2.007961	3	0.137834	TRUE	x_{53}
1.285411	3	0.527111	TRUE	x_{54}
1.248226	4	0.444742	TRUE	x_{55}
0.821385	4	0.113764	TRUE	x_{56}
2.426398	5	0.481401	TRUE	x_{57}
1.799726	5	0.328576	TRUE	x_{58}
2.079124	5	0.375395	TRUE	x_{59}
1.075571	5	0.186621	TRUE	x_{60}
1.79419	5	0.704185	TRUE	x_{61}
0.81289	5	0.175679	TRUE	x_{62}
1.610356	5	0.714769	TRUE	x_{63}
1.491417	6	0.818768	TRUE	x_{64}

Table 9: Tree decision variables

So, for example, x_{37} refers to the tree in plot 1 with a basal area of 0.755 m^2. x_{64} refers to the only removable tree in plot 6 with a basal area of 1.49 m^2.

4 Constraints

This section is a companion to Section 4.6 of the Rmd file. The Rmd contains more thorough explanations of what each constraint means and how it is depicted in the constraint matrix. This section in this document will depict each constraint in mathematical form accompanied by a brief description.

If a variable is represented without an explicit coefficient, the coefficient of that variable in that constraint is 1. If a variable is not represented in a particular constraint, the coefficient of that variable in that constraint is 0.

4.1 Constraints 1-6: 6 row constraints of the density assignment matrix

- 1. $x_1 + x_2 + x_3 = 1$
  - Sum of the first row of the density assignment matrix must equal one (plot 1 can only be assigned to one density bin)
  - The optimization solver will assign some decision variables a value of 1 (if those things are TRUE) and some decision variables a value of 0 (if those things are FALSE). This constraint forces the solver to make one of these three variables equal to 1 and the other two variables equal to 0.
- 2. $x_4 + x_5 + x_6 = 1$
  - Sum of the second row of the density assignment matrix must equal one (plot 2 can only be assigned to one density bin)
- 3. $x_7 + x_8 + x_9 = 1$
  - Sum of the third row of the density assignment matrix must equal one (plot 3 can only be assigned to one density bin)
- 4. $x_{10} + x_{11} + x_{12} = 1$
  - Sum of the fourth row of the density assignment matrix must equal one (plot 4 can only be assigned to one density bin)

- 5. $x_{13} + x_{14} + x_{15} = 1$
  - Sum of the fifth row of the density assignment matrix must equal one (plot 5 can only be assigned to one density bin)
- 6. $x_{16} + x_{17} + x_{18} = 1$
  - Sum of the sixth row of the density assignment matrix must equal one (plot 6 can only be assigned to one density bin)

4.2 Constraints 7-9: 3 lower bound column constraints of the density assignment matrix

- 7. $x_1 + x_4 + x_7 + x_{10} + x_{13} + x_{16} \geq 1$
  - Sum of the first column of the density assignment matrix must be greater than or equal to one (there must be at least one plot assigned to density bin 1)
- 8. $x_2 + x_5 + x_8 + x_{11} + x_{14} + x_{17} \geq 1$
  - Sum of the second column of the density assignment matrix must be greater than or equal to one (there must be at least one plot assigned to density bin 2)
- 9. $x_3 + x_6 + x_9 + x_{12} + x_{15} + x_{18} \geq 1$
  - Sum of the third column of the density assignment matrix must be greater than or equal to one (there must be at least one plot assigned to density bin 3)

4.3 Constraints 10-12: 3 upper bound column constraints of the density assignment matrix

- 10. $x_1 + x_4 + x_7 + x_{10} + x_{13} + x_{16} \leq 3$
  - Sum of the first column of the density assignment matrix must be less than or equal to three (there must no more than three plots assigned to density bin 1)

- 11. $x_2 + x_5 + x_8 + x_{11} + x_{14} + x_{17} \leq 3$
  - Sum of the second column of the density assignment matrix must be less than or equal to three (there must be no more than three plots assigned to density bin 2)
- 12. $x_3 + x_6 + x_9 + x_{12} + x_{15} + x_{18} \leq 3$
  - Sum of the third column of the density assignment matrix must be less than or equal to three (there must be no more than three plots assigned to density bin 3)

4.4 Constraints 13-18: plots must have greater than or equal to the lower limit of the number of trees in the density bin they are assigned to

- 13. $x_1 + 4x_2 + 7x_3$ - x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - $x_{43} \leq 0$
  - Lower bound of the density assignment bins in plot 1. If plot 1 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 1 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 1 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There are no trees in plot 1 that can't be removed.
- 14. $x_4 + 4x_5 + 7x_6$ - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - $x_{49} \leq 0$
  - Lower bound of the density assignment bins in plot 2. If plot 2 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 2 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 2 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There are no trees in plot 2 that can't be removed.
- 15. $x_7 + 4x_8 + 7x_9$ - x_{50} - x_{51} - x_{52} - x_{53} - $x_{54} \leq 2$

- Lower bound of the density assignment bins in plot 3. If plot 3 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 3 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 3 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There are two trees in plot 3 that can't be removed.

- 16. $x_{10} + 4x_{11} + 7x_{12}$ - x_{55} - $x_{56} \leq 1$

    - Lower bound of the density assignment bins in plot 4. If plot 4 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 4 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 4 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There is one tree in plot 4 that can't be removed.

- 17. $x_{13} + 4x_{14} + 7x_{15}$ - x_{57} - x_{58} - x_{59} - x_{60} - x_{61} - x_{62} - $x_{63} \leq 1$

    - Lower bound of the density assignment bins in plot 5. If plot 5 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 5 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 5 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There is one tree in plot 5 that can't be removed.

- 18. $x_{16} + 4x_{17} + 7x_{18}$ - $x_{64} \leq 1$

    - Lower bound of the density assignment bins in plot 6. If plot 6 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 6 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 6 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There is one tree in plot 6 that can't be removed.

4.5 Constraints 19-24: plots must have less than or equal to the upper limit of the number of trees in the density bin they are assigned to

- 19. x_1 + $4x_2$ + $7x_3$ - x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - x_{43} ≥ 0
  - Upper bound of the density assignment bins in plot 1. If plot 1 is assigned to density bin 1, it must contain less than or equal to three trees after thinning. If plot 1 is assigned to density bin 2, it must contain less than or equal to six trees after thinning. If plot 1 is assigned to density bin 3, it must contain less than or equal to nine trees after thinning. There are no trees in plot 1 that can't be removed.

- 20. x_4 + $4x_5$ + $7x_6$ - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - x_{49} ≥ 0
  - Upper bound of the density assignment bins in plot 2. If plot 2 is assigned to density bin 1, it must contain greater than or equal to one tree after thinning. If plot 2 is assigned to density bin 2, it must contain greater than or equal to four trees after thinning. If plot 2 is assigned to density bin 3, it must contain greater than or equal to seven trees after thinning. There are no trees in plot 2 that can't be removed.

- 21. x_7 + $4x_8$ + $7x_9$ - x_{50} - x_{51} - x_{52} - x_{53} - x_{54} ≥ 2
  - Upper bound of the density assignment bins in plot 3. If plot 3 is assigned to density bin 1, it must contain less than or equal to three trees after thinning. If plot 3 is assigned to density bin 2, it must contain less than or equal to six trees after thinning. If plot 3 is assigned to density bin 3, it must contain less than or equal to nine trees after thinning. There are two trees in plot 3 that can't be removed.

- 22. x_{10} + $4x_{11}$ + $7x_{12}$ - x_{55} - x_{56} ≥ 1
  - Upper bound of the density assignment bins in plot 4. If plot 4 is assigned to density bin 1, it must contain less than or equal to three trees after thinning. If plot 4 is assigned to density bin 2, it

must contain less than or equal to six trees after thinning. If plot 4 is assigned to density bin 3, it must contain less than or equal to nine trees after thinning. There is one tree in plot 4 that can't be removed.

- 23. x_{13} + $4x_{14}$ + $7x_{15}$ - x_{57} - x_{58} - x_{59} - x_{60} - x_{61} - x_{62} - x_{63} ≥ 1
  - Upper bound of the density assignment bins in plot 5. If plot 5 is assigned to density bin 1, it must contain less than or equal to three trees after thinning. If plot 5 is assigned to density bin 2, it must contain less than or equal to six trees after thinning. If plot 5 is assigned to density bin 3, it must contain less than or equal to nine trees after thinning. There is one tree in plot 5 that can't be removed.
- 24. x_{16} + $4x_{17}$ + $7x_{18}$ - x_{64} ≥ 1
  - Upper bound of the density assignment bins in plot 6. If plot 6 is assigned to density bin 1, it must contain less than or equal to three trees after thinning. If plot 6 is assigned to density bin 2, it must contain less than or equal to six trees after thinning. If plot 6 is assigned to density bin 3, it must contain less than or equal to nine trees after thinning. There is one tree in plot 6 that can't be removed.

4.6 Constraints 25-30: 6 row constraints of the basal area assignment matrix

- 25. $x_{19} + x_{20} + x_{21} = 1$
  - Sum of the first row of the basal area assignment matrix must equal one (plot 1 can only be assigned to one basal area bin)
- 26. $x_{22} + x_{23} + x_{24} = 1$
  - Sum of the second row of the basal area assignment matrix must equal one (plot 2 can only be assigned to one basal area bin)
- 27. $x_{25} + x_{26} + x_{27} = 1$

- Sum of the third row of the basal area assignment matrix must equal one (plot 3 can only be assigned to one basal area bin)

- 28. $x_{28} + x_{29} + x_{30} = 1$
  - Sum of the fourth row of the basal area assignment matrix must equal one (plot 4 can only be assigned to one basal area bin)

- 29. $x_{31} + x_{32} + x_{33} = 1$
  - Sum of the fifth row of the basal area assignment matrix must equal one (plot 5 can only be assigned to one basal area bin)

- 30. $x_{34} + x_{35} + x_{36} = 1$
  - Sum of the sixth row of the basal area assignment matrix must equal one (plot 6 can only be assigned to one basal area bin)

4.7 Constraints 31-33: 3 lower bound column constraints of the basal area assignment matrix

- 31. $x_{19} + x_{22} + x_{25} + x_{28} + x_{31} + x_{34} \geq 1$
  - Sum of the first column of the basal area assignment matrix must be greater than or equal to one (there must be at least one plot assigned to basal area bin 1)

- 32. $x_{20} + x_{23} + x_{26} + x_{29} + x_{32} + x_{35} \geq 1$
  - Sum of the second column of the basal area assignment matrix must be greater than or equal to one (there must be at least one plot assigned to basal area bin 2)

- 33. $x_{21} + x_{24} + x_{27} + x_{30} + x_{33} + x_{36} \geq 1$
  - Sum of the third column of the basal area assignment matrix must be greater than or equal to one (there must be at least one plot assigned to basal area bin 3)

4.8 Constraints 34-36: 3 upper bound column constraints of the basal area assignment matrix

- 34. $x_{19} + x_{22} + x_{25} + x_{28} + x_{31} + x_{34} \leq 3$
  - Sum of the first column of the basal area assignment matrix must be less than or equal to three (there must be no more than three plots assigned to basal area bin 1)
- 35. $x_{20} + x_{23} + x_{26} + x_{29} + x_{32} + x_{35} \leq 3$
  - Sum of the second column of the basal area assignment matrix must be less than or equal to three (there must be no more than three plots assigned to basal area bin 2)
- 36. $x_{21} + x_{24} + x_{27} + x_{30} + x_{33} + x_{36} \leq 3$
  - Sum of the third column of the basal area assignment matrix must be less than or equal to three (there must be no more than three plots assigned to basal area bin 3)

4.9 Constraints 37-42: plots must have greater than or equal to the lower limit of the total basal area in the basal area bin they are assigned to

- 37. $0.2x_{19} + 5.1x_{20} + 10.1x_{21}$ - $0.755306x_{37}$ - $1.369058x_{38}$ - $1.29647x_{39}$ - $0.346621x_{40}$ - $1.930411x_{41}$ - $1.878314x_{42}$ - $0.368072x_{43} \leq 0$
  - Lower bound of the basal area assignment bins in plot 1. If plot 1 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 1 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 1 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 0 square meters of basal area in plot 1 that can't be removed.

- 38. $0.2x_{22}$ + $5.1x_{23}$ + $10.1x_{24}$ - $1.04935x_{44}$ - $1.112922x_{45}$ - $1.166098x_{46}$ - $0.736422x_{47}$ - $0.884034x_{48}$ - $0.705924x_{49} \leq 0$

  - Lower bound of the basal area assignment bins in plot 2. If plot 2 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 2 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 2 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 0 square meters of basal area in plot 2 that can't be removed.

- 39. $0.2x_{25}$ + $5.1x_{26}$ + $10.1x_{27}$ - $1.798609x_{50}$ - $1.002511x_{51}$ - $0.621592x_{52}$ - -$2.007961x_{53}$ - $1.285411x_{54} \leq 5.824184$

  - Lower bound of the basal area assignment bins in plot 3. If plot 1 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 3 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 3 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 5.824184 square meters of basal area in plot 3 that can't be removed.

- 40. $0.2x_{28}$ + $5.1x_{29}$ + $10.1x_{30}$ - $1.248226x_{55}$ - $0.821385x_{56} \leq 2.645363$

  - Lower bound of the basal area assignment bins in plot 4. If plot 4 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 4 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 4 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 2.645363 square meters of basal area in plot 4 that can't be removed.

- 41. $0.2x_{31}$ + $5.1x_{32}$ + $10.1x_{33}$ - $2.426398x_{57}$ - $1.799726x_{58}$ - $2.079124x_{59}$ - $1.075571x_{60}$ - $1.79419x_{61}$ - $0.81289x_{62}$ - $1.610356x_{63} \leq 2.630840$
  - Lower bound of the basal area assignment bins in plot 5. If plot 5 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 5 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 5 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 2.630840 square meters of basal area in plot 5 that can't be removed.
- 42. $0.2x_{34}$ + $5.1x_{35}$ + $10.1x_{36}$ - $1.491417x_{64} \leq 2.976763$
  - Lower bound of the basal area assignment bins in plot 6. If plot 6 is assigned to basal area bin 1, it must contain greater than or equal to 0.2 square meters of total basal area after thinning. If plot 6 is assigned to basal area bin 2, it must contain greater than or equal to 5.1 square meters of total basal area after thinning. If plot 6 is assigned to basal area bin 3, it must contain greater than or equal to 10.1 square meters of total basal area after thinning. There are 2.976763 square meters of basal area in plot 6 that can't be removed.

4.10 Constraints 43-48: plots must have less than or equal to the upper limit of the total basal area in the basal area bin they are assigned to

- 43. $5x_{19}$ + $10x_{20}$ + $15x_{21}$ - $0.755306x_{37}$ - $1.369058x_{38}$ - $1.29647x_{39}$ - $0.346621x_{40}$ - $1.930411x_{41}$ - $1.878314x_{42}$ - $0.368072x_{43} \geq 0$
  - Upper bound of the basal area assignment bins in plot 1. If plot 1 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 1 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 1 is assigned to basal area bin 3, it must contain less than or equal to

15 square meters of total basal area after thinning. There are 0 square meters of basal area in plot 1 that can't be removed.

- 44. $5x_{22}$ + $10x_{23}$ + $15x_{24}$ - $1.04935x_{44}$ - $1.112922x_{45}$ - $1.166098x_{46}$ - $0.736422x_{47}$ - $0.884034x_{48}$ - $0.705924x_{49} \geq 0$
  - Upper bound of the basal area assignment bins in plot 2. If plot 2 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 2 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 2 is assigned to basal area bin 3, it must contain less than or equal to 15 square meters of total basal area after thinning. There are 0 square meters of basal area in plot 2 that can't be removed.
- 45. $5x_{25}$ + $10x_{26}$ + $15x_{27}$ - $1.798609x_{50}$ - $1.002511x_{51}$ - $0.621592x_{52}$ - -$2.007961x_{53}$ - $1.285411x_{54} \geq 5.824184$
  - Upper bound of the basal area assignment bins in plot 3. If plot 3 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 3 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 3 is assigned to basal area bin 3, it must contain less than or equal to 15 square meters of total basal area after thinning. There are 5.824184 square meters of basal area in plot 3 that can't be removed.
- 46. $5x_{28}$ + $10x_{29}$ + $15x_{30}$ - $1.248226x_{55}$ - $0.821385x_{56} \geq 2.645363$
  - Upper bound of the basal area assignment bins in plot 4. If plot 4 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 4 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 4 is assigned to basal area bin 3, it must contain less than or equal to 15 square meters of total basal area after thinning. There are 2.645363 square meters of basal area in plot 4 that can't be removed.

- 47. $5x_{31}$ + $10x_{32}$ + $15x_{33}$ - $2.426398x_{57}$ - $1.799726x_{58}$ - $2.079124x_{59}$ - $1.075571x_{60}$ - $1.79419x_{61}$ - $0.81289x_{62}$ - $1.610356x_{63} \geq 2.630840$
  - Upper bound of the basal area assignment bins in plot 5. If plot 5 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 5 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 5 is assigned to basal area bin 3, it must contain less than or equal to 15 square meters of total basal area after thinning. There are 2.630840 square meters of basal area in plot 5 that can't be removed.
- 48. $5x_{34}$ + $10x_{35}$ + $15x_{36}$ - $1.491417x_{64} \geq 2.976763$
  - Upper bound of the basal area assignment bins in plot 6. If plot 6 is assigned to basal area bin 1, it must contain less than or equal to 5 square meters of total basal area after thinning. If plot 6 is assigned to basal area bin 2, it must contain less than or equal to 10 square meters of total basal area after thinning. If plot 6 is assigned to basal area bin 3, it must contain less than or equal to 15 square meters of total basal area after thinning. There are 2.976763 square meters of basal area in plot 6 that can't be removed.

5 Objective function

$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9 + 0x_{10} + 0x_{11} + 0x_{12} + 0x_{13} + 0x_{14} + 0x_{15} + 0x_{16} + 0x_{17} + 0x_{18} + 0x_{19} + 0x_{20} + 0x_{21} + 0x_{22} + 0x_{23} + 0x_{24} + 0x_{25} + 0x_{26} + 0x_{27} + 0x_{28} + 0x_{29} + 0x_{30} + 0x_{31} + 0x_{32} + 0x_{33} + 0x_{34} + 0x_{35} + 0x_{36} + 0.379721x_{37} + 0.187526x_{38} + 0.075288x_{39} + 0.336291x_{40} + 0.401934x_{41} + 0.57024x_{42} + 0.980045x_{43} + 0.857089x_{44} + 0.112927x_{45} + 0.166455x_{46} + 0.329131x_{47} + 0.080892x_{48} + 0.753684x_{49} + 0.845185x_{50} + 0.659487x_{51} + 0.537216x_{52} + 0.137834x_{53} + 0.527111x_{54} + 0.444742x_{55} + 0.113764x_{56} + 0.481401x_{57} + 0.328576x_{58} + 0.375395x_{59} + 0.186621x_{60} + 0.704185x_{61} + 0.175679x_{62} + 0.714769x_{63} + 0.818768x_{64}$

6 Putting the optimization problem together

Combining the objective function with the constraints gives us the mathematical form of the problem:

Minimize

$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9 + 0x_{10} + 0x_{11} + 0x_{12} + 0x_{13} + 0x_{14} + 0x_{15} + 0x_{16} + 0x_{17} + 0x_{18} + 0x_{19} + 0x_{20} + 0x_{21} + 0x_{22} + 0x_{23} + 0x_{24} + 0x_{25} + 0x_{26} + 0x_{27} + 0x_{28} + 0x_{29} + 0x_{30} + 0x_{31} + 0x_{32} + 0x_{33} + 0x_{34} + 0x_{35} + 0x_{36} + 0.379721x_{37} + 0.187526x_{38} + 0.075288x_{39} + 0.336291x_{40} + 0.401934x_{41} + 0.57024x_{42} + 0.980045x_{43} + 0.857089x_{44} + 0.112927x_{45} + 0.166455x_{46} + 0.329131x_{47} + 0.080892x_{48} + 0.753684x_{49} + 0.845185x_{50} + 0.659487x_{51} + 0.537216x_{52} + 0.137834x_{53} + 0.527111x_{54} + 0.444742x_{55} + 0.113764x_{56} + 0.481401x_{57} + 0.328576x_{58} + 0.375395x_{59} + 0.186621x_{60} + 0.704185x_{61} + 0.175679x_{62} + 0.714769x_{63} + 0.818768x_{64}$

Subject to

$x_1 + x_2 + x_3 = 1$

$x_4 + x_5 + x_6 = 1$

$x_7 + x_8 + x_9 = 1$

$x_{10} + x_{11} + x_{12} = 1$

$x_{13} + x_{14} + x_{15} = 1$

$x_{16} + x_{17} + x_{18} = 1$

$x_1 + x_4 + x_7 + x_{10} + x_{13} + x_{16} \geq 1$

$x_2 + x_5 + x_8 + x_{11} + x_{14} + x_{17} \geq 1$

$x_3 + x_6 + x_9 + x_{12} + x_{15} + x_{18} \geq 1$

$x_1 + x_4 + x_7 + x_{10} + x_{13} + x_{16} \leq 3$

$x_2 + x_5 + x_8 + x_{11} + x_{14} + x_{17} \leq 3$

x_3 + x_6 + x_9 + x_{12} + x_{15} + $x_{18} \leq 3$

x_1 + $4x_2$ + $7x_3$ - x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - $x_{43} \leq 0$

x_4 + $4x_5$ + $7x_6$ - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - $x_{49} \leq 0$

x_7 + $4x_8$ + $7x_9$ - x_{50} - x_{51} - x_{52} - x_{53} - $x_{54} \leq 2$

x_{10} + $4x_{11}$ + $7x_{12}$ - x_{55} - $x_{56} \leq 1$

x_{13} + $4x_{14}$ + $7x_{15}$ - x_{57} - x_{58} - x_{59} - x_{60} - x_{61} - x_{62} - $x_{63} \leq 1$

x_{16} + $4x_{17}$ + $7x_{18}$ - $x_{64} \leq 1$

x_1 + $4x_2$ + $7x_3$ - x_{37} - x_{38} - x_{39} - x_{40} - x_{41} - x_{42} - $x_{43} \geq 0$

x_4 + $4x_5$ + $7x_6$ - x_{44} - x_{45} - x_{46} - x_{47} - x_{48} - $x_{49} \geq 0$

x_7 + $4x_8$ + $7x_9$ - x_{50} - x_{51} - x_{52} - x_{53} - $x_{54} \geq 2$

x_{10} + $4x_{11}$ + $7x_{12}$ - x_{55} - $x_{56} \geq 1$

x_{13} + $4x_{14}$ + $7x_{15}$ - x_{57} - x_{58} - x_{59} - x_{60} - x_{61} - x_{62} - $x_{63} \geq 1$

x_{16} + $4x_{17}$ + $7x_{18}$ - $x_{64} \geq 1$

$x_{19} + x_{20} + x_{21} = 1$

$x_{22} + x_{23} + x_{24} = 1$

$x_{25} + x_{26} + x_{27} = 1$

$x_{28} + x_{29} + x_{30} = 1$

$x_{31} + x_{32} + x_{33} = 1$

$x_{34} + x_{35} + x_{36} = 1$

$x_{19} + x_{22} + x_{25} + x_{28} + x_{31} + x_{34} \geq 1$

$x_{20} + x_{23} + x_{26} + x_{29} + x_{32} + x_{35} \geq 1$

$x_{21} + x_{24} + x_{27} + x_{30} + x_{33} + x_{36} \geq 1$

$x_{19} + x_{22} + x_{25} + x_{28} + x_{31} + x_{34} \leq 3$

$x_{20} + x_{23} + x_{26} + x_{29} + x_{32} + x_{35} \leq 3$

$x_{21} + x_{24} + x_{27} + x_{30} + x_{33} + x_{36} \leq 3$

$0.2x_{19} + 5.1x_{20} + 10.1x_{21}$ - $0.755306x_{37}$ - $1.369058x_{38}$ - $1.29647x_{39}$ - $0.346621x_{40}$ - $1.930411x_{41}$ - $1.878314x_{42}$ - $0.368072x_{43} \leq 0$

$0.2x_{22} + 5.1x_{23} + 10.1x_{24}$ - $1.04935x_{44}$ - $1.112922x_{45}$ - $1.166098x_{46}$ - $0.736422x_{47}$ - $0.884034x_{48}$ - $0.705924x_{49} \leq 0$

$0.2x_{25} + 5.1x_{26} + 10.1x_{27}$ - $1.798609x_{50}$ - $1.002511x_{51}$ - $0.621592x_{52}$ - - $2.007961x_{53}$ - $1.285411x_{54} \leq 5.824184$

$0.2x_{28} + 5.1x_{29} + 10.1x_{30}$ - $1.248226x_{55}$ - $0.821385x_{56} \leq 2.645363$

$0.2x_{31} + 5.1x_{32} + 10.1x_{33}$ - $2.426398x_{57}$ - $1.799726x_{58}$ - $2.079124x_{59}$ - $1.075571x_{60}$ - $1.79419x_{61}$ - $0.81289x_{62}$ - $1.610356x_{63} \leq 2.630840$

$0.2x_{34} + 5.1x_{35} + 10.1x_{36}$ - $1.491417x_{64} \leq 2.976763$

$5x_{19} + 10x_{20} + 15x_{21}$ - $0.755306x_{37}$ - $1.369058x_{38}$ - $1.29647x_{39}$ - $0.346621x_{40}$ - $1.930411x_{41}$ - $1.878314x_{42}$ - $0.368072x_{43} \geq 0$

$5x_{22} + 10x_{23} + 15x_{24}$ - $1.04935x_{44}$ - $1.112922x_{45}$ - $1.166098x_{46}$ - $0.736422x_{47}$ - $0.884034x_{48}$ - $0.705924x_{49} \geq 0$

$5x_{25} + 10x_{26} + 15x_{27}$ - $1.798609x_{50}$ - $1.002511x_{51}$ - $0.621592x_{52}$ - -2.007961x_{53} - $1.285411x_{54} \geq 5.824184$

$5x_{28}$ + $10x_{29}$ + $15x_{30}$ - $1.248226x_{55}$ - $0.821385x_{56} \geq 2.645363$

$5x_{31}$ + $10x_{32}$ + $15x_{33}$ - $2.426398x_{57}$ - $1.799726x_{58}$ - $2.079124x_{59}$ - $1.075571x_{60}$ - $1.79419x_{61}$ - $0.81289x_{62}$ - $1.610356x_{63} \geq 2.630840$

$5x_{34}$ + $10x_{35}$ + $15x_{36}$ - $1.491417x_{64} \geq 2.976763$