Topic One: Sets

1. Basics

1.1. Definitions.

Definition. $\boxed{\in}$ If x is an element of a set A, we write $x \in A$. This is read "x in A".

Definition. $\boxed{\notin}$ If x is not an element of a set B "$x \notin B$" means that x is not an element of B.

Definition. $\boxed{\mathbb{N}}$ The set of *natural numbers*, denoted $\mathbb{N}$, is the set $\{1, 2, 3, \ldots\}$ [1] For e.g.,

- $\{n^2 : n \in \mathbb{N}\} = \{1, 4, 9, 16, 25, \ldots\}$
- $\{n \in \mathbb{N} : 6|n\} = \{6, 12, 18, 24, 30, \ldots\}$

Definition. $\boxed{\mathbb{Z}}$ The set of *integers*, denoted $\mathbb{Z}$ is the set $\{\ldots$, -3, -2, -1, 0, 1, 2, 3, $\ldots\}$. For e.g.,

- $\{|n| : n \in \mathbb{Z}\} = \{0, 1, 2, 3, \ldots\}$
- $\{n \in \mathbb{Z} : n \textit{ is even}\} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

Definition. $\boxed{\mathbb{Q}}$ The set of *rational numbers*, denoted $\mathbb{Q}$, is the set $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$
This can be read as the following:

$\mathbb{Q}$	$=$	$\{$	$\frac{a}{b}$	$:$	$a, b \in \mathbb{Z}$	,	$b \neq 0\}$
The rational numbers	are defined to be	the set of all	fractions of the form $\frac{a}{b}$	such that	a and b are integers	and	b is nonzero

So the definition for *rational numbers* includes $\frac{2}{3}$ and $\frac{4}{6}$ and $\frac{6}{9}$ and infinitely more representation of this same number. However, the set itself only keeps one of each element, so the duplicates of each rational number would not be included in the set.

Definition. $\boxed{\mathbb{R}}$ The set of *real numbers*, denoted $\mathbb{R}$, is difficult to define (it would take dozens of pages to rigorously define it) but it is effectively all the numbers you can write with a decimal point. However, we can use $\mathbb{R}$ and set notation to generate and define other familiar sets:

- The set of 2 x 2 real matrices can be written:
$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$
- The xy-plane represents the set of *ordered pairs* of real numbers. This set can be written:
$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

[1] Note that it does not include 0.

- The unit circle, which is a circle of radius 1 centered at the origin, is contained inside of $\mathbb{R}^2$ and can be defined as:
$$\mathbb{S}^1 = \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}.$$
- The closed interval [a,b] can be defined as follows:
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$
- The open interval (x,y) can be defined as:
$$(a, b) = \{x \in \mathbb{R} : a < x < b\}.$$
This applies even if $a = -\infty$ and/or $b = \infty$. The definitions for the half open intervals, (a,b] and [a, b), are similar. Also note that the open interval notation (a,b) is the same as an ordered pair, so it will be determined which is which from context.

1.2. **Proving $A \subseteq B$.**

Definition. $\boxed{\subseteq}$ Suppose A and B are sets. If every element in A is also an element of B, then A is a *subset* of B, which is denoted $A \subseteq B$.

In order to prove this we will have to show that if $x \in A$ then $x \in B$. So, here is an outline for a direct proof that a set A is a subset of a set B:

Proposition. *Suppose A and B are sets. It is the case that* [2] $A \subseteq B$

Proof. Assume $x \in A$

« An explanation of what $x \in A$ means »

$\updownarrow$ apply algebra
apply logic
apply techniques

« Oh hey look, that's what $x \in B$ means »

Therefore $x \in B$.
Since $x \in A$ implies that $x \in B$, it follows that $A \subseteq B$.

□

Let's apply this set-up to a proposition where one set is in another:

Proposition. *It is the case that*
$$\{n \in \mathbb{Z} : 12|n\} \subseteq \{n \in \mathbb{Z} : 3|n\}$$

Note. Before writing the proof we need to do some scratch work. Here we can we can write out few of the terms to see what we are dealing with. This may also be helpful in finding a proof.

[2]It is advised to start a sentence with words and not mathematical notation

Scratch Work. Lets write some terms of the first set:

$$\{n \in \mathbb{Z} : 12|n\} = \{\dots, -24, -12, 0, 12, 24, \dots\}.$$

and for the second set:

$$\{n \in \mathbb{Z} : 3|n\} = \{\dots, -27, -24, \dots, -15, -12, \dots, -3, 0, 3, \dots, 9, 12, \dots, 21, 24, \dots\}.$$

So based on this, it seems to be that the elements in the first set are in the second set as well. To write this as a proof, we will use the outline of what a proof looks like above. To start, we need to find an explanation for « An explanation of what $x \in A$ means ». For that, we can rely on the following definition for what it means to say "$12|x$":

Definition. $\boxed{a|b}$ A nonzero integer a is said to divide an integer b if $b = ak$ for some integer k. When a divides b, we write "$a|b$" and when a does not divide b we write "$a \nmid b$."

We can now use this in our proof. Remember, based on our outline we start off by stating "Assume $x \in A$" and then explain what that means:

Proof. Assume $\underbrace{x \in \{n \in \mathbb{Z} : 12|n\}}_{x \in A}$

Thus x is also $\in \mathbb{Z}$ and, therefore, 12 also divides x, i.e., 12|x. By the definition of "a|b", this means $x = 12k$ for some $k \in \mathbb{Z}$.

$\updownarrow$ apply algebra
apply logic
apply techniques

« Oh hey look, that's what $x \in B$ means »

Therefore, $\underbrace{x \in \{n \in \mathbb{Z} : 3|n\}}_{x \in B}$ (This is us saying: "Therefore, $x \in B$").

Since $\overbrace{x \in \{n \in \mathbb{Z} : 12|n\}}^{x \in A}$ implies that $\overbrace{x \in \{n \in \mathbb{Z} : 3|n\}}^{x \in B}$, it follows that $\overbrace{\{n \in \mathbb{Z} : 12|n\} \subseteq \{n \in \mathbb{Z} : 3|n\}}^{A \subseteq B}$

$\square$

Before we look at $\updownarrow$ segment of the proof, lets briefly discuss the « Oh hey look, that's what $x \in B$ means »part. We need to show that, by the definition above, 3 also divides x, i.e., $3|x$, so that x can also be an element of the second set, i.e. set B. If 3 does divide x then, by the definition again, x must equal to $3m$ for some integer m. We can add this to our proof:

Proof. Assume $x \in \{n \in \mathbb{Z} : 12|n\}$
Thus x is also $\in \mathbb{Z}$ and, therefore, 12 also divides x, i.e., $12|x$. By the definition of "$a|b$", this means $x = 12k$ for some $k \in \mathbb{Z}$.

$\updownarrow$ apply algebra
apply logic
apply techniques

Therefore, $x = 3m$ for some $m \in \mathbb{Z}$. Thus, by the definition of "$a|b$", this means $3|x$.

Therefore, $x \in \{n \in \mathbb{Z} : 3|n\}$

Since $x \in \{n \in \mathbb{Z} : 12|n\}$ implies that $x \in \{n \in \mathbb{Z} : 3|n\}$, it follows that $\{n \in \mathbb{Z} : 12|n\} \subseteq \{n \in \mathbb{Z} : 3|n\}$

□

Now, we can tackle the $\updownarrow$ segment. We said $x = 12k$ and $x = 3m$ for some $k, m \in \mathbb{Z}$. Thus $12k = 3m$ or $4k = m$. Since $k \in \mathbb{Z}$ so is $4k$. Lets plug this in to our proof:

Proof. Assume $x \in \{n \in \mathbb{Z} : 12|n\}$
Thus x is also $\in \mathbb{Z}$ and, therefore, 12 also divides x, i.e., $12|x$. By the definition of divisibility, "$a|b$", this means $x = 12k$ for some $k \in \mathbb{Z}$.

Equivalently, $x = 3 \cdot (4k)$. And since $k \in \mathbb{Z}$, it is also true that $4k \in \mathbb{Z}$. Thus, by the definition of divisibility, "$a|b$", this means 3 also divides x, i.e., $3|x$. So, $x \in \{n \in \mathbb{Z} : 3|n\}$

Since $x \in \{n \in \mathbb{Z} : 12|n\}$ implies that $x \in \{n \in \mathbb{Z} : 3|n\}$, it follows that $\{n \in \mathbb{Z} : 12|n\} \subseteq \{n \in \mathbb{Z} : 3|n\}$

□

Thus concludes our first (direct) proof!

Note. It is common to conclude the proofs with the symbol □ or ■ which symbolises *quod erat demonstrandum* meaning "what was to be shown."

Now, lets try another proof, this time not as a direct proof but by case.

Proposition. *Let* $A = \{-1, 3\}$ *and* $B = \{x \in \mathbb{R} : x^3 - 3x^2 - x + 3 = 0\}$. *Then* $A \subseteq B$

Note. Don't forget, for proof what we are trying to show is that if $x \in A$ then $x \in B$.

Scratch Work. For $x \in A$, x can take one of two values only — either -1 or 3. So we can take each of those values as a separate case. In each of those cases we need to show $x \in B$. That means, we need to show that in each case x satisfies $x^3 - 3x^2 - x + 3 = 0$.

Proof. Assume $x \in A$. Then either $x = -1$ or $x = 3$. Consider the two cases separately.

<u>Case 1: $x = -1$</u>.	Note that x is a real number, and $(-1)^3 - 3(-1)^2 - (-1) + 3 = -1 - 3 + 1 + 3 = 0$ which by definition of B implies $x \in B$.
<u>Case 2: $x = 3$</u>.	Note that x is a real number, and $(3)^3 - 3(3)^2 - (3) + 3 = 27 - 27 - 3 + 3 = 0$ which by definition implies $x \in B$.

Since $x \in A$ implies that $x \in B$, it follows that $A \subseteq B$. □

1.3. **Proving $A = B$.** .
Since the two sets contain exactly the same elements, this means that not only every element in A is also in B but also every element in B is in A. This may come across as tautology but this is a way to proove that $A = B$ by showing that $A \subseteq B$ and $B \subseteq A$. This gives an indication to the outline of a proof. We already showed how to prove $A \subseteq B$ above, so the outline will be very similar:

Proposition. *It is the case that $A = B$.*

Proof. Assume $x \in A$

« An explanation of what $x \in A$ means »

$\updownarrow$ apply algebra
apply logic
apply techniques

« Oh hey look, that's what $x \in B$ means »

Therefore $x \in B$.
Since $x \in A$ implies that $x \in B$, it follows that $A \subseteq B$.

Next, assume $x \in B$

« An explanation of what $x \in B$ means »

$\updownarrow$ apply algebra
apply logic
apply techniques

« Oh hey look, that's what $x \in A$ means »

Therefore $x \in B$.
Since $x \in B$ implies that $x \in A$, it follows that $A \subseteq B$.
We have shown that $A \subseteq B$ and $B \subseteq A$. Therefore, $A = B$.

□

2. Set Operations

2.1. **Definitions.**

Definition. $\boxed{\cup}$ The *union* of sets A and B is the set $A \cup B = \{x : x \in A$ or $x \in B\}$.

- If $A_1, A_2, A_3, \ldots, A_n$ are all sets, then the union of all of them is the set $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \{x : x \in A_i \text{for } \underline{\text{some}}\ i\}$. This set is also denoted:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$$

Definition. $\boxed{\cap}$ The *intersection* of sets A and B is the set $A \cap B = \{x : x \in A$ and $x \in B\}$.

- If $A_1, A_2, A_3, \ldots, A_n$ are all sets, then the intersection of all of them is the set $A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \{x : x \in A_i \text{for } \underline{\text{all}}\ i\}$. This set is also denoted:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$$

Set operations include *unions*, *set subtraction*, and *Cartesian product* which are analogous to the operations of addition, subtraction, and multiplications for numbers. Also, just as taking the absolute value of a number tells you how big it is, in set theory one can determine a set's *cardinality*. These are discussed below.

2.2. **Subtraction and Complements.**

Definition. $\boxed{\setminus}$ The *subtraction* of B from A is $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$.

Definition. $\boxed{\text{U}}$ If $A \subseteq U$ then U is called a *universal set* of A.

Definition. $\boxed{A^c}$ The *complement* of A in U is $A^c = U \setminus A$.