

# MATH CONCEPTS 3

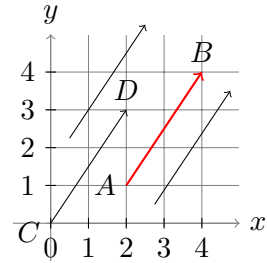
## LINEAR ALGEBRA:

### VECTORS

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#### 1. VECTORS AND VECTOR OPERATIONS

The head of a vector is where the tip of the arrow is and its tail is where it begins at the opposite end of the arrow. On the graph to the left we have point  $A$  located at  $(2,1)$  and point  $B$  located at  $(4,4)$ . The vector joining the two dots is  $\overrightarrow{AB}$  which can be represented as a set of two numbers that indicate the distance  $x$  and  $y$  travels:  $(4,4) - (2,1) = (2,3)$ . However,  $(2,3)$  is the same vector as all the other vectors shown in the graph. This is why it is good to think vectors as *displacements*.



This vector is a displacement of "two over and three up". Wherever they start, they displace by "two over and three up". This is why they are called *free* vectors. These vectors are not similar nor alike, they are *equal*. Generally, vectors are the same if and only if the changes in their individual components are the same. That is, a vector  $\vec{v}$  extending from  $(a_1, a_2)$  to  $(b_1, b_2)$  is equal to vector  $w$  extending from  $(c_1, c_2)$  to  $(d_1, d_2)$  if and only if the change in the first component of  $\vec{v}$ ,  $b_1 - a_1$  is equal to the change in the first component of  $\vec{w}$ ,  $d_1 - c_1$ , as well as if the change in their second components are the same, i.e.,  $b_2 - a_2 = d_2 - c_2$ . That is,  $\vec{v} = \vec{w}$  if and only if

$$\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} = \begin{pmatrix} d_1 - c_1 \\ d_2 - c_2 \end{pmatrix}$$

It should be noted that the coordinates of a vector, such as  $(2,3)$ , specify its length and direction. It does not tell us its location in space. Vectors can be moved around in space without changing the vector itself. To remove this ambiguity about its location, vectors are often drawn as starting at the origin, like the vector  $\overrightarrow{CD}$  in the graph above where it is in its *canonical position*. Textbooks also refer to this as *natural position* or *standard position*.

We can also think of vector  $\overrightarrow{CD}$  representing  $\{(0,0) + (2,3)t : t \in \mathbb{R}\}$  and vector  $\overrightarrow{AB}$  representing  $\{(2,1) + (2,3)t : t \in \mathbb{R}\}$ .

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These notes made liberal use of the following books Johnston, N (2021) *Introduction to Linear and Matrix Algebra* Springer DOI: 10.1007/978-3-030-52811-9; Hefferon, J (2020) *Linear Algebra* 4th ed, available at: <http://joshua.smcvt.edu/linearalgebra>; Vinod H D (2011) *Hands-On Matrix Algebra Using R* World Scientific; and Deisenroth, M P, Faisal, A A, and Ong C S (2020) *Mathematics for Machine Learning* CUP..

Similarly, in  $\mathbb{R}^3$ , the line through (3,1,2) and (0,2,4) is the set of vectors  $\left\{\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix} : t \in \mathbb{R}\right\}$ ; and the plane through the points (1,2,3), (-2,4,-6), and (1,-3,5) consists of the vectors in this set  $\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 2 \\ -9 \end{pmatrix} + s \cdot \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} : s, t \in \mathbb{R}\right\}$ .

*Note.* Notice that the vectors associated with the parameters  $s$  and  $t$  are determined in relation to the particular point (1,2,3). This is because a particle would be free to move on the plane in a combination of two directions. In this sense, the parameters for  $s$  and  $t$  are unrestricted, whereas (1,2,3) is restricted to that point.

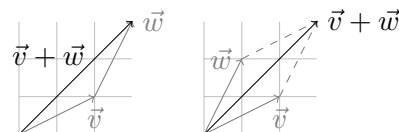
Books usually describe a plane with a linear equation  $Plane = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : 4x - 3y + 2z = 12\right\}$ . In order to translate this to the vector description we can think of this as a linear system and parametrize it:  $x = 3 + \frac{3}{4}y - \frac{1}{2}z$ . Therefore,  $Plane = \left\{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 3/4 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} : y, z \in \mathbb{R}\right\}$ .

If we generalize this, then a set of the form  $\{\vec{p} + t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k : t_1, \dots, t_k \in \mathbb{R}\}$  where  $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$  and  $k \leq n$  is a  $k$ -dimensional linear surface, or  $k$ -flat.

For example, in  $\mathbb{R}^4$ ,  $\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + t \cdot \begin{pmatrix} 7 \\ -2 \\ 13 \\ 0 \end{pmatrix} : t \in \mathbb{R}\right\}$  is a line, whereas  $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + s \cdot \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} : t, s \in \mathbb{R}\right\}$  is a plane, and  $\left\{\begin{pmatrix} -2 \\ 0 \\ -7 \\ 3 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + s \cdot \begin{pmatrix} -3.5 \\ 5 \\ 5 \\ 1 \end{pmatrix} + r \cdot \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix} : t, s, r \in \mathbb{R}\right\}$  is a three dimensional linear surface. When the dimension of the linear surface is one less than the dimension of the space like the three-flat in  $\mathbb{R}^4$  above, that is, when in  $\mathbb{R}^n$  we have an  $(n-1)$ -flat, that surface is called a *hyperplane*.

In general, if we see  $\vec{v}$  and  $\vec{w}$  as displacements, then  $\vec{v} + \vec{w}$  is a combined displacement.

This can be represented geometrically in two different ways: either as a head-to-tail or in standard position with the parallelogram rule as shown on the right.



**Theorem 1.** Suppose  $\vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^n$  are vectors. Then the following properties hold:

- (i) *Commutativity:*  $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- (ii) *Associativity:*  $(\vec{v} + \vec{w}) + \vec{x} = \vec{v} + (\vec{w} + \vec{x})$ .

*Proof.* To prove part (i) we will make use of the definition of vector addition and the fact that addition of real numbers is commutative:

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) = (w_1 + v_1, w_2 + v_2, \dots, w_n + v_n) = \vec{w} + \vec{v}.$$

Proof of part (ii) follows a similar pattern making use of the definition of vector addition and the property of real number addition. □

Another basic operation on vectors called *scalar multiplication* changes a vector's length and can reverse its direction.

**Definition.** Suppose  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ , is a scalar. Then the *scalar multiplication* of real number  $c$  and vector  $\vec{v}$  is:

$$c\vec{v} = c \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix}.$$

*Note.* It should be remembered that scalar multiplication and scalar product do not refer to the same operation.

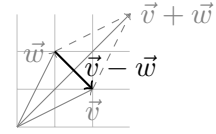
Two special cases of scalar multiplication stand out:

- If  $c = 0$  then  $c\vec{v}$  is called the *zero vector* denoted by  $\vec{0}$  (or  $\mathbf{0}$ ), where all the entries (or components) are all zeros.
- If  $c = -1$  then  $c\vec{v}$ , denoted as  $-\vec{v}$ , whose entries (or components) are the negatives of  $\vec{v}$ 's components.

Using the second special case we can then construct *vector subtraction*:

$$\vec{v} - \vec{w} = \vec{v} + -1 \cdot \vec{w}.$$

This subtraction has a geometric interpretation such that  $\vec{v} - \vec{w}$  is the vector pointing from the head of  $\vec{w}$  to the head of  $\vec{v}$  when both  $\vec{v}$  and  $\vec{w}$  are in canonical, or standard position.



With scalar multiplication, we can expand the commutativity and associativity of vector addition to the following:

**Theorem 2.** -Suppose  $\vec{v}, \vec{w} \in \mathbb{R}^n$  are vectors and  $c, d \in \mathbb{R}$  are scalars. Then the following properties hold:

- (i)  $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ ,
- (ii)  $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ , and
- (iii)  $c(d\vec{v}) = (cd)\vec{v}$ .

*Proof.* To prove the first property, note that we can use the definition of vector addition to add the vectors inside the paranthesis, then use the definition of scalar multiplication as follows.

$$\begin{aligned} c(\vec{v} + \vec{w}) &= c \cdot \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix} = \begin{pmatrix} c(v_1 + w_1) \\ c(v_2 + w_2) \\ \vdots \\ c(v_n + w_n) \end{pmatrix} = \begin{pmatrix} cv_1 + cw_1 \\ cv_2 + cw_2 \\ \vdots \\ cv_n + cw_n \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{pmatrix} + \begin{pmatrix} cw_1 \\ cw_2 \\ \vdots \\ cw_n \end{pmatrix} \\ &= c \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + c \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = c\vec{v} + c\vec{w}. \end{aligned}$$

The proofs of the other two properties would follow a similar pattern of reasoning that utilizes the definitions of vector addition and scalar multiplication.  $\square$

## 2. LENGTHS, ANGLES, AND DOT PRODUCT

When discussing geometric properties of vectors such as lengths and angles, we want our definitions to be independent of dimensions so that we can discuss lengths in higher dimensions as we do in two dimensions, for example.

### 2.1. The Dot Product.

Also referred to as *inner product* or *scalar product*, the dot product is the main tool to extend geometric notions from two or three dimensions to any arbitrary dimension. It is effectively a way of combining two vectors so as to create a single number:

**Definition.** The dot product of two  $n$ -component real vectors is the linear combination of their components. So, suppose  $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$  and  $\vec{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$  are vectors. Then their **dot product**, denoted by  $\vec{v} \cdot \vec{w}$ , is the quantity:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n.$$

*Note.* The dot product is only defined if the two vectors have the same number of components. Some authors require that the first vector be a row vector and the second one be a column vector.