Student Information

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Answer 1

a) X and Y are independent iff $f(x)f(y) = f(x,y)$.

This function has a nonzero value inside

$-\sqrt{1-x^2} < y < \sqrt{1-x^2}$

$-\sqrt{1-y^2} < x < \sqrt{1-y^2}$.

Since integral of $f(x,y)$ outside these boundaries is zero, integrating between these boundaries will be enough.

$f(y) = \displaystyle\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1/\pi \, dx$

$f(y) = \frac{2\sqrt{1-x^2}}{\pi}$

$f(x) = \displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1/\pi \, dy$

$f(x) = \frac{2\sqrt{1-y^2}}{\pi}$

$f(x)f(y) = \frac{2\sqrt{1-y^2}}{\pi}\frac{2\sqrt{1-x^2}}{\pi} = \frac{4\sqrt{(1-x^2)(1-y^2)}}{\pi^2}$

$f(x)f(y) \neq f(x,y)$

Since $f(x)f(y)$ is not equal to $f(x,y)$, random variables X and Y are not independent.

b) From a, marginal pdf for X is $f(x) = \frac{2\sqrt{1-x^2}}{\pi}$ and marginal pdf for Y is $f(y) = \frac{2\sqrt{1-y^2}}{\pi}$

c) Expected value for $f(x)$ is $E(X) = \displaystyle\int_{-1}^{1} x f(x)\, dx = \int_{-1}^{1} \frac{2x\sqrt{1-x^2}}{\pi}\, dx$

Solving this integral we find $E(X) = \frac{-2(1-x^2)^{3/2}}{3\pi}\Big|_{-1}^{1} = 0$

d) $Var(X) = E(X^2) - \mu^2$

$E(X^2) = \int_{-1}^{1} x^2 f(x)\, dx = \int_{-1}^{1} \frac{2x^2\sqrt{1-x^2}}{\pi}\, dx = 0.25$

$\mu = E(X) = 0$ (from part c)

$Var(X) = 0.25 - 0^2 = 0.25$

Answer 2

a) Since these two events are independent $f(t_A, t_B) = f(t_A)f(t_B)$

$f(t_A) = 1/100$ and $f(t_B) = 1/100$

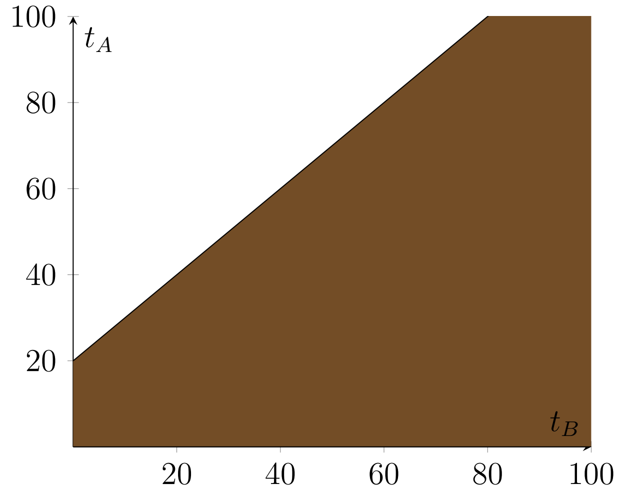
$f(t_A, t_B) = 1/100 \times 1/100 = 1/10000$

b) $P(t_A < 10, t_B > 90) = P(t_A < 10)P(t_B > 90) = \int_0^{10} f(t_A)\, dt \times \int_{90}^{100} f(t_B)\, dt$

$= (\frac{t}{100}|_0^{10}) \times (\frac{t}{100}|_{90}^{100}) = 0.1 \times 0.1 = 0.01$

c) $P(t_A < t_B + 20)$

This probability distribution can be shown with a 2d graph as follows:



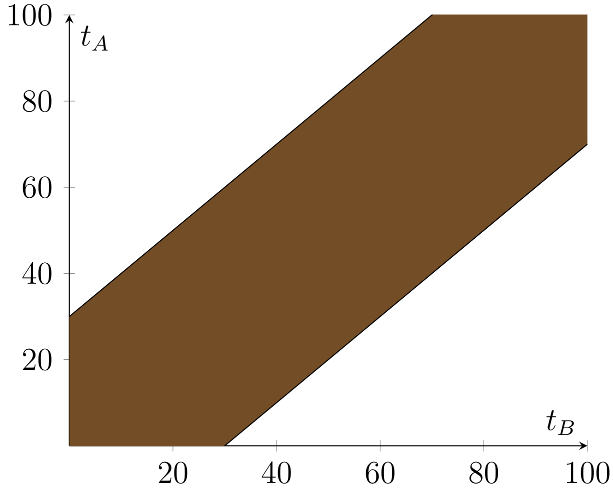
The probability $P(t_A < t_B + 20)$ is the ratio between the area under the curve $t_A < t_B + 20$ to the probability space.

That is $6800/10000 = 0.68$

d) Likewise the previous question, this probability can also be modelled as a 2d graph.

$P(|t_A - t_B| < 30)$:

$|t_A - t_B| < 30 \equiv (t_B - 30 < t_A < t_B + 30)$



$P(|t_A - t_B| < 30) = 5100/10000 = 0.51$

Answer 3

a) Let F_T be the cdf of the random variable T.

$$\begin{aligned}
F_T(t) &= P(T < t) \\
&= 1 - P(T \geq t) \\
&= 1 - P(min\{X_1, X_2, ..., X_n\} \geq t) \\
&= 1 - P(X_1 \geq t, X_2 \geq t, ..., X_N \geq t) \\
&= 1 - P(X_1 \geq t)P(X_2 \geq t)...P(X_N \geq t) \\
&= 1 - e^{-t\lambda_1 - t\lambda_2 - ... - t\lambda_N} \\
&= 1 - e^{-\sum_{n=1}^{N} t\lambda_n}
\end{aligned}$$

b) Let T be the first moment that any of the computers fails and X_n be the random variable that represents the time that computer n fails.

$T = min\{X_1, X_2, ..., X_n\}$

Using the formula from part a, we get $F_T(t) = 1 - e^{-\sum_{n=1}^{N} t\lambda_n}$

$\lambda_n = n/10$

We get, $F_T(t) = 1 - e^{-\sum_{n=1}^{N} \frac{nt}{10}} = 1 - e^{-55/10}$

This cdf in the form of a exponential cdf. Then $\lambda = -(-55/10) = 55/10$

$E(T) = 1/\lambda = 10/55 = 0.18$

Answer 4

a) At least %70 of the participants are undergraduate students means that there are at least 70 undergraduate students among the participants.

The number U of undergraduate students has binomial distribution with $n = 100$ and $p = 0.7$, $\mu = 74$ and $\sigma = \sqrt{100 * 0.74 * (1 - 0.74)} = 4.386$.

Applying the central limit theorem with the continuity correction,

$P(U \geq 70) = P(U > 69.5) = P(\frac{U-\mu}{\sigma} \geq \frac{69.5-\mu}{\sigma})$

$P(Z > \frac{69.5-74}{4.386})$

$P(Z > -1.03) = 1 - P(Z \leq -1.03)$

$P(Z \leq -1.03) = \Phi(-1.03) = 0.1515$

$P(Z > -1.03) = 1 - P(Z \leq -1.03) = 0.8485$

The probability that at least %70 of participants are undergraduate students is 0.8485

b) At most %5 of the participants are pursuing a doctoral degree means that there is at most 5 doctoral students among the participants.

The number D of participants that are pursuing a doctoral degree has binomial distribution with $n = 100$ and $p = 0.1$, $\mu = 10$ and $\sigma = \sqrt{100 * 0.1 * (1 - 0.1)} = 3$

Applying the central limit theorem with the continuity correction,

$P(D \leq 5) = P(D < 5.5) = P(\frac{D-\mu}{\sigma} \leq \frac{5.5-\mu}{\sigma})$

$P(Z \leq \frac{5.5-10}{3}) = \Phi(-1.5) = 0.0668$

The probability that at most %5 of participants are pursuing a doctoral degree is 0.0668