Variation of Matrix Determinant

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1 Maths: Derivation

Let M be an $n \times n$ matrix with components M_{ij} and inverse M^{-1} with components M^{ij}, such that

$$\sum_j M^{-1}_{ij} M_{jk} = \delta_{ik}. \tag{1.1}$$

We are interested in the variation $\delta \det M$. In terms of δM_{ij}, this is

$$\delta \det M = \sum_{ij} \frac{\partial \det M}{\partial M_{ij}} \delta M_{ij}. \tag{1.2}$$

One expression for $\det M$ is

$$\det M = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k M_{k\pi(k)}, \tag{1.3}$$

where π are permutations and S_n is the symmetric group of degree n. Then,

$$\frac{\partial \det M}{\partial M_{ij}} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \frac{\partial}{\partial M_{ij}} \left(\prod_k M_{k\pi(k)} \right) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \delta_{j\pi(i)} \prod_{k \neq i} M_{k\pi(k)}. \tag{1.4}$$

Now, use equation (1.1) to substitute for $\delta_{j\pi(i)}$:

$$\frac{\partial \det M}{\partial M_{ij}} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \left(\sum_\ell M^{-1}_{j\ell} M_{\ell\pi(i)} \right) \prod_{k \neq i} M_{k\pi(k)} = \sum_\ell M^{-1}_{j\ell} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{k \neq i} M_{k\pi(k)} M_{\ell\pi(i)}. \tag{1.5}$$

The trick here is to separate the sum over ℓ into two parts: $\ell = i$ and $\ell \neq i$. The $\ell = i$ part gives

$$M^{-1}_{ji} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{k \neq i} M_{k\pi(k)} M_{i\pi(i)} = M^{-1}_{ji} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_k M_{k\pi(k)} = M^{-1}_{ji} \det M. \tag{1.6}$$

The $\ell \neq i$ part is

$$\sum_{\ell\neq i} M^{-1}_{j\ell} \sum_{\pi\in S_n} \text{sgn}(\pi) \prod_{k\neq i} M_{k\pi(k)} M_{\ell\pi(i)} = \sum_{\ell\neq i} M^{-1}_{j\ell} \sum_{\pi\in S_n} \text{sgn}(\pi) \prod_{k\neq \ell, i} M_{k\pi(k)} M_{\ell\pi(\ell)} M_{\ell\pi(i)}, \quad (1.7)$$

where we separated the ℓ^{th} term from product. Now, here is the key point: the expression is *symmetric* under the exchange $\pi(i) \leftrightarrow \pi(\ell)$ due to the $M_{\ell\pi(\ell)} M_{\ell\pi(i)}$ term.

For any $\pi \in S_n$ with $\text{sgn}(\pi) = +1$, we can construct a unique $\pi' \in S_n$ with $\text{sgn}(\pi') = -1$ by setting

$$\pi'(j) = \begin{cases} \pi(\ell) & j = i, \\ \pi(i) & j = \ell, \\ \pi(j) & \text{otherwise.} \end{cases} \quad (1.8)$$

Moreover, the set of all π' is the set of all odd permutations of degree n. Hence, equation (1.7) is identically zero. This gives the result

$$\frac{\partial \det M}{\partial M_{ij}} = M^{-1}_{ji} \det M \quad \Rightarrow \quad \delta \det M = \det M \sum_{ij} M^{-1}_{ji} \delta M_{ij}. \quad (1.9)$$

2 Applications to GR

2.1 Covariant divergence

Consider the metric $g_{\mu\nu}$ with inverse $g^{\mu\nu}$. Let $g = |\det g_{\mu\nu}|$. Then, we have

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \quad (2.1)$$

In particular, this implies

$$\partial_\lambda g = \frac{\partial g}{\partial g_{\mu\nu}} \partial_\lambda g_{\mu\nu} = g g^{\mu\nu} \partial_\lambda g_{\mu\nu}. \quad (2.2)$$

The Levi-Civita connection $\Gamma^\mu{}_{\nu\lambda}$ obeys

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}. \quad (2.3)$$

Using these, we can obtain a useful expression for the covariant divergence of a vector field

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu{}_{\mu\lambda} V^\lambda. \quad (2.4)$$

Taking a closer look at the connection:

$$\Gamma^\mu{}_{\mu\lambda} = g^{\mu\nu} \Gamma_{\nu\mu\lambda} = \frac{1}{2} g^{\mu\nu} \partial_\lambda g_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\lambda \sqrt{g}. \quad (2.5)$$

Substituting this to the expression above yields the result:

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} V^\mu). \quad (2.6)$$

2.2 Covariant Laplacian of scalar

The Laplacian of a scalar is defined covariantly as

$$\Box\phi = \nabla^\mu \nabla_\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\mu \phi. \quad (2.7)$$

Noting that the metric is covariantly constant, i.e. $\nabla_\lambda g_{\mu\nu} = 0$, equation (2.6) implies

$$g^{\mu\nu} \nabla_\mu \nabla_\mu \phi = \nabla_\mu g^{\mu\nu} \partial_\nu \phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi). \quad (2.8)$$

2.3 Minimal coupling: Klein-Gordon

The special relativistic Klein-Gordon action is

$$\mathcal{S}[\phi] = \int d^4x \left[-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2\right], \tag{2.9}$$

with metric convention $\eta = \mathrm{diag}(-1, +1, +1, +1)$. The resulting equation of motion is

$$(\Box_\eta - m^2)\phi = 0, \tag{2.10}$$

where $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Now, we write generally covariant action:

$$\mathcal{S}[\phi, g_{\mu\nu}] = \int d^4x \sqrt{g}\left[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2\right] \tag{2.11}$$

Varying this action with respect to ϕ yields

$$\begin{aligned}
\delta\mathcal{S} &= -\int d^4x \sqrt{g}\left[g^{\mu\nu}\partial_\nu\phi\partial_\mu\delta\phi + m^2\phi\delta\phi\right] \\
&= \int d^4x \sqrt{g}\left[\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi\right]\delta\phi + \text{surface term} \\
&= \int d^4x \sqrt{g}\left[\Box_g\phi - m^2\phi\right]\delta\phi = 0,
\end{aligned} \tag{2.12}$$

where we used equation (2.8) going from line 2 to 3. Hence, the covariant equation of motion is

$$\left(\Box_g - m^2\right)\phi = 0. \tag{2.13}$$

2.4 Minimal coupling: Maxwell

Vacuum special relativistic Maxwell action is

$$\mathcal{S}[A_\nu] = -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}\int d^4x\, \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}, \tag{2.14}$$

with field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The correct way to proceed is to keep the field tensor definition fixed, i.e. don't replace $\partial \to \nabla$. Instead, simply consider the action

$$\mathcal{S}[A_\nu, g_{\mu\nu}] = -\frac{1}{4}\int \sqrt{g}\, d^4x\, g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu}. \tag{2.15}$$

Varying the action with respect to A_ν:

$$\begin{aligned}
\delta\mathcal{S} &= -\frac{1}{2}\int \sqrt{g}\, d^4x\, g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\delta F_{\mu\nu} \\
&= -\frac{1}{2}\int \sqrt{g}\, d^4x\, g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu) \\
&= -\int \sqrt{g}\, d^4x\, g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\partial_\mu\delta A_\nu \\
&= \int d^4x\, \partial_\mu\big(\sqrt{g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}\big)\delta A_\nu + \text{surface term} \\
&= .\int d^4x\, \partial_\mu(\sqrt{g}F^{\mu\nu})\delta A_\nu = 0.
\end{aligned} \tag{2.16}$$

The resulting equation of motion is

$$\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0. \tag{2.17}$$

At first glance this doesn't look equivalent to $\nabla_\mu F^{\mu\nu} = 0$, which is what we may have expected. It turns out that they are equal due to antisymmetry of $F_{\mu\nu}$:

$$\nabla_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda}F^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda}F^{\mu\lambda}. \tag{2.18}$$

The last term vanishes since $\Gamma^\nu{}_{\mu\lambda}$ is symmetric in $\mu\lambda$, and $F^{\mu\lambda}$ antisymmetric. Hence, we have

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}F^{\mu\nu}) = 0 \quad \Leftrightarrow \quad \partial_\mu(\sqrt{g}F^{\mu\nu}) = 0. \tag{2.19}$$

Note. Using Euler-Lagrange equation to obtain equations of motion instead of varying the action directly is a waste of time and effort. If you don't believe me, have a go at the above derivation.