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Q. 1

Assume that there exists some a such that $a < 1$.
$a < 1$
$a * a < a$
$a^2 < a$
Since a is an element of positive integers, a^2 must also be an element of positive integers. Since, in the beginning we have assumed that a is the smallest positive integer, and we reached a positive integer that is smaller than a, we reached a contradiction. Therefore, there cannot be a positive integer that is smaller than 1. By the well-ordering property of the positive integers, we can conclude that 1 is the smallest positive integer.

Q. 2

Proving S(m, 1):
Basis step:
S(1, 1):
$x_1 = 1$, There is one solution to this problem, $x_1 = 1$.
$f(1,1) = \dfrac{1!}{1 * 1!} = 1$

Induction: Assume that S(j, 1) holds.
for S(j+1, 1):
$x_1 + ... + x_\mathrm{j} + x_\mathrm{j+1} = 1$
There are two different solutions to this problem:
$x_1 + ... + x_\mathrm{j} = 1, x_\mathrm{j+1} = 0$ or $x_1 + ... + x_\mathrm{j} = 0, x_\mathrm{j+1} = 1$
$x_1 + ... + x_\mathrm{j} = 1$ has f(j, 1) $= \dfrac{j!}{(j-1)!} = j$ solutions (by the assumption).
At the end, the equation $x_1 + ... + x_\mathrm{j} + x_\mathrm{j+1} = 1$ has j+1 solutions.
f(j+1, 1) $= \dfrac{(j+1)!}{j!} = j + 1$ holds.
Conclusion:
S(m, 1) proved using induction.

Proving S(1, n):
$x_1 = j$
This kind of an equation always has one solution (independent of j).
f(1, j) $= \dfrac{j!}{j! * 0!} = 1$.
S(m, 1) proved.

Proving $f(m+1, n+1) = f(m+1, n) + f(m, n+1)$:
S(m+1, n+1) states that $x_1 + ... + x_\mathrm{m} + x_\mathrm{m+1} = n + 1$

$x_\mathrm{m+1} == 0 \rightarrow x_1 + ... + x_\mathrm{m} = n$ (f(m,n) different solutions)
$x_\mathrm{m+1} == 1 \rightarrow x_1 + ... + x_\mathrm{m} = n - 1$ (f(m,n-1) different solutions)
......
$x_\mathrm{m+1} == n \rightarrow x_1 + ... + x_\mathrm{m} = 0$ (1 solution)

In total there are $\sum_{n=1}^{n} f(m,i) + 1$ solutions.

f(m+1, n+1) $= \sum_{i=1}^{n} f(m,i) + 1$
$= f(m,n) + \sum_{i=1}^{n-1} f(m,i) + 1$

$= f(m,n) + f(m+1,n-1)$

Conclusion:
$f(m+1,n+1) = f(m+1,n) + f(m,n+1)$ proved.
Proving $f(m+1,n+1) = \dfrac{(n+m)!}{(n+1)!m!}$:

Basis step:
S(m,1) proved before.
S(1,n) proved before.

Induction:
Assume that S(j+1,k) and S(j,k+1) holds.

Then:
$f(j+1,k) = \dfrac{(j+k)!}{k! * j!}$

$f(j,k+1) = \dfrac{(j+k)!}{(k+1)! * (j-1)!}$

$f(j+1,k+1) = f(j+1,k) + f(j,k+1)$

$= \dfrac{(j+k)!}{k! * j!} + \dfrac{(j+k)!}{(k+1)! * (j-1)!}$
$= \dfrac{(k+1) * (j+k)!}{(k+1)! * j!} + \dfrac{j * (j+k)!}{(k+1)! * j!}$
$= \dfrac{(k+1+j) * (j+k)!}{(k+1)! * j!}$
$= \dfrac{(k+1+j)!}{(k+1)! * j!}$

Conclusion: If S(j+1,k) and S(j,k+1) are true, then S(j+1,k+1) is also true. Since the base cases are also true, we can conclude S(m,n) by induction.

Q. 3

a.

There can be 4 orientation for a triangle based on the orientation of its hypotenuse.
For the up-right oriented triangle, there can be 1+2+3+4+5+6+7 = 7*(7+1)/2 = 28 triangles.
For the up-left oriented triangle there can be 1+2+3+4+5+6 = 6*(6+1) = 21 triangles.
For the down-left oriented triangle there can be 1+2+3+4+5+6 = 6*(6+1) = 21 triangles.
For the down-right oriented triangle there can be 1+2+3+4+5+6 = 6*(6+1) = 21 triangles.
Answer: 21+21+21+28 = 91

b.
At least one value in the image should be mapped by more than one values in the domain. There are two possibilities:
1) A value in the image will be mapped by three values, and the rest will be one-to-one (3, 1, 1, 1).
2) 2 different values in the image will be mapped by two values, and the rest will be one-to-one (2, 2, 1, 1).

The number of occurences of the possibility 1 is: C(6, 3)*C(3, 1)*C(2, 1)*C(1, 1)*4 = 480.
The number of occurences of the possibility 2 is: C(6, 2)*C(4, 2)*C(2, 1)*C(1, 1)*6 = 1080.
Total sum: 1080 + 480 = 1560

Q. 4

a. For any string of length n, $n > 1$, if the first n-1 character constitutes a valid string, there are 3 possible states that the last character can take. If, first n-1 character does not constitute a valid string, then there is only one state that the last character can take. Therefore,
$a_{\mathrm{n}} = 2a_{\mathrm{n\text{-}1}} + 3^{\mathrm{n\text{-}1}}$

b.
$a_1 = 0$

c.
Homogenous solution:
$a_{\mathrm{n}} - 2a_{\mathrm{n\text{-}1}} = 0$

Characteristic equation:
$r - 2 = 0$
$r = 2$
${a_{\mathrm{n}}}^{\mathrm{h}} = A * 2^{\mathrm{n}}$

Particular solution:
$a_{\mathrm{n}} = A * 3^{\mathrm{n}}$
$A * 3^{\mathrm{n}} = 2A * 3^{\mathrm{n\text{-}1}} + 3^{\mathrm{n\text{-}1}}$
$3 * A * 3^{\mathrm{n\text{-}1}} = 2A * 3^{\mathrm{n\text{-}1}} + 3^{\mathrm{n\text{-}1}}$
$A * 3^{\mathrm{n\text{-}1}} = 3^{\mathrm{n\text{-}1}}$
$A = 1$
${a_{\mathrm{n}}}^{\mathrm{h}} = 3^{\mathrm{n}}$

General Solution:
$a_{\mathrm{n}} = {a_{\mathrm{n}}}^{\mathrm{h}} + {a_{\mathrm{n}}}^{\mathrm{p}} = A * 2^{\mathrm{n}} + 3^{\mathrm{n}}$
$a_1 = A * 2^1 + 3^1 = 0$
$A = -3/2$
$a_{\mathrm{n}} = -3 * 2^{\mathrm{n\text{-}1}} + 3^{\mathrm{n}}$