EEE 391 Homework 01

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Section 01

Q1

By using Euler's formula, we can get:

(1) $(1-j)\,e^{j\theta} = (1-j)(\cos\theta + j\sin\theta) = \cos\theta + j\sin\theta - j\cos\theta - j^2\sin\theta$

(2) $(1-j)\,e^{-j\theta} = (1-j)(\cos\theta - j\sin\theta) = \cos\theta - j\sin\theta - j\cos\theta + j^2\sin\theta$

Real part of equation (1) gives: $Re[(1)] = \cos\theta + \sin\theta$

Real part of equation (2) gives: $Re[(2)] = \cos\theta - \sin\theta$

Then, main equation becomes

$$\cos\theta + \sin\theta = 1 - (\cos\theta - \sin\theta)$$

$$2\cos\theta = 1$$

$$\cos\theta = \frac{1}{2}$$

Then, θ takes values in: $\theta = \frac{\pi}{3} + 2\pi k\,,\ k \in R$

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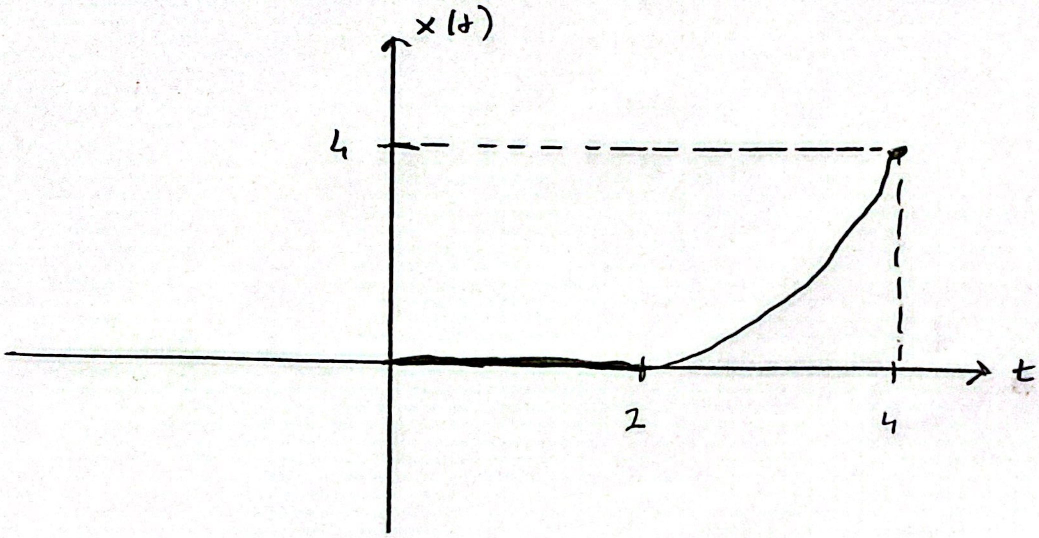
Q2

a) Periodic function $x(t)$ can be drawn as



b) The a_k coefficients for a Fourier series of a periodic function is given by

$$a_k = \frac{2}{T}\int_0^T x(t)\cos\left(\frac{2\pi k t}{T}\right)dt$$

$$a_k = \frac{2}{4}\int_2^4 (t-2)^2 \cos\left(\frac{2\pi k t}{4}\right)dt$$

Note that integral from 0 to 2 gives 0; due to 0 coefficient

$$a_k = \frac{1}{2}\int_2^4 (t-2)^2 \cos\left(\frac{\pi k t}{2}\right)dt$$

Integration by parts : $\int f g' = fg - \int f' g$

$f = t-2, \quad g' = \cos\left(\frac{k\pi t}{2}\right)$

$f' = 1 \qquad g = \dfrac{2\sin\left(\frac{\pi k t}{2}\right)}{\pi k}$

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Q2 - Continued

$$= \frac{2(t-2)\sin\left(\frac{\pi k t}{2}\right)}{\pi k} - \int \frac{2\sin\left(\frac{\pi k t}{2}\right)}{\pi k} dt$$

$\vdots$

$$= \frac{\pi k(t-2)\sin\left(\frac{\pi k t}{2}\right) + 2\cos\left(\frac{\pi k t}{2}\right)}{\pi^2 k^2} + C$$

c) The DC coefficient a_0 is the average value of the function over one period

$$a_0 = \frac{1}{4}\int_2^4 (t-2)\,dt$$

$$a_0 = \frac{1}{4}\int_2^4 (t^2-4t+4)\,dt$$

$$a_0 = \frac{1}{4}\left[\frac{t^3}{3} - 2t^2 + 4t\right]_2^4$$

$$a_0 = \frac{1}{4}\left[\left(\frac{64}{3} - 32 + 16\right) - \left(\frac{8}{3} - 8 + 8\right)\right]$$

$$a_0 = \frac{1}{4}\left[\frac{64}{3} - \frac{8}{3} - 16\right]$$

$$a_0 = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}$$

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Q2 - Continued

d) Let's differentiate $x(t)$ to get $y(t)$

$$y(t) = \frac{d}{dx} x(t) = \frac{d}{dt} (t-2)^2 = 2(t-2)$$

For b_2, we get

$$b_2 = \frac{2}{4} \int_2^4 2(t-2) \sin\left(\frac{2\pi \cdot 2t}{4}\right) dt$$

$$b_2 = \frac{1}{2} \int_2^4 2(t-2) \sin(\pi t)\, dt$$

$$b_2 = \int_2^4 (t-2) \sin(\pi t)\, dt$$

Note that this is a symmetric function around 3, so:

$\underline{b_2 = 0}$

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Q3

a) By using sampling theorem,

$f_s = 500\,Hz$, $\quad 0.26\pi n = \frac{2\pi n f}{f_s}$ (1) ($w = 0.26\pi \frac{radians}{sample}$)

$$0.13 = \frac{f}{f_s} \quad (2)$$

$0.13 f_s = f \longrightarrow f = 65\,Hz$ (3)

So, two possible continuous-time signals could be:

$\longrightarrow x_1(t) = 5\cos(2\pi \cdot 65t + 30°)$

$\longrightarrow x_2(t) = 5\cos(2\pi \cdot (500-65)t + 30°) = 5\cos(2\pi \cdot 435t + 30°)$

b) Based on figure 2, we can construct:

$$y(t) = 2e^{-j\frac{\pi}{2}}\cos(2\pi \cdot 200t) + 8e^{-j\frac{3\pi}{4}}\cos(2\pi \cdot 700t)$$
$$+ 8e^{j\frac{3\pi}{4}}\cos(2\pi \cdot (-700)t) + 2e^{j\frac{\pi}{2}}\cos(2\pi \cdot (-200)t)$$

By using $e^{j\theta} = \cos\theta + j\sin\theta$ equation, we get

$$Re[y(t)] = 2\cos\left(2\pi \cdot 200t - \frac{\pi}{2}\right) + 8\cos\left(2\pi \cdot 700t - \frac{3\pi}{4}\right)$$
$$+ 8\cos\left(2\pi \cdot 700t + \frac{3\pi}{4}\right) + 2\cos\left(2\pi \cdot 200t + \frac{\pi}{2}\right)$$

which simplifies

$$y(t) = 2\cos\left(2\pi \cdot 200t - \frac{\pi}{2}\right) + 8\cos\left(2\pi \cdot 700t - \frac{3\pi}{4}\right)$$

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Q4

a) A cosine signal $\cos(2\pi f t)$ is periodic with period $T = \frac{1}{f}$

$\cos(3\pi(500)t) \longrightarrow T_1 = \frac{2}{1500}$ sec

$\cos(3\pi(1000)t) \longrightarrow T_2 = \frac{2}{3000}$ sec

If T_1/T_2 is a rational number, we can say signal is periodic

$\frac{T_1}{T_2} = \frac{2/1500}{2/3000} = 2$, meaning that signal is periodic.

The period of the signal is the LCM of T_1 and T_2

So period of $x(t)$ is $\frac{2}{1500} = \frac{1}{750}$ sec

b) Note that

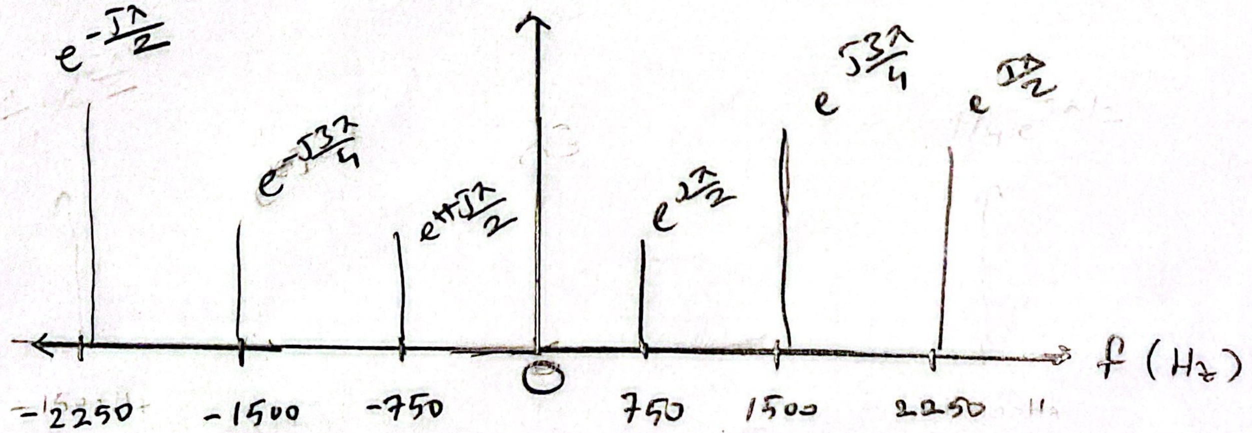
$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

Therefore non-constant part of $x(t)$ becomes

$\frac{1}{2}\left[\cos(-1500\pi t) + \cos(4500\pi t)\right]$

By adding constant '3' in the equation.

$x(t) = 3\cos(3\pi(1000)t) + \frac{1}{2}\left[\cos(-1500\pi t) + \cos(4500\pi t)\right]$



Q4 - Continued

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c) The sampling rate f_s should satisfy Nyquist - Shannon theorem to ensure $y(t) = x(t)$. The sampling rate must be at least twice the maximum frequency present in the signal to avoid aliasing.

$$f_s \geq 2 \cdot f_{max}$$

$$f_s \geq 2 \cdot 2250 \text{ Hz}$$

$$f_s \geq 4500 \text{ Hz}$$

Therefore, the sampling rate should be at least 4500 samples per second to ensure $y(t) = x(t)$.

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Q5

We can use Convolution to solve this problem

n:	...	-3	-2	-1	0	1	2	3	...
$x[n]$:	...	-1	0	1	-1	0	1	-1	...
$h(n)$:					-2	2	4	6	

⋮

So, final output sequence $y[n]$ will be

$$y[n] = \begin{cases} -2 & \text{if } n = 3k \\ 2 & \text{if } n = 3k+1 \\ 0 & \text{if } n = 3k+2 \end{cases}$$

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Q6

- A system is $\boxed{\text{Linear}}$ if it satisfies properties of superposition (additivity and homogenity)
- A system is $\boxed{\text{time-invariant}}$ if a time shift in the input signal causes an identical time shift in the output signal.
- A system is $\boxed{\text{causal}}$ if the output at any time depends on present and past inputs, not future inputs.

a) $y[n] = 2x[n]\cos(\pi n)$

→ Linear: Yes, both additivity and homogeneity satisfied

→ Time-invariant: No, because a time shift in $x[n]$ will only results not the same time shift in $y[n]$

→ Causal: Yes, the output $y[n]$ only depends on the current input $x[n]$.

b) $y[n] = x[n] - x[2n+1]$

→ Linear: Yes

$$y[n] = (\alpha_1 x_1[n] - \alpha_2 x_2[n]) - (\alpha_1 x_1[2n+1] + \alpha_2 x_2[2n+1])$$

$$\alpha_1\underbrace{(x_1[n] - x_1[2n+1])}_{y_1[n]} + \alpha_2\underbrace{(x_2[n] - x_2[2n+1])}_{y_2[n]}$$

→ Time-invariant: Yes.

If $x[n] \to y[n]$, let $V[n] = x[n-n_0]$

output $= V[n] - V[2n+1] = x[n-n_0] - x[2n-2n_0+1]$

$y[n-n_0] = x[n-n_0] - x[2(n-n_0)+1]$.

They are same.

Q6 - Continued

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→ Causal: No

$y[n]$ at $n=n_0$ depends only on $x[n]$ at $n=n_0$, $n=2n_0+1$

Since it depends on the future, it is not causal.

c) $y[n] = -x[n]^2$

→ Linear: No

Let $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$$y[n] = -(\alpha_1 x_1[n] + \alpha_2 x_2[n])^2$$

$$y[n] = -\left(\alpha_1^2 x_1[n]^2 + 2\alpha_1\alpha_2 x_1[n]x_2[n] + \alpha_2^2 x_2[n]^2\right)$$

Which is not equal to $y[n] = \underline{-x[n]^2}$

→ Time-invariant: Yes

The output depeds only on $x[\ ]$ at "n", so

$$y[n-n_0] = -x[n-n_0]^2$$

→ Causal: Yes

The output depends only the current value of $x[n]$

d) $y[n] = x[n] - u[n]$

Assuming that $U[n]$ is a unit step function, which is defined to be 1 for $n \geq 0$ and 0 for $n < 0$

→ Linear: No

$u(n)$ is not a linear operation in this context.

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Q6 - Continued

→ Time-invariant: Yes,

$y[n] = x[n-n_0] + u[n]$

→ Causal: Yes,

The output at any given time depends only on the values of $x[n]$ and $u[n]$ at that time or earlier.

e) $y[n] = 2^{x[n]}$

→ Linear: No, because exp operation violates both additivity and homogeneity

→ Time-invariant: Yes, since the form of system does not change with time shifts in $x[n]$

→ Causal: Yes, because $y[n]$ only depends on the current value of $x[n]$

f) $y[n] = 5 + x[n]$

→ Linear: No,
if $x_1[n] \Rightarrow y_1[n]$, test $2x_1[n] \rightarrow 2y_1[n]$

$$2(y_1[n]) = 2(5 + x_1[n]) = 10 + 2x_1[n] \neq 2x_1[n]$$

→ Time-invariant: Yes
$y[n-n_0] = 5 + x[n-n_0]$ is always true

→ Causal: Yes,
$y[n]$ at $n=n_0$ depends only on $x[n]$ at $n=n_0$

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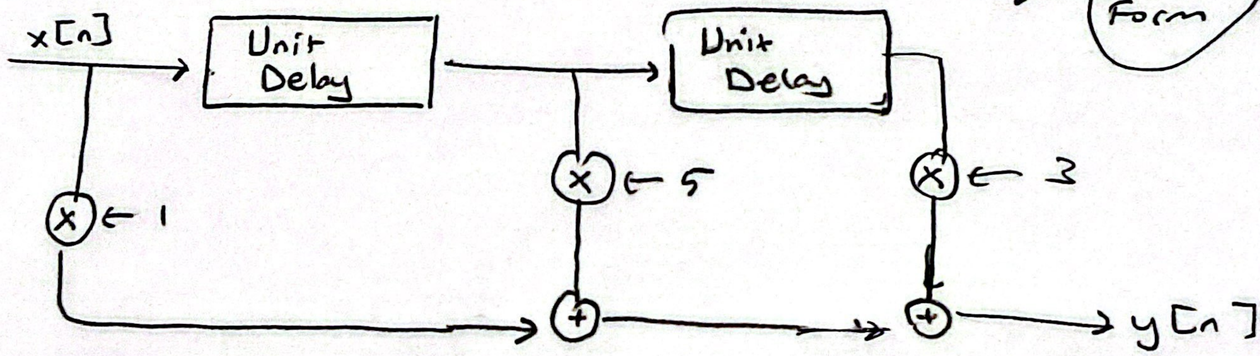
Q7

a) $y[n] = x[n] + 5x[n-1] + 3x[n-3]$

Direct Form



b) Given the system

$y[n] = x[n] + 5x[n-1] + 3x[n-3]$

When we apply the impulse function $\delta[n]$

$h[n] = \delta[n] + 5\delta[n-1] + 3\delta[n-3]$

That means

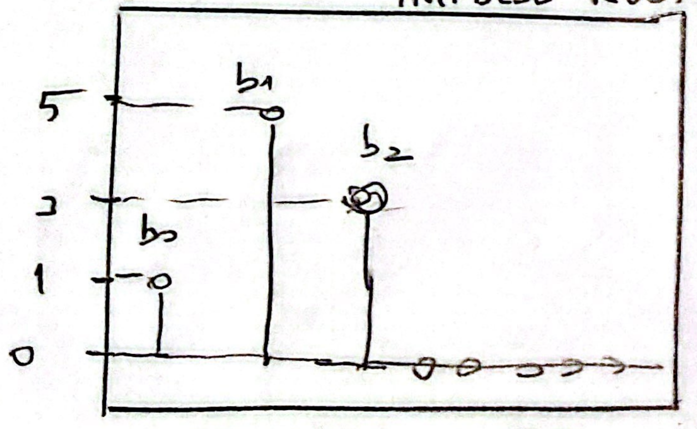
$h[0] = 1$ (since $\delta[0] = 1$)

$h[1] = 5$ (since $\delta[-1] = 0$, but with the shift $\delta[0] = 1$)

$h[3] = 3$ (since $\delta[-3] = 0$, but " " " " ")

$h[n] = 0$ for all other values of n.

IMPULSE RESPONSE h[n]



TIME INDEX (n)