CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 4

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1. (a) $\frac{-1}{8}y[n-2] + \frac{3}{4}y[n-1] + 2x[n] = y[n]$

   (b) Converting the Fourier Transform, we get

   $\frac{-1}{8}Y(e^{jw})e^{-2jw} + \frac{3}{4}Y(e^{jw})e^{-jw} + 2X(e^{jw}) = Y(e^{jw})$

   We can substitute $Y(jw)$ with $X(jw)H(jw)$
   $\frac{-1}{8}X(e^{jw})H(e^{jw})e^{-2jw} + \frac{3}{4}X(e^{jw})H(e^{jw})e^{-jw} + 2X(e^{jw}) = X(e^{jw})H(e^{jw})$

   We can divide both sides with $X(jw)$

   $\frac{-1}{8}H(e^{jw})e^{-2jw} + \frac{3}{4}H(e^{jw})e^{-jw} + 2 = H(e^{jw})$

   $\frac{-1}{8}H(e^{jw})e^{-2jw} + \frac{3}{4}H(e^{jw})e^{-jw} - H(e^{jw}) = -2$

   $H(e^{jw})(\frac{-1}{8}e^{-2jw} + \frac{6}{8}e^{-jw} - 1) = -2$

   $H(e^{jw})(-e^{-2jw} + 6e^{-jw} - 8) = -16$

   $H(e^{jw})((-e^{-jw} + 2)(e^{-jw} - 4)) = -16$

   $H(e^{jw}) = \frac{-16}{((-e^{-jw}+2)(e^{-jw}-4))}$

   (c) We convert the $H(e^{jw})$ to $\frac{A}{(-e^{-jw}+2)} + \frac{B}{(-e^{-jw}+4)}$

   $A = B$ and $-4A + 2B = -16$

   $A = 8$ $B = 8$

   $\frac{8}{(-e^{-jw}+2)} + \frac{8}{(-e^{-jw}+4)}$

   $\frac{4}{(1-\frac{1}{2}e^{-jw})} - \frac{2}{(1-\frac{1}{4}e^{-jw})}$

   From the transform $H(e^{jw}) \xrightarrow{\text{FT}} h[n]$, $H(e^{jw}) = \frac{A}{e^{-jw}+a} \rightarrow h[t] = A(\frac{1}{a})^n u[n]$

   $h[n] = 4(\frac{1}{2})^n u[n] - 2(\frac{1}{4})^n u[n]$

   (d) We transform the $x[n]$ to $X(e^{jw}) = \frac{1}{1-\frac{1}{4}e^{-jw}}$

   $Y(e^{jw}) = X(e^{jw})H(e^{jw}) = \frac{1}{1-\frac{1}{4}e^{-jw}} \frac{-16}{(-e^{-jw}+2)(e^{-jw}-4)}$

   $Y(e^{jw}) = \frac{-64}{(-e^{-jw}+2)(e^{-jw}-4)^2}$

   $Y(e^{jw}) = \frac{A}{(e^{-jw}-4)^2} + \frac{B}{(e^{-jw}-4)} + \frac{C}{(2-e^{-jw})}$

We find from this equation:

$C - B = 0$
$C = B$
$A = -2C$
$A = -32$
$C = B = 16$

$Y(e^{jw}) = \frac{-32}{(e^{-jw}-4)^2} + \frac{16}{(e^{-jw}-4)} + \frac{16}{(2-e^{-jw})}$

$Y(e^{jw}) = \frac{-2}{(1-\frac{1}{4}e^{-jw})^2} - \frac{4}{(1-\frac{1}{4}e^{-jw})} + \frac{8}{(1-\frac{1}{2}e^{-jw})}$

$y[n] = -2(n+1).(\frac{1}{4})^n u[n] - 4(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n]$

2. Since this system is connected in parallel, we know that this system is sum of $h_1[n]$ and $h_2[n]$. We need to make simple $H(e^{jw})$ to find its transform.

   To better understanding, we say that $e^{-jw} = \alpha$

   $H(e^{jw}) = \frac{\alpha-12}{(\alpha-3)(\alpha-4)} = \frac{A}{\alpha-3} + \frac{B}{\alpha-4}$

   $A\alpha - 4\alpha + B\alpha - 3\alpha = 5\alpha - 12$
   $A + B = 5$
   $-4A - 3B = -12$
   $-A - 15 = -12$
   $A = -3$ and $B = 8$

   We get

   $H(e^{jw}) = \frac{-3}{\alpha-3} + \frac{8}{\alpha-4}$

   $H(e^{jw}) = \frac{-3}{e^{-jw}-3} + \frac{8}{e^{-jw}-4}$

   Then, to transform $H(e^{jw})$, we make more simple form from fourier transform.

   $H(e^{jw}) = \frac{1}{1-\frac{1}{3}e^{-jw}} - \frac{2}{1-\frac{1}{4}e^{-jw}}$

   We know that $H(e^{jw}) = H_1(e^{jw})$ $+H_2(e^{jw})$

   We can transform $h_1[n]$ to $H_1(e^{jw}) = \frac{1}{(1-\frac{1}{3}e^{-jw})^2}$

   $H_2(e^{jw}) = \frac{2}{1-\frac{1}{4}e^{-jw}}$

   $h_2[n] = -2(\frac{1}{4})^n u[n]$

3. (a) We can separate $x(t)$ as $x_1(t) = \frac{sin 2\pi t}{\pi t}$ and $x_2(t) = cos 3\pi t$

   We know that

   $$X_1(jw) = \begin{cases} 1, & |w| \le 2\pi \\ 0, & |w| > 2\pi \end{cases}$$

   Also, we transform $x_2(t) = cos 3\pi t$ to $X_2(jw) = \pi(\delta(w - w_0) + \delta(w + w_0))$

   $X_2(jw) = \pi$ when $|w| = 3\pi$

   $X(jw) = X_1(jw) + X_2(jw)$

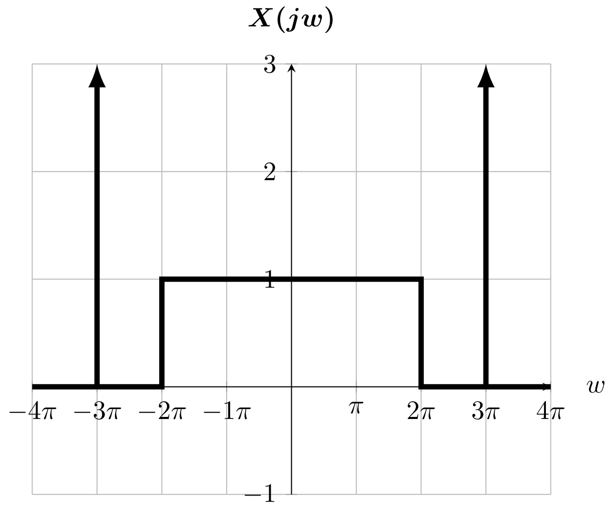

Figure 1: $X(jw)$

(b) $2w_m =$ Nyquist Rate

In $X(jw)$ graph, our $w_m = 3\pi$

So, $2w_m = 6\pi$

Period for sampling must be bigger than $2w_m$.

$w_s > 6\pi$

Since $w_s = \frac{2\pi}{T}$, $T < \frac{1}{3}$

Then, we pick $T = \frac{1}{4}$

Our $w_s = 8\pi$

$P(jw) = \sum \delta(w - k8\pi)$

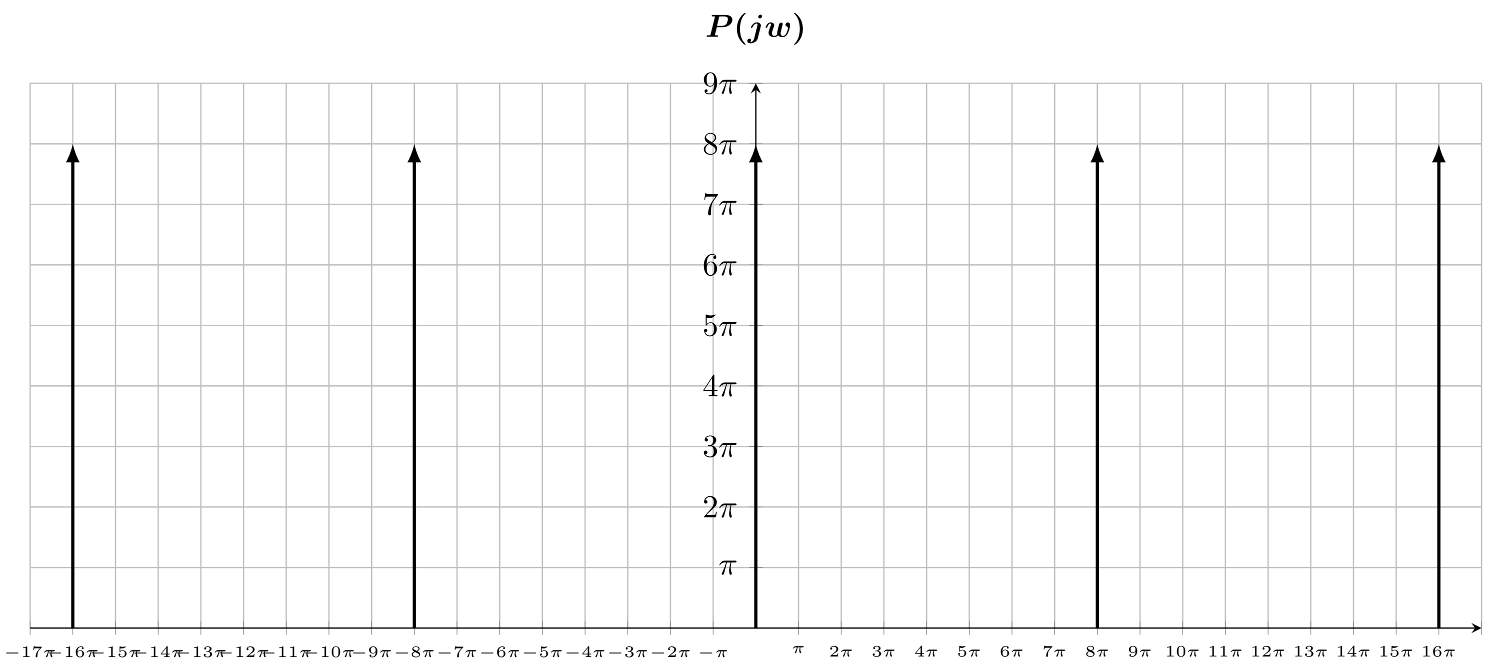

Figure 2: w vs. $P(jw)$.

(c) Then $X_p(jw)$

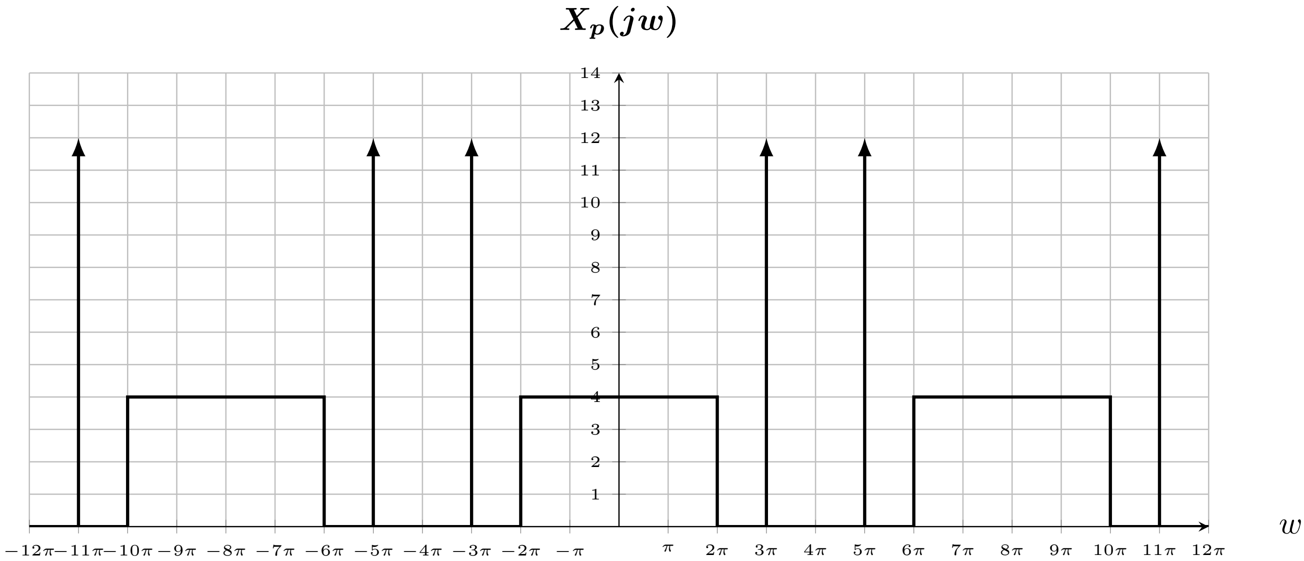

Figure 3: $X_p(jw)$.

4. (a) $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$

Since the transform of $\delta(t - nT)$ is e^{-jwnT}, it follows that

$X_p(jw) = \sum_{n=-\infty}^{\infty} x(nT)e^{-jwnT}$

$X_d(e^j\Omega) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-jn\Omega}$

Using the equation $x_d[n] = x_c(nT)$

$X_d(e^j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-jn\Omega}$

Since the $X_p(\frac{j\Omega}{T}) = X_d(e^{\Omega})$

And we know that $X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w - kw_s)$

Consequently

$X_d(e^j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\Omega - 2k\pi)/T)$

Since the $w_s = \pi$, we found the $T = 2$.

$X_d(e^j\Omega) = \frac{1}{2}\sum_{k=-\infty}^{\infty} X(j(w - k\pi))$

(b) $H(e^{jw}) = \pi\sum_{l=-\infty}^{\infty}(\delta(w - \pi - 2\pi l) + \delta(w + \pi - 2\pi l))$

(c) $Y_d(e^{jw}) = \frac{1}{2\pi}X(e^{jw}) * H(e^{j\theta})$

Substituting $X(e^{jw})$ and $H(e^{j\theta})$,

$Y_d(e^{jw}) = \frac{1}{2\pi}(\frac{1}{2}\sum_{k=-\infty}^{\infty} X(j(w - k\pi))) * (\pi\sum_{l=-\infty}^{\infty}(\delta(w - \pi - 2\pi l) + \delta(w + \pi - 2\pi l)))$

We can move the summations outside and convolve inner parts.

$Y_d(e^{jw}) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}[X(j(w - k\pi)) * \delta(w - \pi - 2\pi l) + X(j(w - k\pi)) * \delta(w + \pi - 2\pi l)]$

We can realize that convolving with impulse is actually a shift in frequency.

$$Y_d(e^{jw}) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}[X(j(w-\pi(k+2l+1))) + X(j(w-\pi(k+2l-1)))]$$