CENG 384 - Signals and Systems for Computer Engineers
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Written Assignment 2

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1. (a) Period N is 4.

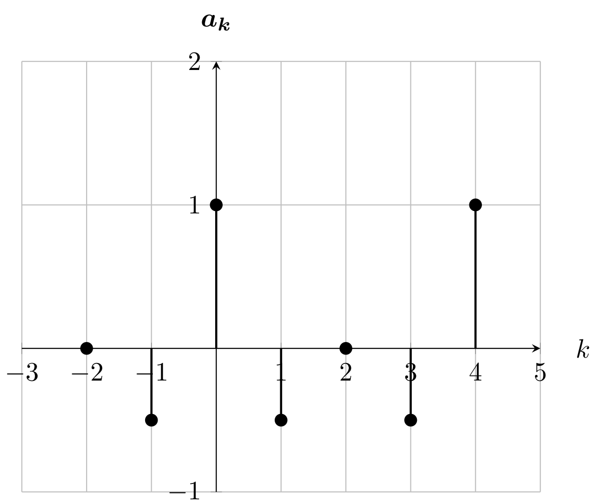
$a_k = \frac{1}{4}\sum_0^3 x[n]e^{-jk\frac{\pi}{2}n}$

We can see that $\sum_0^3 x[n] = 4$ from the figure.

$a_0 = 1$
$a_1 = \frac{1}{4}(0 + e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}})$
$a_2 = \frac{1}{4}(0 + e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi})$
$a_3 = \frac{1}{4}(0 + e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}})$

$a_0 = a_4$ because $e^{j2\pi} = 1$ and $x[n]$ is periodic with period 4.

$a_1 = \frac{1}{4}(\frac{1}{j} - 2 + \frac{-1}{j}) = \frac{-1}{2}$
$a_2 = \frac{1}{4}(-1 + 2 - 1) = 0$
$a_3 = \frac{1}{4}(\frac{-1}{j} - 2 + \frac{1}{j}) = \frac{-1}{2}$



(b) i)
$y[n] = x[n] - \sum_{-\infty}^{\infty}\delta[n - 4k + 1]$

Let c_k is the coefficient of the impulse train $\sum_{-\infty}^{\infty}\delta[n - 4k + 1]$

$c_k = \frac{1}{4}\sum_0^3 \delta[n + 1]e^{-jk\frac{\pi}{2}n} = \frac{1}{4}e^{jk\frac{\pi}{2}}$

Let b_k is the coefficient of $y[n]$

$b_k = a_k - \frac{1}{4}e^{jk\frac{\pi}{2}}$

ii)

$b_k = a_k - \frac{1}{4}e^{jk\frac{\pi}{2}}$

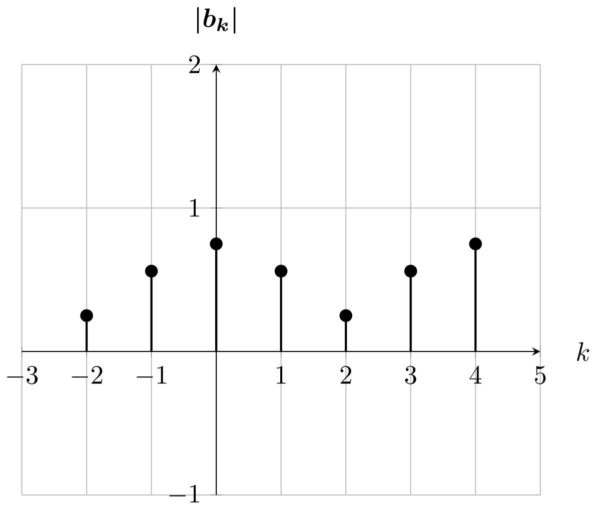
$y[n]$ is also periodic with period 4, thus b_k is also periodic with period 4.

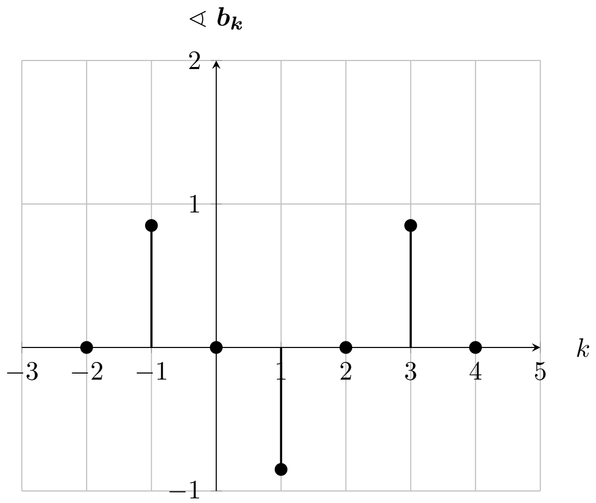
$b_0 = a_0 - \frac{1}{4} = \frac{3}{4}$ $\quad\quad\quad\quad$ $\sphericalangle b_k = 0\pi$

$b_1 = a_1 - \frac{1}{4}e^{j\frac{\pi}{2}} = \frac{-1}{2} - \frac{j}{4}$ $\quad\quad$ $\sphericalangle b_k = -0.85\pi$

$b_2 = a_2 - \frac{1}{4}e^{j\pi} = \frac{1}{4}$ $\quad\quad\quad\quad$ $\sphericalangle b_k = 0\pi$

$b_3 = a_3 - \frac{1}{4}e^{j\frac{3\pi}{2}} = \frac{-1}{2} + \frac{j}{4}$ $\quad\quad$ $\sphericalangle b_k = 0.85\pi$





2. $x[n]$ is real, so $a_k = a^*_{-k}$

$\sum_4^{-3} x[k] = 8$ which covers $2N$ (2 period) values. Thus we can deduce that $\sum_N x[k] = \frac{8}{2} = 4$

$\sum_{k=0}^{3} x[k]e^{-jk\frac{\pi}{2}} + \sum_{k=0}^{3} x[k]e^{-jk\frac{3\pi}{2}} = 4$

we deduce that $\sum_{k=0}^{3} x[k]e^{-jk\frac{\pi}{2}} = 4a_1$ and $\sum_{k=0}^{3} x[k]e^{-jk\frac{3\pi}{2}} = 4a_3$, since $a_k = \frac{1}{4}\sum_N x[n]e^{-jk\frac{\pi}{2}n}$

$4a_1 + 4a_3 = 4 \Rightarrow a_1 + a_3 = 1$

Since we know that $\sum_N x[k] = \frac{8}{2} = 4$, we can say that $a_0 = \frac{1}{4}\sum_4 x[k]e^{-j(0)\frac{\pi}{2}n} = 1 \Rightarrow a_0 = 1$

Since $e^{-jk\frac{\pi}{2}n}$ is periodic with period 4, $a_k = a_{k+4n}$ for all $n \in Z$

$|a_1 - a_{11}| = 1 \Rightarrow \qquad |a_1 - a_{-1}| = 1$ since $a_{-1} = a_{11}$

$a_1 + a_3 = 1 \Rightarrow \qquad a_1 + a_{-1} = 1$ since $a_3 = a_{-1}$

We know that $a_k^* = a_{-k} \quad \& \quad a_k + a_k^* = 2Re\{a_k\}$

$\Rightarrow a_k + a_{-k} = 2Re\{a_k\}$

$a_1 + a_{-1} = 2Re\{a_1\} = 1 \Rightarrow Re\{a_1\} = \frac{1}{2} \qquad (1)$

We know that $a_k^* = a_{-k} \quad \& \quad a_k - a_k^* = 2Im\{a_k\}$

$\Rightarrow a_k - a_{-k} = 2Im\{a_k\}$

$|a_1 - a_{-1}| = |2Im\{a_1\}| = 1 \Rightarrow |Im\{a_1\}| = \frac{1}{2} \qquad (2)$

By using the equations (1) and (2), $\Rightarrow a_1 = \frac{1}{2} + \frac{1}{2}j \quad a_3 = a_{-1} = a_1^* = \frac{1}{2} - \frac{1}{2}j$

Since one of the coefficients is zero (given by question), $a_2 = 0$

$a_0 = 1$, $a_1 = \frac{1}{2} + \frac{1}{2}j$, $a_2 = 0$, $a_3 = \frac{1}{2} - \frac{1}{2}j$

We find the coefficients, then we can find the $x[n]$ values by using these coefficients

$x[n] = \sum_{k=0}^{3} a_k e^{jk\frac{\pi}{2}n}$

$x[n] = a_0 e^{j(0)\frac{\pi}{2}n} + a_1 e^{j\frac{\pi}{2}n} + a_2 e^{j(2)\frac{\pi}{2}n} + a_3 e^{j(3)\frac{\pi}{2}n}$

$x[n] = 1 + (\frac{1}{2} + \frac{1}{2}j)e^{j\frac{\pi}{2}n} + (\frac{1}{2} - \frac{1}{2}j)e^{j3\frac{\pi}{2}n}$

We convert this expression into:

$x[n] = 1 + (\frac{\sqrt{2}}{2})e^{j\frac{\pi}{4}}e^{j\frac{\pi}{2}n} + (\frac{\sqrt{2}}{2})e^{-j\frac{\pi}{4}}e^{j\frac{3\pi}{2}}$

$x[0] = 2$

$x[1] = 1 + (\frac{\sqrt{2}}{2})(e^{j\frac{3\pi}{4}} + e^{j\frac{5\pi}{4}})$

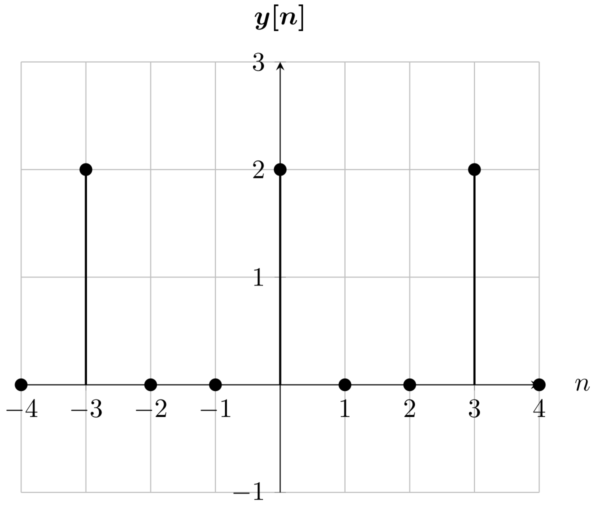
$\Rightarrow 1 - \frac{\sqrt{2}}{2}(\frac{1-j+1+j}{\sqrt{2}}) = 0$

$x[2] = 1 + (\frac{\sqrt{2}}{2})(e^{j\frac{5\pi}{4}} + e^{j\frac{11\pi}{4}})$

$\Rightarrow 1 - \frac{\sqrt{2}}{2}(\frac{1+j}{\sqrt{2}} + \frac{1-j}{\sqrt{2}}) = 0$

$x[3] = 1 + (\frac{\sqrt{2}}{2})(e^{j\frac{7\pi}{4}} + e^{j\frac{17\pi}{4}})$

$\Rightarrow 1 + \frac{\sqrt{2}}{2}(\frac{1-j}{\sqrt{2}} + \frac{1+j}{\sqrt{2}}) = 2$



3. We can realize that this system is a noise filter. In order to eliminate the noise frequencies, we will apply a Low Pass Filter.

   We know that the system can be expressed as $Y(jw) = X(jw)H(jw)$.

   Since, we do not want the high frequencies (caused by the noise), we define a partial function for $H(jw)$ which will have 0 for the high frequencies and 1 for the frequencies we want to have.

$$H(jw) = \begin{cases} 1, & |w| \leq \frac{K2\pi}{T} \\ 0, & |w| > \frac{K2\pi}{T} \end{cases}$$

   We know that $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)e^{jwt}dw$

   When we examine the boundaries and values, we see that $h(t)$ will only have a value between $-\frac{K2\pi}{T}$ and $\frac{K2\pi}{T}$.

   For now, let us call $\frac{K2\pi}{T}$ as w_0, $h(t) = \frac{1}{2\pi}\int_{-w_0}^{w_0} H(jw)e^{jwt}dw$

   $h(t) = \frac{1}{2\pi}\int_{-w_0}^{w_0} e^{jwt}dw$, Since $H(jw)$ is 1 in that region.

   $h(t) = \frac{1}{2\pi}(\frac{e^{jwt}}{jt}\Big|_{-w_0}^{w_0}) \Rightarrow h(t) = \frac{1}{2\pi}(\frac{e^{jw_0t}-e^{-jw_0t}}{jt}) \Rightarrow h(t) = \frac{1}{\pi t}(\frac{e^{jw_0t}-e^{-jw_0t}}{2j}) \Rightarrow h(t) = \frac{sin(w_0t)}{\pi t}$

   Replacing w_0 with $\frac{K2\pi}{T}$ we have $h(t) = \frac{sin(\frac{K2\pi}{T})}{\pi t}$

4. (a) Expression of the system is $y'' + 5y' + 6y = 4x' + x$

Converting the Fourier Transform, we get

$(jw)^2 Y(jw) + 5(jw)Y(jw) + 6Y(jw) = 4jwX(jw) + X(jw)$

We can substitute $Y(jw)$ with $X(jw)H(jw)$

$(jw)^2 X(jw)H(jw) + 5(jw)X(jw)H(jw) + 6X(jw)H(jw) = 4jwX(jw) + X(jw)$

We can divide both sides with $X(jw)$

$(jw)^2 H(jw) + 5(jw)H(jw) + 6H(jw) = 4jw + 1$

$H(jw)((jw)^2 + 5(jw) + 6) = 4jw + 1$

$H(jw) = \frac{4jw+1}{(jw)^2+5(jw)+6} = \frac{A}{jw+3} + \frac{B}{jw+2}$

$A(jw) + 2A + B(jw) + 3B = 4(jw) + 1$

$A + B = 4$ and $2A + 3B = 1 \Rightarrow B = -7$ and $A = 11$

$H(jw) = \frac{11}{(jw)+3} - \frac{7}{jw+2}$

(b) From the transform $H(jw) \xrightarrow{\text{FT}} h(t)$, $H(jw) = \frac{A}{jw+a} \rightarrow h(t) = Ae^{-at}u(t)$

We get $h(t) = (11e^{-3t} - 7e^{-2t})u(t)$

(c) $x(t) = \frac{1}{4}e^{\frac{-t}{4}} \quad \xrightarrow{\text{FT}} \quad X(jw) = \frac{1}{4}\frac{1}{jw+\frac{1}{4}} = \frac{1}{4jw+1}$

$Y(jw) = X(jw)H(jw) = \frac{1}{4jw+1}\frac{4jw+1}{(jw)^2+5jw+6}$

$\Rightarrow \frac{1}{(jw)^2+5jw+6} = \frac{A}{jw+3} + \frac{B}{jw+2}$

$A(jw) + 2A + B(jw) + 3B = 1$

$A + B = 0$ and $2A + 3B = 1 \Rightarrow A = -1$ and $B = 1$

$Y(jw) = \frac{-1}{jw+3} + \frac{1}{jw+2} \quad \xrightarrow{\text{FT}} \quad y(t) = (-e^{-3t} + e^{-2t})u(t)$