Electromagnetism

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preface

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1 Recap

1.1 Charges and Currents

We start with a recap of basic electromagnetism and Maxwell's equations. The basic idea is that electric and magnetic fields are produced by charge and current densities, and particles with charge respond to said fields by a force law. Let's define charge and current densities.

Definition. (Charge density) The *charge density*, $\rho(\mathbf{x}, t)$, is defined as charge per unit volume. Given a volume V, the total charge enclosed is then given by

$$Q = \int_V \rho(\mathbf{x}, t)\, \mathrm{d}^3 x\,.$$

Definition. (Current density) The *movement* of charge is described by the current density. Given a surface S, the total charge per unit time passing through it is given by

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}\,,$$

where $\mathrm{d}\mathbf{S}$ is the unit normal to S. So, the current density can be thought of as the *current per unit area.*

If a charge density $\rho(\mathbf{x}, t)$ has a velocity $\mathbf{v}(\mathbf{x}, t)$, then the current density would be

$$\mathbf{J} = \rho \mathbf{v}.$$

1.2 Continuity and Conservation

Law. (Continuity) Electric charge is *locally conserved.* This means, if the charge density at a point in space is changing, there must be a corresponding local *current.* This is represented by the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

Derivation. This result is simple to obtain. We consider the rate of change of the total charge enclosed in a fixed volume in space, V. By definition, it must be equal to the current *flowing in* the volume through its boundary S. So, we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}\,,$$

where the minus sign is due to the orientation of $\mathrm{d}\mathbf{S}$ being outward. By the divergence theorem, we have

$$\oint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{J}\, \mathrm{d}^3 x\,.$$

Also, by the definition of charge density, we can write

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho\, \mathrm{d}^3 x = \int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}^3 x\,,$$

since the volume is fixed. Hence, we have

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) \mathrm{d}V = 0,$$

which, since V is arbitrary, implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

1.3 Forces and Fields

A *field* is a dynamical quantity that assigns a value to every point in space and time. In the case of electromagnetism, we are concerned with two vector fields:

$$\mathbf{E}(\mathbf{x}, t), \quad \mathbf{B}(\mathbf{x}, t).$$

Each field assigns a *vector* to every point in space and time.

So, charges *create* fields, and fields tell charges how to move. The latter is governed by the Lorentz force law.

Law. (Lorentz force law) Any point charge q experience a force given by

$$\mathbf{F} = q\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right).$$

This can also be expressed in terms of the densities:

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B},$$

where $\mathbf{f}$ is the force experienced by unit volume.

The equations governing the fields are Maxwell's equations.

Law. (Maxwell's equations)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.4}$$

where ϵ_0 and μ_0 are the *electric* and the *magnetic constant*, respectively. We see that there are two terms which are not constants or fields. They are the charge and the current density. Hence, we conclude that electric and magnetic fields are produced by charge and current densities and changing electric and magnetic fields.

2 Electrostatics

In electrostatics, we are interested in a *fixed* charge distribution. Immediately, we can let $\mathbf{J} = 0$. As the charges are fixed, we can also conclude $\partial_t \mathbf{E} = 0$, and the two together imply $\mathbf{B} = 0$. We are now left with two equations to solve:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{2.1}$$

$$\nabla \times \mathbf{E} = 0. \tag{2.2}$$

We are interested in finding $\mathbf{E}$ for a given ρ.

2.1 Gauss' Law

We can rewrite (2.1) in integral form, using the divergence theorem and the definition of the charge density.

Law. (Gauss' law) For a given volume V, enclosed by a surface $S = \partial V$, we have

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\epsilon_0},$$

where Q is the total charge enclosed in V.

We therefore conclude that the electric flux through *any closed surface* is proportional to the charge contained in it.

2.1.1 Coulomb's law

We can use Gauss' law to derive Coulomb's force law. We start by considering a uniform spherical charge distribution of radius R, centered at the origin. Now, we choose a *Gaussian surface* S to be the surface of a sphere of radius $r > R$, also centered on the origin. By symmetry, the electric field cannot have any radial component. Hence, we conclude $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$ where $E(r)$ is a scalar function of the radius of the surface. So, we have

$$\oint_S \mathbf{E}(\mathbf{x}) \cdot \mathrm{d}\mathbf{S} = E(r) \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \, r^2 \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \frac{Q}{\epsilon_0}.$$

Hence, we get

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}},$$

and using the Lorentz force law for a charge q gives

$$\mathbf{F}(\mathbf{x}) = \frac{Qq}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}},$$

which is *Coulomb's law.* By the symmetry of the problem, we didn't need to use (2.2).

2.1.2 Uniform sphere

Consider the same charge distribution ρ, but now let the radius of the Gaussian surface be $r < R$. We have, similarly

$$4\pi r^2 E(r) = \frac{1}{\epsilon_0} \int_{V'} \rho \, \mathrm{d}^3 x' \,,$$

where V' is the volume enclosed by S. Since ρ is uniform, we have

$$\frac{1}{\epsilon_0} \int_{V'} \rho \, \mathrm{d}^3 x' = \frac{1}{\epsilon_0} \frac{4}{3} \pi r^3 \rho = \frac{Q r^3}{\epsilon_0 R^3},$$

where Q is the charge of the entire sphere. Therefore,

$$E(\mathbf{x}) = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}}.$$

Hence, the electric field strength increases linearly inside the sphere, and falls by inverse square law outside.

2.1.3 Line charge

Consider an infinite line of charge, with density λ, oriented along the z-axis. Let the Gaussian surface be a cylinder of length l, oriented along the line. By symmetry, the top and the bottom of the cylinder will not contribute to the flux, and we have again $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$, where $r = \sqrt{x^2 + y^2}$. Hence, we have

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 2\pi r l E(r) = \frac{\lambda l}{\epsilon_0},$$
$$\implies \mathbf{E}(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}.$$

2.1.4 Surface charges and discontinuities

Consider a uniform charge density σ, along the x-y plane. We have, by symmetry,

$$\mathbf{E}(\mathbf{x}) = E(z)\hat{\mathbf{z}}, \quad E(z) = -E(-z).$$

We let S be a cylinder, of height $2z$ and cross-sectional area A. Hence, we have

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E(z)A - E(-z)A = 2E(z)A = \frac{A\sigma}{\epsilon_0}$$
$$\implies E(z) = \frac{\sigma}{2\epsilon_0}.$$

This suggests $\mathbf{E}$ is *discontinuous* at the boundary $z = 0$, since we have

$$\lim_{z\to 0^+} E(z) - \lim_{z\to 0^-} E(z) = \frac{\sigma}{\epsilon_0} \neq 0.$$

So, we conclude the *normal component* of the electric field to the surface is discontinuous on the surface. This result holds for any surface and charge density, as we can take the limit as the area and the height of the cylinder go to zero. So, for a general electric field $\mathbf{E}_\pm$ on either side of the surface, we have

$$\hat{\mathbf{n}} \cdot \mathbf{E}|_+ - \hat{\mathbf{n}} \cdot \mathbf{E}|_- = \frac{\sigma}{\epsilon_0},$$

where $\hat{\mathbf{n}}$ is the unit normal to the surface.

We can look at the tangential component of the electric field at the boundary as well. To do this, we consider a closed loop parallel to the surface on either side, with height a. Then, by Stokes' theorem we have

$$\oint_C \mathbf{E}_\pm \cdot \mathrm{d}\mathbf{r} = \int_S \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{S}.$$

Taking the limit as $a \to 0$, we are only left with the tangential components in the line integral. However, the surface area S clearly approaches zero in the limit, hence we get

$$\hat{\mathbf{n}} \times \mathbf{E}|_+ - \hat{\mathbf{n}} \times \mathbf{E}|_- = 0,$$

hence the tangential electric field is continuous.

An example of the discontinuity of the normal component of $\mathbf{E}$, consider a spherical shell of uniform charge density σ. The field inside the shell is zero, and the field outside is σ/ϵ_0.

2.2 The Electrostatic Potential

For most cases, the symmetry arguments won't be enough and we will have to consider (2.2). The general form of the solution to $\nabla \times \mathbf{E} = 0$ is $\mathbf{E} = -\nabla\phi$ for some scalar field ϕ.

Definition. (Electrostatic potential) If $\mathbf{E} = -\nabla\phi$, then ϕ is the electrostatic (or scalar) potential.

Substituting into (2.1), we have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \implies \nabla^2\phi = -\frac{\rho}{\epsilon_0}. \tag{2.3}$$

This is *Poisson's equation.* For regions of space with zero charge density, the equation reduces to Laplace's equation. In that case, the solution ϕ is said to be a harmonic function.

There are some properties of (2.3) which are interesting. Firstly, we note that it is linear in ϕ and ρ. So, the principle of superposition applies. Secondly, the solutions are unique up to an additive constant. We will require $\phi(\mathbf{x}) \to 0$ as $x \to \infty$.

2.2.1 Point charges

We can express a point charge of magnitude Q located at some $\mathbf{x} = \mathbf{x}_0$ as:

$$\rho(\mathbf{x}) = Q\delta^3(\mathbf{x} - \mathbf{x}_0).$$

We then need the solve the equation

$$\nabla^2\phi = -\frac{Q\delta^3(\mathbf{x} - \mathbf{x}_0)}{\epsilon_0}. \tag{2.4}$$

The solution to (2.4) is essentially the Green's function for the Laplacian. We therefore consider the equation

$$\nabla^2 G(\mathbf{x}; \mathbf{x}_0) = -\delta^3(\mathbf{x} - \mathbf{x}_0).$$

Firstly, since the problem is spherically symmetric, we note that the Green's function must have the form: $G(\mathbf{x}; \mathbf{x}_0) = G(|\mathbf{x} - \mathbf{x}_0|)$, i.e. it is a function only of the distance from the point $\mathbf{x}_0$.

We now integrate over the ball $B_r = \{\mathbf{x} \in \mathbb{R}^3 | |\mathbf{x} - \mathbf{x}_0| \leq r\}$, we have

$$\int_{B_r} \nabla^2 G \, \mathrm{d}^3x = \int_{B_r} -\delta^3(\mathbf{x} - \mathbf{x}_0) \, \mathrm{d}^3x = -1.$$

Using the divergence theorem, we can write

$$\oint_{\partial B_r} \nabla G \cdot \mathrm{d}\mathbf{S} = -1.$$

Since G is a function of $|\mathbf{x} - \mathbf{x}_0|$, the expression simplifies to

$$4\pi r^2 \frac{\mathrm{d}G}{\mathrm{d}r} = -1 \implies G(r) = \frac{1}{4\pi r} + k$$

where $r = |\mathbf{x} - \mathbf{x}_0|$ and $k \in \mathbb{R}$. We can now impose the boundary condition that as $r \to \infty$, $G \to 0$ to conclude $k = 0$.

From (2.4), we can see that

$$\phi(\mathbf{x}) = \frac{Q}{\epsilon_0} G(\mathbf{x}; \mathbf{x}_0) = \frac{Q}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_0|}.$$

Finally, we can calculate the electric fields as

$$\mathbf{E}(\mathbf{x}) = -\nabla\phi = \frac{Q}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_0|^2}\hat{\mathbf{r}},$$

where $\hat{\mathbf{r}} = (\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$.

This seems like a complicated way to reproduce Coulomb's law, however the result can be used to find $\mathbf{E}$ for *any* arbitrary charge distribution. The general solution to Poisson's equation is given by

$$\phi(\mathbf{x}) = \frac{1}{\epsilon_0}\int_\Omega \rho(\mathbf{y})G(\mathbf{y};\mathbf{x})\,\mathrm{d}^3y \;=\frac{1}{\epsilon_0}\int_\Omega \rho(\mathbf{y})\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}^3y\,, \tag{2.5}$$

this is essentially the principle of superposition.

2.2.2 Dipoles

Consider two charges of $+q$ and $-q$ located at $\mathbf{x}_1 = \mathbf{d}$ and $\mathbf{x}_2 = -\mathbf{d}$ respectively. By (2.5) we have

$$\phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{d}|} - \frac{1}{|\mathbf{x}+\mathbf{d}|}\right).$$

We can be interested in the long distance behaviour, hence we look at the potential at $|\mathbf{x}| \gg \mathbf{d}$. The Taylor expansion of a general scalar field to first order is

$$f(\mathbf{x}+\mathbf{d}) = f(\mathbf{x}) + \mathbf{d}\cdot\nabla f(\mathbf{x}) + o(|\mathbf{d}|). \tag{2.6}$$

Applying (2.6) to the terms in the electric potential for a dipole, we have

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{d}|} &\approx \frac{1}{x} + \frac{\mathbf{d}\cdot\mathbf{x}}{x^3},\\ \frac{1}{|\mathbf{x}+\mathbf{d}|} &\approx \frac{1}{x} - \frac{\mathbf{d}\cdot\mathbf{x}}{x^3}.\end{aligned}$$

Hence, the electric potential becomes

$$\phi \approx \frac{2q\,\mathbf{d}\cdot\mathbf{x}}{4\pi\epsilon_0 x^3},$$

where we can define the electric dipole moment $\mathbf{p} = 2q\mathbf{d}$ and obtain

$$\phi \approx \frac{\mathbf{p}\cdot\mathbf{x}}{4\pi\epsilon_0 x^3}. \tag{2.7}$$

We can then calculate the electric field

$$\mathbf{E} = -\nabla\phi = -\frac{\nabla(\mathbf{p}\cdot\mathbf{x}) + (\mathbf{p}\cdot\mathbf{x})\nabla(x^{-3})}{4\pi\epsilon_0}.$$

Since $\mathbf{p}$ is a constant vector, this simplifies to

$$\mathbf{E} = \frac{3(\mathbf{p}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} - \mathbf{p}}{4\pi\epsilon_0 x^3}.$$

2.2.3 General charge distributions - long distance behaviour

Suppose we would like to know what the electric field looks like far away from a general charge distribution over a volume V. So, we are interested in the field at $\mathbf{x}$ with $|\mathbf{x}| \gg |\mathbf{x}'|$ for all $\mathbf{x}' \in V$. We can use (2.5) to find an expression for the scalar potential. Taking the Taylor series:

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{x} - \mathbf{x}'\nabla\frac{1}{x} + \frac{1}{2}(\mathbf{x}'\cdot\nabla)^2\frac{1}{x} + \cdots \\ &= \frac{1}{x} + \frac{\mathbf{x}\cdot\mathbf{x}'}{x^3} + \cdots .\end{aligned}$$

Hence, the potential becomes

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3x'\,\rho(\mathbf{x}')\left(\frac{1}{x} + \frac{\mathbf{x}\cdot\mathbf{x}'}{x^3} + \cdots\right). \tag{2.8}$$

The first term is simply

$$\phi(\mathbf{x}) \approx \frac{Q}{4\pi\epsilon_0 x}$$

where Q is the total charge enclosed in V. The first correction takes the form of a dipole,

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{x}}}{x^2}\cdots\right)$$

where

$$\mathbf{p} = \int_V \mathrm{d}^3x'\,\rho(\mathbf{x}')\mathbf{x}'.$$

Continuing into higher terms, we obtain higher order poles. This is called the *multipole expansion.*

2.3 Electrostatic Energy

Definition. (Electric potential energy) The potential energy $U(\mathbf{x})$ of a particle is defined as the work required to move the particle from infinity to $\mathbf{x}$;

$$U(\mathbf{x}) = -\int_\infty^{\mathbf{x}} \mathbf{F}\cdot\,\mathrm{d}\mathbf{x} = q\phi(\mathbf{x}),$$

where we assumed $\phi(\mathbf{x}) \to 0$ as $x \to \infty$.

The potential energy of an assembly is defined as the total work required to construct the assembly. This can be written as

$$U = \sum_i W_i = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

where the second sum is taken over each pair of particles once. If we sum over all pairs, we get a factor of 1/2, in which case we have

$$U = \frac{1}{2}\sum_i q_i\phi(\mathbf{x_i}), \tag{2.9}$$

since the potential at $\mathbf{x}_i$ due to all the other charges is simply

$$\phi(\mathbf{x}_i) = \frac{1}{4\pi\epsilon_0}\sum_{i\neq j}\frac{q_j}{|\mathbf{x}_i - \mathbf{x}_j|}.$$

Now, we can generalise (2.9) to charge distributions $\rho(\mathbf{x})$. The potential energy associated with a charge distribution is then

$$U = \frac{1}{2}\int_{\mathbb{R}^3} \rho(\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}^3x\,,$$

where we can integrate over $\mathbb{R}^3$ as $\rho(\mathbf{x})$ can approach zero in the limits. We use Gauss' law to rewrite this expression as

$$U = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} (\nabla\cdot\mathbf{E})\phi\,\mathrm{d}^3x\; = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} [\nabla\cdot(\mathbf{E}\phi) - \mathbf{E}\nabla\phi]\,\mathrm{d}^3x\,.$$

The first term vanishes since $\phi \to 0$ as $x \to \infty$. We have found that the electrical potential energy stored in a distribution can be expressed solely in the electric field it creates as

$$U = \frac{\epsilon_0}{2}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\mathbf{E}\cdot\mathbf{E}. \tag{2.10}$$

This derivation is in fact, wrong. But it gives the correct result for the energy stored in the electric field. One way to immediately see that this derivation does not make any sense is to consider the case of placing a single particle. Zero work is done placing the particle anywhere in space, but the final result gives a non-zero potential energy associated with it.

2.3.1 Forces between dipoles

Consider two dipoles, with the first one located at the origin with dipole moment $\mathbf{p}_1$ and the second one with charge Q at $\mathbf{x}$ and $-Q$ at $\mathbf{x}-\mathbf{d}$ such that its moment is $\mathbf{p}_2 = Q\mathbf{d}$. The potential energy, in the limit $|\mathbf{d}| \ll |\mathbf{x}|$, is given by equation (2.9) as

$$U = \frac{Q}{2}\left(\phi(\mathbf{x}) - \phi(\mathbf{x}-\mathbf{d})\right),$$

where

$$\phi(\mathbf{x}) = \frac{\mathbf{p}_1\cdot\mathbf{x}}{4\pi\epsilon_0 x^3}$$

Taking the Taylor series expansion of the second term, we have

$$\begin{aligned} U &= \frac{Q}{8\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{x}}{x^3} - \mathbf{p}_1\cdot\mathbf{x}\left(\frac{1}{x^3} + \frac{3\mathbf{x}\cdot\mathbf{d}}{x^5}\right)\right) \\ &= \frac{1}{8\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{x^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{x})(\mathbf{p}_2\cdot\mathbf{x})}{x^5}\right). \end{aligned}$$

The force is then simply given by

$$\mathbf{F} = -\nabla U = -\frac{1}{8\pi\epsilon_0}\nabla\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{x^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{x})(\mathbf{p}_2\cdot\mathbf{x})}{x^5}\right).$$

2.4 Conductors

Conductors are regions of space where charge can move freely. There are some immediate consequences of this statement:

i. The electric field *inside* a conductor is zero. The charges move such that they reach an equilibrium, and they can only reach an equilibrium if $\mathbf{E} = 0$.

ii. $\rho = 0$ inside the conductor. Follows from $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$.

iii. Any net charge is on the surface.

iv. A conductor is an *equipotential*. If it weren't, $\mathbf{E} = -\nabla\phi \neq 0$, which is not true.

v. The electric field immediately outside the surface is perpendicular to the surface. Any tangential component would result in movement of charge on the surface.

We can also conclude from statement (iii.) that the electrostatic energy of a conductor is minimised when all the charge resides on the surface.

Given any surface charge density σ, we know by discontinuity of $\mathbf{E}$, that the field outside the surface must be

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}.$$

3 Magnetostatics

We are interested in magnetic fields from *steady currents.* We are again looking for time-independent solutions to Maxwell's equations. We let $\rho = 0$ so we have $\mathbf{E} = 0$. Now, we need to solve

$$\nabla \cdot \mathbf{B} = 0, \tag{3.1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{3.2}$$

3.1 Ampere's Law

Given an open surface Ω, integrate equation (3.2) over it.

$$\int_\Omega \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{\Omega} = \mu_0 \int_\Omega \mathbf{J} \cdot \mathrm{d}\mathbf{\Omega}.$$

By Stokes' theorem and the definition of current I, we obtain the integral form of Ampere's law:

$$\oint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I. \tag{3.3}$$

This, by itself is generally not enough to obtain a unique solution unless the system allows for certain symmetry considerations. We look at two such systems now.

3.1.1 Straight wire

Work in cylindrical polar coordinates (ρ, φ, z) where $\rho = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(y/x)$. We let $\mathbf{J} = J\hat{\mathbf{z}}$.

We take an open surface Ω such that $\partial\Omega$ is a circle of radius r centered around the wire. Given a long enough wire, we must have by symmetry $B_z = 0$. Also, $\mathbf{B}$ must only be a function of ρ. So, just from symmetry considerations alone we have

$$\mathbf{B}(\rho) = B_\rho(\rho)\hat{\boldsymbol{\rho}} + B_\varphi(\rho)\hat{\boldsymbol{\varphi}}.$$

Equation (3.1) implies that $B_\rho = 0$, so we are left with

$$\begin{aligned}\oint_{\partial\Omega} \mathbf{B} \cdot \mathrm{d}\mathbf{\Omega} &= \int_0^{2\pi} B(\rho)\rho \,\mathrm{d}\varphi \\ &= 2\pi\rho B(\rho) = \mu_0 I.\end{aligned}$$

Letting $\rho = r$, we get the solution:

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}}.$$

3.1.2 Solenoid

By similar symmetry arguments, we conclude that for a solenoid oriented along $\hat{\mathbf{z}}$, we have $\mathbf{B} = B_z(\rho)\hat{\mathbf{z}}$.

Outside the solenoid, we have $\nabla \times \mathbf{B} = 0$ which implies $\mathrm{d}B_z / \mathrm{d}\rho = 0$ so B_z is constant. Since as $\rho \to \infty$, $B \to 0$, the magnetic field is zero everywhere outside the solenoid.

Inside the solenoid, we take a curve of side length L, parallel to the solenoid. We have

$$\oint \mathbf{B} \cdot \mathrm{d}\mathbf{r} = BL = \mu_0 INL$$

where N is the number of turns per unit length. So, we have

$$\mathbf{B} = \mu_0 IN\hat{\mathbf{z}}.$$

3.1.3 Surface currents and discontinuities

Consider an infinite plane with surface current density $\mathbf{K}$, with units of current per length. Let $\mathbf{K} = K\hat{\mathbf{x}}$. By symmetry, we must have

$$\mathbf{B} = -B(z)\hat{\mathbf{y}},$$

with $B(z) = -B(-z)$. By considering a curve of length L in the y direction, we have

$$\begin{aligned} 2LB(z) &= \mu_0 KL \\ \implies B(z) &= \frac{\mu_0 K}{2}, \quad z > 0. \end{aligned}$$

The tangential component of the magnetic field to the surface is discontinuous, with discontinuity given by

$$B(z \to 0^+) - B(z \to 0^-) = \mu_0 K.$$

However, the magnetic field normal to the surface is continuous. This is similar to the discontinuities with the electric field, but the other way around.

3.2 The Vector Potential

In general, symmetry consideration are not enough to justify $\nabla \cdot \mathbf{B} = 0$.

Proposition. Given any vector field $\mathbf{F}$, we have

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

So, we can guarantee (3.1) is satisfied if we write, for some $\mathbf{A}$,

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{3.4}$$

The vector field $\mathbf{A}$ is *the vector potential*. Ampere's law, written with the vector potential is then

$$\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A} + \nabla (\nabla \cdot \mathbf{A}) = \mu_0 \mathbf{J}.$$

3.2.1 Gauge transformations

The choice for the vector potential in (3.4) is not unique. In fact, for any vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = 0$, we can have

$$\mathbf{B} = \nabla \times (\mathbf{A} + \mathbf{F}) = \nabla \times \mathbf{A}.$$

The vector field $\mathbf{F}$ can be written as the gradient of *any scalar field* χ. So, we have

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \implies \nabla \times \mathbf{A}' = \nabla \times \mathbf{A}.$$

Such a change in $\mathbf{A}$ is called a *gauge transformation.* By choosing specific gauges, we can simplify things a lot.

Proposition. We can always find a gauge transformation χ such that $\nabla \cdot \mathbf{A}' = 0$. This is called the *Coulomb gauge.*

Proof. We require

$$\begin{aligned} \nabla \cdot \mathbf{A}' = \nabla \cdot (\mathbf{A} + \nabla\chi) &= \nabla \cdot \mathbf{A} + \nabla^2\chi = 0 \\ \implies \nabla^2\chi &= -\nabla \cdot \mathbf{A} \end{aligned}$$

This is nothing but Poisson's equation for some field χ, and we know there is a unique solution.

Using the Coulomb gauge, we find that Ampere's law becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

In Cartesian coordinates, this is simply Poisson's equation for each coordinate:

$$\nabla^2 A_i = -\mu_0 J_i, \quad (i = 1, 2, 3).$$

The general solution is given by the Green's function solution, equation (2.5), as

$$A_i(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \frac{J_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

or, equivalently,

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

We have to make sure this solution satisfies the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. We have

$$\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \mathbf{J}(\mathbf{x}') \nabla \cdot \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)$$

Now, we can use a trick to substitute ∇' where we differentiate with respect to $\mathbf{x}'$, in which case we have

$$\begin{aligned} \nabla \cdot \mathbf{A} &= -\frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \mathbf{J}(\mathbf{x}') \nabla' \cdot \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\ &= -\frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \left[\nabla' \cdot \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) - \nabla' \cdot \mathbf{J}(\mathbf{x}') \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \end{aligned}$$

where we have used integration by parts. From the continuity equation, $\rho = 0$ implies $\nabla \cdot \mathbf{J} = 0$ so the second term vanishes. We also require that the current density is zero on the boundary of V so that it is localised. This makes the first term zero and we have the required result.

Now that we know that we have the correct solution to Maxwell's equations, let's calculate the magnetic field. By definition,

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Doing some calculation with the general solution, we obtain the *Biot-Savart law*:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \mathrm{d}^3 x' \, \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \tag{3.5}$$

If the current density is restricted to a wire of constant cross sectional area that traces out a curve C, the law reduces to a different form. We have, for a volume including a section of the wire, $\mathbf{J}\delta V = (JA)\delta\mathbf{x}$ where A is the cross sectional area of the wire. So, equation (3.5) becomes

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathrm{d}\mathbf{x}' \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{3.6}$$

3.3 Magnetic Dipoles

Consider a loop of current, centered at the origin *(the shape of the loop really doesn't matter.)* To compute the magnetic field far away, we refer to the general solution to the vector potential. So, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint_C \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

Taylor expanding the denominator about $\mathbf{r}' = 0$, we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \dots$$

where $r = |\mathbf{r}|$. Putting this into the integral and ignoring higher terms yields

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_C \mathrm{d}\mathbf{r}' \,\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} = \frac{\mu_0 I}{4\pi r^3}\oint_C \mathrm{d}\mathbf{r}' \,\mathbf{r}\cdot\mathbf{r}'.$$

Now, we need to put this into a form where we can use Stokes' theorem. Consider taking the dot product with a constant vector $\mathbf{u}$.

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi r^3}\oint_C (\mathbf{r}\cdot\mathbf{r}')\mathbf{u}\cdot\,\mathrm{d}\mathbf{r}' \\
&= \frac{\mu_0 I}{4\pi r^3}\int_S \mathrm{d}\mathbf{S}\,\cdot(\nabla'\times(\mathbf{u}(\mathbf{r}\cdot\mathbf{r}'))) \\
&= \frac{\mu_0 I}{4\pi r^3}\mathbf{u}\cdot\int_S \mathrm{d}\mathbf{S}\,\times\mathbf{r}.
\end{aligned}$$

Since the constant vector $\mathbf{u}$ is arbitrary, we have the general result:

$$\oint_{\partial S} \mathrm{d}\mathbf{r}'\,(\mathbf{r}\cdot\mathbf{r}') = \int_S \mathrm{d}\mathbf{S}\,\times\mathbf{r}.$$

We define the *vector area* $\boldsymbol{\mathcal{S}}$ of the surface S as

$$\boldsymbol{\mathcal{S}} = \int_S \mathrm{d}\mathbf{S}\,.$$

We define the *magnetic moment* as $\mathbf{m} = \boldsymbol{\mathcal{S}} I$, so we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \mathbf{m}\times\mathbf{r}}{4\pi r^3}.$$

The magnetic field is then given by

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}}{r^3}\right).$$

We see that we have the same form as for electric field due to a dipole.

3.3.1 General current distribution (multipole expansion)

Note: Horrible calculation - reader discretion advised.

We start with the vector potential from a general current distribution and consider the Taylor expansion about $\mathbf{r}' = 0$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathrm{d}^3 r' \,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\int_V \mathrm{d}^3 r' \,\mathbf{J}(\mathbf{r}')\left(\frac{1}{r}+\frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}\dots\right) \tag{3.7}$$

We ignore higher terms. Let's look at the first term. We now introduce the condition that the current density is localised. This means $\mathbf{J}\to 0$ as $r\to\infty$. We also note that, since we are dealing with magnetostatics, $\nabla\cdot\mathbf{J} = 0$ from continuity. So, we have

$$\nabla\cdot\mathbf{J} = 0 \Longleftrightarrow \mathbf{J} = \nabla\times(\mathbf{F}+\nabla\varphi) \tag{3.8}$$

for some $\mathbf{F}$ and any φ. The first term of (3.7) becomes

$$\begin{aligned}\frac{\mu_0}{4\pi}\int_V \mathrm{d}^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{r} &= \frac{\mu_0}{4\pi r}\int_V \mathrm{d}^3r'\,\nabla\times(\mathbf{F}+\nabla\varphi)\\ &= \frac{\mu_0}{4\pi r}\oint_{\partial V}\mathrm{d}\mathbf{S}\,\times(\mathbf{F}+\nabla\varphi).\end{aligned}$$

We make use of the following the identity,

$$\oint_{\partial S}\psi\,\mathrm{d}\mathbf{x} = \int_S \mathrm{d}\mathbf{S}\,\times\nabla\psi,$$

$$\therefore \oint_S \mathrm{d}\mathbf{S}\,\times\nabla\psi = 0 \quad \forall\psi, S.$$

This makes the $\nabla\varphi$ term in the integral vanish. To make sense of the first term, we take the volume of integration to be $\mathbb{R}^3$. Since $\mathbf{J}$ is localised, we have

$$\mathbf{J}|_{\partial V} = \nabla\times(\mathbf{F}+\nabla\varphi)|_{\partial V} = 0 \Longleftrightarrow \mathbf{F}|_{\partial V} = \nabla\chi,$$

for some χ. Therefore, with the same argument, the first term of the integral vanishes as well. Hence, we have

$$\int_V \mathrm{d}^3r'\,\mathbf{J} = 0.$$

This is, in fact, a statement that *magnetic monopoles do not exist.*

This was the easy part of the calculation. Now, let's look at the second term in the Taylor expansion. We have

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi r^3}\int_V \mathrm{d}^3r'\,\mathbf{J}(\mathbf{r}\cdot\mathbf{r}'). \tag{3.9}$$

In order to simplify, we make use of the vector triple product identity:

$$(\mathbf{r}\cdot\mathbf{r}')\mathbf{J} = (\mathbf{r}\cdot\mathbf{J})\mathbf{r}' - \mathbf{r}\times(\mathbf{r}'\times\mathbf{J}).$$

Now, we will show that

$$\int_V \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{r}')\mathbf{J} = -\int_V \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{J})\mathbf{r}'. \tag{3.10}$$

Using (3.8), we obtain for the left hand side:

$$\begin{aligned}\int_V \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{r}')\mathbf{J} &= \int_V \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{r}')\nabla'\times\mathbf{F}\\ &= \int_V \mathrm{d}^3r'\,\{-\mathbf{r}\times\mathbf{F}+\nabla'\times((\mathbf{r}\cdot\mathbf{r}')\mathbf{F})\}\\ &= -\int_V \mathrm{d}^3r'\,(\mathbf{r}\times\mathbf{F}) + \int_V \nabla'\times((\mathbf{r}\cdot\mathbf{r}')\mathbf{F})\end{aligned}$$

where we have used the identity *(for any scalar and vector field ψ and $\mathbf{A}$)*:

$$\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) + \nabla\psi\times\mathbf{A}.$$

Let's look at the second integral. It reduces to a closed surface integral:

$$\int_V \nabla'\times((\mathbf{r}\cdot\mathbf{r}')\mathbf{F}) = \oint_{\partial V}\mathrm{d}\mathbf{S}\,\times\mathbf{F}(\mathbf{r}\cdot\mathbf{r}'),$$

where we can make use of the boundary condition by taking the volume to be $\mathbb{R}^3$ and write $\mathbf{F} = \nabla\chi$ for some χ. We take the dot product with some constant vector $\mathbf{u}$ and use Stokes' theorem:

$$\begin{aligned}
\mathbf{u}\cdot\oint_{\partial V} \mathrm{d}\mathbf{S}\times\mathbf{F}(\mathbf{r}\cdot\mathbf{r}') &= \oint_{\partial V} \mathrm{d}\mathbf{S}\cdot(\nabla'\chi\times\mathbf{u}(\mathbf{r}\cdot\mathbf{r}')) \\
&= \oint_{\partial V} \mathrm{d}\mathbf{S}\cdot\nabla'\times(\chi\mathbf{u}(\mathbf{r}\cdot\mathbf{r}')) \\
&= \oint_{\partial S} (\mathrm{d}\mathbf{r}'\cdot\mathbf{u})\chi(\mathbf{r}\cdot\mathbf{r}') \equiv 0
\end{aligned}$$

where the equivalence to zero is due to the surface being closed. This holds for all constant vectors $\mathbf{u}$. We have used the same curl identity. Hence, we have

$$\int_V \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{r}')\mathbf{J} = -\int_V \mathrm{d}^3r'\,(\mathbf{r}\times\mathbf{F}).$$

Now, let's look at the right hand side of (3.10). Again, by (3.8) we have

$$\mathbf{r}\cdot\mathbf{J} = \mathbf{r}\cdot(\nabla\times\mathbf{F}) = \nabla\cdot(\mathbf{F}\times\mathbf{r}) + \mathbf{F}\cdot(\nabla\times\mathbf{r}) = \nabla\cdot(\mathbf{F}\times\mathbf{r})$$

which implies

$$\int \mathrm{d}^3r'\,(\mathbf{r}\cdot\mathbf{J})\mathbf{r}' = \int \mathrm{d}^3r'\,\nabla\cdot(\mathbf{F}\times\mathbf{r})\mathbf{r}'.$$

Hence, (3.10) reduces to

$$-\int \mathrm{d}^3r'\,(\mathbf{r}\times\mathbf{F}) = \int \mathrm{d}^3r'\,\nabla\cdot(\mathbf{r}\times\mathbf{F})\mathbf{r}'.$$

We will not prove it here, but this statement is true. So, we have

$$\int \mathrm{d}^3r'\,\mathbf{J}(\mathbf{r}\cdot\mathbf{r}') = -\frac{1}{2}\mathbf{r}\times\int \mathrm{d}^3r'\,(\mathbf{r}'\times\mathbf{J}) \tag{3.11}$$

We see that the vector potential takes the form of a dipole, such that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}$$

where now, we can define *a general magnetic dipole moment*

$$\mathbf{m} = \frac{1}{2}\int_V \mathrm{d}^3r'\,\mathbf{r}'\times\mathbf{J}. \tag{3.12}$$

3.4 Magnetic Forces

A charge, moving with velocity $\mathbf{v}$ will experience a force due to the magnetic field given by the Lorentz fore law

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}. \tag{3.13}$$

Since currents are composed of moving charges, this implies two currents will exert forces on each other.

3.4.1 Force between currents

Consider two parallel wires with current densities $\mathbf{J}_1$ and $\mathbf{J}_2$ along the z axis. The separation of the wires is d. Let's look at the force on wire 2 due to wire 1:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q\mathbf{v} \times \frac{\mu_0 I_1}{2\pi d}\hat{\mathbf{y}}.$$

Now, we note that $\mathbf{J}_2 = nq\mathbf{v}$ where n is the electron number density. Hence, we have $I_2 = AJ_2$. We now look at the force per unit length,

$$\mathbf{f} = nA\mathbf{F} = \frac{\mu_0 I_1 I_2}{2\pi d}\hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{\mathbf{x}}.$$

If the two current densities are in the same direction, we have an attractive force. If they are in opposite directions, the force is repulsive.

The general force between current densities

Consider two current densities $\mathbf{J}_1$ and $\mathbf{J}_2$, localised on curves C_1 and C_2 parameterised by $\mathbf{r}_1$ and $\mathbf{r}_2$. Let's look at the force on 2 due to 1. The magnetic field generated by $\mathbf{J}_1$ is given by the Biot-Savart law:

$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \mathrm{d}^3r' \, \frac{\mathbf{J}_1(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0 I_1}{4\pi}\oint_{C_1} \frac{\mathrm{d}\mathbf{r}_1 \times (\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}.$$

The force on 2 is given by

$$\begin{aligned}
\mathbf{F} &= \int_V \mathrm{d}^3r \, \mathbf{J}_2(\mathbf{r}) \times \mathbf{B}_1(\mathbf{r}) \\
&= I_2 \oint_{C_2} \mathrm{d}\mathbf{r}_2 \times \mathbf{B}_1(\mathbf{r}_2) \\
&= \frac{\mu_0 I_1 I_2}{4\pi}\oint_{C_1}\oint_{C_2} \frac{\mathrm{d}\mathbf{r}_2 \times (\mathrm{d}\mathbf{r}_1 \times (\mathbf{r}_2 - \mathbf{r}_1))}{|\mathbf{r}_2-\mathbf{r}_1|^3}.
\end{aligned}$$

3.4.2 Force and energy for a dipole

Consider a current distribution $\mathbf{J}$, localised around some $\mathbf{r} = \mathbf{R}$. Let $\mathbf{B}$ be a magnetic field which only slowly changes around $\mathbf{r} = \mathbf{R}$. The total force on the current distribution $\mathbf{J}$ is

$$\mathbf{F} = \int_V \mathrm{d}^3r \, \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \tag{3.14}$$

We Taylor expand $\mathbf{B}$ around $\mathbf{r} = \mathbf{R}$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{R}) + (\mathbf{r}\cdot\nabla)\mathbf{B}(\mathbf{R}) + \ldots$$

Ignoring higher terms, (3.14) becomes

$$\mathbf{F} \approx -\mathbf{B}(\mathbf{R}) \times \int_V \mathrm{d}^3r \, \mathbf{J}(\mathbf{r}) + \int_V \mathrm{d}^3r \, \mathbf{J}(\mathbf{r}) \times (\mathbf{r}\cdot\nabla)\mathbf{B}(\mathbf{R}). \tag{3.15}$$

We know, from section 3.3.1, that the first term is zero. For the second term, we write

$$(\mathbf{r}\cdot\nabla)\mathbf{B}(\mathbf{R}) = (\mathbf{r}\cdot\nabla')\mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=\mathbf{R}}.$$

Using the fact that $\nabla' \times \mathbf{B}$ is zero in the vicinity of $\mathbf{R}$, we have

$$(\mathbf{r}\cdot\nabla')\mathbf{B}(\mathbf{r}')\big|_{\mathbf{r}'=\mathbf{R}} = \nabla'(\mathbf{r}\cdot\mathbf{B}')\big|_{\mathbf{r}'=\mathbf{R}}.$$

$$\implies \mathbf{J} \times (\mathbf{r} \cdot \nabla')\mathbf{B}' = -\nabla' \times (\mathbf{J}(\mathbf{r} \cdot \mathbf{B}')).$$

Hence, equation (3.15) becomes

$$\mathbf{F} \approx -\nabla' \times \int_V \mathrm{d}^3 r \, \mathbf{J}(\mathbf{r} \cdot \mathbf{B}').$$

From section 3.3.1, we have

$$\int_V \mathrm{d}^3 r \, (\mathbf{B}' \cdot \mathbf{r})\mathbf{J} = \frac{1}{2}\mathbf{B}' \times \int_V \mathrm{d}^3 r \, \mathbf{J} \times \mathbf{r} = -\mathbf{B} \times \mathbf{m}$$

$$\implies \mathbf{F} = \nabla \times (\mathbf{B} \times \mathbf{m})$$

Using $\nabla \cdot \mathbf{B} = 0$, this is equivalent to

$$\mathbf{F} = \nabla(\mathbf{B} \cdot \mathbf{m}) \tag{3.16}$$

This is the force on a dipole. From this, we can simply read the energy stored by the dipole in the magnetic field as

$$U = -\mathbf{B} \cdot \mathbf{m} \tag{3.17}$$

3.4.3 Force between dipoles

Consider two dipoles with dipole moments $\mathbf{m}_1$ and $\mathbf{m}_2$. The force on dipole 1 due to 2 is given by

$$\mathbf{F} = \nabla(\mathbf{m}_1 \cdot \mathbf{B}_2)$$

where $\mathbf{B}_2$ is the field created by the second dipole. We know that

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m}_2 \times \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2}{r^3}$$

where $\mathbf{r}$ is the displacement from dipole 2 to dipole 1. Doing some calculation, it is straightforward to show that

$$\mathbf{F} = \frac{3\mu_0}{4\pi r^4} \left[(\mathbf{m}_1 \cdot \hat{\mathbf{r}}) \, \mathbf{m}_2 + (\mathbf{m}_2 \cdot \hat{\mathbf{r}}) \, \mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2) \, \hat{\mathbf{r}} - 5 \, (\mathbf{m}_1 \cdot \hat{\mathbf{r}}) \, (\mathbf{m}_2 \cdot \hat{\mathbf{r}}) \, \hat{\mathbf{r}} \right]. \tag{3.18}$$

4 Electrodynamics