Special Relativity

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Preface.

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Chapter 1

The Principle of Relativity

$(c = 1)$

1.1 Postulates

Definition 1. A *reference frame* is a system of coordinates (labels) which associates a position $\vec{x}$ and a time t for *every point in spacetime.*

Definition 2. A frame of reference in which a free body moves with constant velocity is said to be *inertial.*

It follows that if two reference frames move uniformly relative to each other, and if one of them is inertial then so is the other one. Now, we state two experimental facts which we call the postulates of relativity.

Postulate 1. *The laws of physics are the same in all inertial frames.*

Postulate 2. *The speed of light in vacuum is the same in all inertial frames.*

1.2 Interval

Definition 3. An *event* is a point in space time.

We now express the second postulate in a mathematical form. Consider two events $\mathscr{P}_1$ and $\mathscr{P}_2$, connected by a light beam. Let $\mathcal{S}$ be an inertial frame, in which the events have coordinates

$$\mathscr{P}_1 = (t_1, \vec{x}_1), \quad \mathscr{P}_2 = (t_2, \vec{x}_2).$$

Since the two events are connected by a light beam, we have

$$|\vec{x}_2 - \vec{x}_1|^2 = (t_2 - t_1)^2.$$

We write this in the form

$$(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0.$$

Let $\mathcal{S}'$ be another inertial frame. By postulate two, we immediately have

$$(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2 = 0.$$

This motivates us to define an *interval.*

Definition 4. Given an inertial frame $\mathcal{S}$ and two events with coordinates $(t_1, \vec{x}_1), (t_2, \vec{x}_2)$ the interval between them is defined as

$$s_{12} = \left[(t_2 - t_1)^2 - (\vec{x}_2 - \vec{x}_1)^2\right]^{\frac{1}{2}}. \tag{1.1}$$

Note that, by the second postulate, if in any given inertial frame the interval is zero, it must be zero in *every inertial frame*:

$$s = 0 \Longleftrightarrow s' = 0. \tag{1.2}$$

1.2.1 Invariance of the interval

We will now prove that the interval between *any two events* is invariant between inertial frames. But first, we need to show that inertial frames must be related to each other by *linear transformations*. From now on, we denote coordinates by the convention

$$(t, x, y, z) \longrightarrow (x^0, x^1, x^2, x^3) \longrightarrow x^\mu$$

with $\mu = 0, 1, 2, 3$. Similarly for frame $\mathcal{S}'$,

$$(t', x', y', z') \longrightarrow (x^{0'}, x^{1'}, x^{2'}, x^{3'}) \longrightarrow x^{\mu'}.$$

Proposition. The coordinate transformations from an inertial frame to another are *linear*.

Proof. Let $\mathcal{S}$ and $\mathcal{S}'$ be two inertial frames. Consider an arbitrary clock, reading time τ, moving at uniformly in frame $\mathcal{S}$. By homogeneity, equal ticks in τ correspond equal intervals in coordinates $(t, \vec{x})$. Therefore,

$$\frac{dx^\mu}{d\tau} = \text{constant}, \qquad \frac{d^2x^\mu}{d\tau^2} = 0.$$

In general, the coordinates of the clock in $\mathcal{S}'$ is given by some function of the coordinates x^μ:

$$x^{\mu'} = x^{\mu'}(x).$$

By chain rule, we have

$$\frac{dx^{\mu'}}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial x^{\mu'}}{\partial x^\mu}$$

and similarly,

$$\begin{aligned}
\frac{d^2x^{\mu'}}{d\tau^2} &= \frac{d}{d\tau}\left[\frac{dx^\mu}{d\tau}\frac{\partial x^{\mu'}}{\partial x^\mu}\right] \\
&= \underbrace{\frac{d^2x^\mu}{d\tau^2}}_{=0}\frac{\partial x^{\mu'}}{\partial x^\mu} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\nu} \\
&= \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\nu} = 0.
\end{aligned}$$

Since this must hold for all inertial frames $\mathcal{S}'$, we must have

$$\frac{\partial^2 x^{\mu'}}{\partial x^\mu \partial x^\nu} \equiv 0,$$

and so the coordinate transformation $x^{\mu'} = x^{\mu'}(x)$ is linear. □

Proposition. Given any two inertial frames $\mathcal{S}$ and $\mathcal{S}'$, the interval between any two events in $\mathcal{S}$ is equal to the interval in $\mathcal{S}'$.

Proof. Our starting point will be to show that the infinitesimal interval

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

is of the same order in any two inertial frames. Let's state this precisely. Consider two arbitrary inertial frames $\mathcal{S}$ and $\mathcal{S}'$. Suppose we parameterize the interval between two arbitrary points by some parameter ϵ (this can be done by parameterizing one of the points), such that

$$\lim_{\epsilon\to 0} s(\epsilon) = 0 \iff \lim_{\epsilon \to 0} s'(\epsilon) = 0. \tag{$*$}$$

As $\epsilon \to 0$, $s(\epsilon) \to ds$ and $s'(\epsilon) \to ds'$ assuming the two points approach each other in the limit $\epsilon \to 0$. Now, the infinitesimal intervals ds and ds' are of the same order if

$$\lim_{\epsilon\to 0} \frac{s(\epsilon)}{s'(\epsilon)} = A \neq 0, \tag{$**$}$$

meaning they approach zero in same order in ϵ. Let's show explicitly that this is the case.

As we are dealing with the interval between two points, we can fix one of the points without loss of generality. So, let's define one of the points to be the origin of $\mathcal{S}$ and $\mathcal{S}'$, denoted $\mathscr{O}$. Now, choose any point in spacetime $\mathscr{P}$ and consider some parameterization $\mathscr{P}(\epsilon)$ such that

$$\lim_{\epsilon\to 0} \mathscr{P}(\epsilon) = \mathscr{O}.$$

Clearly, with such parameterization, we satisfy ($*$). In frame $\mathcal{S}$, the point $\mathscr{P}(\epsilon)$ will have some coordinates

$$\mathscr{P}(\epsilon) = (t(\epsilon), \vec{x}(\epsilon)),$$

where we consider parameterizations such that the functions $t(\epsilon)$ and $\vec{x}(\epsilon)$ are analytic in ϵ, which we are allowed to do in a general sense because spacetime does not have gaps in it. Then, near $\epsilon = 0$, in general we will have

$$t(\epsilon) = \mathcal{O}(\epsilon^n), \quad x(\epsilon) = \mathcal{O}(\epsilon^m), \quad y(\epsilon) = \mathcal{O}(\epsilon^k), \quad z(\epsilon) = \mathcal{O}(\epsilon^\ell),$$

for some $n, m, k, \ell \geq 0$. It then follows that

$$s(\epsilon) = \mathcal{O}\Big(\epsilon^{\min(n,m,k,\ell)}\Big).$$

We note that if one of the indices, (say k), equals zero, then the condition $\mathscr{P}(\epsilon \to 0) = \mathscr{O}$ can only be satisfied if the corresponding coordinate (say y), is identically zero. In this case, it does not contribute to the behaviour of $s(\epsilon \to 0)$. So, when we write $\min(n, m, k, \ell)$ we only consider the non-zero powers.

Now, since the transformation from $x^\mu \to x^{\mu'}$ is linear, we can write it as

$$J^{\mu'}{}_{\mu} x^\mu = x^{\mu'}.$$

We know that we are able to invert this coordinate transformation since all inertial frames are treated on equal footing, hence the determinant of the Jacobian is non-zero. It then follows that there exists at least one coordinate $x^{\mu'}$ such that

$$x^{\mu'} = \mathcal{O}\Big(\epsilon^{\min(n,m,k,\ell)}\Big),$$

and there doesn't exist any $x^{\nu'}$ such that

$$x^{\nu'} = \mathcal{O}(\epsilon^p) \quad \text{with} \quad p < \min(n, m, k, \ell).$$

This is all we need to state that

$$s'(\epsilon) = \mathcal{O}\Big(\epsilon^{\min(n,m,k,\ell)}\Big) = \mathcal{O}(s(\epsilon)).$$

Therefore, for an arbitrary parameterization which is analytic in the coordinates, $(**)$ is satisfied.

Now, let's consider a particular parameterization: $\mathscr{P}(\epsilon) = (\epsilon t_0, \epsilon \vec{x}_0)$ in $\mathcal{S}$. The interval between $\mathscr{P}(\epsilon)$ and $\mathscr{O}$ in frame $\mathcal{S}$ is

$$s^2(\epsilon) = (\epsilon t_0)^2 - (\epsilon \vec{x}_0)^2 \implies \lim_{\epsilon\to 0} s^2(\epsilon) = dt^2 - dx^2 - dy^2 - dz^2 = ds^2,$$

where as $\epsilon \to 0$, $\epsilon x^\mu \to dx^\mu$. From $(**)$ we have

$$ds^2 = A ds'^2.$$

We don't know what the coefficient A might be, but since it relates two inertial frames the only parameters it can depend on are the coordinates $x^\mu, x^{\mu'}$ and the relative velocity $\vec{v}$ of the frames. By *homogeneity* of space and time, we immediately conclude that there cannot be any dependence on the coordinates - *there are no special points in spacetime.* Also, by *isotropy* of space it cannot depend on the direction of $\vec{v}$ - *space has no preferred direction.* Hence we conclude $A = A(v)$ can only be a function of the magnitude of the relative velocity between frames $\mathcal{S}$ and $\mathcal{S}'$. Note that by choosing a particular parameterization (path) for $\mathscr{P}(\epsilon)$, we essentially convert any dependence of A on the path parameterized by ϵ to the coordinates $(t_0, \vec{x}_0)$, so don't have to worry about A depending on the path we take.

Let v be the speed of $\mathcal{S}$ relative to $\mathcal{S}'$. Then,

$$ds^2 = A(v) ds'^2.$$

But there is nothing special about frame $\mathcal{S}$, and since A only depends on the relative speed, by symmetry we must also have

$$ds'^2 = A(v) ds^2.$$

Together, these imply $A(v) \equiv \pm 1$. Considering a third frame moving relative to $\mathcal{S}$ and $\mathcal{S}'$ it becomes clear that $A(v) \equiv 1$, hence

$$ds^2 = ds'^2. \tag{1.3}$$

Since the infinitesimal intervals are invariant, clearly finite intervals must remain invariant. □

1.2.2 Time-like and space-like separations

Two events are said to be *timelike* separated if their interval is real, meaning $\Delta s^2 > 0$. This immediately implies that there exists an inertial frame in which the two events occur at the same position, setting $\Delta x = 0$,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = \Delta t^2 > 0.$$

Similarly, two events are said to be *spacelike* if their interval is imaginary, meaning $\Delta s^2 < 0$. Again, it follows that there exists an inertial frame in which the two events happen simultaneously, setting $\Delta t = 0$,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = -\Delta x^2 < 0.$$

Since the interval is invariant, a timelike interval remains timelike in all inertial frames. Similarly, a spacelike interval remains spacelike. This is

1.3 Proper Time

Imagine a particle, following an arbitrary path $x^\mu(\lambda)$ through spacetime. How is the time measured by the particle related to the time an inertial observer measures?

At any given instant, the particle can be regarded as an inertial frame, in which case it travels a distance $|d\vec{x}|$ in time dt in some reference frame $\mathcal{S}$. Let $d\tau$ be the time experienced by the particle. In the frame of the particle, the displacement is zero since the particle sees itself at rest, so we have

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = d\tau^2,$$

from which we obtain

$$d\tau = dt\sqrt{1 - \frac{d\vec{x}^2}{dt}} = dt\sqrt{1 - v^2},$$

where v is the velocity of the particle in frame $\mathcal{S}$. If we want to obtain the time experienced by the particle over some path, we simply integrate this

$$\tau = \int_\lambda d\tau = \int_{t(0)}^{t(\lambda)} dt\sqrt{1 - v^2(t)}.$$

The time experienced by the moving particle is called the *proper time* of the particle. It equals the interval ds in natural units. Furthermore, it is always the less than the time measured by a different observer. Moving clocks tick slower. As a consequence, since the proper time equals the interval, we obtain that the maximum value for

$$\int_a^b ds$$

is obtained if it is taken along the straight world line joining the two points together.

1.4 Lorentz Transformations

We can obtain transformations which take us from one inertial frame $\mathcal{S}$ with coordinates x^μ to another $\mathcal{S}'$ with coordinates $x^{\mu'}$ by simply considering the invariant interval. This mathematically translates to finding the Jacobian matrix $J^{\mu'}{}_{\mu}$. Landau motivates the form of the group of transformations that leaves the interval invariant (Lorentz group) as follows:

> *"...we may say that the required transformation must leave unchanged all distances in the x, y, z, t space. But such transformations consist only of parallel displacements, and rotations of the coordinate system. Of these the displacement of the coordinate system parallel to itself is of no interest, since it leads only to a shift in the origin of the space coordinates and a change in the time reference point. Thus the required transformation must be expressible mathematically as a rotation of the four-dimensional x, y, z, t coordinate system.*
>
> *"Every rotation in the four-dimensional space can be resolved into six rotations, in the planes xy, zy, xz, tx, ty, tz. The first three of these rotations transform only the space coordinates; they correspond to the usual space rotations."*

So, we expect six transformations in our group, three of which we already now. Let's consider a rotation in the tx plane, which corresponds to a boost in the $x-$direction.

Assuming $\mathcal{S}$ and $\mathcal{S}'$ share their origins, we have a general linear transformation:

$$\begin{aligned} t' &= At + Bx, \\ x' &= Ct + Dx. \end{aligned} \tag{1.4}$$

By the invariance of the interval, we must have

$$s^2 = {t'}^2 - {x'}^2 = t^2\big(A^2 - C^2\big) - x^2\big(D^2 - B^2\big) + 2xt(AB - CD) = t^2 - x^2.$$

$$\implies A^2 - C^2 = 1, \quad D^2 - B^2 = 1, \quad AB = CD.$$

The most general solution is given by

$$A = \pm\cosh\psi, \quad B = \pm\sinh\psi, \quad C = \pm\sinh\psi, \quad D = \pm\cosh\psi$$

for some $\psi \in \mathbb{R}$. If the two frames are identical, we require $x = x'$ and $t = t'$. This immediately gets rid of two minus signs:

$$A = \cosh\psi, \quad D = \cosh\psi.$$

As for the plus and minus ambiguity in B and C, we note that since ψ is arbitrary, we can absorb the sign into ψ. For now, let's take $B = C = -\sinh\psi$ and solve for ψ. So far, we have

$$\begin{aligned} t' &= t\cosh\psi - x\sinh\psi, & t &= t'\cosh\psi + x'\sinh\psi, \\ x' &= x\cosh\psi - t\sinh\psi, & x &= x'\cosh\psi + t'\sinh\psi. \end{aligned} \tag{$*$}$$

Let $\mathcal{S}'$ move with velocity v in $+x$ direction in $\mathcal{S}$. As both frames coincide at the origin, we have

$$x' = 0 \iff x = vt.$$

Substituting this into $(*)$, we obtain

$$\begin{aligned} t &= t'\cosh\psi \\ vt &= t'\sinh\psi \end{aligned} \implies v = \tanh\psi.$$

By hyperbolic identities, we have

$$\cosh^2\psi - \sinh^2\psi = 1 \implies 1 - \tanh^2\psi = \frac{1}{\cosh^2\psi} \implies \cosh\psi = \sqrt{\frac{1}{1-v^2}},$$

and similarly we have

$$\sinh\psi = \tanh\psi\cosh\psi = v\sqrt{\frac{1}{1-v^2}}.$$

Hence, we obtain the Lorentz transformation for a rotation in the tx plane

$$\begin{aligned} t' &= \gamma(v)(t - vx), & t &= \gamma(v)(t' + vx'), \\ x' &= \gamma(v)(x - vt), & x &= \gamma(v)(x' + vt'), \\ dy' &= dy & dy &= dy', \\ dz' &= dz & dz &= dz', \end{aligned} \tag{1.5}$$

where we defined $\gamma(v) = (1 - v^2)^{-\frac{1}{2}}$.

Note that $\gamma(v \to 1)$ diverges. This sets the speed of light as a natural speed limit. Finally, note that in general, Lorentz transformations *do not commute.* As they are rotations in the four dimensional space, the order in which two rotations are performed matters (unless the axis of rotation remains the same).

1.4.1 Length contraction

Suppose there is a rod in frame $\mathcal{S}$, parallel to the $x-$axis. Let its length be $\ell = x_2 - x_1$. We want to determine the rod's length in $\mathcal{S}'$, so we need to measure x_2' and x_1' at the same time t'. We have

$$x_1 = \gamma(x_1' + vt'), \quad x_2 = \gamma(x_2' + vt') \implies \ell = \gamma\ell'.$$

Since $\gamma > 1$, the length ℓ' in the moving frame is contracted. This is *Lorentz contraction.*

1.4.2 Addition of velocities

Suppose $\mathcal{S}'$ moves with velocity v relative to $\mathcal{S}$ along the x direction. Let $\vec{u}$ be the velocity of a particle in the $\mathcal{S}$ and $\vec{u}'$ be its velocity in $\mathcal{S}'$. How are the two related? We have

$$u^i = \frac{dx^i}{dt}, \quad u^{i'} = \frac{dx^{i'}}{dt'}.$$

By (1.5), we have

$$u_x = \frac{dx}{dt} = \frac{dx' + vdt'}{dt' + vdx'} = \frac{u'_x + v}{1 + vu'_x} \tag{1.6}$$

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + vdx')} = \frac{\sqrt{1-v^2}u'_y}{1 + vu'_x}, \tag{1.7}$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + vdx')} = \frac{\sqrt{1-v^2}u'_z}{1 + vu'_x}, \tag{1.8}$$

Now, suppose that the particle moves on the $x-y$ plane such that we can decompose its velocity into $u_x = u\cos\theta$ and $u_y = u\sin\theta$. Then, by (1.6) and (1.7) we have

$$\tan\theta = \frac{u'\sqrt{1-v^2}\sin\theta'}{u'\cos\theta' + v.} \tag{1.9}$$

This describes the change in the direction of the velocity. Finally, we consider the special case of the deviation of light from one frame to another. This is called *aberration*. In this case, $u = u' = 1$, so $u_x = \cos\theta$ and $u_y = \sin\theta$. From (1.6) and (1.7) we directly obtain

$$\cos\theta = \frac{\cos\theta' + v}{1 + v\cos\theta'}, \quad \sin\theta = \frac{\sin\theta'\sqrt{1-v^2}}{1 + v\cos\theta'}.$$

For small v, these reduce to the classical expression $\theta' - \theta = v\sin\theta'$.

1.4.3 Doppler effect

Suppose we have two inertial frames $\mathcal{S}'$ and $\mathcal{S}$. Let a light source sit at $\vec{x}' = \vec{0}$ emitting light with wavelength (period) λ'. When an observer sat at $\vec{x} = \vec{0}$ observes the light beam, what wavelength λ will he measure? Let's look at different cases.

Longitudinal

We may imagine two signals, separated in time $\Delta t' = \lambda'$ and in space by $\Delta x' = 0$ in $\mathcal{S}'$. Now, we ask: *how far apart in time are these two signals observed at $x = 0$ in $\mathcal{S}$?* Let's break the whole problem down into four events:

- $\mathscr{A}_1(x = 0, t = t_0)$: the first signal is observed, coordinates given in $\mathcal{S}$.
- $\mathscr{A}_2(x = 0, t = t_0 + \lambda)$: the second signal is observed, coordinates given in $\mathcal{S}$.
- $\mathscr{B}'_1(x' = 0, t' = t'_0)$: first signal is emitted, coordinates given in $\mathcal{S}'$.
- $\mathscr{B}'_2(x' = 0, t' = t'_0 + \lambda')$: second signal is emitted, coordinates given in $\mathcal{S}'$.

To simplify our lives, let's fix the origins of our coordinates such that $t_0 = 0$ in $\mathcal{S}$ and $t'_0 = 0$ in $\mathcal{S}'$. Now, let's write the coordinates of $\mathscr{B}_2$ in $\mathcal{S}$:

$$\mathscr{B}_2(x_2, t_2) = (\gamma(\Delta x' + v\Delta t'), \gamma(\Delta t' + v\Delta x')) = (\gamma v\lambda', \gamma\lambda').$$

All we need to do is calculate when a light signal emitted from $\mathscr{B}_2$ reaches the t-axis in $\mathcal{S}$. Now, we have to specify whether the source $x' = 0$ lies in the $x > 0$ or $x < 0$ region, as this determines the orientation of the light beam. The choice we make will not matter in the end, as long as we are consistent. Suppose the source lies on the $x > 0$ half of the xt plane. Then, the signal emitted from $\mathscr{B}_2(x_2, t_2)$ travels along the line

$$(x - x_2) = -(t - t_2),$$

from which we can read off the time it reaches the observer at $x = 0$:

$$t = x_2 + t_2 = \lambda'\gamma(1 + v) = \lambda'\sqrt{\frac{1+v}{1-v}}.$$

Referring back to the point $\mathscr{A}_2$, we see that $t = \lambda$ and so

$$\lambda = \lambda'\sqrt{\frac{1+v}{1-v}}. \tag{1.10}$$

Transverse

This is a simpler case. We imagine the light beam traveling along the y axis, perpendicular to the relative motion of the two frames. Let the source sit at some $Y > 0$ in the $\mathcal{S}$ frame. By the same construction, we write down the events:

- $\mathscr{A}_1(x, y, t) = (0, 0, t_0)$,
- $\mathscr{A}_2(x, y, t) = (0, 0, t_0 + \lambda)$,
- $\mathscr{B}_1'(x', y', t') = (0, 0, 0)$,
- $\mathscr{B}_2'(x', y', t') = (0, 0, \lambda')$.

Now, the first signal takes time to travel a distance Y. So, let's fix our origin $t = 0$ such that we have $\mathscr{B}_1(x, y, t) = (0, Y, 0)$. This implies $t_0 = Y$. As before, we find B_2 in $\mathcal{S}$ coordinates:

$$\mathscr{B}_2(x_2, y_2, t_2) = (\gamma(\Delta x' + v\Delta t'), Y, \gamma(\Delta t' + v\Delta x')) = (\gamma v\lambda', Y, \gamma\lambda').$$

Now, we need to find the time at which a light beam emitted from $\mathscr{B}_2$ reaches the t-axis. We do the approximation: $Y \gg \gamma v\lambda'$. *This approximation essentially means we are just looking at the transverse component.* If the y separation of the source and the observer is large enough, we can simply ignore the distance the source travels in the x direction in one period. So, we solve the equation for $y = 0$:

$$-(y - Y) = (t - t_2) \implies Y = t_0 + \lambda - t_2 \implies \lambda = t_2 = \gamma\lambda'.$$

Hence, we obtain the result

$$\lambda = \lambda'\sqrt{\frac{1}{1-v^2}}. \tag{1.11}$$

General

For the general case, we imagine the source at a distance L away from the observer, at an angle θ from the x axis, moving along the x axis. The events are

- $\mathscr{A}_1(x, y, t) = (0, 0, t_0)$,
- $\mathscr{A}_2(x, y, t) = (0, 0, t_0 + \lambda)$,

- $\mathscr{B}_1'(x', y', t') = (0, 0, 0)$,
- $\mathscr{B}_2'(x', y', t') = (0, 0, \lambda')$.

As before, we first calculate event $\mathscr{B}_1$ in $\mathcal{S}$ coordinates. Defining $t_1 = 0$, and noting that the source is a displacement $\vec{x}_1 = (L\cos\theta, L\sin\theta)$ away from the observer, we obtain:

$$\mathscr{B}_1(x_1, y_1, t_1) = (L\cos\theta, L\sin\theta, 0).$$

Now, we can solve for t_0 by noticing that the light has to travel a distance L to reach the observer. Hence, $t_0 = L$ and so $\mathscr{A}_1 = (0, 0, L), \mathscr{A}_2 = (0, 0, L + \lambda)$. Now, we write down $\mathscr{B}_2$ in $\mathcal{S}$:

$$\mathscr{B}_2(x_2, y_2, t_2) = (\gamma v\lambda' + L\cos\theta, L\sin\theta, \gamma\lambda').$$

The light emitted from the source travels with velocity $\vec{u} = (-\cos\theta, -\sin\theta)$, so the time it takes to reach the observer is $x_2\cos\theta + y_2\sin\theta$. Hence, the time at which the light beam reaches the observer is

$$x_2\cos\theta + y_2\sin\theta + t_2 = L\cos^2\theta + \gamma v\lambda'\cos\theta + L\sin^2\theta + \gamma\lambda' = L + \lambda'\gamma(1 + v\cos\theta).$$

But, we know from $\mathscr{A}_2$ that this time equals $L + \lambda$. Hence we obtain the general result

$$\lambda = \lambda' \frac{1 + v\cos\theta}{\sqrt{1 - v^2}}. \tag{1.12}$$

Let's perform a sanity check. When we set $\theta = 0$, we obtain (1.10). This corresponds to longitudinal shift. When we set $\theta = \pi/2$, we obtain (1.11), which corresponds to the transverse shift. Finally, when $v > 0$, the source is moving away from the observer, so $\lambda > \lambda'$ and the light is red-shifted. When $v < 0$, for θ not too large the light will get blue-shifted. The condition for blue-shift is

$$\frac{1 + v\cos\theta}{\sqrt{1 - v^2}} < 1,$$

where remember that $v < 0$. $\mathbb{1}$