Module 1D - Differential Equations

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June 11, 2023

Contents

Objectives: Review key differential equations concepts as they pertain to subsequent modules.

Prerequisite Knowledge: N/A

Prerequisite Modules: 1A - Calculus

Difficulty: Easy

Summary: This module reviews key differential equations concepts that will be used for future modules.

1 Theory

We will be looking at solving second order ordinary differential equations (ODE) in one dimension. The standard form of second order ODE is

$$\frac{d^2u}{dx^2} = f(u, x). \tag{1.1}$$

The function $f(u, x)$ can be anything such as quadratic, cubic, trigonometric, etc. The ultimate goal to solving this differential equation is finding u as a function of x: $u(x)$. There are two cases in which Eq. 1.1 can take form. Based on those two cases, that will tell us what our method of solving should be.

1.1 Case 1: Simple

Our first case, which will be the more simple of the two, is going to be the second derivative of u in respect to x being equal to a function that is *only* in terms of x.

$$\frac{d^2u}{dx^2} = f(x) \tag{1.2}$$

The reason this is simple is because our solution u is not present on the right hand side, therefore, this can be solved directly through integration. Generally, you will start by integrating Eq. 1.2 in respect to x:

$$\int \frac{d^2u}{dx^2}\ dx = \int f(x)\ dx$$

$$\frac{du}{dx} = g(x) + C_1 \tag{1.3}$$

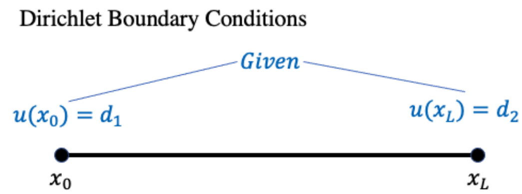
where $g(x) + C_1$ is equal to $\int f(x)\ dx$ given that C_1 is the constant of integration. Integrating both with respect to x again we will obtain the general form of our solution:

$$\int \frac{du}{dx}\ dx = \int (g(x) + C_1)\ dx$$

$$u(x) = h(x) + C_1x + C_2 \tag{1.4}$$

where $h(x)$ is equal to $\int g(x)\ dx$. Since C_1 is a constant, integrating it will result in C_1x and C_2 is another constant of integration resulting from the second integration.

At this point, we will apply our boundary conditions to help us solve for constants C_1 and C_2 to obtain our specific form of our solution for this differential equation.



1.2 Types of Boundary Conditions

There are two types of boundary conditions you will come across which are called Dirichlet and Neumann.

The Dirichlet boundary condition is which our solution is directly given at specified locations. Let's say, for example, a 1D domain spanning from x_0 to x_L, the solutions at those points are given such as $u(x_0) = d_1$ and $u(x_L) = d_2$.

At this point, given we have reached our general solution to our differential equation back in Eq. 1.4 and the value of the left hand side at two points, then we will have to solve a system of equations with two equations and two unknowns. Where the setup will be

$$u(x_0) = h(x_0) + C_1 x_0 + C_2$$
$$u(x_L) = h(x_L) + C_1 x_L + C_2$$

From here, we can solve for C_1 and C_2 directly.

The second type of boundary condition is the Neumann boundary condition. Unlike Dirichlet where your solution is directly given at certain points, instead the solution *derivative* (commonly called "flux") is given at specified locations. For example $\frac{du}{dx}\Big|_{x_0} = d_3$ and $\frac{du}{dx}\Big|_{x_L} = d_4$. With this type of boundary condition we will substitute values into the derivative of our solution $\frac{du}{dx}$ which is Eq. 1.3.

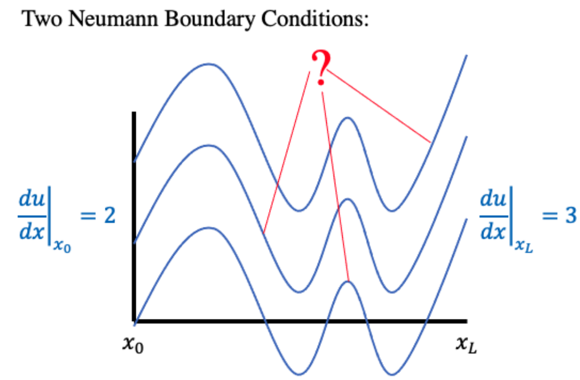
There are cases where you will have to solve equations with both types of boundary conditions, where Dirichlet can be on one side and Neumann can be applied on the other. The procedure will be the same as mentioned above when plugging values into their respective forms of the general solution (Eq. 1.3 or 1.4).

NOTE: A second order differential equation requires:

(a) Two distinct boundary conditions

(b) At least one Dirichlet boundary condition

The reason why we need two distinct boundary conditions is because we have two constants of integration to solve for, C_1 and C_2 which is a result of integrating our differential equation twice. Therefore, for each time we integrate, we will need a distinct number of boundary conditions to solve for the constants of integration. As for the second requirement, if there is not a definitive reference solution to the differential equation, there can potentially be many solutions and the problem is deemed ill-posed. For example, given you have two Neumann boundary conditions $\frac{du}{dx}\Big|_{x_0} = 2$ and $\frac{du}{dx}\Big|_{x_L} = 3$, which can be considered the "slopes" of the solution at domain ends x_0 and x_L, the solution can not be anchored down to a certain position along the y axis. This is illustrated in the figure below.



1.3 Case 2: More difficult

Our second case, which will be more complicated than the first case, is going to be the second derivative of u in respect to x being equal to a function that is in terms of x and the solution itself, $u(x)$.

$$\frac{d^2u}{dx^2} = f(u,x) \tag{1.5}$$

For this module, we will only consider differential equations where u and x are separable, meaning we have the ability to write $f(u,x)$ in terms of two functions, $p(u)$ and $g(x)$. The general form for this case can now be written as

$$\frac{d^2u}{dx^2} + p(u) = g(x). \tag{1.6}$$

In this case we will use the Method of Undetermined Coefficients. For further detail into theory see Paul's Online Differential Equations Notes Section 3-9 or go to `https://tutorial.math.lamar.edu/Classes/DE/UndeterminedCoefficients.aspx`. We will go through an example on how to apply this method.

Example:

$$u'' - 4u' - 12u = 3e^{5x}$$

with boundary conditions

$$u(0) = \frac{18}{7} \quad \text{(Dirichlet)}$$
$$u'(0) = -\frac{1}{7} \quad \text{(Neumann)}$$

Step 1: Find Homogeneous Solution
To find the homogeneous solution u_h, we will set $g(x)$ equal to 0. In this case $g(x)$ is $3e^{5x}$, therefore

$$u_h'' - 4u_h' - 12u_h = 0.$$

In this example, our equation resembles a quadratic equation in the form of

$$r^2 - 4r - 12 = 0.$$

where the order of the derivative corresponds to the power of the variable r:
$u_h'' \to r^2$, $u_h' \to r$, and $u_h \to r^0 = 1$.

We will then find the roots of our characteristic equation:

$$(r-6)(r+2)=0$$
$$r_1=-2,\ r_2=6.$$

The general form of a homogeneous solution of this form is

$$u_h(x)=C_1e^{r_1x}+C_2e^{r_2x}.$$

Plugging in our values of r_1 and r_2 we will have our homogeneous solution to be:

$$u_h(x)=C_1e^{-2x}+C_2e^{6x}.$$

Step 2: Find Particular Solution

For this step, we will reintroduce the original value of $g(x)$ to solve for the particular solution, u_p.

$$u_p''-4u_p'-12u_p=3e^{5x}\equiv g(x) \tag{1.7}$$

Based on the value of $g(x)$ we will guess a solution guided by the table below:

g(x)	u_p **guess**
$ae^{\beta x}$	$Ae^{\beta x}$
$a\cos(\beta x)$	$A\cos(\beta x)+B\sin(\beta x)$
$b\sin(\beta t)$	$A\cos(\beta x)+B\sin(\beta x)$
$acos(\beta x)+bsin(\beta x)$	$A\cos(\beta x)+B\sin(\beta x)$
n^{th} degree polynomial	$A_nt^n+A_{n-1}t^{n-1}+\cdots A_1t+A_0$

Since our $g(x)$ closely resembles that of the first row, our particular solution will have the form of $Ae^{\beta x}$. Therefore, we guess

$$u_p(x)=Ae^{5x}. \tag{1.8}$$

Where A is unknown. To solve for A, we will now take our guess (Eq. 1.8) and calculate the first and second derivatives of u_p

$$u_p'(x)=5Ae^{5x}$$
$$u_p''(x)=25Ae^{5x}$$

and plugging them into our differential equation (Eq. 1.7) to solve for A.

$$u_p''-4u_p'-12u_p=25Ae^{5x}-20Ae^{5x}-12Ae^{5x}$$
$$3e^{5x}=\ -7Ae^{5x}\longrightarrow A=-\frac{3}{7}$$

The particular solution turns out to be

$$u_p(x)=-\frac{3}{7}e^{5x}. \tag{1.9}$$

Step 3: Find General Solution

The general form of our solution comes from combining the homogeneous part and the particular part. This takes the form of

$$\begin{aligned}u(x)&=u_h(x)+u_p(x)\\&=C_1e^{-2x}+C_2e^{6x}-\frac{3}{7}e^{5x}\end{aligned} \tag{1.10}$$

Taking the derivative of this we obtain

$$u'(x) = -2C_1e^{-2x} + 6C_2e^{6x} - \frac{15}{7}e^{5x} \tag{1.11}$$

We may now plug in our Dirichlet and Neumann boundary conditions that were given to us into Eq. 1.11 and Eq. 1.10 and solve for the constants of integration C_1 and C_2.

$$\begin{aligned} \frac{18}{7} &= u(0) = C_1 + C_2 - \frac{3}{7} \\ -\frac{1}{7} &= u'(0) = 2C_1 + 6C_2 - \frac{15}{7} \\ &C_1 = 2 \qquad C_2 = 1 \end{aligned}$$

The final answer to our differential equation is:

$$u(x) = 2e^{-2x} + e^{6x} - \frac{3}{7}e^{5x} \tag{1.12}$$

2 Examples

EXAMPLE 1: Find the solution $u(x)$ analytically to the boundary value problem (BVP). Show all steps of your derivation. (This will give you the true solution, u^{true}, to the problem).

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) = 2kcos\left(\frac{\pi kx}{L}\right) + \sqrt{k}sin\left(\frac{4\pi kx}{L}\right) \tag{2.1}$$

$$\begin{gathered}\text{Domain: } \Omega = (x_0, L) \text{ where } x_0 = 0.1 \text{ and } L = 1.2 \\ E = 0.7 \\ k = 10 \\ \text{Boundary Conditions: } E\frac{du}{dx}\Big|_{x=L} = -2 \text{ and } u(x_0) = 1\end{gathered}$$

SOLUTION FOR EXAMPLE 1: Integrate both sides:

$$\int \frac{d}{dx}\left(E\frac{du}{dx}\right)dx = \int \left[2kcos\left(\frac{\pi kx}{L}\right) + \sqrt{k}sin\left(\frac{4\pi kx}{L}\right)\right]dx$$

$$E\frac{du}{dx} = \frac{2L}{\pi}sin\left(\frac{\pi kx}{L}\right) - \frac{L}{4\pi\sqrt{k}}cos\left(\frac{4\pi kx}{L}\right) + C_1 \tag{2.2}$$

Integrate both sides again:

$$\int E\frac{du}{dx}dx = \int \left(\frac{2L}{\pi}sin\left(\frac{\pi kx}{L}\right) - \frac{L}{4\pi\sqrt{k}}cos\left(\frac{4\pi kx}{L}\right) + C_1\right)dx$$

$$Eu = -\frac{2L^2}{\pi^2 k}cos\left(\frac{\pi kx}{L}\right) - \frac{L^2}{16\pi^2 k^{3/2}}sin\left(\frac{4\pi kx}{L}\right) + C_1 x + C_2$$

$$u^{true} = -\frac{2L^2}{E\pi^2 k}cos\left(\frac{\pi kx}{L}\right) - \frac{L^2}{16E\pi^2 k^{3/2}}sin\left(\frac{4\pi kx}{L}\right) + \frac{C_1 x}{E} + \frac{C_2}{E} \tag{2.3}$$

Apply Neumann boundary condition $E\frac{du}{dx}\Big|_{x=L} = -2$ to Eq. 2.2:

$$-2 = \frac{2L}{\pi}sin\left(\frac{\pi kL}{L}\right) - \frac{L}{4\pi\sqrt{k}}cos\left(\frac{4\pi kL}{L}\right) + C_1$$

$$C_1 = \frac{L}{4\pi\sqrt{k}}cos(4\pi k) - \frac{2L}{\pi}sin(\pi k) \tag{2.4}$$

Apply Dirichlet boundary condition $u(x_0) = 1$ to Eq. 2.3:

$$1 = -\frac{2L^2}{E\pi^2 k}cos\left(\frac{\pi kx_0}{L}\right) - \frac{L^2}{16E\pi^2 k^{3/2}}sin\left(\frac{4\pi kx_0}{L}\right) + \frac{C_1 x_0}{E} + \frac{C_2}{E}$$

$$C_2 = \frac{2L^2}{\pi^2 k}cos\left(\frac{\pi kx_0}{L}\right) + \frac{L^2}{16\pi^2 k^{3/2}}sin\left(\frac{4\pi kx_0}{L}\right) + C_1 x_0 + E \tag{2.5}$$

Combining Eq. 2.4 and Eq. 2.5 into Eq. 2.3 and substituting known values of $x_0 = 0.1$, $L = 1.2$, $E = 0.7$ and $k = 10$:

$$u^{true} = -\frac{72}{175\pi^2}\cos\left(\frac{25\pi x}{3}\right) - \frac{9}{700\cdot 10^{\frac{1}{2}}\pi^2}\sin\left(\frac{100\pi x}{3}\right) + \frac{3x}{7\sqrt{10}\pi} + \frac{1400\cdot 10^{\frac{1}{2}}\pi^2 - 9\sqrt{3}\pi - 60\pi - 288\cdot 30^{\frac{1}{2}}}{1400\cdot 10^{\frac{1}{2}}\pi^2} \tag{2.6}$$

3 Assignment

PROBLEM 1:
Given the 1-D 2nd order differential equation find $\theta(x)$. ρ_o, C, K, a, J, and E are constants.

$$0 = \frac{1}{\rho_o C}\left(K\frac{d^2\theta}{dx^2} + aJE\right) \tag{3.1}$$

with the Dirichlet boundary condition at $x_0 = 0$:

$$\theta(x_0) = \theta_0$$

You will use four boundary conditions at $x_L = 0.1$. Solve the differential equation for each condition given below.

- First, use a Dirichlet condition identical to the condition on the other end:

$$\theta(x_L) = \theta_0$$

- Next, use a Dirichlet condition that does not match the condition on the other end:

$$\theta(x_L) = \theta_0 + 250$$

- Next, use a Neumann (fixed-derivative / fixed flux) boundary condition. First, you will use a "zero flux".

$$K\frac{d\theta}{dx}\bigg|_{x_L} = 0$$

- Finally, use a fixed flux Neumann boundary condition.

$$K\frac{d\theta}{dx}\bigg|_{x_L} = -10^7$$

PROBLEM 2: Solve the 2nd order ordinary differential equation:

$$\frac{d^2u}{dx^2} - C_1 u + C_2 = 0$$

with boundary conditions $u(0) = 0$ and $\frac{du}{dx}\Big|_L = 0$

4 Solutions

PROBLEM 1 Solution:

From equation 3.1 we get the 2nd order spatial derivative to one side:

$$\frac{d^2\theta}{dx^2} = \frac{-aJE}{K} \tag{4.1}$$

Integrating throughout x.

$$\int \frac{d^2\theta}{dx^2}dx = \int \frac{-aJE}{K}dx$$

$$\frac{d\theta}{dx} = \frac{-aJE}{K}x + C_1 \tag{4.2}$$

And integrate throughout x again.

$$\int \frac{d\theta}{dx}dx = \int \frac{-aJE}{K}x + C_1 dx$$

$$\theta(x) = \frac{-aJE}{K}x^2 + C_1 x + C_2 \tag{4.3}$$

Solving for constants C_1 and C_2 with boundary conditions for Case 1:

$$\theta_0 = \frac{-aJE}{2K}(0)^2 + C_1(0) + C_2 \Longrightarrow C_2 = \theta_0 \tag{4.4}$$

$$\theta_0 = \frac{-aJE}{2K}(x_L)^2 + C_1(x_L) + C_2 \Longrightarrow C_1 = \frac{aJE}{2K}x_L \tag{4.5}$$

We obtain our analytical solution for Case 1:

$$\theta(x) = \frac{-aJE}{2_o}x^2 + \frac{aJE}{2_o}Lx + \theta_0 \tag{4.6}$$

For Case 2:
Left Dirichlet Boundary Condition: $\theta(x_0) = \theta_0$.
Right Dirichlet Boundary Condition : $\theta(x_L) = \theta_0 + 250$.
We start again from 3.1 and do the same operations from equation 4.1 to equation 4.4 for solving constant $C_2 = \theta_0$. Next we will now solve for C_1 using the right boundary condition:

$$\theta_0 + 250 = \frac{-aJE}{2K}(x_L)^2 + C_1(x_L) + \theta_0 \tag{4.7}$$

$$C_1 = \frac{250}{x_L} + \frac{aJE}{2K}x_L \tag{4.8}$$

We obtain the analytical solution for Case 2:

$$\theta(x) = \frac{-aJE}{2K}x^2 + \left(\frac{250}{x_L} + \frac{aJE}{2K}x_L\right)x + \theta_0 \tag{4.9}$$

For Case 3:
Left Dirichlet Boundary Condition: $\theta(0) = \theta_0$.
Right Neumann Boundary Condition : $K\frac{d\theta}{dx}\Big|_{x_L} = 0$.
We start again from 3.1 and isolate the flux term on one side and set $x = L$ to use our boundary condition value:

$$K_0\frac{d\theta}{dx} = 0 = -aJEL + C_1 K \tag{4.10}$$

Solving for C_1:

$$C_1 = \frac{aJEL}{K} \tag{4.11}$$

We integrate equation 4.10 and solve for C_2 by first plugging in our Dirichlet Boundary condition:

$$\theta_0 = \frac{-aJE}{2K}(0)^2 + C_1(0) + C_2 \Longrightarrow C_2 = \theta_0 \tag{4.12}$$

We obtain the analytical solution for Case 3:

$$\theta(x) = \frac{-aJE}{2K}x^2 + \frac{aJE}{K}x_L x + \theta_0 \tag{4.13}$$

For Case 4:
Left Dirichlet Boundary Condition: $\theta(0) = \theta_0$.
Right Neumann Boundary Condition : $K\frac{d\theta}{dt} = -10^7$.
Starting from equation 3.1, we work ourselves through similar to equation 4.10 where we set our the flux term to the boundary condition.

$$K\frac{d\theta}{dx} = -aJEx + C_1 \tag{4.14}$$

$$K\frac{d\theta}{dx} = -10^7 = -aJEL + C_1 \tag{4.15}$$

$$C_1 = aJEL - 10^7 \tag{4.16}$$

$$\frac{d\theta}{dx} = \frac{-aJE}{K}x + \frac{C_1}{K} \tag{4.17}$$

Integrate again and apply the left Dirichlet Boundary condition:

$$\int \frac{d\theta}{dx}dx = \int -\frac{aJE}{K}x + \frac{C_1}{K}dx \tag{4.18}$$

$$\theta = -\frac{aJE}{2K}x^2 + \frac{C_1}{K}x + C_2 \tag{4.19}$$

$$\theta = -\frac{aJE}{2K}(0)^2 + \frac{C_1}{K}(0) + C_2 \longrightarrow C_2 = \theta_o \tag{4.20}$$

PROBLEM 2 Solution: First step is to compute the homogeneous part:

$$\frac{d^2u_h}{dx^2} - C_1u_h = 0 \longrightarrow \textbf{guess } u_h = e^{\beta x}$$

Homogeneous solution: $(\beta^2 - C_1)e^{\beta x} = 0 \longrightarrow \beta = \pm\sqrt{C_1}$

$$\longrightarrow (\beta_1 = \sqrt{C_1}, \beta_2 = -\sqrt{C_1}) \longrightarrow u_h = K_1e^{\beta_1 x} + K_2e^{\beta_2 x}$$

Compute the particular part:

$$\frac{d^2u_p}{dx^2} - C_1u_p = -C_2 \longrightarrow \textbf{guess } u_p = K_3x^3 + K_4x^2 + K_5x + K_6$$

Compute the second derivative of the u_p guess and plug it into our particular part:

$$6K_3x + 2K_4 - C_1(K_3x^3 + K_4x^2 + K_5x + K_6) = -C_2$$

Solving for values of K_3, K_4, K_5, and K_6 by combining like terms of x^3, x^2, x, and x^0 we get: $K_3 = 0$, $K_4 = 0$, $K_5 = 0$, and $K_6 = \frac{C_2}{C_1}$.

Combine the particular and homogeneous parts to obtain:

$$u(x) = u_h(x) + u_p(x) = K_1 e^{\sqrt{C_1}x} + K_2 e^{-\sqrt{C_2}x} + \frac{C_2}{C_1}$$

Apply the boundary conditions:

$$\begin{aligned} u(0) = 0 &\longrightarrow K_1 + K_2 + C_2 = 0 \\ \frac{du}{dx}\bigg|_L = 0 &\longrightarrow \sqrt{C_1}K_1 e^{\sqrt{C_1}L} + \sqrt{C_2}K_2 e^{-\sqrt{C_1}L} = 0 \end{aligned}$$

We are left now with 2 equations and 2 unknowns. It is left up to the reader to solve for those constants C_1 and C_2

5 References

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