Classical Mechanics

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Preface.

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1 Rigid Body Rotation

1.1 Basics of rigid bodies

A body is a collection of particles. We consider a set of N particles with positions $\vec{r}_a(t)$ and masses m_a, where $a = 1, \ldots, N$. Each particle obeys the equation of motion

$$m_a \ddot{\vec{r}}_a = \sum_{b \neq a} \vec{F}_{ab} + \vec{F}_a^{\text{ext}}, \tag{1.1}$$

where $\vec{F}_{ab}$ is the force on particle a due to b. It is mathematically convenient to define $\vec{F}_{aa} \equiv 0$ and to take the sum over all b instead of over $b \neq a$.

The total mass M of the body is

$$M = \sum_a m_a. \tag{1.2}$$

The centre of mass $\vec{R}$ is defined as

$$\vec{R} = \frac{1}{M} \sum_a m_a \vec{r}_a \quad \Rightarrow \quad \dot{\vec{R}} = \frac{1}{M} \sum_a m_a \dot{\vec{r}}_a, \tag{1.3}$$

which is just a weighted average with weights of m_a/M. In the continuum limit, we have

$$m_a \to \rho(\vec{r}) \quad \Rightarrow \quad M = \int_V \rho(\vec{r})\, \mathrm{d}^3 r\,, \quad \text{and} \quad \vec{R} = \frac{1}{M} \int_V \rho(\vec{r}) \vec{r}\, \mathrm{d}^3 r\,. \tag{1.4}$$

Similarly, the total momentum $\vec{P}$ of the body is

$$\vec{P} = \sum_a \vec{p}_a = \sum_a m_a \dot{\vec{r}}_a = M \dot{\vec{R}}. \tag{1.5}$$

It then follows that

$$\dot{\vec{P}} = \sum_a m_a \ddot{\vec{r}}_a = \sum_a \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{\text{ext}} \right) = \sum_{ab} \vec{F}_{ab} + \sum_a \vec{F}_a^{\text{ext}}. \tag{1.6}$$

By Newton's third law, we have

$$\vec{F}_{ab} = -\vec{F}_{ba} \quad \Rightarrow \quad \sum_{ab} \vec{F}_{ab} \equiv 0. \tag{1.7}$$

Hence, we have

$$\dot{\vec{P}} = M \ddot{\vec{R}} = \sum_a \vec{F}_a^{\text{ext}}. \tag{1.8}$$

So, total momentum of the body is conserved if no external forces act on the body with internal forces having no effect.

The total angular momentum $\vec{L}$ is

$$\vec{L} = \sum_a \vec{\ell}_a = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a. \tag{1.9}$$

The rate of change of total angular momentum is then

$$\begin{aligned}\dot{\vec{L}} = \sum_a m_a \frac{\mathrm{d}}{\mathrm{d}t}\Big(\vec{r}_a \times \dot{\vec{r}}_a\Big) &= \sum_a m_a \Big(\dot{\vec{r}}_a \times \dot{\vec{r}}_a + \vec{r}_a \times \ddot{\vec{r}}_a\Big) \\ &= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a \\ &= \sum_a \vec{r}_a \times \left(\sum_b \vec{F}_{ab} + \vec{F}_a^{\text{ext}}\right) \\ &= \sum_{ab} \vec{r}_a \times \vec{F}_{ab} + \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}}.\end{aligned} \tag{1.10}$$

Let's inspect the first term. It follows from Newton's third law that

$$\sum_{ab} \vec{r}_a \times \vec{F}_{ab} = -\sum_{ab} \vec{r}_a \times \vec{F}_{ba}.$$

Since the indices a and b are dummy indices, we can relabel $a \leftrightarrow b$ to obtain

$$\sum_{ab} \vec{r}_a \times \vec{F}_{ab} = \sum_{ab} -\vec{r}_b \times \vec{F}_{ab} \quad \Rightarrow \quad \sum_{ab} \vec{r}_a \times \vec{F}_{ab} = \frac{1}{2}\sum_{ab} (\vec{r}_a - \vec{r}_b) \times \vec{F}_{ab}. \tag{1.11}$$

If the inter-particle forces are *central*, the force $\vec{F}_{ab}$ points in the direction of the separation vector $\vec{r}_{ab} \equiv \vec{r}_a - \vec{r}_b$. Hence, the contribution above becomes identically zero and we get

$$\dot{\vec{L}} = \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}} \equiv \vec{\tau}, \tag{1.12}$$

where $\vec{\tau}$ is the total torque on the body due to external forces. We again see that the inter-particle forces are *averaged out* as far as the total angular momentum is concerned.

Finally, we look at how the motion of the body decouples into centre of mass motion and relative motion about the centre of mass. We define the position relative to the centre of mass:

$$\vec{r}^{\,*} = \vec{r} - \vec{R}. \tag{1.13}$$

Then, we have

$$\sum_a m_a \vec{r}_a^{\,*} = \sum_a m_a \Big(\vec{r}_a - \vec{R}\Big) = M\vec{R} - M\vec{R} \equiv 0, \tag{1.14}$$

as expected. The total linear momentum about the centre of mass is also identically zero,

$$\vec{P}^{\,*} = \sum_a m_a \vec{r}_a^{\,*} = \sum_a m_a(\vec{r}_a - \vec{R}) = M\vec{R} - M\vec{R} \equiv 0 \quad \Rightarrow \quad \dot{\vec{P}}^{\,*} \equiv 0. \tag{1.15}$$

The total angular momentum is expressed in terms of $\vec{r}_a^{\,*}$ as follows:

$$\begin{aligned}\vec{L} = \sum_a m_a \vec{r}_a \times \dot{\vec{r}}_a &= \sum_a m_a \Big(\vec{r}_a^{\,*} + \vec{R}\Big) \times \Big(\dot{\vec{r}}_a^{\,*} + \dot{\vec{R}}\Big) \\ &= \sum_a m_a \Big[\vec{r}_a^{\,*} \times \dot{\vec{r}}_a^{\,*} + \vec{r}_a^{\,*} \times \dot{R} + \vec{R} \times \dot{\vec{r}}_a^{\,*} + \vec{R} \times \dot{\vec{R}}\Big] \\ &= \sum_a m_a \vec{r}_a^{\,*} \times \dot{\vec{r}}_a^{\,*} + \left(\sum_a m_a \vec{r}_a^{\,*}\right) \times \dot{R} + \vec{R} \times \left(\sum_a m_a \dot{\vec{r}}_a^{\,*}\right) + M\vec{R} \times \dot{\vec{R}} \\ &= \sum_a m_a \vec{r}_a^{\,*} \times \dot{\vec{r}}_a^{\,*} + \vec{R} \times \vec{P} \\ &\equiv \vec{L}^{\,*} + \vec{R} \times \vec{P},\end{aligned} \tag{1.16}$$

where we defined the angular momentum about the central mass $\vec{L}^* = \sum_a m_a \vec{r}_a^* \times \dot{\vec{r}}_a^*$. The rate of change of $\vec{L}^*$ is given by

$$\begin{aligned}
\dot{\vec{L}}^* = \sum_a m_a \vec{r}_a^* \times \ddot{\vec{r}}_a^* &= \sum_a m_a \Big(\vec{r}_a - \vec{R}\Big) \times \Big(\ddot{\vec{r}}_a - \ddot{\vec{R}}\Big) \\
&= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a - \left(\sum_a m_a \vec{r}_a\right) \times \ddot{\vec{R}} - \vec{R} \times \left(\sum_a m_a \ddot{\vec{r}}_a\right) + M\vec{R} \times \ddot{\vec{R}} \\
&= \sum_a m_a \vec{r}_a \times \ddot{\vec{r}}_a - M\vec{R} \times \ddot{\vec{R}} \\
&= \sum_a \vec{r}_a \times \vec{F}_a^{\text{ext}} = \vec{R} \times \left(\sum_a \vec{F}_a^{\text{ext}}\right) \\
&= \sum_a \Big(\vec{r}_a - \vec{R}\Big) \times \vec{F}_a^{\text{ext}} \\
&= \sum_a \vec{r}_a^* \times \vec{F}_a^{\text{ext}},
\end{aligned} \tag{1.17}$$

where we assumed central inter-particle forces.

The total kinetic energy of the body is

$$\begin{aligned}
T = \frac{1}{2}\sum_a m_a \Big(\dot{\vec{r}}_a \cdot \dot{\vec{r}}_a\Big) &= \frac{1}{2}\sum_a m_a \Big(\dot{\vec{r}}_a^* + \dot{\vec{R}}\Big) \cdot \Big(\dot{\vec{r}}_a^* + \dot{\vec{R}}\Big) \\
&= \frac{1}{2}\sum_a m_a \Big(\dot{\vec{r}}_a^* \cdot \dot{\vec{r}}_a^* + 2\dot{\vec{r}}_a^* \cdot \dot{\vec{R}} + \dot{\vec{R}} \cdot \dot{\vec{R}}\Big) \\
&= \frac{1}{2}\sum_a m_a \dot{\vec{r}}_a^* \cdot \dot{\vec{r}}_a^* + \frac{1}{2} M \dot{\vec{R}} \cdot \dot{\vec{R}}.
\end{aligned} \tag{1.18}$$

The first term is the kinetic energy about the centre of mass and the second term the kinetic energy of the centre of mass. Hence, we see that both the kinetic energy and the angular momentum decouple into a centre of mass part and a part about the centre of mass.

We conclude the section by stating that for a rigid body, $|\vec{r}_a - \vec{r}_b| \equiv 0$ for all a and b by definition.

1.2 Rotation

Consider a rigid body rotating about a point P. Define a fixed space frame $\{\hat{e}'_a\}$ and moving body frame $\{\hat{e}_a(t)\}$ which rotates with the body. We choose the basis vectors to be orthonormal:

$$\hat{e}'_a \cdot \hat{e}'_b = \delta_{ab}, \quad \text{and} \quad \hat{e}_a(t) \cdot \hat{e}_b(t) = \delta_{ab} \ \forall\, t. \tag{1.19}$$

Proposition. *For all t, there exists a unique, orthogonal matrix $R(t)$ such that*

$$\hat{e}_a(t) = R_{ab}(t)\hat{e}'_b. \tag{1.20}$$

Proof. Construct $R(t)$ by the rule

$$\hat{e}_a(t) = R_{ab}(t)\hat{e}'_b \iff R_{ab}(t) = \hat{e}_a(t) \cdot \hat{e}'_b.$$

Then, by orthonormality we have

$$\delta_{ab} = \hat{e}_a(t) \cdot \hat{e}_b(t) = R_{ac}(t)\hat{e}'_c \cdot R_{bd}(t)\hat{e}'_d = R_{ac}(t)R_{bd}(t)\delta_{cd} = R_{ac}R_{bc} = R_{ac}R^T_{cb} = (RR^T)_{ab}.$$

Therefore $RR^T = 1$ and hence R is orthogonal. □

So, the rotation of a rigid body may be completely specified by a 3×3 orthogonal matrix $R(t)$.

1.3 Angular velocity

Any point $\vec{r}$ in the body can be written in space or body frame:

$$\begin{aligned} \vec{r}(t) &= \tilde{r}_a(t)\tilde{\hat{e}}_a \quad \text{space frame,} \\ &= r_a\hat{e}_a(t) \quad \text{body frame.} \end{aligned}$$

By using the rotation matrix, we can relate the components $\tilde{r}_a(t)$ and r_a as follows:

$$\begin{aligned} & \vec{r} = r_a\hat{e}_a(t) = r_aR_{ab}(t)\tilde{\hat{e}}_b = \tilde{r}_b(t)\tilde{\hat{e}}_b \\ \Rightarrow \quad & \tilde{r}_b(t) = r_aR_{ab}(t) \Longleftrightarrow R_{ab}(t)\tilde{r}_b(t) = r_a. \end{aligned}$$

The velocity is

$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}\tilde{r}_a(t)}{\mathrm{d}t}\tilde{\hat{e}}_a = r_a\frac{\mathrm{d}\hat{e}_a(t)}{\mathrm{d}t}. \tag{1.21}$$

Let's have a look at the $\dot{\hat{e}}_a(t)$ term:

$$\frac{\mathrm{d}\hat{e}_a}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[R_{ab}(t)\tilde{\hat{e}}_b\Big] = \frac{\mathrm{d}R_{ab}}{\mathrm{d}t}\tilde{\hat{e}}_b. \tag{1.22}$$

Substituting $\tilde{\hat{e}}_b = R^{-1}_{bc}\hat{e}_c = R_{cb}\hat{e}_c$ yields

$$\frac{\mathrm{d}\hat{e}_a}{\mathrm{d}t} = \dot{R}_{ab}R_{cb}\hat{e}_c \equiv \omega_{ac}\hat{e}_c, \tag{1.23}$$

where we defined $\omega_{ac} \equiv \dot{R_{ab}}R_{cb}$.

Proposition. *ω is antisymmetric.*

Proof. Since R is orthogonal, we have

$$\delta_{ac} = R_{ab}R_{cb} \quad \Rightarrow \quad 0 \equiv \dot{\delta}_{ac} = \dot{R}_{ab}R_{cb} + R_{ab}\dot{R}_{cb} = \omega_{ac} + \omega_{ca},$$

hence $\omega_{ac} = -\omega_{ca}$. □

Since ω is antisymmetric, it has only 3 independent components. We can therefore define a vector with components:

$$\omega_a \equiv \frac{1}{2}\epsilon_{abc}\omega_{bc}. \tag{1.24}$$

We treat ω_a as the components of a vector in the *body frame*, so that

$$\vec{\omega} = \omega_a\hat{e}_a. \tag{1.25}$$

The reason we choose the body frame instead of the space frame will become apparent shortly. First, let's write ω_{ac} in terms of ω_a:

$$\epsilon_{ade}\omega_a = \frac{1}{2}\epsilon_{ade}\epsilon_{abc}\omega_{bc} = \frac{1}{2}(\delta_{db}\delta_{ec} - \delta_{dc}\delta_{eb})\omega_{bc} = \frac{1}{2}(\omega_{de} - \omega_{ed}) = \omega_{de}.$$

Relabeling the indices we get:

$$\omega_{ac} = \epsilon_{bac}\omega_b = -\epsilon_{abc}\omega_b. \tag{1.26}$$

Now, we can rewrite $\mathrm{d}\hat{e}_a/\mathrm{d}t$ as:

$$\frac{\mathrm{d}\hat{e}_a}{\mathrm{d}t} = -\epsilon_{abc}\omega_b\hat{e}_c = \omega_b(\hat{e}_b \times \hat{e}_a) = \vec{\omega} \times \hat{e}_a, \tag{1.27}$$

where we used that for right-handed coordinates $\hat{e}_a \times \hat{e}_b = \epsilon_{abc}\hat{e}_c$. Obtaining this relation was the reason we defined the components of $\vec{\omega}$ to be in the body frame. The vector $\vec{\omega}$ is the *instantaneous angular velocity*.

An important corollary of relation (1.27) is that given any vector $\vec{A}$ we have

$$\dot{\vec{A}} = \dot{A}_a\hat{e}_a + A_a\dot{\hat{e}}_a = \dot{A}_a\hat{e}_a + A_a\vec{\omega} \times \hat{e}_a = \dot{A}_a\hat{e}_a + \vec{\omega} \times \vec{A}. \tag{1.28}$$

A special case is for the position vector, since $\dot{r}_a = 0$,

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}. \tag{1.29}$$

1.4 Inertia tensor

Let's write the angular momentum and kinetic energy of a rotating body using (1.29). The angular momentum is

$$\begin{aligned}
\vec{L} = \sum_i m_i\vec{r}_i \times \dot{\vec{r}}_i &= \sum_i m_i\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\
&= \sum_i m_i\hat{e}_a\epsilon_{abc}r_{i,b}\epsilon_{cde}\omega_d r_{i,e} \\
&= \hat{e}_a\sum_i m_i(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd})r_{i,b}\omega_d r_{i,e} \\
&= \hat{e}_a\sum_i m_i\big(r_i^2\omega_a - r_{i,a}r_{i,b}\omega_b\big) \\
&= \hat{e}_a\sum_i m_i\big(r_i^2\delta_{ab} - r_{i,a}r_{i,b}\big)\omega_b.
\end{aligned} \tag{1.30}$$

Hence, we see that the components of $\vec{L}$ are

$$L_a = \sum_i m_i\big(r_i^2\delta_{ab} - r_{i,a}r_{i,b}\big)\omega_b \equiv I_{ab}\omega_b, \tag{1.31}$$

where we defined the *inertia tensor* I with components

$$I_{ab} \coloneqq \sum_i m_i\big(r_i^2\delta_{ab} - r_{i,a}r_{i,b}\big). \tag{1.32}$$

Now, consider the kinetic energy:

$$\begin{aligned}
T = \frac{1}{2}\sum_i m_i\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i &= \frac{1}{2}\sum_i m_i(\omega \times \vec{r}_i) \cdot (\omega \times \vec{r}_i) \\
&= \frac{1}{2}\sum_i m_i(\epsilon_{abc}\omega_b r_{i,c})(\epsilon_{ade}\omega_d r_{i,e}) \\
&= \frac{1}{2}\sum_i m_i(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd})\omega_b\omega_d r_{i,c}r_{i,e} \\
&= \frac{1}{2}\sum_i m_i\big(r_i^2\delta_{bc} - r_{i,c}r_{i,b}\big)\omega_b\omega_c \\
&= \frac{1}{2}\omega_b I_{bc}\omega_c = \frac{1}{2}\vec{\omega} \cdot I\vec{\omega}.
\end{aligned} \tag{1.33}$$

From the definition of the inertia tensor (1.32), it follows that I is symmetric since

$$I_{ab} = \sum_i m_i\big(r_i^2\delta_{ab} - r_{i,a}r_{i,b}\big) = \sum_i m_i\big(r_i^2\delta_{ba} - r_{i,b}r_{i,a}\big) = I_{ba}. \tag{1.34}$$

The components of the inertia tensor are measured in the body frame because we used the relation (1.29). Since the position vector $\vec{r}_i$ is time-independent in the body frame, so are the components of I.

The inertia tensor generalizes to continuous bodies as follows:

$$I = \int \mathrm{d}^3r\,\rho(\vec{r}) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & z^2+x^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}, \tag{1.35}$$

where we denote the position vector in the body frame as

$$\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3.$$

Since the inertia tensor is a real, symmetric matrix, we can diagonalise it by an orthogonal transformation: $I' = OIO^T$ and $\hat{e}'_a = O\hat{e}_a$ where O is an orthogonal matrix. This would yield

$$I = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}. \tag{1.36}$$

The body axes in which I is diagonal is called the *principal axes*. Eigenvalues $I_{1,2,3}$ are called the *principal moments*. The angular momentum and kinetic energy take particularly simple forms under this choice of axes:

$$\vec{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3 \quad \text{and} \quad T = \frac{1}{2}\big(I_1\omega_1^2\hat{e}_1 + I_2\omega_2^2\hat{e}_2 + I_3\omega_3^2\hat{e}_3\big) \tag{1.37}$$

So far, we haven't specified about which point we are calculating the inertia tensor. If the body is not free, it will not necessarily rotate about its centre of mass. But the inertia tensor about the centre of mass has a nice property which allows us to compute the inertia tensor about an arbitrary point more easily. We write the position vector as $\vec{r}_i = \vec{r}_i^{\,*} + \vec{R}$, where $\vec{R}$ is the centre of mass position. Then, the inertia tensor is

$$\begin{aligned} I_{ab} =& \sum_i m_i\big(r_i^2\delta_{ab} - r_{i,a}r_{i,b}\big) \\ =& \sum_i m_i\Big(\Big[\vec{r}_i^{\,*} + \vec{R}\Big]^2\delta_{ab} - (r^*_{i,a}+R_a)(r^*_{i,b}+R_b)\Big) \\ =& \sum_i m_i\big(\big[(r_i^*)^2 + 2r_i^*R + R^2\big]\delta_{ab} - r^*_{i,a}r^*_{i,b} - r^*_{i,a}R_b - R_ar^*_{i,b} - R_aR_b\big) \\ =& \sum_i m_i\big((r_i^*)^2\delta_{ab} - r^*_{i,a}r^*_{i,b}\big) + 2R\delta_{ab}\underbrace{\sum_i m_ir_i^*}_{\equiv 0} - R_b\underbrace{\sum_i m_ir^*_{i,a}}_{\equiv 0} - R_a\underbrace{\sum_i m_ir^*_{i,b}}_{\equiv 0} \\ &+ \sum_i m_i\big(R^2\delta_{ab} - R_aR_b\big) \\ =& I^*_{ab} + M\big(R^2\delta_{ab} - R_aR_b\big). \end{aligned} \tag{1.38}$$

So the inertia tensor decouples into the inertia tensor about the centre of mass I^* and the inertia tensor of the centre of mass. This is known as the *parallel axis theorem.*

1.5 Euler's equations

We are interested the equation of motion for a rotating rigid body under external torque $\vec{\tau}$. As we can completely specify the rotation of the body by the rotation matrix $R(t)$, and since it is possible to obtain $R(t)$ from $\vec{\omega}(t)$, we may look for an equation relating $\vec{\omega}$ to $\vec{\tau}$. We start by stating

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(L_a\hat{e}_a) = \dot{L}_a\hat{e}_a + L_a\vec{\omega}\times\hat{e}_a = \vec{\tau}. \tag{1.39}$$

Now, we work in the principal axes and write the principal moments as $I^{(1,2,3)}$ to avoid confusion with the summation convention. Then, we have $L_a = I^{(a)}\omega_a$ and so

$$\dot{\vec{L}} = I^{(a)}\dot{\omega}_a\hat{e}_a + I^{(a)}\omega_a\omega_b\hat{e}_b\times\hat{e}_a = I^{(a)}\dot{\omega}_a\hat{e}_a + I^{(a)}\omega_a\omega_b\epsilon_{bac}\hat{e}_c. \tag{1.40}$$

Substituting this yields

$$\dot{\vec{L}} = \Big(I^{(c)}\dot{\omega}_c + I^{(a)}\omega_a\omega_b\epsilon_{bac}\Big)\hat{e}_c = \vec{\tau} \tag{1.41}$$

$$\Rightarrow \quad \dot{L}_c = I^{(c)}\dot{\omega}_c + \epsilon_{cba}\omega_a\omega_b I^{(a)} = \tau_c. \tag{1.42}$$

We may explicitly write the set of three equations:

$$\dot{L}_1 = I^{(1)}\dot{\omega}_1 + \omega_2\omega_3\Big(I^{(3)} - I^{(2)}\Big) = \tau_1, \tag{1.43}$$

$$\dot{L}_2 = I^{(2)}\dot{\omega}_2 + \omega_3\omega_1\Big(I^{(1)} - I^{(3)}\Big) = \tau_2, \tag{1.44}$$

$$\dot{L}_3 = I^{(3)}\dot{\omega}_3 + \omega_1\omega_2\Big(I^{(2)} - I^{(1)}\Big) = \tau_3. \tag{1.45}$$

These are called the *Euler equations.*

1.6 Euler angles

We want an explicit parametrisation of the rotation which takes us from the space to the body frame. Any arbitrary rotation may be expressed as a composition of 3 rotations about 3 different axes. We want rotation R such that $\hat{e}_a = R_{ab}\tilde{\hat{e}}_b$. Let's split this into the three steps:

$$\{\tilde{\hat{e}}_a\} \xrightarrow{R_3(\phi)} \{\hat{e}'_a\} \xrightarrow{R_1(\theta)} \{\hat{e}''_a\} \xrightarrow{R_3(\psi)} \{\hat{e}_a\}. \tag{1.46}$$

First, we rotate by ϕ about $\tilde{\hat{e}}_3$. So, $\hat{e}'_a = R_3(\phi)_{ab}\tilde{\hat{e}}_b$ where

$$R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.47}$$

Then, we rotate by θ around $\hat{e}'_1$ so that $\hat{e}''_a = R_1(\theta)_{ab}\hat{e}'_b$, where

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{1.48}$$

Finally, we rotate by ψ about $\hat{e}''_3$ so that $\hat{e}_a = R_3(\psi)_{ab}\hat{e}''_b$ where

$$R_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.49}$$

Putting everything together, we have $R(\phi,\theta,\psi) = R_3(\psi)R_1(\theta)R_3(\phi)$.

Let's now express the angular velocity in terms of the Euler angles ϕ,θ,ψ. In time $\mathrm{d}t$, the angles change as

$$(\phi,\theta,\psi) \longrightarrow (\phi+\mathrm{d}\phi\,,\theta+\mathrm{d}\theta\,,\psi+\mathrm{d}\psi),$$

which means the angular displacement is

$$\vec{\omega}\,\mathrm{d}t = \mathrm{d}\phi\,\tilde{\hat{e}}_3 + \mathrm{d}\theta\,\hat{e}'_1 + \mathrm{d}\psi\,\hat{e}''_3,$$

and hence the angular velocity is given by

$$\vec{\omega} = \dot{\phi}\,\tilde{\hat{e}}_3 + \dot{\theta}\,\hat{e}'_1 + \dot{\psi}\,\hat{e}''_3. \tag{1.50}$$

Now, we need to write $\tilde{\hat{e}}_3, \hat{e}'_1$ and $\hat{e}''_3$ in terms of $\hat{e}_{1,2,3}$. This is a straightforward computation:

$$\begin{aligned}
\hat{e}''_3 &= \hat{e}_3,\\
\hat{e}'_1 = \hat{e}''_1 &= \cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2,\\
\tilde{\hat{e}}_3 = \hat{e}'_3 = \sin\theta\,\hat{e}''_2 + \cos\theta\,\hat{e}''_3 &= \sin\theta(\sin\psi\,\hat{e}_1 + \cos\psi\,\hat{e}_2) + \cos\theta\,\hat{e}_3.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\vec{\omega} &= \dot{\phi}(\sin\theta(\sin\psi\,\hat{e}_1 + \cos\psi\,\hat{e}_2) + \cos\theta\,\hat{e}_3) + \dot{\theta}(\cos\psi\,\hat{e}_1 - \sin\psi\,\hat{e}_2) + \dot{\psi}\,\hat{e}_3\\
&= \hat{e}_1\Big(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\Big) + \hat{e}_2\Big(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\Big) + \hat{e}_3\Big(\dot{\phi}\cos\theta + \dot{\psi}\Big).
\end{aligned} \tag{1.51}$$

2 Lagrangian Mechanics

2.1 Generalised coordinates

We are interested in formulating classical mechanics in a way more general than Newton's laws which restrict us to deal with components of vectors in inertial frames. A sensible first step would be to propose a way of specifying the configuration of a system.

Physical systems must always have a finite number of degrees of freedom. The configuration of any given system with N degrees of freedom may then be completely specified by a finite set of coordinates $\{q^1, q^2, \ldots, q^N\}$. Such a set of coordinates is called *generalised coordinates.* For example, a set of n free particles require $3n$ generalised coordinates to specify the position of each. A free rigid body is described by 6 coordinates: 3 for centre of mass position and 3 Euler angles.

We can go one step further and define a *configuration space* $\mathcal{C}$ as the space of all possible configurations of the system. A choice of generalised coordinates $\{q^a\}$ then becomes an explicit parametrisation of $\mathcal{C}$. The configuration of a system at any given time corresponds to a *point* in $\mathcal{C}$. As the system evolves in time, it traces out a *curve* in $\mathcal{C}$.

All we need now is a method to determine which particular curve is traversed.

2.2 Action principle

Since we are interested in finding a particular curve out of all possible curves in the configuration space, it is obvious that we will need to use *variational calculus.* Let's define a *functional* of the generalised coordinates such that a stationary point of the functional corresponds to the path traversed in the configuration space. This functional $\mathcal{S}[q]$ is called the *action*, defined as

$$\mathcal{S}[q] := \int_{t_i}^{t_f} \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t\,, \tag{2.1}$$

where $\mathcal{L}(q, \dot{q}, t)$[1] is the *Lagrangian*, defined as the difference between the kinetic and potential energy of the system:

$$\mathcal{L}(q, \dot{q}, t) := T(\dot{q}) - V(q, t). \tag{2.2}$$

Out of all curves in $\mathcal{C}$ with fixed endpoints

$$q(t_i) = q_{\text{initial}} \quad \text{and} \quad q(t_f) = q_{\text{final}}, \tag{2.3}$$

the system follows the one which makes the action $\mathcal{S}$ stationary. This is called the *action principle.*

Note that in equation (2.1) we assumed that the curve is parametrised by time so that $q^a = q^a(t)$. Equivalently, we could have used an arbitrary parametrisation. If the Lagrangian has explicit time dependence, using another parametrisation would imply that time itself should be taken as a generalised coordinate. In relativity, since space and time are treated on an equal footing, the curve may not always be parametrised by time.

2.3 Euler-Lagrange equations

To find the path which stationarises the action, we consider the variation

$$\delta\mathcal{S} = \mathcal{S}[q + \delta q] - \mathcal{S}[q], \tag{2.4}$$

[1] The set of all generalised coordinates $\{q^1, \ldots, q^N\}$ is denoted by q, and similarly $\dot{q}$ denotes the set $\{\dot{q}^1, \ldots, \dot{q}^N\}$.

where δq is a small perturbation to the trajectory. The boundary condition (2.3) implies

$$\delta q(t_i) = \delta q(t_f) \equiv 0. \tag{2.5}$$

The change in the Lagrangian is

$$\delta\mathcal{L} = \mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}, t) - \mathcal{L}(q,\dot{q},t) = \frac{\partial\mathcal{L}}{\partial q^a}\delta q^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\delta\dot{q}^a + o(\delta q). \tag{2.6}$$

The change in the action is then given by

$$\delta\mathcal{S}[q,\delta q] = \int_{t_i}^{t_f} \delta\mathcal{L}(q,\delta q,\dot{q},\delta\dot{q},t)\,\mathrm{d}t = \int_{t_i}^{t_f} \frac{\partial\mathcal{L}}{\partial q^a}\delta q^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\delta\dot{q}^a + o(\delta q).$$

Integrating second term by parts:

$$\int_{t_i}^{t_f} \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\delta\dot{q}\,\mathrm{d}t = \underbrace{\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\delta q^a\Big|_{t_i}^{t_f}}_{=0} - \int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\delta q^a\,\mathrm{d}t = -\int_{t_i}^{t_f} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\delta q^a\,\mathrm{d}t\,.$$

Hence, we have

$$\delta\mathcal{S}[q,\delta q] = \int_{t_i}^{t_f}\left[\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\right]\delta q^a\,\mathrm{d}t = 0. \tag{2.7}$$

Since this must hold for all δq, we must have

$$\frac{\delta\mathcal{S}}{\delta q^a} = \frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) = 0 \quad \forall\, a. \tag{2.8}$$

This is known as the *Euler-Lagrange* equation. The expression is also the functional derivative $\delta\mathcal{S}/\delta q^a$, which equals zero at a stationary point.

As a side note, the *functional derivative* $\delta\mathcal{S}/\delta q$ is defined such that

$$\delta\mathcal{S} = \int \frac{\delta\mathcal{S}}{\delta q^a}\delta q^a + o(\delta q)\,\mathrm{d}t\,. \tag{2.9}$$

So, for a Lagrangian which depends on $q, \dot{q}$ and t, the functional derivative is

$$\frac{\delta\mathcal{S}}{\delta q^a} = \frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right). \tag{2.10}$$

We may now show that the action principle is equivalent to Newton's laws. Choosing Cartesian coordinates, we have

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

for a single particle. Then,

$$\frac{\partial\mathcal{L}}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) = -m\ddot{x}^a - \frac{\partial V}{\partial x^a} = 0 \quad \Rightarrow \quad m\ddot{\vec{r}} = -\vec{\nabla}V.$$

2.4 Coordinate transformations

The beauty of the Lagrangian formalism is *coordinate invariance.* The Euler-Lagrange equations are coordinate invariant - meaning they look the same in any coordinate system for the configuration space. This should be obvious as the action principle deals with paths in the

configuration space. How we choose to parametrise the configuration space should be irrelevant. However, we will gain greater insight if we look at how functional derivatives transform.

Consider a coordinate transformation which may depend on time with non-zero Jacobian determinant:

$$\bar{q}^a = \bar{q}^a(q_1, \ldots, q_N, t) \quad \text{and} \quad q^a = q^a(\bar{q}_1, \ldots, \bar{q}_N, t). \tag{2.11}$$

The Lagrangian in the transformed coordinates is

$$\overline{\mathcal{L}}(\bar{q}, \dot{\bar{q}}, t) = \mathcal{L}(q(\bar{q}, t), \dot{q}(\bar{q}, \dot{\bar{q}}, t, t)), \tag{2.12}$$

where

$$\dot{q}^a = \frac{\mathrm{d}q^a}{\mathrm{d}t} = \frac{\partial q^a}{\partial \bar{q}^b}\dot{\bar{q}}^b + \frac{\partial q^a}{\partial t}. \tag{2.13}$$

First, look at $\partial\overline{\mathcal{L}}/\partial\bar{q}^a$:

$$\begin{aligned}
\frac{\partial \overline{\mathcal{L}}}{\partial \bar{q}^a} &= \frac{\partial}{\partial \bar{q}^a}\left[\mathcal{L}(q(\bar{q}, t), \dot{q}(\bar{q}, \dot{\bar{q}}, t, t))\right] \\
&= \frac{\partial \mathcal{L}}{\partial q^b}\frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial \dot{q}^b}{\partial \bar{q}^a} \\
&= \frac{\partial \mathcal{L}}{\partial q^b}\frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial}{\partial \bar{q}^a}\left(\frac{\partial q^b}{\partial \bar{q}^c}\dot{\bar{q}}^c + \frac{\partial q^b}{\partial t}\right) \\
&= \frac{\partial \mathcal{L}}{\partial q^b}\frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\left(\frac{\partial^2 q^b}{\partial \bar{q}^a \partial \bar{q}^c}\dot{\bar{q}}^c + \frac{\partial^2 q^b}{\partial \bar{q}^a \partial t}\right).
\end{aligned} \tag{2.14}$$

Now, look at $\partial\overline{\mathcal{L}}\Big/\partial\dot{\bar{q}}^a$

$$\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\bar{q}}^a} = \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial \dot{q}^b}{\partial \dot{\bar{q}}^a} = \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial}{\partial \dot{\bar{q}}^a}\left(\frac{\partial q^b}{\partial \bar{q}^c}\dot{\bar{q}}^c + \frac{\partial q^a}{\partial t}\right) = \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial q^b}{\partial \bar{q}^c}\delta^c_a = \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial q^b}{\partial \bar{q}^a}. \tag{2.15}$$

Hence, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\bar{q}}^a}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\partial q^b}{\partial \bar{q}^a}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^b}\right)\frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial q^b}{\partial \bar{q}^a}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^b}\right)\frac{\partial q^b}{\partial \bar{q}^a} + \frac{\partial \mathcal{L}}{\partial \dot{q}^b}\left(\frac{\partial^2 q^b}{\partial \bar{q}^a \partial \bar{q}^c}\dot{\bar{q}}^c + \frac{\partial^2 q^b}{\partial \bar{q}^a \partial t}\right).
\end{aligned} \tag{2.16}$$

Putting (2.14) and (2.16) together, we obtain the transformation rule for the functional derivative:

$$\frac{\delta \mathcal{S}}{\delta \bar{q}^a} = \frac{\partial \overline{\mathcal{L}}}{\partial \bar{q}^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \overline{\mathcal{L}}}{\partial \dot{\bar{q}}^a}\right) = \left[\frac{\partial \mathcal{L}}{\partial q^b} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^b}\right)\right]\frac{\partial q^b}{\partial \bar{q}^a} = \frac{\partial q^b}{\partial \bar{q}^a}\frac{\delta \mathcal{S}}{\delta q^b}. \tag{2.17}$$

It follows from this that if the Euler-Lagrange equations hold in one set of coordinates q, it holds in all sets of coordinate related to q by some transformation of the form (2.11). But that's not it! Interestingly enough, the functional derivative transforms as a *covariant tensor!* Although none of the two terms in the functional derivative transforms as a tensor, their combination does.

2.5 Dissipation in Lagrangian formalism

Dissipative systems are more tricky to deal with using the Lagrangian formalism simply due to the fact that dissipative forces are not conservative and so not derived from potentials.

Usually, dissipation in systems is most easily expressed in terms of *dissipative forces*. We can make use of the transformation property of the functional derivative to incorporate dissipative forces into the Lagrangian formalism.

Consider a system with 3 degrees of freedom which may be expressed in Cartesian coordinates. Let $\vec{\gamma}$ be the dissipative force. The Newtonian equation of motion reads:

$$m\ddot{\vec{r}} = -\vec{\nabla}V + \vec{\gamma}, \tag{2.18}$$

where we naturally split up to forces into conservative and non-conservative components. In component form:

$$m\ddot{x}^a = -\frac{\partial V}{\partial x^a} + \gamma_a \iff -\gamma_a = -m\ddot{x}^a - \frac{\partial V}{\partial x^a} = \frac{\delta \mathcal{S}}{\delta x^a}, \tag{2.19}$$

where γ_a denotes the covariant component of $\vec{\gamma}$. Now, we have a tensor equation since both terms transform covariantly. Then, the equation of motion in any coordinate system q is:

$$-\frac{\partial x^b}{\partial q^a}\gamma_b = \frac{\partial x^b}{\partial q^a}\frac{\delta \mathcal{S}}{\delta x^b} = \frac{\delta \mathcal{S}}{\delta q^a}. \tag{2.20}$$

This may be generalised for systems with more (or fewer) degrees of freedom. For constrained systems, describing the system with Cartesian coordinates may require the use of Lagrange multipliers.

I'm sure there is a better approach to deal with dissipation, this is the one I could come up with.

2.6 Holonomic constraints

Holonomic constraints are relationships between the coordinates of the form:

$$f_\alpha(x_1, \ldots, x_M, t) = 0, \quad \alpha = 1, \ldots, M - N. \tag{2.21}$$

These can be solved for N generalised coordinates $\{q_1, \ldots, q_N\}$. The system is said to have N degrees of freedom and so the configuration space is N-dimensional.

Holonomic constraints arise naturally if we try to describe the configuration space with more coordinates than degrees of freedom. An example would be to use Cartesian coordinates (x, y) to describe a simple pendulum swinging along a plane.

We can deal with holonomic constraints without explicitly solving for the generalised coordinates. We consider instead the Lagrangian

$$\mathcal{L}'(x, \dot{x}, \lambda, t) = \mathcal{L}(x, \dot{x}, t) + \sum_\alpha \lambda_\alpha f_\alpha(x, t), \tag{2.22}$$

where we treat λ as additional coordinates. These are known as *Lagrange multipliers*. The Euler-Lagrange equation for λ_α reads:

$$\frac{\partial \mathcal{L}'}{\partial \lambda_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}'}{\partial \dot{\lambda}_\alpha}\right) = f_\alpha(x, t) = 0, \tag{2.23}$$

which is the constraint itself. The Euler-Lagrange equation for the coordinates are

$$\frac{\partial \mathcal{L}'}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}'}{\partial \dot{x}^a}\right) = \frac{\partial \mathcal{L}}{\partial x^a} + \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) = 0 \iff \frac{\delta \mathcal{S}}{\delta x^a} = -\sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial x^a}. \tag{2.24}$$

Constraint forces may be found by choosing Cartesian coordinates, then

$$m\ddot{x}^a = -\frac{\partial V}{\partial x^a} + \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial x^a}. \tag{2.25}$$

2.7 Lagrangian transformations

Some transformations of the Lagrangian leave the dynamics invariant, but may make the equations easier to obtain. Let's have a look at a few of them.

1. For any $\lambda \in \mathbb{R}$ and $f(t)$, the Lagrangian

$$\mathcal{L}' = \lambda\mathcal{L} + f(t)$$

describes the same dynamics as the Lagrangian $\mathcal{L}$. To see this, simply consider the functional derivative:

$$\frac{\delta\mathcal{S}'}{\delta q^a} = \frac{\partial\mathcal{L}'}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}'}{\partial\dot{q}^a}\right) = \lambda\frac{\partial\mathcal{L}}{\partial q^a} - \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) = \lambda\frac{\delta\mathcal{S}}{\delta q^a}.$$

2. Adding a total derivative to the Lagrangian shifts the action by a constant which leaves the dynamics unchanged. Consider the action associated with the Lagrangian $\mathcal{L}' = \mathcal{L} + \mathrm{d}f/\mathrm{d}t$:

$$\mathcal{S}' = \int_{t_i}^{t_f} \mathcal{L} + \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \int_{t_i}^{t_f} \mathcal{L}\,\mathrm{d}t + f\Big|_{t_i}^{t_f} = \mathcal{S} + \text{constant}.$$

Note that f can be a function of time and any of the generalised coordinates and their derivatives, so that $f = f(q, \dot{q}, \ddot{q}, \ldots, t)$.

3. If the Lagrangian is *totally conserved,* meaning

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = \frac{\partial\mathcal{L}}{\partial q^a}\dot{q}^a + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\ddot{q}^a + \frac{\partial\mathcal{L}}{\partial t} = 0,$$

then *any* (nice enough) function of the Lagrangian $F(\mathcal{L})$ describes the same dynamics. To see why, simply look at the Euler-Lagrange equations for $F(\mathcal{L})$:

$$\begin{aligned}
\frac{\partial F(\mathcal{L})}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F(\mathcal{L})}{\partial\dot{q}^a}\right) &= \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) \\
&= \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\right)\frac{\partial\mathcal{L}}{\partial\dot{q}^a} - \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) \\
&= \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\left[\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\right] - \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\frac{\mathrm{d}^2F}{\mathrm{d}\mathcal{L}^2}\underbrace{\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t}}_{=0} \\
&= \frac{\mathrm{d}F}{\mathrm{d}\mathcal{L}}\left[\frac{\partial\mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\right].
\end{aligned}$$

The solution of the Euler-Lagrange equation for Lagrangian $\mathcal{L}$ makes the term in the square brackets vanish, meaning the transformed Lagrangian $F(\mathcal{L})$ also obeys its Euler-Lagrange equation. *This trick is used in relativity to consider $\mathcal{L}^2$ instead of $\mathcal{L}$.*

2.8 Conserved quantities

A function $F(q, \dot{q}, t)$ is called a *constant of motion* (or a *conserved quantity*) if its total time derivative vanishes,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q^a}\dot{q}^a + \frac{\partial F}{\partial\dot{q}^a}\ddot{q}^a + \frac{\partial F}{\partial t} = 0, \tag{2.26}$$

along the path in configuration space taken by the system. Hence, F remains constant as the system evolves.

Suppose $\partial\mathcal{L}/\partial q^a = 0$ for some q^a. Then, by Euler-Lagrange:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) = \frac{\partial\mathcal{L}}{\partial q^a} = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{q}^a} \quad \text{is conserved.} \tag{2.27}$$

The quantity $\partial\mathcal{L}/\partial\dot{q}^a$ defines the *generalised momentum* associated with coordinate q^a:

$$p_a := \frac{\partial\mathcal{L}}{\partial\dot{q}^a} \quad \Rightarrow \quad \dot{p}_a = \frac{\partial\mathcal{L}}{\partial q^a}. \tag{2.28}$$

In Cartesian coordinates, p_a corresponds to *linear momentum*:

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{x}^a} = m\dot{x}_a.$$

Now suppose the Lagrangian has no explicit time dependent. Then, the quantity

$$H = \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\dot{q}^a - \mathcal{L} = p_a\dot{q}^a - \mathcal{L} \tag{2.29}$$

is conserved, since

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(p_a\dot{q}^a - \mathcal{L}) = \dot{p}_a\dot{q}^a + p_a\ddot{q}^a - \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\ddot{q}^a - \frac{\partial\mathcal{L}}{\partial q^a}\dot{q}^a - \frac{\partial\mathcal{L}}{\partial t} = -\frac{\partial\mathcal{L}}{\partial t} = 0. \tag{2.30}$$

H is the *Hamiltonian*, usually identified with the total energy of the system.

2.9 Noether's theorem

Conserved quantities are related to continuous symmetries of the Lagrangian. First, let's define what we mean by a continuous symmetry.

Consider a transformation of the coordinates q which depends continuously on some parameter s such that

$$q^a(t) \longmapsto Q^a(s,t) \quad \text{where} \quad Q^a(0,t) = q^a(t). \tag{2.31}$$

Then, this transformation is said to be a *continuous symmetry* of the Lagrangian $\mathcal{L}$ if

$$\frac{\partial}{\partial s}\mathcal{L}(Q^a(s,t), \dot{Q}^a(s,t), t) = 0. \tag{2.32}$$

In other words, the Lagrangian remains constant to linear order as we dial up or down the parameter s. Noether's theorem states that for each such symmetry, there exists a *conserved quantity*.

The proof is as follows: First, note that

$$\frac{\partial\mathcal{L}}{\partial s} = \frac{\partial\mathcal{L}}{\partial Q^a}\frac{\partial Q^a}{\partial s} + \frac{\partial\mathcal{L}}{\partial\dot{Q}^a}\frac{\partial\dot{Q}^a}{\partial s}. \tag{2.33}$$

Then, evaluating this at $s = 0$ yields

$$\begin{aligned} 0 = \left.\frac{\partial\mathcal{L}}{\partial s}\right|_{s=0} &= \left(\frac{\partial\mathcal{L}}{\partial Q^a}\frac{\partial Q^a}{\partial s} + \frac{\partial\mathcal{L}}{\partial\dot{Q}^a}\frac{\partial\dot{Q}^a}{\partial s}\right)\Bigg|_{s=0} \\ &= \frac{\partial\mathcal{L}}{\partial q^a}\left.\frac{\partial Q^a}{\partial s}\right|_{s=0} + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\left.\frac{\partial\dot{Q}^a}{\partial s}\right|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\left.\frac{\partial Q^a}{\partial s}\right|_{s=0} + \frac{\partial\mathcal{L}}{\partial\dot{q}^a}\left.\frac{\partial\dot{Q}^a}{\partial s}\right|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\left.\frac{\partial Q^a}{\partial s}\right|_{s=0}\right), \end{aligned} \tag{2.34}$$

where we used

$$\frac{\partial \dot{Q}^a}{\partial s} = \frac{\partial}{\partial s}\frac{\mathrm{d}Q^a}{\mathrm{d}t} = \frac{\partial}{\partial s}\left(\frac{\partial Q^a}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\partial Q^a}{\partial t}\right) = \frac{\partial^2 Q^a}{\partial s \partial t} = \frac{\partial^2 Q^a}{\partial t \partial s} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial Q^a}{\partial s}.$$

Hence, the quantity $p_a \ \partial Q^a/\partial s \ \big|_{s=0}$ is a constant of motion.

Let's look at some examples. Suppose the Lagrangian has translation symmetry, so that

$$\mathcal{L}(\vec{r} + s\vec{n}, \dot{\vec{r}}, t) = \mathcal{L}(\vec{r}, \dot{\vec{r}}, t) \tag{2.35}$$

for some constant vector $\vec{n}$. Then, the associated conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^a} n^a = p_a n^a = \vec{p} \cdot \vec{n}. \tag{2.36}$$

Hence, linear momentum in the $\vec{n}$ direction is conserved. This may be generalised to many particle systems.

Suppose now that we work in cylindrical coordinates (r, ϕ, z) and the Lagrangian has rotational symmetry about the z-axis, so that

$$\mathcal{L}(r, \phi + s, z, \dot{r}, \dot{\phi}, \dot{z}, t) = \mathcal{L}(r, \phi, z, \dot{r}, \dot{\phi}, \dot{z}, t). \tag{2.37}$$

Then, the conserved quantity is

$$\frac{\partial \mathcal{L}}{\partial \phi} = p_\phi. \tag{2.38}$$

For a kinetic energy of the form

$$T = \frac{1}{2}m\big(\dot{r}^2 + \dot{z}^2\big) + \frac{1}{2}I\dot{\phi}^2 \quad \Rightarrow \quad p_\phi = I\dot{\phi} = L_z,$$

so the angular momentum about z is conserved.

Finally, consider time translation invariance, meaning the Lagrangian remains constant as $t \mapsto t + s$. This implies $\partial \mathcal{L}/\partial t = 0$. We know already that this implies that the conserved quantity is the Hamiltonian.

2.10 Kinetic matrix

Consider rewriting the set of Euler-Lagrange equation as:

$$\frac{\delta \mathcal{S}}{\delta q^a} = 0 \quad \Leftrightarrow \quad \mathcal{Z}_{ab}\ddot{q}^b + \mathcal{M}_a = 0. \tag{2.39}$$

This can be done explicitly:

$$\frac{\partial \mathcal{L}}{\partial q^a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \frac{\partial \mathcal{L}}{\partial q^a} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^b \partial \dot{q}^a}\ddot{q}^b - \frac{\partial^2 \mathcal{L}}{\partial q^b \partial \dot{q}^a}\dot{q}^b - \frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{q}^a} \tag{2.40}$$

$$\Rightarrow \quad \mathcal{Z}_{ab} = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^a \partial \dot{q}^b} \tag{2.41}$$

$$\Rightarrow \quad \mathcal{M}_a = \frac{\partial^2 \mathcal{L}}{\partial q^b \partial \dot{q}^a}\dot{q}^b + \frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{q}^a} - \frac{\partial \mathcal{L}}{\partial q^a}. \tag{2.42}$$

Provided $\det(Z) \neq 0$, we can invert the equation to yield a set of coupled second order equations:

$$\ddot{q}^a = -(\mathcal{Z}^{-1})^{ab}\mathcal{M}_b. \tag{2.43}$$

Although the kinetic matrix may seem useless now, it will become useful when we talk about stability.

2.11 Stability and oscillations

We can deduce a lot about the behaviour of a system around its equilibrium points assuming the Lagrangian is of the form:

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - V(q) = \frac{1}{2}\alpha_{ab}(q)\dot{q}^a\dot{q}^b - V(q). \tag{2.44}$$

Natural systems

A system is said to be *natural* if the kinetic energy can be written as a quadratic, homogeneous function of $\dot{q}$:

$$T = \frac{1}{2}\alpha_{ab}(q)\dot{q}^a\dot{q}^b \tag{2.45}$$

for some $\alpha_{ab}(q)$ which can be a function of the coordinates q. For a single particle of mass m, the kinetic energy is given by

$$T = \frac{1}{2}mg_{ab}\dot{q}^a\dot{q}^b, \tag{2.46}$$

where $g_{ab}(q)$ is the metric tensor. We therefore see that α_{ab} contains information about the metric associated with the coordinates q, as well as the inertias of each constituent of the system. Note that α can be chosen to be *symmetric*.

Assuming a Lagrangian of the form (2.44), we see that the kinetic matrix is given by

$$\mathcal{Z}(q)_{ab} = \frac{\partial^2\mathcal{L}}{\partial\dot{q}^a\partial\dot{q}^b} = \frac{1}{2}\alpha_{cd}(q)\frac{\partial^2}{\partial\dot{q}^a\partial\dot{q}^b}(\dot{q}^c\dot{q}^d) = \alpha_{ab}(q). \tag{2.47}$$

From now on we will work with the kinetic matrix instead and simply write

$$T(q,\dot{q}) = \frac{1}{2}\mathcal{Z}_{ab}\dot{q}^a\dot{q}^b. \tag{2.48}$$

Equilibrium points

An *equilibrium point* q_{eq} is defined as a point in the configuration space which satisfies the Euler-Lagrange equations for all time t. In other words, the curve $q(t) = q_{\text{eq}}$ is a solution of the Euler-Lagrange equations. This implies $\dot{q} \equiv 0$. We can then find an equilibrium point by solving:

$$\left.\frac{\partial\mathcal{L}}{\partial q^a}\right|_{q_{\text{eq}},\dot{q}\equiv 0} - \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right)\right|_{q_{\text{eq}},\dot{q}\equiv 0} = 0 \quad \forall\, a. \tag{2.49}$$

For a Lagrangian of the form (2.44), we have

$$\frac{\partial\mathcal{L}}{\partial q^a} = \left.\frac{1}{2}\frac{\partial\mathcal{Z}_{bc}}{\partial q^a}\dot{q}^b\dot{q}^c\right|_{\dot{q}\equiv 0} - \left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}} = -\left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}},$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{q}^a} = \frac{1}{2}\mathcal{Z}_{ab}\dot{q}^b \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^a}\right) = \left.\frac{1}{2}\frac{\mathrm{d}\mathcal{Z}_{ab}}{\mathrm{d}t}\dot{q}^b\right|_{\dot{q}\equiv 0} + \left.\frac{1}{2}\mathcal{Z}_{ab}\ddot{q}^b\right|_{\dot{q}\equiv 0} = 0,$$

where $\dot{q} \equiv 0 \Rightarrow \ddot{q} \equiv 0$. Hence, an equilibrium point may be found by simply solving

$$\left.\frac{\partial V}{\partial q^a}\right|_{q_{\text{eq}}} = 0 \quad \forall\, a. \tag{2.50}$$

Small oscillations

Now, let's try to describe the system about an equilibrium point q_{eq}. We start by considering a small perturbation in the form

$$q^a(t) = q^a_{\text{eq}} + \eta^a(t) \tag{2.51}$$

for all a. Since we want to obtain equation of motion linear in η, we have to expand the Lagrangian about q_{eq} up to second order. This yields

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\left(\mathcal{Z}_{ab}(q_{\text{eq}}) + \left.\frac{\partial \mathcal{Z}_{ab}}{\partial q^c}\right|_{q_{\text{eq}}} \eta^c + \dots\right)\dot{\eta}^a\dot{\eta}^b - \frac{1}{2}\left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_{q_{\text{eq}}} \eta^a\eta^b + \mathcal{O}\big(\eta^3\big), \tag{2.52}$$

where we omitted the constant term $V(q_{\text{eq}})$. Noting that $\dot{\eta} = \mathcal{O}(\eta)$, only the first constant term in the expansion for $\mathcal{Z}$ survives. Then, we have

$$\frac{\partial \mathcal{L}}{\partial q^a} = -\frac{1}{2}\left.\frac{\partial^2 V}{\partial q^b \partial q^c}\right|_{q_{\text{eq}}} \frac{\partial}{\partial \eta^a}\big(\eta^b\eta^c\big) = -\left.\frac{\partial^2 V}{\partial q^a \partial q^b}\right|_{q_{\text{eq}}} \eta^b = -V_{ab}\eta^b,$$

where we defined $V_{ab} = \partial^2 V/\partial q^a \partial q^b$ evaluated at q_{eq}. Similarly, we have

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^a} = \frac{1}{2}\mathcal{Z}_{bc}(q^{\text{eq}})\frac{\partial}{\partial \dot{\eta}^a}(\dot{\eta}^b\dot{\eta}^c) = \mathcal{Z}_{ab}\dot{\eta}^b \tag{2.53}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^a}\right) = \mathcal{Z}_{ab}\ddot{\eta}^b. \tag{2.54}$$

Hence, the Euler-Lagrange equations read

$$\mathcal{Z}_{ab}\ddot{\eta}^b + V_{ab}\eta^b = 0 \quad \Rightarrow \quad \ddot{\eta}^a = -(\mathcal{Z}^{-1}V)^a{}_b\eta^b = F^a{}_b\,\eta^b, \tag{2.55}$$

where we defined the matrix $F = \mathcal{Z}^{-1}V$. In matrix form:

$$\ddot{\vec{\eta}} = F\vec{\eta}. \tag{2.56}$$

The solution is given in terms of the eigenvectors $\vec{\mu}_i$ of F, assuming they exist with real eigenvalues $\lambda_i^2 \in \mathbb{R}$:

$$F\vec{\mu}_i = \lambda_i^2\vec{\mu}. \tag{2.57}$$

The general solution is then given by:

$$\vec{\eta}(t) = \sum_i \vec{\mu}_i\big(A_i e^{\lambda_i t} + B_i e^{-\lambda_i t}\big), \tag{2.58}$$

where A_i and B_i are integration constants. The stability is determined by the eigenvalues λ_i^2 as follows:

1. If $\lambda_i^2 < 0$, then $\lambda_i = \pm i\omega_i$ for some real ω_i. This corresponds to simple harmonic motion with frequency ω_i and the system is stable in the $\vec{\eta} = \vec{\mu}_i$ direction.
2. If $\lambda_i^2 > 0$, the perturbation $\vec{\eta}$ grows exponentially in the $\vec{\mu}_i$ direction since $\lambda_i \in \mathbb{R}$. The system is unstable.

The eigenvectors $\vec{\mu}_i$ are called *normal modes*. The system is said to be stable around the equilibrium point if all the normal modes of oscillation are stable, meaning $\lambda_i^2 < 0$ for all i.

2.12 Continuous systems

What happens when the system we want to describe doesn't have a discrete set of generalised coordinates, but a continuum? This is the case for any field or composite system in the continuum limit. Such systems have infinite degrees of freedom. Let's first look at a simple example.

One-dimensional string

Consider a string of length ℓ_0 with fixed endpoints at $x = (0, \ell_0)$. The state of the system is specified by the displacement from equilibrium $\phi(x,t)$ for each position in the continuum $x \in [0, \ell_0]$. Our set of generalised coordinates is now a continuous function of position! Let's write the Lagrangian nonetheless. We use the notation

$$\phi_x = \frac{\partial \phi}{\partial x}, \quad \phi_t = \frac{\partial \phi}{\partial t}.$$

To find the kinetic energy, we have to be specific about how we define the mass density μ. The more thorough approach is to define it so that it is homogeneous throughout the string:

$$\mu = \frac{M}{\ell} \quad \Rightarrow \quad \mathrm{d}m = \mu \, \mathrm{d}\ell \,,$$

where M is the total mass and ℓ is the total length of the string. Note that by this definition, the mass density decreases as the string is stretched. The total length of the string is

$$\ell = \int \mathrm{d}\ell = \int \sqrt{\mathrm{d}x^2 + \mathrm{d}\phi^2} = \int_0^{\ell_0} \sqrt{1 + \phi_x^2} \, \mathrm{d}x \,. \tag{2.59}$$

Then, the kinetic energy is given by

$$T = \frac{1}{2} \int \phi_t^2 \, \mathrm{d}m = \frac{1}{2} \int \mu \phi_t^2 \, \mathrm{d}\ell \,. \tag{2.60}$$

This is a horrible expression which depends both on ϕ_t and ϕ_x (through $\mathrm{d}\ell$ and μ). We can simply this by assuming the displacement is small and smooth so that $\phi_x \ll 1$. Then,

$$\mathrm{d}\ell = \sqrt{1 + \phi_x^2} \, \mathrm{d}x = \left(1 + \frac{1}{2}\phi_x^2 + \dots\right) \mathrm{d}x = \mathrm{d}x + \mathcal{O}\big(\phi_x^2\big) \approx \mathrm{d}x \,, \tag{2.61}$$

$$\mu = \frac{M}{\ell} \approx \frac{M}{\ell_0} = \mu_0 = \text{constant}, \tag{2.62}$$

where we ignore terms of order $\mathcal{O}\big(\phi_x^2\big)$ because they are summed with terms of order $\mathcal{O}(1)$. With this approximation, the kinetic energy is

$$T = \frac{1}{2} \int_0^{\ell_0} \mu_0 \phi_t^2 \, \mathrm{d}x \,. \tag{2.63}$$

The potential energy is due to the stretching of the string and is given in terms of the *tension* k as

$$V = k(\ell - \ell_0) \tag{2.64}$$

where we assume the tension remains constant. The length difference $(\ell - \ell_0)$ is given by

$$\ell - \ell_0 = \int_0^{\ell_0} \left(1 + \frac{1}{2}\phi_x^2 + \dots\right) \mathrm{d}x - \ell_0 = \frac{1}{2} \int_0^{\ell_0} \phi_x^2 + \mathcal{O}\big(\phi_x^4\big) \, \mathrm{d}x \,, \tag{2.65}$$

where this time we have to keep terms of order $\mathcal{O}\big(\phi_x^2\big)$ otherwise we get no potential. The potential is therefore

$$V = \frac{1}{2} \int_0^{\ell_0} k \phi_x^2 \, \mathrm{d}x \,. \tag{2.66}$$

The Lagrangian is

$$\mathcal{L} = T - V = \int_0^{\ell_0} \frac{1}{2} \big(\mu \phi_t^2 - k \phi_x^2\big) \, \mathrm{d}x \,, \tag{2.67}$$

which is itself a *functional* of the field ϕ. The integrand is called the *Lagrangian density*, denoted $\mathscr{L}$, which is sort of like the "Lagrangian" of the functional $\mathcal{L}$. In terms of $\mathscr{L}$, we have

$$\mathcal{L} = \int_0^{\ell_0} \mathscr{L}\,\mathrm{d}x\,, \quad \text{where} \quad \mathscr{L} = \frac{1}{2}\big(\mu\phi_t^2 - k\phi_x^2\big). \tag{2.68}$$

The action for a general continuous system in one-dimension with Lagrangian density $\mathscr{L}(\phi, \phi_t, \phi_x)$ is

$$\mathcal{S}[\phi] = \int \mathrm{d}t \int \mathrm{d}x\, \mathscr{L}(\phi, \phi_t, \phi_x), \tag{2.69}$$

$$\begin{aligned} \Rightarrow \quad \delta\mathcal{S} &= \int \mathrm{d}t \int \mathrm{d}x \left[\frac{\partial\mathscr{L}}{\partial\phi}\delta\phi + \frac{\partial\mathscr{L}}{\partial\phi_t}\delta\phi_t + \frac{\partial\mathscr{L}}{\partial\phi_x}\delta\phi_x\right] \\ &= \int \mathrm{d}t \int \mathrm{d}x \left[\frac{\partial\mathscr{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\phi_t}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathscr{L}}{\partial\phi_x}\right)\right]\delta\phi = 0 \end{aligned} \tag{2.70}$$

$$\Rightarrow \quad \frac{\partial\mathscr{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\phi_t}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathscr{L}}{\partial\phi_x}\right) = 0. \tag{2.71}$$

This is the Euler-Lagrange equation for a continuous field $\phi(x,t)$, given in terms of the Lagrangian density $\mathscr{L}(\phi, \phi_t, \phi_x)$.

For the string, this yields

$$\mu\frac{\partial^2\phi}{\partial t^2} - k\frac{\partial^2\phi}{\partial x^2} = 0, \tag{2.72}$$

which is the wave equation.

Klein-Gordon equation

We can generalise this formalism to fields ϕ of time and any number of space dimensions. Let $x^\mu = \{t, x, y, z\}$ with $\mu = 0, 1, 2, 3$. Then, $\phi = \phi(x^\mu)$ and $\phi_\mu = \partial\phi/\partial x^\mu$. The Lagrangian density is in general:

$$\mathscr{L} = \mathscr{L}(\phi, \phi_\mu, x^\mu).$$

The Euler-Lagrange equations take the form:

$$\frac{\partial\mathscr{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathscr{L}}{\partial\phi_\mu}\right) = 0. \tag{2.73}$$

Consider a Lagrangian density of the form

$$\mathscr{L} = \frac{1}{2}\phi_t^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi), \tag{2.74}$$

$$\frac{\partial\mathscr{L}}{\partial\phi} = -\frac{\partial V}{\partial\phi}, \tag{2.75}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\phi_t}\right) = \frac{\partial^2\phi}{\partial t^2}, \tag{2.76}$$

$$\vec{\nabla}\cdot\left(\frac{\partial\mathscr{L}}{\partial\vec{\nabla}\phi}\right) = -\nabla^2\phi. \tag{2.77}$$

This yields a Klein-Gordon equation:

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - \frac{\partial V}{\partial\phi} = \Box\phi - \frac{\partial V}{\partial\phi} = 0, \tag{2.78}$$

where we defined the d'Alembertian operator $\Box = -\partial_t^2 + \nabla^2$.

3 Hamiltonian Mechanics

Throughout this section we will use the summation convention whenever there are repeated indices. However, there is no point in bothering with up and down indices as Hamiltonian mechanics is not covariant.

3.1 Legendre transform

Suppose we have a function of two variables $f = f(x, y)$. Now, introduce variable u, related to x, y by

$$u = \frac{\partial f}{\partial x}. \tag{3.1}$$

We will need to invert u for x to write some $x = x(u, v)$. This puts a constraints on our function $f(x, y)$, in the sense that $\partial f/\partial x$ needs to be a *monotonic function* of u for all y. This is equivalent to the statement that $f(x, y)$ is a convex function of x for all y.

We want a transformation of the form:

$$f(x, y) \longmapsto g(u, y) \quad \text{where} \quad x = \frac{\partial g}{\partial u}, \tag{3.2}$$

so that u and x are treated symmetrically. In other words, u is related to x through f in the same way as x is related to u through g.

The simple substitution $g(u, y) = f(x(u, y), y)$ won't satisfy the symmetry property. So, consider:

$$g(u, y) = ux(u, y) - f(x(u, y), y) \tag{3.3}$$

$$\Rightarrow \quad \frac{\partial g}{\partial u} = x(u, y) + u\frac{\partial x(u, y)}{\partial u} - \underbrace{\frac{\partial f(x, y)}{\partial x}}_{=u} \frac{\partial x(u, y)}{\partial u} = x(u, y). \tag{3.4}$$

This satisfies our condition. Does the inverse transform exist? Consider the transform $g(u, y) \longmapsto h(x, y)$ where $x = \partial g/\partial u$:

$$h(x, y) = xu - g(u, y) = xu - (xu - f(x, y)) = f(x, y). \tag{3.5}$$

Hence, not only does the inverse transform exist, it is obtained by repeated application of the original transformation in the sense that

$$f(x, y) \xrightarrow{x \to u} g(u, y) \xrightarrow{u \to x} f(x, y). \tag{3.6}$$

This transformation is called the *Legendre transform.* We will apply it to the Lagrangian to obtain the Hamiltonian.

As a last note, we note that the Legendre transform generalises to multiple variables as follows: suppose we have a function $f(x_1, \ldots, x_n, y)$ and a set $u_1, \ldots, u_n$ with

$$u_a = u_a(x_1, \ldots, x_n, y) = \frac{\partial f}{\partial x_a} \quad \text{invertible so that} \quad x_a = x_a(u_1, \ldots, u_n, y) = x_a(u, y).$$

The Legendre transform is given by

$$g(u_1, \ldots, u_n, y) = \sum_{a=1}^{n} x_a(u, y)u_a - f(x_1(u, y), \ldots, x_n(u, y), y). \tag{3.7}$$

3.2 Hamilton's equations

The point of the Hamiltonian formalism is to reformulate the Lagrangian formalism in terms of the generalised momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i},$$

instead of $\dot{q}_i$. So, we define the Hamiltonian $H(q, p, t)$ to be the Legendre transform of the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ with respect to $\dot{q}$:

$$H(q, p, t) = p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t), \tag{3.8}$$

where $\dot{q}$ is eliminated by inverting the generalised momenta $p_i = \partial \mathcal{L}/\partial \dot{q}_i$ to get $\dot{q}_i = \dot{q}_i(q, p, t)$.

There are many ways to recover the equations of motion. An easy way is to simply consider the variation of H:

$$\begin{aligned} \mathrm{d}H &= \frac{\partial H}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial H}{\partial p_i}\,\mathrm{d}p_i + \frac{\partial H}{\partial t}\,\mathrm{d}t \\ &= -\frac{\partial \mathcal{L}}{\partial q_i}\,\mathrm{d}q_i + \left(\dot{q}_i + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i}\right)\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\mathrm{d}q_i + \dot{q}_i\,\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t \\ &= -\dot{p}_i\,\mathrm{d}q_i + \dot{q}_i\,\mathrm{d}p_i - \frac{\partial \mathcal{L}}{\partial t}\,\mathrm{d}t \end{aligned} \tag{3.9}$$

We can see, by comparison, that

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{3.10}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{3.11}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{3.12}$$

These are *Hamilton's equations*. Equation (3.10) describes the time evolution of the momenta, (3.11) the time evolution of the coordinates. Note the symmetry between q and p.

Let's look at a few simple conservation laws. If the Hamiltonian has no explicit time dependence so that $\partial H/\partial t = 0$, then H is a constant of motion:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = 0. \tag{3.13}$$

If the Lagrangian doesn't explicitly depend on some coordinate q_i, then so doesn't Hamiltonian by construction. The associated momentum p_i is conserved:

$$p_i = \frac{\partial H}{\partial q_i} = 0. \tag{3.14}$$

We already knew these conservation laws from the Lagrangian formalism. Later, we will see a greater class of transformations called *canonical transformation*, and associated symmetries and conserved quantities.

3.3 Phase space

In the Lagrangian formalism, the time evolution is described by a set of N Euler-Lagrange equations, each of which is second order in time. In the Hamiltonian formalism, this set of N

second order equations are replaced by a set of $2N$ first order equations: N for momenta and N for coordinates.

Recall that in the Lagrangian formalism, the system is described by a point in an N dimensional configuration space $\mathcal{C}$. Time evolution traces out a curve in $\mathcal{C}$. There is a subtle point here: the path in $\mathcal{C}$ depends on a set of $2N$ initial conditions $(q_0, \dot{q}_0)$. A point in $\mathcal{C}$ only gives half of the initial conditions - it doesn't specify $\dot{q}_0$. In this sense, a point in configuration space does not give a *complete* description - *the state* - of the system.

In the Hamiltonian formalism, the N dimensional configuration space is replaces by a $2N$ dimensional *phase space*. A pair (p, q) specifies a point in phase space. Since Hamilton's equations are first order in time, a point in phase space not only describes the system, but *completely* determines the time evolution. It gives a complete set of $2N$ initial conditions and completely specifies the state of the system.

As a corollary, since any point in phase space completely determines the time evolution of the system, trajectories in phase space can never cross due to uniqueness. The evolution is said to be described by a *flow* in the phase space.

3.4 Action principle

The action is "modified" in a subtle way. Instead of considering an action of the form

$$\mathcal{S}[q] = \int \mathcal{L}(q, \dot{q}, t)\, \mathrm{d}t\,,$$

which is a functional of a curve in an N dimensional configuration space, we consider

$$\mathcal{S}[q, p] = \int \left(p_i \dot{q}_i(q, p) - H(q, p, t)\right) \mathrm{d}t\,, \tag{3.15}$$

which is a functional of a curve in a $2N$ dimensional phase space. This means q and p are varied *independently*. Note that the integrand doesn't explicitly depend on $\dot{q}$ or $\dot{p}$.

The two actions describe the same dynamics. To obtain the equations of motion, we look at the variation of action with respect to q and p:

$$\begin{aligned}
\delta\mathcal{S} &= \int_{t_1}^{t_2} \left\{\delta(p_i \dot{q}_i) - \delta H(q, p, t)\right\} \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left\{\dot{q}_i \delta p_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right\} \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left\{\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) + \delta q_i \left(-\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\right\} \mathrm{d}t + p_i \delta q_i \Big|_{t_1}^{t_2}\,.
\end{aligned} \tag{3.16}$$

Requiring $\delta\mathcal{S} = 0$ yields Hamilton's equations, provided the variation obeys $\delta q_i(t_1) = \delta q_i(t_2) = 0$ for all i. There is no such constraint for δp_i.

3.5 Poisson brackets

Suppose we have some quantity $F(q, p, t)$ defined over phase space. The total time derivative is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial F}{\partial t} + \{F, H\}, \tag{3.17}$$

where we defined the *Poisson bracket* of two functions $f(q, p, t)$ and $g(q, p, t)$ to be

$$\{f, g\} := \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{3.18}$$

Let's look at some properties:

1. *Antisymmetry:* $\{f,g\} = -\{f,g\}$.
2. *Linearity:* $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$.
3. *Leibniz:* $\{fg,h\} = f\{g,h\} + g\{f,h\}$.
4. *Jacobi:* $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.

A corollary of equation (3.17) is that for any $F(q,p)$, we have

$$\{F,H\} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}F}{\mathrm{d}t} = 0 \quad \Rightarrow \quad F \text{ conserved.} \tag{3.19}$$

For example, if H doesn't depend on some q_i, then

$$\{p_i, H\} = \frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j} = 0,$$

so p_i is a constant of motion.

Suppose F and G are constants of motion, Then, $\{F,G\}$ is also a constant of motion since

$$\{\{F,G\},H\} = \{F,\{G,H\}\} + \{G,\{H,F\}\} = 0$$

by Jacobi. The constants of motion form a closed algebra under the Poisson bracket.

Hamilton's equations can be written in terms of Poisson brackets as

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\}. \tag{3.20}$$

Finally, the Poisson brackets of the coordinates are:

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}. \tag{3.21}$$

Angular momentum

Consider the angular momentum in Cartesian coordinates:

$$\vec{L} = \vec{r} \times \vec{p} \quad \Leftrightarrow \quad L_i = \epsilon_{ijk} r_j p_k.$$

Let's look at the Poisson bracket structure:

$$\{L_1, L_2\} = \{r_2p_3 - r_3p_2, r_3p_1 - r_1p_3\} = \{r_2p_3, r_3p_1\} + \{r_3p_2, r_1p_3\} = -r_2p_1 + r_1p_2 = L_3.$$

This generalises easily to:

$$\{L_i, L_j\} = \epsilon_{ijk} L_k.$$

Furthermore, we have

$$\left\{L^2, L_i\right\} = 2L_j\{L_j, L_i\} = 2\epsilon_{jik} L_j L_k = 0.$$

The similarity to quantum mechanics is not an accident.

3.6 Liouville's theorem

We will present two *almost* equivalent statements of *Liouville's theorem*: one in terms of the time evolution of a volume in phase space; the other in terms of the evolution of a density function.

3.6.1 Phase space volumes

Theorem (Liouville 1)**.** *Consider some bounded region Ω in phase space. As time evolves, although the shape of the region may change, the total volume of it remains constant.*

Proof. Consider a $2n$ dimensional phase space. The volume element at a given point (q,p) is

$$\mathrm{d}V = \mathrm{d}q_1 \dots \mathrm{d}q_n \, \mathrm{d}p_1 \dots \mathrm{d}p_n = \mathrm{d}^n q \, \mathrm{d}^n p \,. \tag{3.22}$$

In time δt, the coordinates transform as

$$q_i \xrightarrow{\delta t} \tilde{q}_i(q,p) = q_i + \dot{q}_i(q,p)\delta t + \mathcal{O}\big(\delta t^2\big), \quad p_i \xrightarrow{\delta t} \tilde{p}_i(q,p) = p_i + \dot{p}_i(q,p)\delta t + \mathcal{O}\big(\delta t^2\big), \tag{3.23}$$

where we note that each $\dot{q}_i$ and $\dot{p}_i$ are functions of all coordinates (q,p). The volume element transforms with the Jacobian:

$$\mathrm{d}V \xrightarrow{\delta t} \mathrm{d}\tilde{V} = \mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p} = \det(J(\delta t))\,\mathrm{d}^n q\,\mathrm{d}^n p\,, \tag{3.24}$$

where the Jacobian is given by

$$J(\delta) = \begin{pmatrix} \frac{\partial\tilde{q}_1}{\partial q_1} & \cdots & \frac{\partial\tilde{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\tilde{p}_n}{\partial q_1} & \cdots & \frac{\partial\tilde{p}_n}{\partial p_n} \end{pmatrix} = \begin{pmatrix} 1+\delta t\frac{\partial\dot{q}_1}{\partial q_1} & \cdots & \delta t\frac{\partial\dot{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \delta t\frac{\partial\dot{p}_n}{\partial q_1} & \cdots & 1+\delta t\frac{\partial\dot{p}_n}{\partial p_n} \end{pmatrix} = 1 + \delta t \begin{pmatrix} \frac{\partial\dot{q}_1}{\partial q_1} & \cdots & \frac{\partial\dot{q}_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\dot{p}_n}{\partial q_1} & \cdots & \frac{\partial\dot{p}_n}{\partial p_n} \end{pmatrix}. \tag{3.25}$$

Now, we have to take a short linear algebra aside. For any matrix M, the determinant can be expressed in terms of the eigenvalues as

$$\det(M) = \prod_i \lambda_i \quad \Rightarrow \quad \det(1+\epsilon M) = \prod_i (1+\epsilon\lambda_i) = 1 + \epsilon\,\mathrm{Tr}(M) + \mathcal{O}\big(\epsilon^2\big), \tag{3.26}$$

where we used $\sum_i \lambda_i = \mathrm{Tr}(M)$. Now, combining equations (3.25) and (3.26), we get that

$$\mathrm{d}\tilde{V} = \left(1 + \delta t \sum_{i=1}^{n} \left[\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i}\right] + \mathcal{O}\big(\delta t^2\big)\right) \mathrm{d}^n q\,\mathrm{d}^n p\,. \tag{3.27}$$

By Hamilton's equations, the terms in square brackets vanish:

$$\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i} = \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0.$$

Hence, we get

$$\mathrm{d}\tilde{V} = \det(J(\delta t))\,\mathrm{d}V = \big(1 + \mathcal{O}\big(\delta t^2\big)\big)\,\mathrm{d}V\,. \tag{3.28}$$

Noting that $J(t=0) = 1$, we have

$$\det(J(\delta t)) = \det(J(0)) + \mathcal{O}\big(\delta t^2\big) \quad \Rightarrow \quad \frac{\mathrm{d}(\det(J))}{\mathrm{d}t} = 0. \tag{3.29}$$

We're basically done at this stage. But, for completeness, we can consider the finite volume V of a region Ω:

$$V(t) = \int_\Omega \mathrm{d}V\,,$$

which, after time δt becomes

$$V(t+\delta t) = \int_{\tilde{\Omega}} \mathrm{d}\tilde{V} = \int_\Omega \big(1 + \mathcal{O}\big(\delta t^2\big)\big)\,\mathrm{d}V = V(t) + \mathcal{O}\big(\delta t^2\big) \quad \Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

This completes our proof. □

3.6.2 Distributions over phase space

The second, *almost* equivalent statement of the theorem is concerned with distribution functions over phase space. Suppose we have an ensemble of N identical, independent systems distributed in a $2n$ dimensional phase space by some distribution $\rho(q,p,t)$. Each system is described by a corresponding point in phase space at a given time. Also suppose the time evolution for all systems is governed by the same Hamiltonian $H(q,p,t)$.

The normalisation condition is

$$N = \int \rho(q,p,t)\,\mathrm{d}^n q\,\mathrm{d}^n p \quad \forall\, t, \tag{3.30}$$

where the integral is taken over all phase space. Correspondingly, the number of systems in a volume Ω of phase space is given by

$$N_\Omega = \int_\Omega \rho(q,p,t)\,\mathrm{d}^n q\,\mathrm{d}^n p\,. \tag{3.31}$$

Theorem (Liouville 2)**.** *The distribution ρ is constant along every trajectory in phase space. That is,* $\mathrm{d}\rho/\mathrm{d}t = 0$.

Proof. From equation (3.31), it follows that

$$\frac{\mathrm{d}N_\Omega}{\mathrm{d}t} = \int_\Omega \frac{\partial \rho}{\partial t}\,\mathrm{d}^n q\,\mathrm{d}^n p\,. \tag{3.32}$$

We can express this in terms of some *phase space flux* $\vec{J}$ so that

$$\frac{\mathrm{d}N_\Omega}{\mathrm{d}t} = -\oint_{\partial\Omega} \vec{J}\cdot\mathrm{d}\vec{S} = -\int_\Omega \vec{\nabla}\cdot\vec{J}\,\mathrm{d}^n q\,\mathrm{d}^n p\,, \tag{3.33}$$

where we used the divergence theorem. Note that since we are dealing with a $2n$ dimensional phase space, the $\vec{\nabla}$ operator is given by

$$\vec{\nabla} = \left(\frac{\partial}{\partial q_1},\ldots,\frac{\partial}{\partial q_n},\frac{\partial}{\partial p_1},\ldots\frac{\partial}{\partial p_n}\right).$$

Combining equations (3.32) and (3.33) and noting we can take an infinitesimally small region Ω yields

$$-\int_\Omega \vec{\nabla}\cdot\vec{J}\,\mathrm{d}^n q\,\mathrm{d}^n p = \int_\Omega \frac{\partial\rho}{\partial t}\,\mathrm{d}^n q\,\mathrm{d}^n p \quad\Rightarrow\quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0.$$

This is the continuity equation for a distribution over phase space.

Since $\rho(q,p,t)$ is the number density of systems, the flux can be expressed as

$$\vec{J}(q,p,t) = \rho(q,p,t)\vec{v}(q,p,t),$$

where $\vec{v}$ is the velocity of the flow in phase space:

$$\vec{v} = (\dot{q}_1,\ldots,\dot{q}_n,\dot{p}_1,\ldots,\dot{p}_n) = \left(\frac{\partial H}{\partial p_1},\ldots,\frac{\partial H}{\partial p_n},-\frac{\partial H}{\partial q_1},\ldots,-\frac{\partial H}{\partial q_n}\right)$$

This implies

$$\vec{\nabla}\cdot\vec{v} = \sum_{i=1}^{n}\left(\frac{\partial^2 H}{\partial q_i\partial p_i} - \frac{\partial^2 H}{\partial p_i\partial q_i}\right) = 0.$$

Hence, we have

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot(\vec{v}\rho) = -\vec{v}\cdot\vec{\nabla}\rho = -\sum_{i=1}^{n}\left(\dot{q}_i\frac{\partial\rho}{\partial q_i} + \dot{p}_i\frac{\partial\rho}{\partial p_i}\right) \quad\Rightarrow\quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0. \tag{3.34}$$

This completes our proof. □

Although the two theorems seem different, we can infer one from the other. Consider the number of systems in some region Ω. As we follow Ω along its flow in phase space, we expect the number of systems enclosed in the region to be conserved, that is

$$N_\Omega(t) = N_{\tilde{\Omega}}(t+\delta t) \quad \Rightarrow \quad \int_\Omega \rho(q,p,t)\,\mathrm{d}^n q\,\mathrm{d}^n p = \int_{\tilde{\Omega}} \rho(\tilde{q},\tilde{p},t+\delta t)\,\mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p}\,. \tag{3.35}$$

By the first statement of the theorem, we have

$$\mathrm{d}^n\tilde{q}\mathrm{d}^n\tilde{p} = \big(1+\mathcal{O}\big(\delta t^2\big)\big)\,\mathrm{d}^n q\,\mathrm{d}^n p\,,$$

which implies

$$\int_{\tilde{\Omega}} \rho(\tilde{q},\tilde{p},t+\delta t)\,\mathrm{d}^n\tilde{q}\,\mathrm{d}^n\tilde{p} = \int_\Omega \rho(\tilde{q},\tilde{p},t+\delta t) + \mathcal{O}\big(\delta t^2\big)\,\mathrm{d}^n q\,\mathrm{d}^n p = \int_\Omega \rho(q,p,t)\,\mathrm{d}^n q\,\mathrm{d}^n p\,.$$

Substituting $\tilde{q} = q + \dot{q}\delta t$ and $\tilde{p} = p + \dot{p}\delta t$ and taking Ω to be small, we conclude

$$\rho(q+\dot{q}\delta t, p+\dot{p}\delta t, t+\delta t) - \rho(q,p,t) = \mathcal{O}\big(\delta t^2\big) \quad \Rightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0.$$

The reason I've said the two statements were *almost* equivalent is that the first statement merely a statement about volume flows in phase space and doesn't introduce any distribution function. In this sense, it is more fundamental and, as it turns out, it is a direct consequence of the symplectic manifold structure of phase space.

3.6.3 Liouville equation

We can rewrite $\mathrm{d}\rho/\mathrm{d}t = 0$ in the form

$$\frac{\partial\rho}{\partial t} = -\sum_{i=1}^n \frac{\partial\rho}{\partial q_i}\dot{q}_i + \frac{\partial\rho}{\partial p_i}\dot{p}_i = \sum_{i=1}^n \frac{\partial\rho}{\partial p_i}\frac{\partial H}{\partial q_i} - \frac{\partial\rho}{\partial q_i}\frac{\partial H}{\partial p_i} = \{H,\rho\}. \tag{3.36}$$

This is known as the *Liouville equation.* It is a statement of Liouville's theorem in terms of the distribution function.

3.6.4 Time independent distributions

3.6.5 Poincare Recurrence Theorem

3.7 Canonical Transformations

Recall that the Euler-Lagrange equations are coordinate invariant, meaning they take the same form under coordinate transformations of the form $q_i \longmapsto Q_i(q)$. Now that we are dealing with the Hamiltonian formalism, we are interested in the class of transformations of the form

$$q_i \longmapsto Q_i(q,p), \quad \text{and} \quad p_i \longmapsto P_i(q,p) \tag{3.37}$$

which leave Hamilton's equations invariant. By this, we mean that we require

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \text{and} \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}. \tag{3.38}$$

Such transformations are called *canonical transformations.* We may explicitly write out this condition:

$$\dot{Q}_i = \{Q_i,H\} = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} = \frac{\partial H}{\partial P_i}, \tag{3.39}$$

$$\dot{P}_i = \{P_i,H\} = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j} = -\frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_i} = -\frac{\partial H}{\partial Q_i}. \tag{3.40}$$

Comparing terms, this gives us a set of four equations relating q, p, Q, P. These put constraints on the forms of the functions Q_i and P_i.

As you can see, things tend to become too tedious, too quickly with this notation. Instead, consider the $2n$-tuple $\vec{x} = (\vec{q}, \vec{p})$ and the $2n \times 2n$ matrix J:

$$J = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{3.41}$$

With these, Hamilton's equations read:

$$\dot{\vec{x}} = J\frac{\partial H}{\partial \vec{x}} \quad \Leftrightarrow \quad \dot{x}_i = J_{ij}\frac{\partial H}{\partial x_j}. \tag{3.42}$$

Now, consider the transformation

$$\vec{x} \longmapsto \vec{y}(\vec{x}) \quad \text{where} \quad \vec{y}(\vec{x}) = \Big(\vec{Q}(q,p), \vec{P}(q,p)\Big). \tag{3.43}$$

Then, we have

$$\dot{y}_i = \sum_j \frac{\partial y_i}{\partial x_j}\dot{x}_j = \sum_{jk} \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial x_k} = \sum_{jk\ell} \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial y_\ell}\frac{\partial y_\ell}{\partial x_k}. \tag{3.44}$$

Now, note that $\partial y_i/\partial x_j$ is the Jacobian associated with the transformation. Denote this by $2n \times 2n$ matrix M:

$$M = \frac{\partial \vec{y}}{\partial \vec{x}} = \begin{pmatrix} [\partial Q/\partial q]_{n\times n} & [\partial Q/\partial p]_{n\times n} \\ [\partial P/\partial q]_{n\times n} & [\partial P/\partial p]_{n\times n} \end{pmatrix}. \tag{3.45}$$

Now, we can write equation (3.44) as

$$\dot{y}_i = M_{ij}J_{jk}M_{\ell k}\frac{\partial H}{\partial y_\ell} \quad \Leftrightarrow \quad \dot{\vec{y}} = \big(MJM^T\big)\frac{\partial H}{\partial \vec{y}}. \tag{3.46}$$

For the transformation to be canonical, we require

$$\dot{\vec{y}} = J\frac{\partial H}{\partial \vec{y}} \quad \Rightarrow \quad MJM^T = J. \tag{3.47}$$

So, $\vec{x} \longmapsto \vec{y}$ is a canonical transformation if and only if $MJM^T = J$. This is know as the *symplectic* condition. Matrices M which obey this are said to be *symplectic.*

Let's have a closer look at the symplectic condition. Suppose M is any matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each entry A, B, C, D is an $n \times n$ matrix. Then,

$$MJM^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} B & D \\ -A & -C \end{pmatrix} = \begin{pmatrix} 0 & AD - BC \\ BC - AD & 0 \end{pmatrix}.$$

Then, our requirement simply reduces to $AD - BC = 1$. Expressed in terms of the Jacobian, this is

$$\frac{\partial Q_i}{\partial q_j}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_j}\frac{\partial P_j}{\partial q_k} = \delta_{ik}. \tag{3.48}$$

3.7.1 Poisson brackets

How do Poisson brackets transform under canonical transformations? Consider the Poisson bracket of two arbitrary functions f and g:

$$\{f,g\}_{\vec{x}} = \frac{\partial f}{\partial x_i} J_{ij} \frac{\partial g}{\partial x_j} = \frac{\partial f}{\partial y_k} M_{ki} J_{ij} \frac{\partial g}{\partial y_\ell} M_{\ell j} = \frac{\partial f}{\partial y_k} \big(MJM^T\big)_{k\ell} \frac{\partial g}{\partial y_\ell} = \frac{\partial f}{\partial y_k} J_{k\ell} \frac{\partial g}{\partial y_\ell} = \{f,g\}_{\vec{y}}, \tag{3.49}$$

Hence, Poisson brackets are left invariant under canonical transformations. This can equivalently be taken as a defining property of a canonical transformation.

We can do one better and even claim the following: *A transformation is canonical if and only if the Poisson bracket structure between the coordinates and momenta are unchanged:*

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad \text{and} \quad \{Q_i, P_j\} = \delta_{ij}. \tag{3.50}$$

Proof. First, note that the Poisson bracket structure between q and p can be written as

$$\{x_i, x_j\} = \frac{\partial x_i}{\partial x_k} J_{k\ell} \frac{\partial x_j}{\partial x_\ell} = J_{ij}.$$

Following the same method, we have

$$\{y_i, y_j\} = \big(MJM^T\big)_{ij} = J_{ij} \quad \Leftrightarrow \quad MJM^T = J. \tag{3.51}$$

This completes the proof. □

Note that

$$MJM^T = \begin{pmatrix} \{Q_i, Q_j\}_{n\times n} & \{Q_i, P_j\}_{n\times n} \\ \{P_i, Q_j\}_{n\times n} & \{P_i, P_j\}_{n\times n} \end{pmatrix}. \tag{3.52}$$

So, canonical transformations preserve the Poisson bracket structure, and whenever the Poisson bracket structure is preserved, the transformation must be canonical.

Transformation of coordinates

We know from the Lagrangian formalism that under a coordinate transformation $q_i \longmapsto Q_i(q)$, the Euler-Lagrange equations are left invariant. Let's check what constraint this puts on the form of P_i in the Hamiltonian formalism.

We refer to equation (3.48) and note that $\partial Q/\partial p \equiv 0$, so

$$\frac{\partial Q_i}{\partial q_j} \frac{\partial P_j}{\partial p_k} = \delta_{ik} \quad \Rightarrow \quad P_i = \frac{\partial q_j}{\partial Q_i} p_j.$$

This is equivalent to the statement $P_i = \partial \mathcal{L} \Big/ \partial \dot{Q}_i$, as we would expect.

3.7.2 Infinitesimal canonical transformations

Suppose we consider transformations of the form

$$q_i \longmapsto Q_i(q,p;\theta) \quad \text{and} \quad p_i \longmapsto P_i(q,p;\theta), \tag{3.53}$$

which depend continuously on some parameter θ. We set $\theta = 0$ to correspond to our original coordinates such that $Q_i(q,p,0) = q_i$ and $P_i(q,p,0) = p_i$. We can write such an infinitesimal transformation as

$$q_i \longmapsto Q_i = q_i + \theta \left.\frac{\mathrm{d}Q_i}{\mathrm{d}\theta}\right|_{\theta=0} + \mathcal{O}\big(\theta^2\big), \tag{3.54}$$

$$p_i \longmapsto P_i = p_i + \theta \left.\frac{\mathrm{d}P_i}{\mathrm{d}\theta}\right|_{\theta=0} + \mathcal{O}\big(\theta^2\big). \tag{3.55}$$

Requiring that this is canonical puts constraints on the functions $\mathrm{d}Q_i/\mathrm{d}\theta$ and $\mathrm{d}P_i/\mathrm{d}\theta$. From equation (3.48) we find that this constraint, to linear order in θ, is

$$\frac{\partial}{\partial q_j}\left(\frac{\mathrm{d}Q_i}{\mathrm{d}\theta}\right) = -\frac{\partial}{\partial p_j}\left(\frac{\mathrm{d}P_i}{\mathrm{d}\theta}\right), \tag{3.56}$$

which is satisfied if

$$\frac{\mathrm{d}Q_i}{\mathrm{d}\theta} = \frac{\partial G}{\partial p_i} \quad \text{and} \quad \frac{\mathrm{d}P_i}{\mathrm{d}\theta} = -\frac{\partial G}{\partial q_i} \tag{3.57}$$

for some function $G(q,p)$ which is called the *generator*. An infinitesimal canonical transformation must then look like:

$$q_i \longmapsto Q_i = q_i + \theta\frac{\partial G}{\partial p_i} + \mathcal{O}\big(\theta^2\big), \tag{3.58}$$

$$p_i \longmapsto P_i = p_i - \theta\frac{\partial G}{\partial q_i} + \mathcal{O}\big(\theta^2\big). \tag{3.59}$$

Note that so far we have been interpreting transformations in the *passive* sense: relabeling same points on phase space with different labels (choice of coordinates). A one-parameter family of canonical transformations can instead be viewed in an *active* sense: as taking us from one point in phase space (q,p) to another $(q(\theta),p(\theta))$. As we vary the parameter θ, we flow along trajectories in phase space. The tangent to the trajectories are given by

$$\frac{\mathrm{d}q_i}{\mathrm{d}\theta} = \frac{\partial G}{\partial p_i}, \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}\theta} = -\frac{\partial G}{\partial q_i}. \tag{3.60}$$

This is just Hamilton's equations with the Hamiltonian replaced by generator G and time replaced by parameter θ. Then, time evolution itself can be thought of as a canonical transformation of coordinates

$$(q(t_0),p(t_0)) \longmapsto (q(t),p(t)),$$

generated by the Hamiltonian.

3.7.3 Noether's theorem

Consider an infinitesimal canonical transformation generated by G. The generator G is called a *symmetry* of the Hamiltonian if, under the transformation, the Hamiltonian is left invariant. To see when this happens, simply consider the variation δH:

$$\delta H = \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i = \theta\left(\frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i}\right) = \theta\{H,G\}. \tag{3.61}$$

G is a symmetry if $\{H,G\}=0$. But we know already that $\dot{G}=\{G,H\}$. So if G is a symmetry, it is obviously a conserved quantity. Conversely, if we have a conserved quantity, we can generate with it infinitesimal canonical transformations.

3.7.4 Generating functions

Generating functions provide a different approach to obtaining canonical transformations. Now, let's consider transformations which can have explicit time dependence so that

$$q_i \longmapsto Q_i = Q_i(q,p,t) \quad \text{and} \quad p_i \longmapsto P_i = P_i(q,p,t).$$

In general such transformations modify the Hamiltonian (as we will see). In this case, the transformations are said to be canonical if there exists a new Hamiltonian $K = K(Q,P,T)$ such that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Let's now look at the action in both coordinates:

$$\mathcal{S}[q,p] = \int p_i\dot{q}_i - H(q,p,t)\,\mathrm{d}t \quad \text{and} \quad \mathcal{S}[Q,P] = \int P_i\dot{Q}_i - K(Q,P,t)\,\mathrm{d}t\,.$$

We require that $\delta\mathcal{S}[q,p] = 0 \Leftrightarrow \delta\mathcal{S}[Q,P] = 0$. This implies

$$\lambda(p_i\dot{q}_i - H(q,p,t)) = \int P_i\dot{Q}_i - K(Q,P,t) + \dot{F}. \tag{3.62}$$

for some constant λ and function F. If $\lambda = 1$, the transformation is said to be canonical. Otherwise, if $\lambda \neq 1$ it is *extended canonical*. Finally, if $Q = Q(q,p)$ and $P = P(q,p)$ without explicit time dependence, the transformation is *restricted canonical* - which is what we considered so far. Now, we only impose $\lambda = 1$.

If F depends on a mix of old and new phase space coordinates it is called a *generating function*. There are four types:

1. Let $F = F_1(q,Q,t)$. This relates q to Q. Then,

$$\dot{q}_ip_i - H = \dot{Q}_iP_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i. \tag{3.63}$$

   Matching the terms yields

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}, \quad K = H + \frac{\partial F_1}{\partial t}. \tag{3.64}$$

   This gives us the rest of the transformation $p_i = p_i(q,Q,t)$, $P_i = P_i(q,Q,t)$.

2. Now, we simply take Legendre transforms. Let $F = F_2(q,P,t) - Q_iP_i$, where F_2 relates q to P. Then,

$$\dot{q}_ip_i - H = -Q_i\dot{P}_i - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i, \tag{3.65}$$

   which yields

$$p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i}, \quad K = H + \frac{\partial F_2}{\partial t}. \tag{3.66}$$

3. $F = q_ip_i + F_3(p,Q,t)$:

$$q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i}, \quad K = H + \frac{\partial F_3}{\partial t}. \tag{3.67}$$

4. $F = q_ip_i - Q_iP_i + F_4(p,P,t)$:

$$q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i}, \quad K = H + \frac{\partial F_4}{\partial t}. \tag{3.68}$$

3.8 Integrability

Symmetries lead to conserved quantities. Conserved quantities impose additional constraints on the system. To see this, let $F(q,p)$ be a conserved quantity. Then, the constraint is of the form

$$\dot{F} = \{F,H\} = 0.$$

Since the Hamiltonian is a function of (q,p), each such symmetry gives a constraint between the coordinates and momenta. The phase space flow is constrained to be on a surface of constant F.

If there are enough symmetries so that a solution to Hamilton's equations can be obtained by just integration, the system is said to be *integrable.*

For a $2n$ dimensional phase space, integrability requires n independent conserved quantities F_i with the additional conditions

$$\{F_i, F_j\} = 0 \quad \forall\, i, j.$$

Such functions F_i and F_j are said to be *in involution.* As it turns out, the topology of the surface we are constrained on in the phase space is that of an n-torus.

3.9 Action-Angle Variables

If a system is integrable, we expect n conserved quantities. One way to naturally obtain n conserved quantities is to have n ignorable coordinates. Supposing the Hamiltonian has no explicit q dependence, the n generalised momenta are conserved:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \equiv 0.$$

We are interested in a transformation $(q, p) \longmapsto (\theta, I)$ which satisfy:

1. New momenta are functions of the conserved quantities: $I_i = I_i(F)$.
2. New coordinates are all *ignorable* - Hamiltonian has no explicit θ dependence.
3. Transformation is restricted canonical:

$$\begin{aligned} H(q,p) &\longmapsto H(\theta, I) = H(I), \\ \dot{q}_i = \frac{\partial H}{\partial p_i} &\longmapsto \dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i(I), \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} &\longmapsto \dot{I}_i = -\frac{\partial H}{\partial \theta_i} \equiv 0, \end{aligned}$$

   where $\omega_i(I)$ are functions of I.

The variables (θ, I) are called the *angle-action variables.* Since I_i are constant, all angle variables evolve at *uniform* rates $\omega_i(I)$. The equations of motion are integrated to give:

$$\theta_i = \omega_i(I)t + \beta_i, \quad I_i = \text{constant}, \tag{3.69}$$

for integration constants β_i.