TOPIC TWO: FUNCTIONS

In Topic One we studied the static properties of sets. Things get more dynamic when we start applying functions to those sets.

1. Definitions

Definition. $\boxed{f : A \to B}$ Given a pair of sets A and B, suppose that each element $x \in A$ is associated, in some way, to a unique element of B, which we denote $f(x)$. Then f is said to be a *function* from A to B. This is often denoted

$$f : A \to B$$

which is typically read "f from A to B."

Furthermore, A is called the *domain* of f, (can be though of as inputs of f)

B is called the *codomain* of f, and (. . . as a set to which all outputs belong)

The set $\{f(x) : x \in A\}$ is called the *range* of f. (. . . as outputs of f)

All three correspond to each other via f but they are just sets.
Range consists only of elements in the codomain that gets mapped. That is, y is in the range if there is an x in the domain that maps to it: $f(x) = y$. For e.g.,

- if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 2x$ then the range is $\mathbb{R}$.
- if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$ then the range is the set of nonnegative real numbers: ie, the interval $[0, \infty)$.

A function's domain and codomain can each be any set. Here is a graphical way to write som function f:

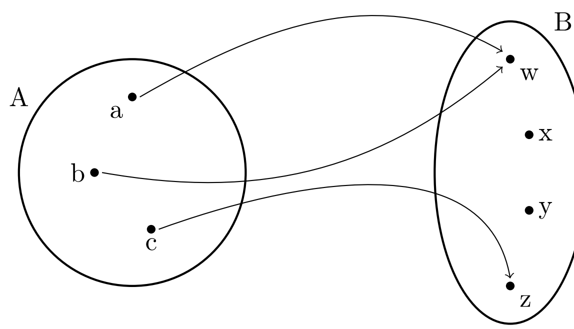


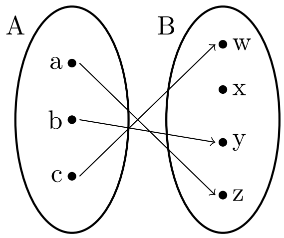
Figure 1. A function $f : A \to B$

Figure 1 above illustrates a function with *domain* {a, b, c}, *codomain* {w, x, y, z}, and *range* {w, z}.

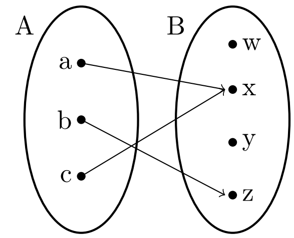
Note. There does not exist any $k \in \{a, b, c\}$ such that $f(k) = y$, which is why y is not in the range.

For diagrams such as Figure 1 to represent a function, it would have to satisfy both existence and uniqueness aspects of a function. If it fails to do so, then it would not represent a function. For e.g., if b did not map onto any of the elements in B in Figure 1 then the diagram would not satisfy the existence condition - since b is being sent to nowhere - and thus not represent a function. Similarly, if b not only mapped onto w but also onto y simultaneously then it would not satisfy the uniqueness condition and the diagram would again not represent a function.

Definition. $\boxed{\textit{injection}}$ A function $f : A \to B$ is *injective*, or one-to-one, if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$. For e.g.,



An injection from A to B



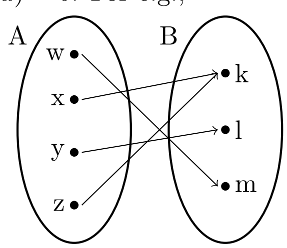
Not an injection from A to B

Both of these are functions. However, the second one is not injective because $f(a) = x$ and $f(b) = x$, which means $f(a) = f(b)$ while $a \neq b$, as these are distinct elements in the domain. Basically, to be injective means that you do not have two arrows pointing at the same point.
This can also be shown via contrapositive:

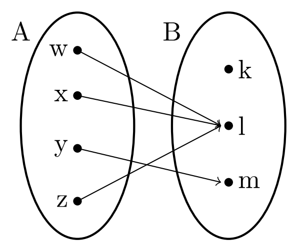
A function $f : A \to B$ is *injective*, or one-to-one, if $a_1 \neq a_2$ implies that $f(a_1) \neq f(a_2)$.

So a function is *injective* if different points in the domain are sent to different points in the codomain. No two arrowheads collide.

Definition. $\boxed{\textit{surjection}}$ A function $f : A \to B$ is *surjective*, or onto, if, for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. For e.g.,



A surjection from A to B



Not a surjection from A to B

The diagram on the right does not satisfy the definition of *surjection* because it is not true that for every $b \in \{k, l, m\}$ there ought to exist some $a \in \{w, x, y, z\}$ such that $f(a) = b$, but $b = k$ does not have this property.
In terms of arrows, this means every dot in B has at least one arrow pointing at it.
As was the case for injection, surjection can also be defined using contrapositive:

A function $f : A \to B$ is *surjective*, or onto, if there does not exist any $b \in B$ for which $f(a) \neq b$ for all $a \in A$.

Note. When defining a function $f : A \to B$, the ideas of *existence* and *uniqueness* were focused on A — for every $x \in A$, we demand that $f(x)$ exists and be unique. To be injective and surjective, the attention shifts to B. To be *surjective* means that B has an existence criterion (for every $b \in B$, there *exists* some $a \in A$ that maps to it). To be *injective* means that B has a uniqueness-type criterion (for every $b \in B$, there is *at most one* $a \in A$ that maps to it).

Definition. $\boxed{\textit{bijection}}$ A function $f : A \to B$ is *bijective* if it is both injective and surjective. Being bijective means that every element in A is paired up with precisely one element in B.

If we use the pairing analogy, simply by being a function means everyone in A has found someone to be in a relationship with. Surjectivity guarantees that everyone in B has also found someone to pair up with. Injectivity means all the relationships are monogamous. Therefore being bijective means everyone is in a monogamous relationship.

In terms of arrows, being a bijection means that every dot on the left has precisely one arrow emanating from it, and every dot on the right has precisely one arrow entering it.

Note. Defining a function $f : A \to B$ placed an existence and uniqueness criteria on A. If f is both injective and surjective, then this adds existence and uniqueness criteria to B. Thus, if f is a bijection, then it has these criteria on both sides: Every $a \in A$ is mapped to precisely one $b \in B$, and every $b \in B$ is mapped to precisely one $a \in A$. In effect, this pairs up each element of A with an element of B; namely, a is paired with $f(a)$ in this way.

1.1. Proving $x$jectiveness, for $x \in$ {in, sur, bi}.

In order to prove injectiveness and surjectiveness we will use their definitions. The outline to prove a function is injective is as follows.

Proposition. $f : A \to B$ is an injection.

Proof. Assume $x, y \in A$ and $f(x) = f(y)$.

« An explanation of what $x, y \in A$ and $f(x) = f(y)$ means »

$\updownarrow$ apply algebra
apply logic
apply techniques

« Oh hey look, that's what $x = y$ means »

Therefore $x = y$.
Since $x, y \in A$ and $f(x) = f(y)$ implies that $x = y$, it follows that f is injective.

$\square$

Alternatively, we could also use the contrapositive where we would start by assuming $x \neq y$, and then conclude that $f(x) \neq f(y)$.

The outline for proving surjectiveness similarly uses its definition:

Proposition. $f : A \to B$ is a surjection.

Proof. Assume $b \in B$.

$\updownarrow$ Find an $a \in A$ where $f(a) = b$,
and show how it works

Since every $b \in B$ has an $a \in A$ where $f(a) = b$, it follows that f is surjective.

$\square$

To prove a function is a bijection, we can prove both above.
Let's apply these to some examples.

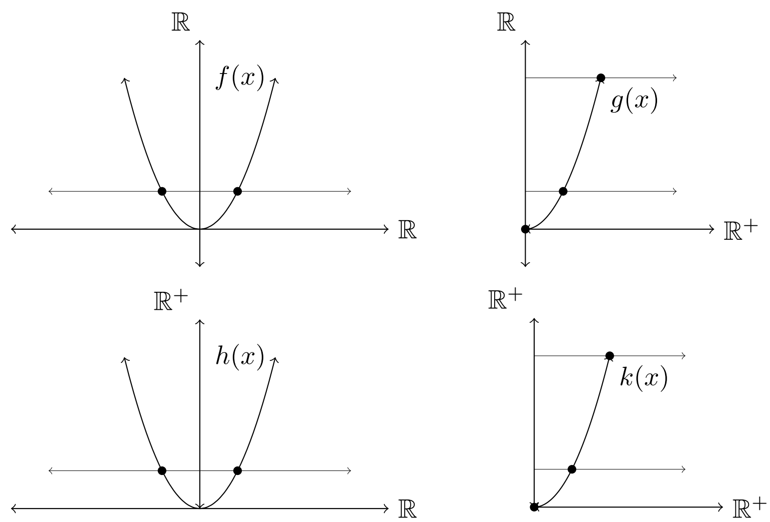
1.1.1. *Proving injectiveness, surjectiveness, or bijectiveness of x^2.*

Let $\mathbb{R}^+$ denote the nonnegative real numbers. Lets prove the following:

(a) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ is not injective, surjective, or bijective.
(b) $g : \mathbb{R}^+ \to \mathbb{R}$ where $g(x) = x^2$ is injective, but not surjective or bijective.
(c) $h : \mathbb{R} \to \mathbb{R}^+$ where $h(x) = x^2$ is surjective, but not injective, or bijective.
(d) $k : \mathbb{R}^+ \to \mathbb{R}^+$ where $k(x) = x^2$ is injective, surjective, and bijective.

Notice that even though each function squares its input x, they are all different. This is because a function is not only the operation, but the domain and codomain as well. Since their domains and/or codomains do not match, they are all different functions. This allows them to have different properties, as we are proving here.

Scratch Work. The injective property is essentially a "horizontal line test". If every horizontal line through the range hits the function in only one place, the function is injective. But if any horizontal line through the range hits the function in more than one place, then the function is not injective because it wouldn't satisfy the uniqueness criterion.



Looking at the graphs, it makes sense that f and h are not injective since we can show that any two points from the domain do map to the same value in the codomain (for example, -1 and 1). On the other hand, g and k seems to be injective.

To prove that g and k are injective, we will assume that, say, $g(x) = g(y)$, and we will try to prove $x = y$:

$$\begin{gathered} g(x) = g(y) \\ x^2 = y^2 \\ \sqrt{x^2} = \sqrt{y^2} \\ x = y \text{ or } x = -y \end{gathered}$$

While x take on two values, y and $-y$, and thus violates the uniqueness criterion for f and h, this is not the case for g and k since their domains do not include negative numbers. So, for g and k this scratch work should prove $x = y$, which would in turn prove these functions are injective.

For surjectivity, or at least to show that a function is not surjective, we need to find some y in the codomain that nothing maps to. Since functions f and g has ys that are negative, and there aren't any xs square of which would be negative, we can easily show that f and g are not surjective. On the other hand, the codomains of h and k do not include negative numbers so they will be surjective. To show this, we will pick any y in the codomain, and find the specific x in the domain such that $h(x) = y$ and $k(x) = y$. The value of $x = \sqrt{y}$ should work in both cases.

Proof. Part i. Observe that $f(-2) = f(2)$ while $-2 \neq 2$, showing that f is not injective.
Also observe that $f(x) = x^2 \geq 0$ for all $x \in \mathbb{R}$ showing that there does not exist $x \in \mathbb{R}$ for which $f(x) = -4$ — or for which $f(x) < 0$. Since -4, and all negative numbers, are in f's codomain, this proves that f is not surjective, either.
Since f is not surjective or injective, it cannot be bijective either.

Part ii. Observe that $g(x) = x^2 \geq 0$ for all $\mathbb{R}^+$, there does not exist $x \in \mathbb{R}$ for which $g(x) = -4$ — or for which $f(x) < 0$. Since -4, and all negative numbers, are in g's codomain, this proves that g is not surjective. By definition, this also means g is not bijective.
To see g is injective, assume $x, y \in \mathbb{R}^+$ and $g(x) = g(y)$. Then,

$$g(x) = g(y)$$
$$x^2 = y^2$$
$$\sqrt{x^2} = \sqrt{y^2}$$
$$x = y \text{ or } x = -y$$

Since $x, y \in \mathbb{R}^+$, the only option is that both x and y can only be positive, so $x = y$.
We have shown that $g(x) = g(y)$ implies $x = y$, thus g is an injection.

Part iii. Observe that $h(x)$ is not injective for the same reasons as f: $f(-2) = f(2)$ but $-2 \neq 2$ showing that h is not injective.
This means, by definition, it cannot be bijective, either.
To show that h is a surjection, pick any y in its codomain, $\mathbb{R}^+$, and note that its positive square root exists since $y \geq 0$. If that positive square root is x, then $x \in \mathbb{R}^+$ since $x = \sqrt{y}$. Therefore,

$$h(x) = x^2 = (\sqrt{y}^2) = y.$$

We have shown that for every $b \in \mathbb{R}^+$ there exists an $x \in \mathbb{R}^+$ such that $h(x) = y$. This proves that h is a surjection.

Part iii. The fact that k is an injection follows the same reasoning as with g in Part ii, and the fact that k is a surjection follows the exact same reasoning as with h in Part iii. Since k is an injection and a surjection, it is also a bijection.

$\square$