CSE 321

Introduction To Algorithm Design - HW 1

1) $\boxed{T_3 > T_5 > T_2 > T_1 > T_6 > T_4}$

$T_3 > T_5 \Rightarrow \lim_{n \to \infty} \frac{n!}{4^n} = \frac{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n}{4^n \ln 4}$ $\quad \left(\frac{n}{e}\right)^n > 4^n$

growth rate

$= \infty \Rightarrow T_5 = O(T_3)$

$T_5 > T_2 \Rightarrow \lim_{n \to \infty} \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n = \infty \Rightarrow T_5 > T_2$

$\Rightarrow T_2 = O(T_5)$

$T_2 > T_1 \Rightarrow \lim_{n \to \infty} \frac{3^n}{n^4} = \frac{3^n \cdot \ln 3}{4n^3} = \infty$ $\quad 3^n > n^3$

growth rate

$\Rightarrow T_1 = O(T_2)$

$T_1 > T_6 \Rightarrow \lim_{n \to \infty} \frac{n^4}{n^{1/3}} = n^{11/3} = \infty$

$\Rightarrow T_6 = O(T_1)$

$T_6 > T_4 \Rightarrow \lim_{n \to \infty} \frac{n^{1/3}}{\ln^2 n} = \frac{\frac{1}{3} n^{-2/3}}{2\ln n \cdot \frac{1}{n}} = \frac{n^{4/3}}{6 \ln n} = \frac{4}{18} \cdot \frac{n^{1/3} \cdot n}{1} = \infty$

$\Rightarrow T_4 = O(T_6)$

2) a) Algoritma, verilen dizideki en büyük ve en küçük sayıyı bulup, bu iki sayının ortalamasına en yakın sayıyı bulup return eder.

fruits → dizi

plum → en küçük sayı

watermelon → en büyük sayı

orange → (plum + watermelon) / 2 ye en yakın değer

orangeTime → döngüyü sağlayan flag

fruit → for each'te her bir sayı

b) i. worst case; En küçüğün en sonda olma durumu,

bu durumda 2n defa çalışır.

$O(2n) = \boxed{\Theta(n)}$

ii. Best case: Hiç shift edilmezse, küçükten büyüğe doğru sıralı olduğundan n defa çalışır.

$\Rightarrow \boxed{\Theta(n)}$

iii. Average case: $\boxed{\Theta(n)}$ dir.

3) a) $\sum_{i=0}^{n-1}(i^2+1)^2$

$$\int_0^{n-2}(x^2+1)^2\,dx \le f(n) \le \int_1^{n}(x^2+1)^2\,dx$$

$$\int_0^{n-2}(x^4+2x^2+1)\,dx \le f(x) \le \int_1^{n}(x^4+2x^2+1)\,dx$$

$$\frac{x^5}{5}+\frac{2x^3}{3}+x\Big|_0^{n-2} \le f(n) \le \frac{x^5}{5}+\frac{2x^3}{3}+x\Big|_1^{n}$$

$$\frac{(n-2)^5}{5}+\frac{2(n-2)^3}{3}+(n-2)-1 \le f(n) \le \frac{(n-2)^5}{5}+\frac{2(n-2)^3}{3}+(n-2)-4$$

$$f(n)\in\Theta(n^5)$$

b) $\sum_{i=2}^{n-1}\log i^2$

$$\int_1^{i-1}\log x^2\,dx \le f(n) \le \int_2^{i}\log x^2\,dx$$

$\log x^2 = u$ $\quad dx = dv$

$\frac{2\log x}{x} = dx$ $\quad x = v$

$x\log x^2 - 2\int\frac{\log x}{x}\,dx$

$\Downarrow$

$\log x = u$

$\frac{1}{x}dx = du$

$\int u\,du$ $\Downarrow$ $u^2/2$

$$x\log x^2 - 2\log^2 x = 2x\log x - 2\log^2 x = \boxed{2\log x(x-\log x)}$$

$\Downarrow$

$O(x\log x)$

$$2\log(n-1)(n-1-\log(n-1)) \le f(x) \le 2\log n(n-\log n) - 2\log 2(2-\log 2)$$

$$\boxed{f(n)\in\Theta(n\log n)}$$

c) $\sum_{i=1}^{n}(i+1)\,2^{i-1}$

$$\int_0^{n}((x+1)\,2^{x-1})\,dx \le f(x) \le \int_1^{n+1}((x+1)\,2^{x-1})\,dx$$

$x+1=u$ $\quad 2^{x-1}dx = dv$

$dx = du$ $\quad \frac{2^{x-1}}{\ln 2} = v$

$(x+1)\frac{2^{x-1}}{\ln 2} - \int\frac{2^{x-1}}{\ln 2}\,dx$

$\rightsquigarrow (x+1)\frac{2^{x-1}}{\ln 2} - \frac{2^{x-1}}{\ln^2 2}$ $\qquad \boxed{f(n)\in\Theta(n2^n)}$

$$(n+1)\frac{2^{n-1}}{\ln 2} - \frac{2^{n-1}}{\ln^2 2} - \frac{x+1}{\ln 2} - \frac{1}{\ln^2 2} \le f(n) \le (x+2)\frac{2^{n+1}}{\ln 2} - \frac{2^{n+1}}{\ln^2 2} - \frac{4}{\ln 2} - \frac{2}{\ln^2 2}$$

3-d) $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+j)$

$$\Downarrow$$

$$\sum_{i=0}^{n-1} \left((i-1)i + \frac{(i-1)i}{2} \right) = 3/2 \sum_{i=0}^{n-1} (i^2 - i)$$

$$3/2 \int_0^{n-2} (x^2 - x)\,dx \le f(x) \le 3/2 \int_1^{n} (x^2 - x)\,dx$$

$$3/2 \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_0^{n-2} \le f(x) \le 3/2 \left(\frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_1^{n}$$

$$3/2 \left(\frac{(n-2)^3}{3} - \frac{(n-2)^2}{2} \right) \le f(x) \le 3/2 \left(\frac{n^3}{3} - \frac{n^2}{2} \right) - \frac{15}{12}$$

$$\boxed{f(n) \in \Theta(n^3)}$$

3-) Code

a-)
```
for i in range(0, n-1):
    sum += (i * i + 1) * (i * i + 1);
```

d-)
```
for i in range(0, n):
    for j in range(0, i):
        sum += i + j;
```

4e)

$$n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \dots + \frac{n}{2^n}$$

$$\underbrace{n\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n}\right)} \rightarrow \boxed{n \cdot \sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^i}$$

$$\boxed{n \cdot \frac{1-r^n}{1-r}}$$

$$n\left(\frac{1-\frac{1}{2}^n}{1-1/2}\right) = 2n\left(1-\tfrac{1}{2}^n\right)$$

$$\Rightarrow O\left(2n\left(1-\tfrac{1}{2}^n\right)\right)$$

$$\boxed{= O(n)}$$

5) a) $n^3 \in O(3^{2n})$ ✓

$$\lim_{n\to\infty} \frac{n^3}{3^{2n}} = \frac{3n^2}{9^n \ln 9} = \frac{6n}{9^n \ln 9 \cdot \ln 9} = \frac{6}{3^n} = 0$$

b) $n \in o(\log \log n)$ ✗

$$\lim_{n\to\infty} \frac{\log \log n}{n} = \frac{\frac{1}{n \cdot \ln 10}}{\frac{\log n \ln 10}{1}} = \frac{1}{n \log n} = \frac{1}{\infty} = 0$$

$$\boxed{\log \log n \in o(n)}$$

c) $n^2 \log^2 n \in o(n!)$ ✓

$$\lim_{n\to\infty} \frac{n^2 \log^2 n}{n!} = \frac{2n \log^2 n + n^2\, 2 \log n}{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \ln 10} = \frac{\log^2 n + 2n \log n}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \ln 10}$$

$$= 0 \quad \left(\left(\frac{n}{e}\right)^n > n \log n\right)$$

d) $\sqrt{10n^2 + 7n + 3} \in \Theta(n)$ ✓

$$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n} = \frac{2n+7}{2\sqrt{10n^2+7n+3}} = c \text{ (sabit)}$$