EEE 391 - Analytical HW 02

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EEE391-01

Q1

a) z transforms of $x_1[n]$ and $x_2[n]$ are as follows:

- The z transform of $x_1[n]$

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

This is a geometric series and converges to $\dfrac{1}{1-\frac{1}{2}z^{-1}}$ for $|z| > \frac{1}{2}$

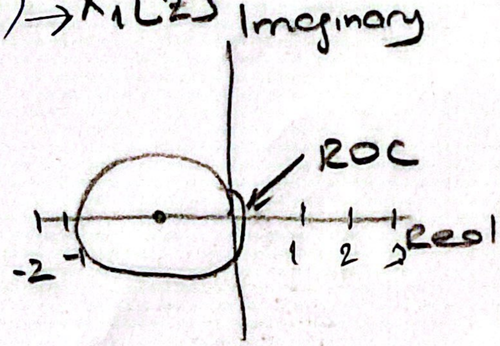
- The z transform of $x_2[n] = -\left(\frac{1}{2}\right)^n u[n-1]$ will be:

$$X_2(z) = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n}$$

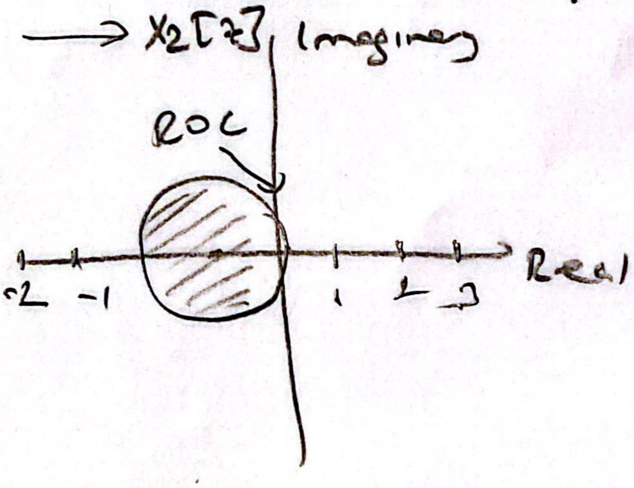
→ This series converges to $-\dfrac{1}{1-2z}$ for $|z| < \frac{1}{2}$.

(*) It's worth to note that $X_2[n]$ is a time reversed and negated version of $X_1[n]$. It's z transform will be similar but reflected about the origin and ROC will be inverse, which can be seen in part (b).

b) → $X_1[z]$



→ $X_2[z]$



Q1 - Continued

c) For $x_3[n] = 2u[n]$, the z transform is

$$X_3(z) = 2\sum_{n=0}^{\infty} z^{-n}$$

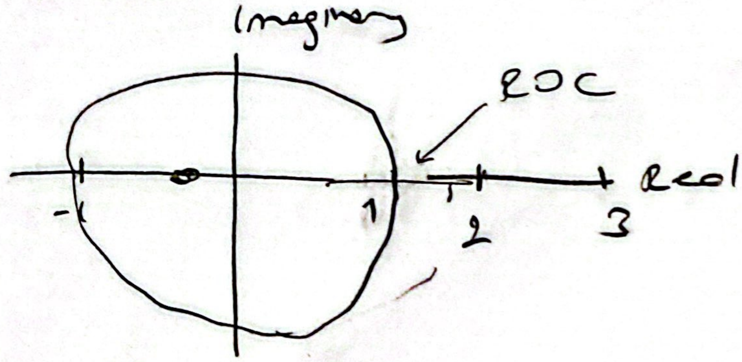
This is a geometric series that converges to $\frac{2}{1-z^{-1}}$ for $|z|>1$.

For $x_4[n] = -(2)^n u[-n-1]$

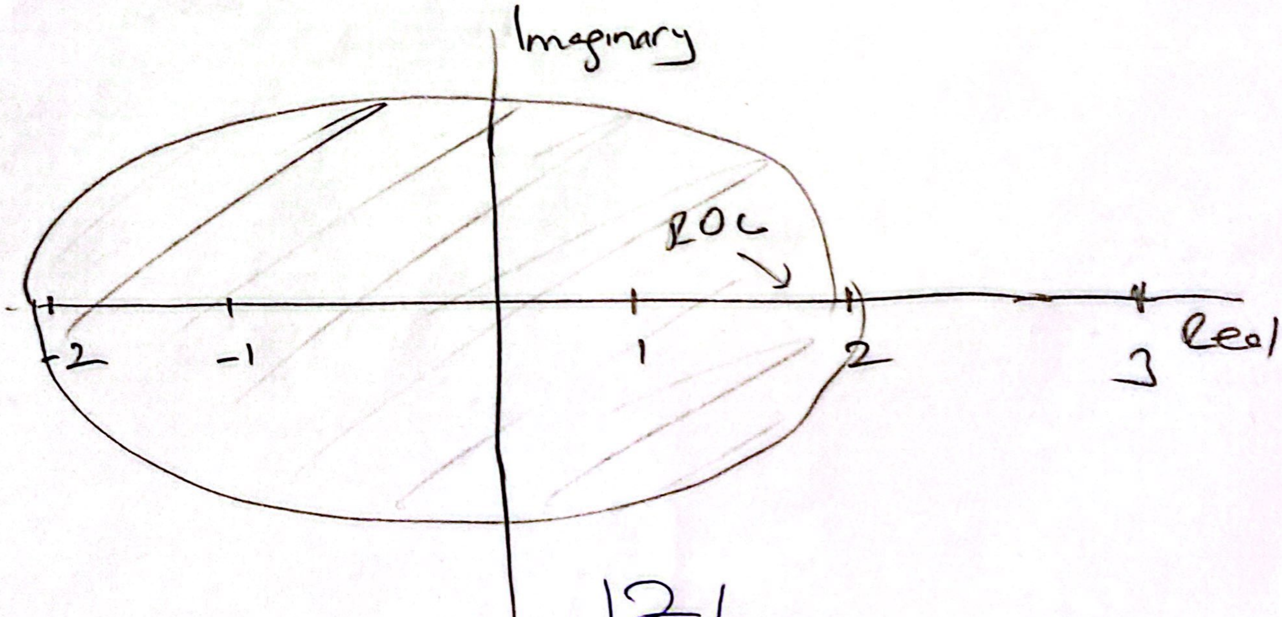
$$X_4(z) = -2\sum_{n=-\infty}^{-1} z^{-n}$$

The convergence is for $|z|<1$.

→ $X_3[z]$



→ $X_4[z]$



$|z|$

Q2

a) Given the frequency response

$$H(e^{j\omega}) = (1 + e^{-j\omega})\,(1 - e^{-j2\pi/3} e^{-j\omega})(1 + e^{-j2\pi/3} e^{-j\omega})$$

We can express each term in time domain

(1) (*) $1 + e^{-j\omega}$ corresponds to $\delta[n] + \delta[n-1]$

(2) (*) $1 \; e^{-j2\pi/3} e^{-j\omega}$ corresponds to $\delta[n] - e^{-j2\pi/3}\,\delta[n-1]$

(3) (*) $1 + e^{-j2\pi/3} e^{-j\omega}$ corresponds to $\delta[n] + e^{-j2\pi/3}\,\delta[n-1]$

Then, we get $h[n]$ as, which is simply multiplication of (1), (2) and (3

$$h[n] = (\delta[n] + \delta[n-1]) * [\delta[n] - e^{-j2\pi/3}\,\delta[n-1]]\,.$$
$$* \left(\delta[n] + e^{-j2\pi/3}, \delta[n-1]\right)$$

$$\rightarrow y[n] = \sum_k h[k] \cdot x[n-k]$$

$$y[n] = x[n] - e^{-j\frac{4\pi}{3}}\, x[n-2]$$

b) Note that

$$y[n] = (h[n] * x[n]) = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

which simplifies as

$$y[n] = h[n] * \delta[n] = h[n]$$

From the result obtained from part (a), we get:

$$h[n] = \delta[n] - e^{-j\frac{4\pi}{3}}\,\delta[n-2]$$

$$y[n] = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n=1 \\ -e^{-j\frac{4\pi}{3}} & \text{for } n=2 \\ 0 & \text{for } \forall n \neq 0,1,2 \end{cases}$$

Q2 - Continued

Q2 - C

Checking the equation:

$1+e^{-jw}=0$ implies $e^{-jw}=-1$, $w=\pi$

$1-e^{-j2\pi/3}e^{-jw}=0$ implies $e^{-jw}=e^{j2\pi/3}$, $w=-2\pi/3$

$1+e^{-j2\pi/3}e^{-jw}=0$ implies $e^{-jw}=-e^{j2\pi/3}$, $w=2\pi/3$

We must consider the span $-\pi \le w \le \pi$

So, the complete set of w values for which $y[n]=0$ when the input is form of $Ae^{j\theta}e^{jwn}$ are

$$w=\left\{-\pi,-\frac{2\pi}{3},\frac{2\pi}{3},\pi\right\}$$

Q3

a) $X(z)=\dfrac{1}{1+\frac{1}{2}z^{-1}}$ with ROC $|z|>\frac{1}{2}$

$$X(z)=\frac{1}{1-rz^{-1}} \Longleftrightarrow x[n]=r^n u[n]$$

For this case, $r=-\frac{1}{2}$, so the inverse z-transform is

$$\rightarrow x[n]=\left(-\frac{1}{2}\right)^n u[n]$$

b) $$X(z)=\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}=\frac{\cancel{\left(1-\frac{1}{2}z^{-1}\right)}}{\cancel{\left(1-\frac{1}{2}z^{-1}\right)}\left(1+\frac{1}{2}z^{-1}\right)}$$

$$X(z)=\frac{1}{1+\frac{1}{2}z^{-1}} \Longrightarrow x[n]=\left(\frac{1}{2}\right)^n u[n]$$

Q3-Continued

Q3-C

$$X(z) = \frac{1-az^{-1}}{z^{-1}-a} = \frac{1-az^{-1}}{-a\left(1-\left(\frac{1}{a}\right)z^{-1}\right)} = \frac{az^{-1}-1}{a\left(1-\left(\frac{1}{a}\right)\right)z^{-1}}$$

$$= \frac{az^{-1}}{a\left(1-\left(\frac{1}{a}\right)z^{-1}\right)} - \frac{1}{a\left(1-\left(\frac{1}{a}\right)z^{-1}\right)} = \frac{z^{-1}}{1-\left(\frac{1}{a}\right)z^{-1}} - \frac{1}{a\left(1-\left(\frac{1}{a}\right)z^{-1}\right)}$$

$$x[n] = \boxed{\left(\frac{1}{a}\right)^{n-1} u[n-1] - \frac{1}{a}\left(\frac{1}{a}\right)^{n} u[n]}$$

Q4

Calculating the z-transforms of $x_1[n]$ and $x_2[n]$

$$X_1(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad \text{for } |z| > \frac{1}{2}$$

$$X_2(z) = \frac{1}{1+\frac{1}{3}z^{-1}} \quad \text{for } |z| > \frac{1}{3}$$

for $x_2[-n+1]$, we need to apply the time reversal and shifting property in z-domain

$$x[-n] \leftrightarrow x\left(\frac{1}{z}\right) \quad \text{and} \quad x[n-k] \leftrightarrow z^{-k} x(z)$$

so for $x_2[n+1]$, its z transform will be $z X_2\left(\frac{1}{z}\right)$

Therefore $x_2[n+1] = \dfrac{z}{1+3z}$

For x_1, involves a time shift, therefore

$$x_1[n+3] \Rightarrow z^3 X_1(z)$$

$$Y(z) = z^3 X_1(z) - \frac{z}{1+3z}$$

$$Y(z) = z^3 \cdot \frac{1}{1+\frac{1}{2z}} - \frac{z}{1+3z}$$

$$Y(z) = z^3 \cdot \frac{2z}{2z+1} \cdot \frac{z}{1+3z} = \frac{2z^5}{6z^2+5z+1}$$

$$Y(z) = \frac{2z^5}{6z^2+5z+1} \quad \text{for } (2z+1).(1+3z) \neq 0$$

$$z \neq -\frac{1}{2} \text{ and } z \neq -\frac{1}{3}$$

Q5

a) Given the difference equation for the casual LTI System

$$y[n] - \frac{\sqrt{2}}{2} y[n-1] + \frac{1}{4} y[n-2] = x[n] - x[n-1]$$

And it's worth to note that

$$\delta[n] = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

for

$$y[n] - \frac{\sqrt{2}}{2} y[n-1] + \frac{1}{4} y[n-2] = \delta[n] - \delta[n-1]$$

→ For $n=0$, the equation simplifies to

$$y[0] - \frac{\sqrt{2}}{2} y[-1] + \frac{1}{4} y[-2] = 1 - 0$$

(*) System is causal, $y[n] = 0$ for $n<0$, therefore we get $y[0] = 1$.

→ For $n=1$,

$$y[1] - \frac{\sqrt{2}}{2} y[0] + \frac{1}{4} y[-1] = 0 - 1$$

$$y[1] = -1 + \frac{\sqrt{2}}{2}$$

and for $n=2$, $y[2] = \frac{\sqrt{2}}{2} y[1] - \frac{1}{4}$

→ In general form $n>1$, the equation is

$$y[n] = \frac{\sqrt{2}}{2} y[n-1] - \frac{1}{4} y[n-2]$$

Assume $y[n] = z^n$

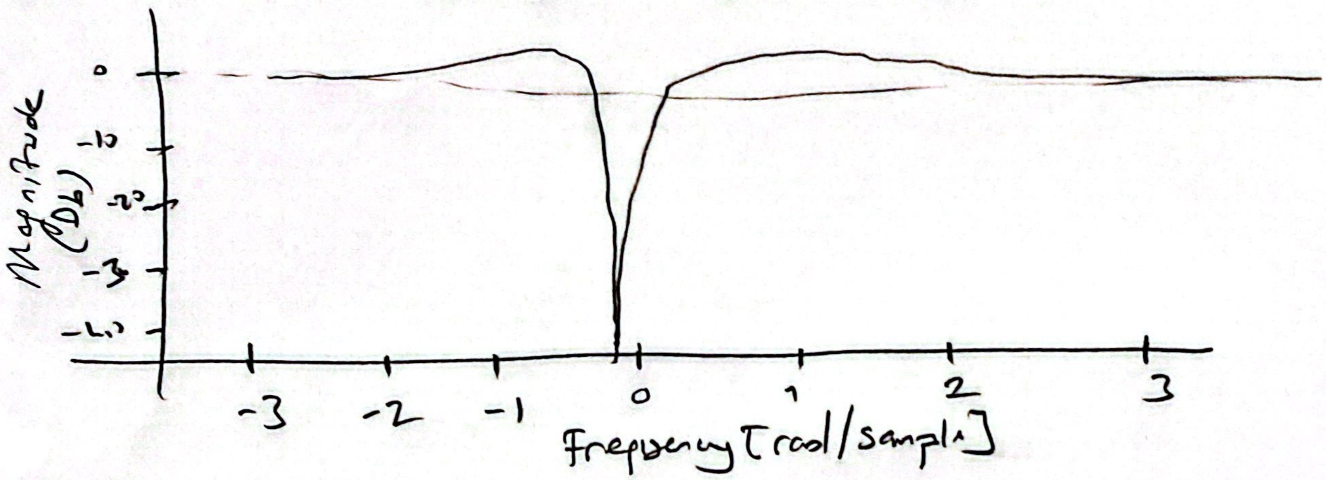
$$z^n - \frac{\sqrt{2}}{2} z^{n-1} + \frac{1}{4} z^{n-2} = 0$$

Q5 – Continued

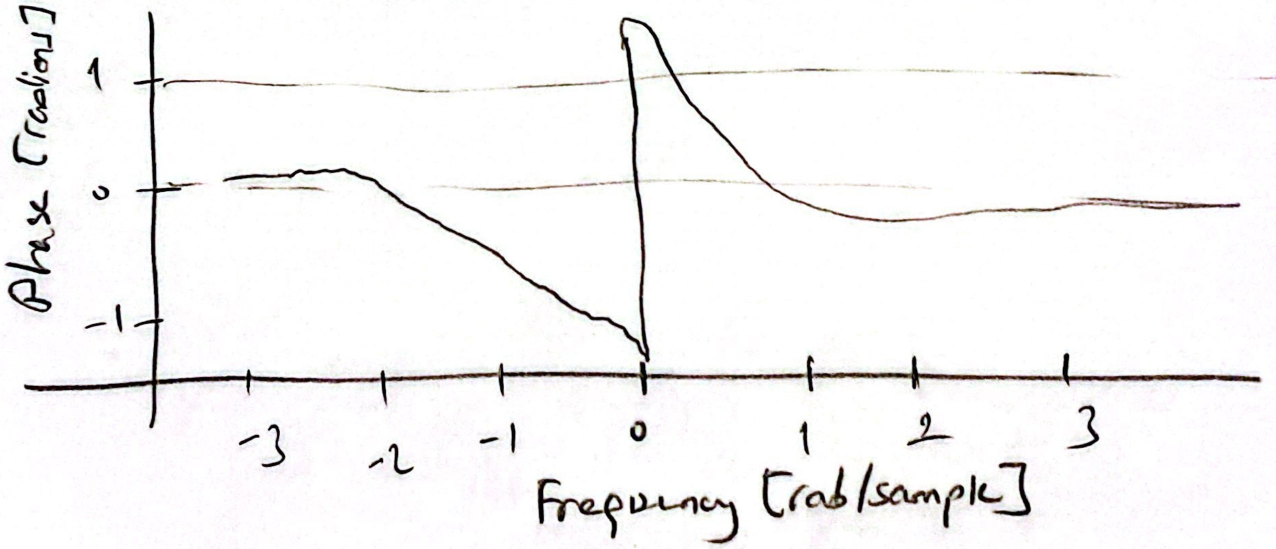
After the calculations done in MATLAB to calculate roots, at the final; we got

$$h[n] = (0.5)^n \left(\cos\left(-\frac{\pi}{4} n\right) + 1.8284 \sin\left(-\frac{\pi}{4} n\right)\right)$$

b) → Log Magnitude of Frequency Response



→ Phase of Frequency Response.



Q6

The Nyquist rate given for the $x(t)$ as w_s

a) $x(t) + x(t-1) \longrightarrow$ This is a time shifted version of $x(t)$ added to $x(t)$. Time shifting does not affect the frequency content of a signal, so the Nyquist rate remains w_s.

b) $\frac{d\,x(t)}{dt} \rightarrow$ Differentiating a signal in time domain corresponds to multiplying its spectrum by $j\omega$ in the frequency domain. This operation can increase the maximum frequency of the signal, potentially doubling it. So, the new Nyquist rate will be up to $2w_s$

c) $x^2(t) \rightarrow$ Squaring the signal in the time domain corresponds to convolving its spectrum with itself in the frequency domain. This operation can extend the bandwidth up to twice the original maximum frequency. Thus, Nyquist rate could be up to $2w_s$.

d) $x(t)\cos(w_0 t) \rightarrow$ Multiplying $x(t)$ by a cosine function of frequency results sum and difference frequencies. The highest frequency component will be the sum of w_0 and the highest frequency in $x(t)$ if $x(t)$ contains frequencies up to $w_0/2$, then Nyquist rate after modulation will be $3w_0/2$. If the max frequency $x(t)$ is w_0, the Nyquist rate would be $2w_0$.

Q7

a) $x_1(t) = x(1-t) + x(-1-t)$

- The term $x(1-t)$ represent a time reversal and a shift, which in the fourier domain translates to $X(-jw) \cdot e^{-jw}$
- The term $x(-1-t)$ represents a time reversal and a shift in opposite direction, which translate to $X(-jw)\, e^{jw}$

$$X_1(jw) = X(-jw) \cdot 2\cos(w)$$

b) $x_2(t) = x(3t-6)$

- The term $x(3t)$ is scaling in a time by $\frac{1}{3}$, which in the Fourier domain is represented by scaling of $\frac{1}{|3|}$ and a multiplication of $X(j\frac{w}{3})$, so $\frac{1}{3}X(j\frac{w}{3})$
- The -6 represents a shift, which translates to $e^{-jw(-6)} = e^{j6w}$

$$X_2(jw) = \frac{1}{3} X\left(j\frac{w}{3}\right) \cdot e^{j6w}$$

Q8

a) Given the equation

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

The z-transform of $x[n] = u[n]$ is $X(z) = \frac{1}{1-z^{-1}}$ for $|z|>1$.

Taking the z-transform of both sides of the difference equation, we get:

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z)$$

$$Y(z) = \frac{X(z)}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-\frac{1}{2}z^{-1}}$$

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Q8 - Continued

Applying partial fraction expansion on $Y(z)$, we get

$$y[n] = 1 - 0.5\,(0.5)^n\, u[n] + 2\cdot(1)^n \cdot u[n]$$

$$\boxed{y[n] = 3u[n] - 0.5^n\, u[n]}$$

b) For the input $x[n] = e^{j(\pi/4)n}\, u[n]$, we need to find output $y[n]$.

The z-transform of $x[n] = e^{j(\pi/4)}\, u[n]$ is:

$$X(z) = \frac{1}{1 - e^{j\pi/4} z^{-1}} \quad \text{for } |z| > |e^{j\pi/4}|$$

Using the same difference equation and substituting $X(z)$ we get:

$$Y(z) = \frac{X(z)}{1 - \frac{1}{2} z^{-1}} = \frac{1}{1 - e^{j\pi/4} z^{-1}} \cdot \frac{1}{1 - \frac{1}{2} z^{-1}}$$

We can simplify this expression and then perform an inverse z transform to find $y[n]$. Calculations are left to be done in MATLAB.

C) By calculating exact $y[n]$, We can compare its magnitude and phase to be frequency response at $w = \frac{\pi}{4}$ by substituting $z = e^{jw}$ into $Y(z)$ and evaluating at $w = \frac{\pi}{4}$.