# DON'T OVERFIT THE HISTORY RECURSIVE TIME SERIES DATA AUGMENTATION

# A PREPRINT

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### **ABSTRACT**

Time series observations can be seen as realizations of an underlying dynamical system governed by rules that we typically do not know. For time series learning tasks, we need to understand that we fit our model on available data, which is "a unique realized history." Training on a single realization often induces severe over-fitting lacking generalization. To address this issue, we introduce a general recursive framework for time series augmentation, which we call Recursive Interpolation Method (RIM). New samples are generated using a recursive interpolation function of all previous values in such a way that the enhanced samples preserve the original inherent time series dynamics. We perform theoretical analysis to characterize the proposed RIM and to guarantee its test performance. We apply RIM to diverse real world time series cases to achieve strong performance over non-augmented data on regression, classification, and reinforcement learning tasks.

# 1 Introduction

The recent success of machine learning (ML) algorithms depends on the availability of a large amount of data and a prodigious computing power, which in practice are not always available. For example, in real world reinforcement learning, it is often impossible to interact indefinitely with the environment and the RL agent ideally should learn a good policy after limited interactions with its environment. To overcome these issues, we can exploit additional information such as the structure or invariance in the data that help the ML algorithms efficiently learn and focus on the most important features for solving the task. In ML, the exploitation of structure in the data has been handled using four different yet complementary approaches: 1) Architecture design, 2) Transfer learning, 3) Data representation, and 4) Data augmentation. Our focus in this work is on data augmentation approaches in the context of time series learning.

One-dimensional time series (1D-TS) representations do not expose the full information of the true dynamical system [1] in a way that ML can easily recognize. For instance, in financial time series data, there are patterns at various scales that can be learned to improve performance. At a more fundamental level, 1D-TS are one-dimensional projections of a hypersurface of data called the phase space of a dynamical system. This projection results in a loss of information regarding the dynamics of the system. However, we can still make inferences about the dynamical system that projects a 1D-TS realization. Our approach is to build richer alternate representations to improve time series pattern identification by using these inferences to generate additional 1D-TS data from the original realization resulting in more optimal parameters and reduced variance. We show that our methodology is applicable to a variety of ML algorithms.

Time series learning problems depend on the observed historical data used for training. However, the fixed historical data is only one particular realization of the true underlying stochastic dynamical systems in the real world that we are trying to learn. Our work focuses on problems where only one realization is available such as stock prices as opposed to problems with multiple realizations such as speech recognition where many audio clips are available for training. Let us consider the stock price prediction problem. The task is to predict future price changes. Ideally we want our prediction model to perform well by capturing the stochastic dynamics of stock markets. However, we only train the model using historical data, and there is no uncertainty in the data. As a result, we do not truly capture the characteristic behaviour of the underlying dynamical system. Using the original training data and hence one realization of the true dynamical system usually induces over-fitting. This is not ideal as we want our model to perform well in the stochastic system instead of just a specific realization of that system.

**Contributions.** The contributions of our work are as follows:

- We present a time series augmentation technique based on recursive interpolation.
- We provide a theoretical analysis of learning improvement for the proposed time series augmentation method:
  - We show that our recursive augmentation not only preserves the original time series dynamics but also allows us to control by how much the augmented time series trajectory deviates from the original time series trajectory (Theorem 4.1).
  - We demonstrate that the augmented loss function induced by our method naturally acts as a regularization effect over the original non-augmented loss function (Theorems 4.2 and 4.3).
  - We believe that this work is the first to offer a theoretical ML framework for time series data augmentation with guarantees for variance reduction in the learned model (Theorems 4.4 and 4.5).
- We empirically demonstrate learning improvements on both supervised learning (regression and classification) and reinforcement learning using real world time series datasets.

**Outline of the paper.** Section 2 presents the literature review. Section 3 defines the notations and the problem setting. Section 4 provides the main theoretical results. Section 5 describes the experimental results and Section 6 concludes with a summary and a discussion of future work.

# 2 Related Work

**Augmentation for Computer Vision.** In the computer vision context, there are multiple ways to augment image data like cropping, rotation, translation, flipping, noise injection and so on. Among them, the mixup technique proposed in [3] is similar to our approach. They train a neural network on convex combinations of pairs of examples and their labels. However, just applying such a technique to time series data is not appropriate due to the stochastic character inherent in time series. We don't know whether the properties inherent in the time series are preserved in the augmented data. In fact, some properties inherent in time series data are non-stationary, so considering simply a convex combination of two sample data points, old time series data and recent time series data, would be unreasonable in the time series context.

Augmentation for RL. [5] presented the Reinforcement Learning with Augmented Data (RAD) module which can augment most RL algorithms. They have demonstrated that augmentations such as random translate, random convolutions, crop, patch cutout, amplitude scale, and color jitter can enable simple RL algorithms to outperform complex advanced methods on standard benchmarks. [6] presented a data augmentation method that can be applied to conventional model-free reinforcement learning (RL) algorithms, enabling learning directly from pixels without the need for pre-training or auxiliary losses. The inclusion of this augmentation method improves performance substantially, enabling a Soft Actor-Critic agent to reach advanced functioning capability on the DeepMind control suite, outperforming model-based methods and contrastive learning. [5] and [6] show the benefit of RL-augmentation using convolutional neural networks (CNNs) for static data but do not handle dynamic data such as time series.

**Augmentation for Time Series.** There is an exhaustive list of transformations applied to time series that are usually used as data augmentation [7]. We first mention the transformations in the time domain such as time warping and time permutation [8]. We also have methods that belong to the magnitude domain such as magnitude warping, Gaussian noise injection, quantization, scaling, and rotation [9]. There exists other transformations on time series in the frequency and time-frequency domains that are based on Discrete Fourier Transform (DFT). In this context, we apply transformations in the amplitude and phase spectrum's of the time series and apply the reverse DFT to generate a new time series signal [10]. Besides the transformations in different domains for time series, there are also more advanced methods, including decomposition-based methods such as the Seasonal and Trend decomposition using Loess (STL) method and its variants [11, 12], statistical generative models [13], and learning-based methods. The learning-based methods can be further divided into embedding space [14], and deep generative models (DGMs) [15, 16]. These approaches are problem dependent and they do not offer guaranteed learning improvement.

Ideally, we would like to have access to more data that is representative of the true dynamics of the system or the regime under which we operate. However, we can not simply randomly add more data as there is a probability that the added data might not be representative of our stochastic system. To ensure that we are able to add meaningful data without disturbing the properties of the original data, we introduce a new approach called Recursive augmentation. Our paper proposes a recursive interpolation method for time series as a tool for generating data augmentations that preserve the dynamic structure underlying the time series data.

# 3 Recursive Interpolation Method (RIM)

Notations. We observe only one realization of a time series from the true dynamical system. The realization consists of features along the time axis. Let d be a dimension of features at each time t and T is the horizon of the given time series. Then at each time step t, features belong to  $\mathbb{R}^d$  for some  $d \in \mathbb{N}$ . Since it is not easy to keep track of the whole time series from t=0 to t=T, we usually decompose the whole sequence of the time series into several pieces of time series with labels using window w. In other words, the whole sequence of time series in  $\mathbb{R}^{d\times T}$  is transformed into a set of samples such that each sample is contained in  $\mathbb{R}^{d\times w}$  and labels belong to  $\mathbb{R}$ . Let  $S=\{s_0,s_1,\ldots,s_T\}$  be the transformed original historical data such that  $s_i=(x_i,y_i)$  where  $x_i\in\mathbb{R}^{d\times w}$  consists of features of the data and  $y_i\in\mathbb{R}$  is a label corresponding to  $x_i$ . Let  $\mathcal{D}$  be a distribution with support [0,1) and  $\lambda_i$  be drawn from  $\mathcal{D}$  independently denoted by  $\lambda_i\sim\mathcal{D}$  for  $i\in[1:T]$ . Now we have a vector  $(\lambda_1,\ldots,\lambda_T)$  and let  $\lambda$  be the vector. For a given vector  $\lambda$ , we generate an augmented time series  $S_{\lambda}$  created from the transformed original historical data  $\lambda$ . Set  $\lambda$  in  $\lambda$  is  $\lambda$  in  $\lambda$ 

$$x_{i,\lambda_i} = (1 - \lambda_i)x_i + \lambda_i x_{i-1,\lambda_{i-1}} \quad \text{and} \quad y_{i,\lambda_i} = (1 - \lambda_i)y_i + \lambda_i y_{i-1,\lambda_{i-1}}. \tag{1}$$

This recursive definition allows us to generate a new time series realization that preserves the trajectory of the original data within some bound (Theorem 4.1).

Case 1. Classification. For the classification problem, we augment data  $x_{\vec{\lambda}}$  using itself, x, as follows. Take one sample (x,y) where  $x=(x_0,x_1,\ldots,x_T)\in\mathbb{R}^{T+1}$  and  $y\in\{0,1\}$ . For each  $\vec{\lambda}=(\lambda_1,\ldots,\lambda_T)\in[0,1]^T$  and  $\lambda_0$  which is a dummy value, we define an augmented data  $x_{\vec{\lambda}}=(x_0,x_{1,\lambda_1},\ldots,x_{T,\lambda_T})$  such that

$$x_{i,\lambda_i} = \lambda_i x_i + (1 - \lambda_i) x_{i-1,\lambda_{i-1}} \quad \text{and} \quad x_{0,\lambda_0} = x_0$$
 (2)

We put the augmented  $x_{\vec{\lambda}}$  in the same class with x, so  $(x_{\vec{\lambda}}, y)$  will be a new sample.

Case 2. Regression. For each  $\vec{\lambda} = (\lambda_1, \dots, \lambda_T) \in [0, 1]^T$ , let  $S_{\vec{\lambda}} = \{s_{0, \lambda_0}, \dots, s_{T, \lambda_T}\}$  where  $s_{i, \lambda_i} = (x_{i, \lambda_i}, y_{i, \lambda_i})$  such that

$$x_{i,\lambda_i} = \lambda_i x_i + (1 - \lambda_i) x_{i-1,\lambda_{i-1}} \quad \text{and} \quad y_{i,\lambda_i} = \lambda_i y_i + (1 - \lambda_i) y_{i-1,\lambda_{i-1}}. \tag{3}$$

**Definition of loss functions.** Let l be a loss function on each sample  $(x_i,y_i)$  and  $\theta$  the model parameter. Now consider a total loss function L on one realization  $\{(x_i,y_i)\}_{i\in[0:T]}$ . Then  $L(\{f_{\theta}(x_i),y_i\}_{i\in[0:T]}) = \sum_{i=0}^{T} l(f_{\theta}(x_i),y_i)$  where  $f_{\theta}$  is a function approximator (e.g. Neural Networks). Considering a total loss function L on the augmented data  $S_{\vec{\lambda}}$ , we see that the loss function L depends on  $\vec{\lambda}$ ,  $\theta$  and  $\{(x_{i,\lambda_i},y_{i,\lambda_i})\}_{i\in[0:T]}$ . Since  $L(\{f_{\theta}(x_{i,\lambda_i}),y_{i,\lambda_i}\}_{i\in[0:T]}\}) = \sum_{i=0}^{T} l(f_{\theta}(x_{i,\lambda_i}),y_{i,\lambda_i})$ , we denote by  $L_{\theta}(\vec{\lambda})$  as  $L(\{f_{\theta}(x_{i,\lambda_i}),y_{i,\lambda_i}\}_{i\in[0:T]}\})$ . Then  $L_{\theta}(\vec{0}) = L(\{f_{\theta}(x_i),y_i\}_{i\in[0:T]}\})$  is the total loss function on the original historical data. Depending on the type of problems, we use two loss functions:

**Regression loss.** We use mean squared error loss function. For each augmented sample  $s=(x_{i,\lambda_i},y_{i,\lambda_i})$ , let  $l(f_{\theta}(x_{i,\lambda_i}),y_{i,\lambda_i})=\|f_{\theta}(x_{i,\lambda_i})-y_{i,\lambda_i}\|_2^2$  and denote by  $l_{\theta}(\lambda_i)=l(f_{\theta}(x_{i,\lambda_i}),y_{i,\lambda_i})$ .

**Classification loss.** We use binary cross entropy loss function. For each augmented sample  $s=(x_{\vec{\lambda}},y_{\vec{\lambda}})$ , let  $l(f_{\theta}(x_{\vec{\lambda}}),y_{\vec{\lambda}})=y_{\vec{\lambda}}\log(f_{\theta}(x_{\vec{\lambda}}))+(1-y_{\vec{\lambda}})\log(1-f_{\theta}(x_{\vec{\lambda}}))$  and denote by  $l_{\theta}(\vec{\lambda})=l(f_{\theta}(x_{\vec{\lambda}}),y_{\vec{\lambda}})$ .

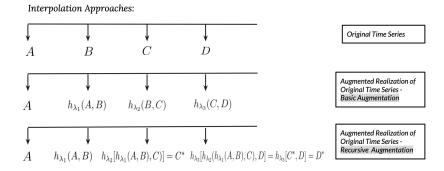


Figure 1: Intuition behind our Recursive Interpolation Method (RIM):  $h_{\lambda}(A, B) = (1 - \lambda)A + \lambda B$ .

In Figure 1 we always start with the same initial data point A. Points A, B, C, D, define our original time series. We denote by "Basic Augmentation" the usual interpolation technique for time series which is restricted by the Markovian assumption. However for RIM, we do consider all the lagged data points. Because we are not restricted to just one lagged value as in Basic Augmentation, our approach relaxes the Markovian assumption. Furthermore, our approach allows the information to flow through time thereby enabling us to handle any time series data without any underlying assumptions including non-stationarity.

# 4 Theoretical Framework for Recursive Interpolation Method

The three most important aspects of our theoretical framework are the characterization of our recursive time series augmentation (Section 4.1), the induction of regularization effects (Section 4.2), and better parameter learning with reduced variance (Section 4.3).

# 4.1 Connection Between Original Time Series and Augmented Time Series

We define the recursively interpolated time series and show that the augmented time series has some bound on distance with an original time series. Moreover, the bound depends on features of the time series. This basically means that the augmented time series does not deviate from the original one and preserves some information inherent about the original time series. Theorem 4.1 shows that RIM generates augmented time series' that are close to the original time series.

$$\mathrm{Let}\,\mathrm{sign}(t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } t > 0 \text{, } \delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \text{, and } \mathcal{D} \text{ is a distribution with support } [0,1).$$

**Theorem 4.1.** (Characterization of recursive augmentation) If  $\lambda_t \sim \mathcal{D}$  and  $g(\lambda_t) = (1 - \lambda_t)(1 - \delta_{0t}) + (1 - \operatorname{sign}(t))$ , then the following holds.

(1) 
$$x_{t,\lambda_t} = \sum_{k=0}^t (\prod_{i=k+1}^t \lambda_i) g(\lambda_k) x_k$$
 where  $\lambda_j \sim \mathcal{D}$  for  $j \geq 1$  and  $\lambda_0$  is a dummy value.

(2) Let 
$$\|\cdot\|$$
 be a norm,  $e = \mathbb{E}[\mathcal{D}]$ ,  $m' = \max_{i \in [1:t]} \{\|x_i - x_{i-1}\|\}$  and  $m = \max_{i \in [0:t]} \{\|x_i\|\}$ . Then 
$$\|\mathbb{E}_{\lambda_1, \dots, \lambda_t}[(x_{t, \lambda_t} - x_t)]\| \le \min\{(1 + e)m, \frac{e}{1 - e}m'\}. \tag{4}$$

# 4.2 The Regularzation Effect of the Recursive Time Series Augmentation

We treat regression and classification as two separate cases. In the regression case using neural networks constructed with an activation function such as ReLU, we see the clear relation between the total loss function on the original data and the total loss function on the augmented data. Theorem 4.2 shows regularization effects on updating parameters of

neural networks. Because the one realization is not enough to capture the true distribution of time series data, it would be difficult for the model to update its parameter so as to cause the total loss function on the original data to converge to zero. Compared to traditional regularization methods like  $l_2$ -regularization, our regularization term is naturally induced with the augmentation which preserves the original time series information.

**Theorem 4.2.** (Regularization effects in Regression) The regularized total loss function can be characterized as:

$$(2T+1)L_{\theta}(\vec{0}) \le L_{\theta}(\vec{\lambda}) \le (T+1)L_{\theta}(\vec{0}) + A, \quad \text{where}$$
(5)

$$A = \sum_{i=1}^{T} ((a_i f_{\theta}(x_{i-1,\lambda_{i-1}}) - y_{i-1,\lambda_{i-1}})^2) \text{ and}$$

$$a_i = \nabla f_{\theta}(x_i)^T (\nabla f_{\theta}(x_{i-1,\lambda_{i-1}}) \nabla f_{\theta}(x_{i-1,\lambda_{i-1}})^T)^{-1} \nabla f_{\theta}(x_{i-1,\lambda_{i-1}})$$
(6)

In particular, let

$$\theta \in \{\Theta \mid \text{For } i \in [0:T], f_{\theta}(x_i) = y_i \text{ and } f_{\theta}(x_{i-1,\lambda_{i-1}}) = y_{i-1,\lambda_{i-1}}\}$$
 (7)

which means  $\theta$  is an optimal parameter for both  $L_{\theta}(\vec{0})$  and  $L_{\theta}(\vec{\lambda})$ . Then

$$L_{\theta}(\vec{\lambda}) = \sum_{i=1}^{T} ((a_i - 1) f_{\theta}(x_{i-1,\lambda_{i-1}}))^2.$$
 (8)

For the classification problem, Theorem 4.3 reveals that there are also natural regularization effects on the parameter update induced by our recursive augmentation.

**Theorem 4.3.** (Regularization effects in Classification) The regularized total loss function can be characterized as:

$$L_{\theta}(\vec{0}) - \sum_{i=1}^{T} \left( |a_i| + \frac{17}{8} a_i^2 + \frac{1}{2} |b_i| \right) \le L_{\theta}(\vec{\lambda}) \le L_{\theta}(\vec{0}) + \sum_{i=1}^{T} \left( |a_i| + \frac{5}{2} a_i^2 + \frac{1}{2} |b_i| \right), \quad \text{where}$$
 (9)

$$a_i = (x_{i-1,\lambda_{i-1}} - x_i)^T \nabla g_{\theta}(x_i)$$
 and  $b_i = (x_{i-1,\lambda_{i-1}} - x_i)^T \nabla^2 g_{\theta}(x_i) (x_{i-1,\lambda_{i-1}} - x_i)$ . (10)

$$|b_{i}| = |(x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla^{2} g_{\theta}(x_{i}) (x_{i-1,\lambda_{i-1}} - x_{i})| \le ||x_{i-1,\lambda_{i-1}} - x_{i}||^{2} \max |\sigma_{j}|$$
and  $|a_{i}| \le ||x_{i-1,\lambda_{i-1}} - x_{i}|| ||\nabla g_{\theta}(x_{i})||,$ 

$$(11)$$

where  $\{\sigma_j\}$  is the set of eigenvalues of  $\nabla^2 g_{\theta}(x_i)$  and  $g_{\theta}(x_i)$  is the preactivation used to compute  $f_{\theta}(x_i)$ . We see that the bound depends on the deviation  $\|x_{i-1,\lambda_{i-1}} - x_i\|$ 's and parameters of a function  $g_{\theta}$ . The terms  $a_i$  and  $b_i$  are similar to the computer vision technique of noise injection or adding perturbations as they are added to the original time series so as to overcome overfitting.

Since learning using only one realization of the time series might lead to overfitting, the above two theorems 4.2 and 4.3 show that our recursive time series augmentation method naturally induces regularization effects that can help to prevent overfitting.

# 4.3 Effect Derived by the Recursive Time Series Augmentation with Loss-Averaging

We investigate the Rademacher complexity of our method. Using the augmented data and average loss function induced by the augmented data, we see that the rate of convergence of the average augmented loss function is faster than the rate of convergence of the non-augmented loss function. We assume that the parameter space is large (Assumption 2 in Appendix A.5) which is reasonable because of the Universal Approximation Theorem. This means that the parameter space is big enough to generate a function which outputs the values that we want to get for finite samples. There is the true sample space  $S_{True}$ , which is unknown. We assume that the sample space  $S = \bigcup_{i=0}^{T} S_{(x_i,y_i)} \subseteq S_{True}$ . There is the true distribution  $P_{\theta_*}$  on the true sample space  $S_{True}$ . We have one realization from the true distribution  $P_{\theta_*}$  and

this is  $\{(x_i, y_i)\}_{i \in [0:T]}$ . We set:

$$\theta_{*} = \underset{\theta}{\operatorname{argmin}} \mathbb{E}[l(f_{\theta}(x), y)]$$

$$\theta_{n} = \underset{\theta}{\operatorname{argmin}} \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_{i}), y_{i})$$

$$\theta_{\operatorname{aug}} = \underset{\theta}{\operatorname{argmin}} \mathbb{E}[\mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}]]$$

$$\theta_{\operatorname{aug},n} = \underset{\theta}{\operatorname{argmin}} \frac{1}{n+1} \sum_{i=0}^{n+1} \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x_{i}, y_{i})}],$$

$$(12)$$

$$\bar{l}(f_{\theta}(x), y) = \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}] \text{ and } \mathcal{R}_n(l \circ \Theta) = \mathbb{E}\left[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n \epsilon_i l(f_{\theta}(x), y) \right| \right]$$
(13)

**Theorem 4.4.** (Augmented optimal parameter bound with  $\bar{l}$ ) Let l be a loss function such that  $l(\cdot, \cdot) \in [0, 1]$ . Then with probability at least  $1 - \delta$  over the samples  $\{(x_i, y_i)\}_{i \in [0:T]}$ , we have

$$\mathbb{E}[l(f_{\theta_{\text{aug},n}}(x), y)] - \mathbb{E}[l(f_{\theta_*}(x), y))] < 2\mathcal{R}_n(\bar{l} \circ \Theta) + \sqrt{\frac{2\log(2/\delta)}{n+1}}.$$
(14)

With Assumption 2, we have

$$\mathcal{R}_n(\bar{l} \circ \Theta) \le \mathcal{R}_n(l \circ \Theta) \tag{15}$$

Theorem 4.5 describes the asymptotic behaviour of the learning parameters. We show that our recursive time series augmentation reduces the variance of the learning parameters.

**Theorem 4.5.** (Asymptotic normality and variance reduction) We assume the regularity of both the population risk minimizer and the loss function (Assumptions 3 and 4 in Appendix A.6), and assume  $\Theta$  is open  $S_{True} = S$ , then  $\theta_n$  and  $\theta_{\text{aug},n}$  admit the following Bahadur representation;

$$\sqrt{n+1}(\theta_n - \theta_*) = \frac{1}{\sqrt{n+1}} V_{\theta_*}^{-1} \sum_{i=0}^n \nabla l(f_{\theta_*}(x_i), y_i) + o_{\mathcal{P}}(1)$$

$$\sqrt{n+1}(\theta_{\text{aug},n} - \theta_*) = \frac{1}{\sqrt{n+1}} V_{\theta_*}^{-1} \sum_{i=0}^n \nabla \bar{l}(f_{\theta_*}(x_i), y_i) + o_{\mathcal{P}}(1)$$
(16)

Therefore, both  $\theta_n$  and  $\theta_{\text{aug},n}$  are asymptotically normal

$$\sqrt{n+1}(\theta_n - \theta_*) \to N(0, \Sigma_0)$$
 and  $\sqrt{n+1}(\theta_{\text{aug},n} - \theta_*) \to N(0, \Sigma_{\text{aug}})$  (17)

where the covariance is given by

$$\Sigma_{0} = V_{\theta_{*}}^{-1} \mathbb{E}[\nabla l(f_{\theta_{*}}(x), y) \nabla l(f_{\theta_{*}}(x), y)^{T}] V_{\theta_{*}}^{-1}$$

$$\Sigma_{\text{aug}} = \Sigma_{0} - V_{\theta_{*}}^{-1} \mathbb{E}[\nabla \bar{l}(f_{\theta_{*}}(x), y) \nabla \bar{l}(f_{\theta_{*}}(x), y)^{T}] V_{\theta_{*}}^{-1}$$
(18)

As a consequence, the asymptotic relative efficiency of  $\theta_{\text{aug},n}$  compared to  $\theta_n$  is  $\text{RE} = \frac{tr(\Sigma_0)}{tr(\Sigma_{\text{aug}})} \geq 1$ .

### **Algorithm 1:** RIM: Classification Training

```
input :Observed time series data S = \{s_0, s_1, \dots, s_N\} where s_i = (x_i, y_i) for x_i \in \mathbb{R}^{T+1} and y_i \in \{0, 1\}.
 1 Initialize \theta parameter for a neural network, Y epochs, and a
     distribution \mathcal{D} with support [0,1).
    for e = 1 to Y/2 do
           Initialize \vec{\lambda} = (\lambda_1, \dots, \lambda_T) with \lambda_i \sim \mathcal{D} and L = 0 //
           Initialize interpolation coefficients vector and the total loss
           function L
 3
           for i = 0 to N do
                  Sample s = (x, y) uniformly from S \cup S_{\vec{x}} where
                     S_{\vec{\lambda}} = \{s_{0,\vec{\lambda}}, s_{1,\vec{\lambda}}, \dots, s_{N,\vec{\lambda}}\}
                    l(f_{\theta}(x), y) = (1 - y) \log(1 - f_{\theta}(x)) + y \log f_{\theta}(x)
                    // Error on augmented states L = L + l(f_{\theta}(x), y) #
                     Update total loss
 5
           \theta = \theta - \alpha \nabla_{\theta} L_{\theta}(\vec{\lambda}) where L_{\theta}(\vec{\lambda}) = L
 6
7
    end
    for e = Y/2 to Y do
 8
           Initialize L = 0 // Initialize total loss L
           for i = 0 to N do
10
11
                  l(f_{\theta}(x_i), y_i) =
                    (1-y_i)\log(1-f_{\theta}(x_i))+y_i\log f_{\theta}(x_i) // Error
                     on augmented states L = L + l(f_{\theta}(x_i), y_i) \#
                     Update total loss
12
           \theta = \theta - \alpha \nabla_{\theta} L_{\theta}(\vec{\lambda}) \text{ where } L_{\theta}(\vec{\lambda}) = L
13
14 end
```

### **Algorithm 2:** RIM: Regression Training

```
input: Observed time series data S = \{s_0, s_1, \dots, s_T\} where
                 s_i = (x_i, y_i) for x_i \in \mathbb{R}^d and y_i \in \mathbb{R} for i \in [0:T].
 1 Initialize \theta parameter for a neural network, Y epochs, and a
     distribution \mathcal{D} with support [0,1).
    for e = 1 to n do
           Initialize L = 0 // the total loss function L
           for i = 0 to T do
 3
                  L = L + l(f_{\theta}(x_i), y_i) where
                     l(f_{\theta}(x_i), y_i) = ||f_{\theta}(x_i) - y_i||_2^2
 4
           \theta = \theta - \alpha \nabla_{\theta} L_{\theta}(\vec{\lambda}) \text{ where } L_{\theta}(\vec{\lambda}) = L
 5
 6 end
 7 for e = n + 1 to Y do
           Initialize \vec{\lambda} = (\lambda_1, \dots, \lambda_T) with \lambda_i \sim \mathcal{D} and
            L = l(f_{\theta}(x_0), y_0) = ||f_{\theta}(x_0) - y_0||_2^2 // \text{Initialize}
            interpolation coefficients vector and the total loss function \boldsymbol{L}
           Augmented Path Simulator Generate an augmented
           trajectory S_{\vec{\lambda}} = \{s_0, s_{1,\lambda_1}, \dots, s_{T,\lambda_T}\} where
            s_{i,\lambda_i} = (x_{i,\lambda_i}, y_{i,\lambda_i})
           for i = 1 to T do
                  L = L + l(f_{\theta}(x_{i,\lambda_i}), y_{i,\lambda_i})
 9
                     l(f_{\theta}(x_{i,\lambda_{i}}), y_{i,\lambda_{i}}) = ||f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}||_{2}^{2} / 
                     Update total loss
10
           \theta = \theta - \alpha \nabla_{\theta} L_{\theta}(\vec{\lambda}) where L_{\theta}(\vec{\lambda}) = L
11
12 end
```

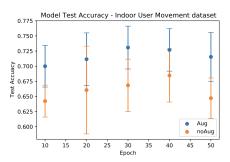
# 5 Experiments

We performed experiments in five application areas including classification, regression, and reinforcement learning to demonstrate improved generalization learning performance. The pseudocodes 1 and 2 show the RIM training methodology for classification and for regression problems (pseudocode for RL can be found in Appendix C). For classification where we have limited but labeled time series, we first use augmented sample to learn the extrinsic features of the underlying system and then train it on the original data to learn the intrinsic features. For regression, we could have first used augmented sample for training as in classification. However, since in regression we only have one realization, it is not sufficient to learn the intrinsic features. As a result, in regression, we first train the ML model using the historical original time series to learn the intrinsic features and then use augmented sample to learn the extrinsic features for a better generalization.

Classification Task: Indoor User Movement from the Radio Signal Strength (RSS) Data. This binary classification task from [17] is associated with predicting the pattern of user movements in real world office environments from time series generated by a Wireless Sensor Network (WSN). The input data contains RSS measured between the nodes of a WSN, comprising of 5 sensors: 4 in the environment and 1 for the user. Data has been collected during movement of the users and labelled to indicate whether the user's trajectory will lead to a change in the room or not. For our experiments, we use a subset of the data to form a small training set to challenge our algorithm. We achieved better and more robust accuracy with smaller test loss when using augmented data as reflected in Figure 2.

Classification Task: Ford Engine Condition. For this classification task, we use a subset of the FordA dataset from 2008 WCCI Ford classification challenge [18]. This dataset contains time series corresponding to measurements of engine noise captured by a motor sensor. The goal is to detect the presence of a specific issue with the engine by classifying each time series into issue/no issue classes. We sample 100 time series from FordA to form a small training set to challenge our algorithm and 100 time series for testing. As shown in Figure 3, the augmentation trained accuracy is better than the non augmentation accuracy only after 40 epochs. The variance of the test loss decreases dramatically as the number of epochs increases which corroborates with our theoretical results.

Regression Task: Predicting Stock Price Movement. This regression task consists of predicting the next day SPY500 index Open price from historical SPY500 index using data from July 2008 to December 2012 as training data and data from January 2013 to March 2014 as testing data. The input data contains the last 7 days' historical Open, Close, High, Low, and Volume of the SPY500 index. After predicting the next day's Open price, we take a long position if the predicted next day Open is larger than today's Open, short otherwise. On comparing the results on the test data for



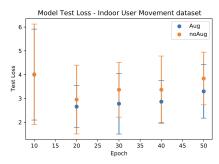
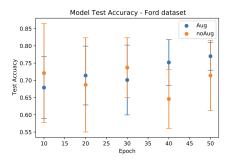


Figure 2: Test Accuracy and Test Loss for the Indoor User Movement Classification using a Convolutional Neural Network with kernel size=3, filter=32, batch size=16, using BatchNorm and Adam optimizer. The plots indicate the resulting mean  $\pm$  standard deviation from 10 runs.



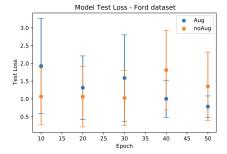
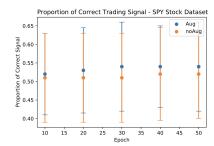
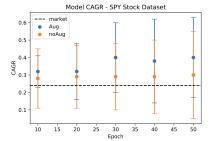


Figure 3: Test Accuracy and Test Loss for the Ford Engine Classification with a Convolutional Neural Network with kernel size=3, filter=32, batch size=16, using BatchNorm and Adam optimizer. The plots indicate the resulting mean  $\pm$  standard deviation from 10 runs.

the augmented case and the original case, we observe that the proportion of profitable trading signals is higher in the augmentation-trained model as observed in Figure 4. Using these trading signals, we also calculate the trading system's CAGR (Compound Annual Growth Rate) which we observe to be higher in the augmentation trained model. The test loss plot shows that the MSE for the augmentation trained model is consistently lower than the non-augmentation trained model.





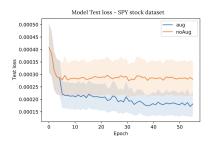
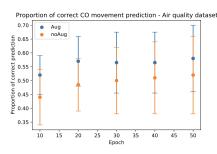


Figure 4: Profitable trading signals (left), test set CAGR (middle), test set MSE (right) for the SPY500 Dataset using an LSTM model with 2 LSTM layers (200 neurons), 2 dense layers (100 neurons), lr=1e-4, batch size=16. The plots indicate resulting mean  $\pm$  standard deviation from 10 runs.

**Regression Task: Predicting Air Quality.** The restricted air quality dataset contains 1200 instances of hourly averaged responses from an array of 5 metal oxide chemical sensors. This is a time series regression task where the target is the next time step's CO concentration. The input data contains the last six time steps' 10 features as used in [19] and [20]. Figure 5 shows that the test MSE of the augmentation trained model remains lower than the non-augmentation trained model during the training epochs validating our claims about the robustness of the approach. Accordingly, the proportion of correct predictions of CO up/down also remains higher for the augmented case.

**Reinforcement Learning Task: Portfolio Management.** We identified financial portfolio management as a case study where we have multiple 1D-TS as input. [21, 22, 23] summarize the state-of-the-art RL models in finance. Based



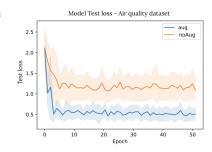


Figure 5: Test accuracy (left) and Test MSE (right) for the Air Quality Dataset using an LSTM model with 2 LSTM layers (200 neurons), 2 dense layers (100 neurons), lr=1e-4, and batch size=16. The plots indicate the resulting mean  $\pm$  standard deviation from 10 runs.

on the work of [23], which indicated that policy-based RL works better than value-based RL for portfolio management, we studied two policy-based RL algorithms, namely deterministic policy gradient (DPG) and Deep Deterministic Policy Gradient (DDPG), to demonstrate the benefits of our recursive time series augmentation approach which can be used in conjunction with any RL algorithm. Our RL models are trained over 600 trading days from 2010-08-09 to 2012-12-25 and tested over 200 trading days from 2012-12-26 to 2013-10-11 on a portfolio consisting of ten selected stocks. To promote diversification of the portfolio, these stocks were selected from different sectors of S&P 500 so that they are uncorrelated as much as possible as shown in Figure 9 in appendix C. Figure 6 shows the RL performance in terms of the cumulative total return for both training and testing. For both agents, our recursive augmentation leads to higher cumulative total return.

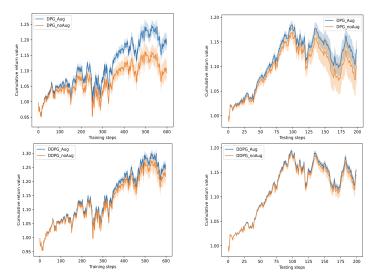


Figure 6: Training and testing results for DPG (above) and DDPG (below). The plots indicate the resulting mean  $\pm$  standard deviation from 20 runs with different seeds.

# 6 Conclusion

We developed a Recursive Interpolation Method (RIM) for time series as a data augmentation technique to learn accurate models with limited data. The RIM is simple yet effective supported by theoretical analysis guaranteeing faster convergence. Specifically, we proved that the RIM has regularization effects, and guarantees better parameter convergence with reduced variance. Empirically, we demonstrated that our methodology is promising, as we obtain substantial improvements in classification, regression, and reinforcement learning tasks in different real world problem domains. Because our approach operates on the input time series data, it is invariant to the choice of the ML algorithm. The methodology described in this paper can be used to enhance ML solutions to a wide variety of time series learning problems.

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# **Appendix**

# A Proofs

Appendix A contains proofs of the theorems mentioned in the paper and additional theoretical results.

# A.1 Proof of Theorem 4.1

*Proof.* (1) Note that  $\lambda_0$  is a dummy value for mathematical convenience and  $\prod_{i=k+1}^n \lambda_i = 1$  if n < k+1. We prove this by induction on n. The case n=1 shows that  $x_{1,\lambda_1} = (1-g(\lambda_1))x_1 + \lambda_1 x_0 = (1-\lambda_1)x_1 + \lambda_1 x_0$  since  $g(\lambda_0) = 0$  and  $g(\lambda_1) = 1 - \lambda_1$ . We now assume that the inequality holds for n-1 and prove it for n. By construction of  $x_{n,\lambda_n}$ ,

$$\begin{aligned} x_{n,\lambda_n} &= (1 - \lambda_n) x_n + \lambda_n x_{n-1,\lambda_{n-1}} \\ &= (1 - \lambda_n) x_n + \lambda_n \sum_{k=0}^{n-1} \left( \prod_{i=k+1}^{n-1} \lambda_i \right) g(\lambda_k) x_k \\ &= \left( \prod_{i=n+1}^n \lambda_i \right) g(\lambda_n) x_n + \sum_{k=0}^{n-1} \left( \prod_{i=k+1}^{n+1} \lambda_i \right) g(\lambda_k) x_k \\ &= \sum_{k=0}^{n+1} \left( \prod_{i=k+1}^{n+1} \lambda_i \right) g(\lambda_k) x_k \end{aligned}$$

Thus (1) holds by mathematical induction.

(2) Consider the equation  $||x_{n,\lambda_n} - x_n||$ . Then by (1), the first bound can be found by

$$\|\mathbb{E}[(x_{n,\lambda_n} - x_n)]\| = \|\mathbb{E}[\sum_{k=0}^n (\prod_{i=k+1}^n \lambda_i) g(\lambda_k) x_k - x_n]\|$$

$$= \|\mathbb{E}[\sum_{k=0}^{n-1} (\prod_{i=k+1}^n \lambda_i) g(\lambda_k) x_k - \lambda_n x_n]\|$$

$$= \|\sum_{k=0}^{n-1} e^{n-k-1} (1-e) x_k - e x_n\|$$

$$\leq \sum_{k=0}^{n-1} e^{n-k-1} (1-e) \|x_k\| + e \|x_n\|$$

$$\leq (\sum_{k=0}^{n-1} e^{n-k-1} - \sum_{k=0}^{n-1} e^{n-k} + e) m \leq (1+e) m$$

where  $e = \mathbb{E}[\mathcal{D}]$  and  $m = \max_{i \in [0:n]} \{ ||x_i|| \}$ .

Now we prove the second bound. Since

$$||x_{n,\lambda_n} - x_n|| = ||x_n - ((1 - \lambda_n)x_n + \lambda_n x_{n-1,\lambda_{n-1}})||$$

$$= \lambda_n ||x_n - x_{n-1} + x_{n-1} - x_{n-1,\lambda_{n-1}}||$$

$$\leq \lambda_n (||x_n - x_{n-1}|| + ||x_{n-1} - x_{n-1,\lambda_{n-1}}||),$$
(19)

By recursively applying equation 19, we obtain

$$||x_{n,\lambda_n} - x_n|| \le \sum_{k=1}^n (\prod_{i=k}^n \lambda_i) ||x_k - x_{k-1}||.$$
(20)

By Jensen's inequality and equation 20, we obtain the second bound

$$\|\mathbb{E}[(x_{n,\lambda_{n}} - x_{n})]\| \leq \mathbb{E}[\|x_{n,\lambda_{n}} - x_{n}\|]$$

$$\leq \mathbb{E}[\sum_{k=1}^{n} (\prod_{i=k}^{n} \lambda_{i}) \|x_{k} - x_{k-1}\|]$$

$$\leq \sum_{k=1}^{n} e^{n-k+1} m' \leq \frac{e}{1-e} m'$$
(21)

where  $e = \mathbb{E}[\mathcal{D}]$  and  $m' = \max_{i \in [1:n]} \{ ||x_i - x_{i-1}|| \}$ .

For Theorems 4.2 and 4.3, we address the problems given that the whole time series data is  $(p_0,\ldots,p_H)$  without labels and where H is the horizon of the time series. We first, decompose the whole sequence into several pieces of data with window size w as follows. Let  $p_0 \in \mathbb{R}^d$  and  $w \in \mathbb{N}$ . Then  $x_0 = (p_0,\ldots,p_{w-1}) \in \mathbb{R}^{d\times w}$  and  $y_0 = \text{label}(p_0,\ldots,p_{w-1},p_w)$ . Similarly, with a time step t, the ith sample is defined as follows  $x_i = (p_{ti},\ldots,p_{ti+w-1})$  and  $y_i = \text{label}(p_{ti},\ldots,p_{ti+w-1},p_{ti+w})$ . Note that the label function determines labels based on types of problems (regression problems or classification problems). Now we have decomposed the time series data  $S = \{s_0,s_1,\ldots,s_T\}$  such that  $s_i = (x_i,y_i)$  where  $x_i \in \mathbb{R}^{d\times w}$  and  $y_i \in \mathbb{R}$ . The goal is to find a predictor or a classifier f for data f. Just using this fixed historical data will lead to an overfitting problem. So, we generate an augmented time series f0 and f1. The series f2 and f3 are f4 and f5 and f5 are f6 and f7 and f8. The goal is to find a predictor or a classifier f6 for data f7. The series f8 are f9 and f9 are f9 and f

**Binary Trend Prediction.**  $y_{i,\lambda_i}$  is defined as follows. For each  $\vec{\lambda} = (\lambda_1, \dots, \lambda_T) \in [0,1]^T$ , we define the augmented data  $S_{\vec{\lambda}} = \{s_{0,\lambda_0}, \dots, s_{T,\lambda_T}\}$  where  $s_{i,\lambda_i} = (x_{i,\lambda_i}, y_{i,\lambda_i})$  such that

$$x_{i,\lambda_i} = (1 - \lambda_i)x_i + \lambda_i x_{i-1,\lambda_{i-1}} \text{ and } y_{i,\lambda_i} = \begin{cases} y_i & \text{if } \lambda_i < \frac{1}{2} \\ y_{i-1,\lambda_{i-1}} & \text{if } \lambda_i \ge \frac{1}{2} \end{cases}$$
 (22)

Note that  $S_{\vec{0}} = D$ . In terms of  $y_{i,\lambda_i}, y_i = 1$  and  $y_{i-1,\lambda_{i-1}} = 1$  implies  $y_{i,\lambda_i} = 1$ . Similarly,  $y_i = 0$  and  $y_{i-1,\lambda_{i-1}} = 0$  implies  $y_{i,\lambda_i} = 0$ . So this basically means that if the data is augmented with data whose trends are the same (go up or go down), then the result also has the same trend. In the case  $y_i \neq y_{i-1,\lambda_{i-1}}$ , we set the label as follows. If  $\lambda_i < \frac{1}{2}$ , then  $y_{i,\lambda_i} = y_i$ . Otherwise,  $y_{i,\lambda_i} = y_{i-1,\lambda_{i-1}}$ . Intuitively,  $\lambda_i$  controls the degree to which each sample influences the augmented sample. So,  $\lambda_i < \frac{1}{2}$  means  $y_i$  affects the augmented sample more than  $y_{i-1,\lambda_{i-1}}$  due to the form of augmentation.

**Regression.**  $y_{i,\lambda_i}$  is recursively defined. For each  $\vec{\lambda}=(\lambda_1,\ldots,\lambda_T)\in[0,1]^T$ , we define the augmented data  $S_{\vec{\lambda}}=\{s_{0,\lambda_0},\ldots,s_{T,\lambda_T}\}$  where  $s_{i,\lambda_i}=(x_{i,\lambda_i},y_{i,\lambda_i})$  such that

$$x_{i,\lambda_i} = (1 - \lambda_i)x_i + \lambda_i x_{i-1,\lambda_{i-1}} \text{ and } y_{i,\lambda_i} = (1 - \lambda_i)y_i + \lambda_i y_{i-1,\lambda_{i-1}}$$
(23)

For both cases, binary trend prediction and regression, we regard the augmented data as one of the possible realizations. Hence whenever  $x_{i,\lambda_i}$  is observed, we assume that  $x_{j,\lambda_j}$  is constant for j < i not depending on  $\lambda_j$ . Theorems 4.2 and 4.3 follow the assumption that  $x_{i,\lambda_i}$  only depends on  $\lambda_i$ .

**Example A.1.** We observe univariate (d=1) time series sequence (1,3,5,7,10,2). Set up window size w as 3 and time step t as 1. Then the whole sequence is decomposed into the set  $S=\{s_0,s_1,s_2\}$  such that  $s_0=(x_0,y_0),s_1=(x_1,y_1)$  and  $s_2=(x_2,y_2)$  where  $x_0=(1,3,5),x_1=(3,5,7)$  and  $x_2=(5,7,10)$ . Depending on the type of problems, labels are determined in a different way. For a regression problem,  $y_0=7,y_1=10$  and  $y_2=2$ . For a binary trend prediction problem,  $y_0=(\sin(7-5)+1)/2=1,y_1=(\sin(10-7)+1)/2=1,$  and  $y_2=(\sin(2-10)+1)/2=0$ . For  $(0.5,0.2)\in[0,1]^2$ , we generate augmented data  $S_{(0.5,0.2)}=\{s_0,s_{1,\lambda_1},s_{2,\lambda_2}\}$  such that  $x_{1,0.5}=(0.5)x_1+(0.5)x_0=(2,4,6)$  and  $x_{2,0.2}=(0.7)x_2+(0.3)x_1=(2.4,4.4,6.4)$ . If we want to solve a regression problem, then  $y_1=(0.5)y_1+(0.5)y_0=8.5$  and  $y_2=(0.7)y_2+(0.3)y_1=4.4$ . If we want to solve a binary trend prediction problem, which means finding a predictor for the trend (go up or go down),  $y_1=(0.5)y_1+(0.5)y_0=1$  and  $y_2=(0.7)y_2+(0.3)y_1=y_2=0$ .

Since a total loss function L is a differential function on each  $\lambda$ ,

$$L_{\theta}(\vec{\lambda}) = L_{\theta}(\vec{0}) + \sum_{i=1}^{T} \frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}} \bigg|_{\vec{\lambda} = \vec{0}} \lambda_{i} + \frac{1}{2} \sum_{i,j} \frac{\partial^{2} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i} \partial \lambda_{j}} \bigg|_{\vec{\lambda} = \vec{0}} \lambda_{i} \lambda_{j} + r(\vec{\lambda}) \|\vec{\lambda}\|^{2}$$
(24)

where  $\lim_{\|\vec{\lambda}\|\to 0} r(\vec{\lambda}) = 0$ . We see that  $\frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_i}$  is not dependant on  $\lambda_j$  where  $j \neq i$ . Thus  $\frac{\partial^2 L_{\theta}(\vec{\lambda})}{\partial \lambda_i \lambda_j} = 0$  whenever  $i \neq j$ . Hence, under the assumption that  $L_{\theta}(\vec{\lambda})$  is infinitely differentiable, we get

$$L_{\theta}(\vec{\lambda}) = L_{\theta}(\vec{0}) + \sum_{j=1}^{\infty} \frac{1}{j!} \sum_{i=1}^{T} \frac{\partial^{j} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}^{j}} \bigg|_{\vec{\lambda} = \vec{0}} \lambda_{i}^{j}$$
(25)

Mean Squared Error Loss. Consider mean squared error loss function  $l(s_{i,\lambda_i})$ . Then  $L_{\theta}(\vec{\lambda}) = \sum_{i=1}^{T} l(s_{i,\lambda_i})$  where  $l(s_{i,\lambda_i}) = \|f_{\theta}(x_{i,\lambda_i}) - y_{i,\lambda_i}\|_2^2$ . Since we assume that  $x_{i,\lambda_i}$  and  $y_{i,\lambda_i}$  depend only on  $\lambda_i$ , then  $l(s_{i,\lambda_i})$  only depends on  $\lambda_i$ . Hence

$$\frac{\partial L_{\theta}(\lambda)}{\partial \lambda_{i}} = 2\left(\frac{\partial f_{\theta}(x_{i,\lambda_{i}})}{\partial \lambda_{i}} - \frac{\partial y_{i,\lambda_{i}}}{\partial \lambda_{i}}\right) \left(f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}\right) 
= 2\left(\frac{\partial (x_{i,\lambda_{i}})}{\partial \lambda_{i}} \frac{\partial f_{\theta}(x_{i,\lambda_{i}})}{\partial (x_{i,\lambda_{i}})} - (y_{i-1,\lambda_{i-1}} - y_{i})\right) \left(f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}\right) 
= 2\left(\left(x_{i-1,\lambda_{i-1}} - x_{i}\right)^{T} \nabla f_{\theta}(x_{i,\lambda_{i}}) - (y_{i-1,\lambda_{i-1}} - y_{i})\right) \left(f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}\right)$$
(26)

and

$$\frac{\partial^{2} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}^{2}} = \frac{\partial}{\partial \lambda_{i}} 2((x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i,\lambda_{i}}) - (y_{i-1,\lambda_{i-1}} - y_{i}))(f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}) 
= 2(x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla^{2} f_{\theta}(x_{i,\lambda_{i}})(x_{i-1,\lambda_{i-1}} - x_{i})(f_{\theta}(x_{i,\lambda_{i}}) - y_{i,\lambda_{i}}) + 
2((x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i,\lambda_{i}}) - (y_{i-1,\lambda_{i-1}} - y_{i}))^{2}$$
(27)

**Binary Cross Entropy Loss.** Consider a binary cross entropy for a loss function. Then  $L_{\theta}(\vec{\lambda}) = \sum_{i=1}^{T} l(s_{i,\lambda_i})$  where  $l(s_{i,\lambda_i}) = y_{i,\lambda_i} \log(f_{\theta}(x_{i,\lambda_i})) + (1-y_{i,\lambda_i}) \log(1-f_{\theta}(x_{i,\lambda_i}))$ . Since we assume that  $x_{i,\lambda_i}$  and  $y_{i,\lambda_i}$  are constant, only  $l(s_{i,\lambda_i})$  depends on  $\lambda_i$ . Hence

$$\frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}} = y_{i,\lambda_{i}} \frac{\partial \log(f_{\theta}(x_{i,\lambda_{i}}))}{\partial \lambda_{i}} + (1 - y_{i,\lambda_{i}}) \frac{\partial \log(1 - f_{\theta}(x_{i,\lambda_{i}}))}{\partial \lambda_{i}} 
= y_{i,\lambda_{i}} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{i}} \frac{\partial \log(f_{\theta}(x_{i,\lambda_{i}}))}{\partial (x_{i,\lambda_{i}})} + (1 - y_{i,\lambda_{i}}) \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{i}} \frac{\partial \log(1 - f_{\theta}(x_{i,\lambda_{i}}))}{\partial (x_{i,\lambda_{i}})} 
= \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{i}} (y_{i,\lambda_{i}} \nabla \log(f_{\theta}(x_{i,\lambda_{i}})) + (1 - y_{i,\lambda_{i}}) \nabla \log(1 - f_{\theta}(x_{i,\lambda_{i}}))) 
= (x_{i-1,\lambda_{i-1}} - x_{i})^{T} (\nabla \log(1 - f_{\theta}(x_{i,\lambda_{i}})) + y_{i,\lambda_{i}} \frac{\nabla f_{\theta}(x_{i,\lambda_{i}})}{f_{\theta}(x_{i,\lambda_{i}})(1 - f_{\theta}(x_{i,\lambda_{i}}))}) 
= (x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i,\lambda_{i}}) (\frac{-f_{\theta}(x_{i,\lambda_{i}}) + y_{i,\lambda_{i}}}{(1 - f_{\theta}(x_{i,\lambda_{i}}))f_{\theta}(x_{i,\lambda_{i}})})$$
(28)

and

$$\frac{\partial^{2} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}^{2}} = \frac{\partial}{\partial \lambda_{i}} (x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i,\lambda_{i}}) (\frac{-f_{\theta}(x_{i,\lambda_{i}}) + y_{i,\lambda_{i}}}{(1 - f_{\theta}(x_{i,\lambda_{i}})) f_{\theta}(x_{i,\lambda_{i}})}) 
= (x_{i-1,\lambda_{i-1}} - x_{i})^{T} \frac{\partial}{\partial \lambda_{i}} \nabla f_{\theta}(x_{i,\lambda_{i}}) (\frac{-f_{\theta}(x_{i,\lambda_{i}}) + y_{i,\lambda_{i}}}{(1 - f_{\theta}(x_{i,\lambda_{i}})) f_{\theta}(x_{i,\lambda_{i}})}) 
= (x_{i-1,\lambda_{i-1}} - x_{i})^{T} (\nabla^{2} f_{\theta}(x_{i,\lambda_{i}}) (\frac{-f_{\theta}(x_{i,\lambda_{i}}) + y_{i,\lambda_{i}}}{(1 - f_{\theta}(x_{i,\lambda_{i}})) f_{\theta}(x_{i,\lambda_{i}})}) + 
+ \frac{\nabla f_{\theta}(x_{i,\lambda_{i}}) \nabla f_{\theta}(x_{i,\lambda_{i}})^{T} (f_{\theta}(x_{i,\lambda_{i}})^{2} + f_{\theta}(x_{i,\lambda_{i}}) - 1)}{(1 - f_{\theta}(x_{i,\lambda_{i}}))^{2} f_{\theta}(x_{i,\lambda_{i}})^{2}} ) (x_{i-1,\lambda_{i-1}} - x_{i}).$$
(29)

Neural Networks with ReLU Activations. Using ReLU activation functions, neural networks are constructed by piecewise linear functions of an input. This implies that for a neural network  $f_{\theta}$ ,  $f_{\theta}(x) = \nabla f_{\theta}(x)^T x + b$  where x is an input,  $\nabla f_{\theta}(x)$  is the gradient of  $f_{\theta}(x)$  along x,  $\nabla^2 f_{\theta}(x) = 0$ , and b is a bias. We derive the relationship between  $L_{\theta}(\vec{\lambda})$  and  $L_{\theta}(\vec{0})$  for ReLU Networks. This relationship remains the same if we use linear activation functions or CNNs with max pooling.

### A.2 Proof of Theorem 4.2

Proof. Note that

$$(x_{i-1,\lambda_{i-1}} - x_i)^T \nabla f_{\theta}(x_i) - (y_{i-1,\lambda_{i-1}} - y_i) = \nabla f_{\theta}(x_i)^T (x_{i-1,\lambda_{i-1}} - x_i) - y_{i-1,\lambda_{i-1}} + y_i)$$

$$= -\nabla f_{\theta}(x_i)^T x_i + y_i + \nabla f_{\theta}(x_i)^T x_{i-1,\lambda_{i-1}} - y_{i-1,\lambda_{i-1}}$$

$$= -(f_{\theta}(x_i) - y_i) + \nabla f_{\theta}(x_i)^T (\nabla f_{\theta}(x_{i-1,\lambda_{i-1}}) \nabla f_{\theta}(x_{i-1,\lambda_{i-1}})^T)^{-1} \cdot$$

$$\nabla f_{\theta}(x_{i-1,\lambda_{i-1}}) \nabla f_{\theta}(x_{i-1,\lambda_{i-1}})^T x_{i-1,\lambda_{i-1}} + b - y_{i-1,\lambda_{i-1}}$$

$$= -(f_{\theta}(x_i) - y_i) + a f_{\theta}(x_{i-1,\lambda_{i-1}}) - y_{i-1,\lambda_{i-1}}$$

Since

$$L_{\theta}(\vec{\lambda}) = L_{\theta}(\vec{0}) + \sum_{i=1}^{T} \frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i} + \frac{1}{2} \sum_{i=1}^{T} \frac{\partial^{2} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}^{2}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i}^{2}$$

$$= L_{\theta}(\vec{0}) + \sum_{i=1}^{T} 2((x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i}) - (y_{i-1,\lambda_{i-1}} - y_{i}))(f_{\theta}(x_{i}) - y_{i})\lambda_{i} + \frac{1}{2} \sum_{i=1}^{T} 2((x_{i-1,\lambda_{i-1}} - x_{i})^{T} \nabla f_{\theta}(x_{i}) - (y_{i-1,\lambda_{i-1}} - y_{i}))^{2} \lambda_{i}^{2}$$

and

$$-2\sum_{i=1}^{T} (f_{\theta}(x_{i}) - y_{i})^{2} \lambda_{i} \leq 0 \text{ and } \sum_{i=1}^{T} ((a_{i}f_{\theta}(x_{i-1,\lambda_{i-1}}) - y_{i-1,\lambda_{i-1}})^{2} + (f_{\theta}(x_{i}) - y_{i})^{2}) =$$

$$2\sum_{i=1}^{T} (a_{i}f_{\theta}(x_{i-1,\lambda_{i-1}}) - y_{i-1,\lambda_{i-1}})(f_{\theta}(x_{i}) - y_{i}) +$$

$$\sum_{i=1}^{T} (a_{i}f_{\theta}(x_{i-1,\lambda_{i-1}}) - y_{i-1,\lambda_{i-1}} - (f_{\theta}(x_{i}) - y_{i}))^{2}$$

we conclude the result.

In terms of binary trend prediction problems using neural networks with ReLU activations, we use a sigmoid function as the activation function for the last layer. So, a neural network  $f_{\theta}(x) = \sigma(g_{\theta}(x))$  where  $g_{\theta}(x) = \nabla g_{\theta}(x)^T x + b$ .

### A.3 Proof of Theorem 4.3

Proof. Let

$$a_i = (x_{i-1,\lambda_{i-1}} - x_i)^T \nabla g_{\theta}(x_i) \text{ and } b_i = (x_{i-1,\lambda_{i-1}} - x_i)^T \nabla^2 g_{\theta}(x_i) (x_{i-1,\lambda_{i-1}} - x_i)$$
$$c_i = (x_{i-1,\lambda_{i-1}} - x_i)^T \nabla^2 f_{\theta}(x_i) (x_{i-1,\lambda_{i-1}} - x_i).$$

Since

$$\nabla f_{\theta}(x) = \nabla g_{\theta}(x) f_{\theta}(x) (1 - f_{\theta}(x)) \text{ and}$$

$$\nabla^2 f_{\theta}(x) = \nabla^2 g_{\theta}(x) f_{\theta}(x) (1 - f_{\theta}(x)) + \nabla g_{\theta}(x) \nabla g_{\theta}(x)^T f_{\theta}(x) (1 - f_{\theta}(x))^2 - \nabla g_{\theta}(x) \nabla g_{\theta}(x)^T f_{\theta}(x)^2 (1 - f_{\theta}(x)),$$

we have

$$c_i = b_i f_{\theta}(x) (1 - f_{\theta}(x)) + a_i^2 f_{\theta}(x) (1 - f_{\theta}(x))^2 - a_i^2 f_{\theta}(x)^2 (1 - f_{\theta}(x)).$$

Hence

$$\frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_i}\bigg|_{\vec{\lambda}=\vec{0}} = a_i(y_{i,\lambda_i} - f_{\theta}(x_i))$$
 and

$$\begin{split} \frac{\partial^2 L_{\theta}(\vec{\lambda})}{\partial \lambda_i^2} \bigg|_{\vec{\lambda} = \vec{0}} &= c_i (\frac{y_{i,\lambda_i} - f_{\theta}(x_i)}{(1 - f_{\theta}(x_i))f_{\theta}(x_i)}) + a_i^2 (f_{\theta}(x_i)^2 + f_{\theta}(x_i) - 1) \\ &= c_i (\frac{y_{i,\lambda_i} - f_{\theta}(x_i)}{(1 - f_{\theta}(x_i))f_{\theta}(x_i)}) + (a_i (f_{\theta}(x_i) + \frac{1}{2}))^2 - \frac{5a_i^2}{4} \\ &= (y_{i,\lambda_i} - f_{\theta}(x_i))(b_i + a_i^2 (1 - f_{\theta}(x_i)) - a_i^2 f_{\theta}(x_i)) + (a_i (f_{\theta}(x_i) + \frac{1}{2}))^2 - \frac{5a_i^2}{4} \\ &= -2a_i^2 f_{\theta}(x_i)(y_{i,\lambda_i} - f_{\theta}(x_i)) + (a_i (f_{\theta}(x_i) + \frac{1}{2}))^2 - \frac{5a_i^2}{4} \end{split}$$

by equations (28) and (29). Since

$$L_{\theta}(\vec{\lambda}) = L_{\theta}(\vec{0}) + \sum_{i=1}^{T} \frac{\partial L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i} + \frac{1}{2} \sum_{i=1}^{T} \frac{\partial^{2} L_{\theta}(\vec{\lambda})}{\partial \lambda_{i}^{2}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i}^{2}$$

$$= L_{\theta}(\vec{0}) + \sum_{i=1}^{T} a_{i}(y_{i,\lambda_{i}} - f_{\theta}(x_{i}))\lambda_{i} + \frac{1}{2} \sum_{i=1}^{T} (-2a_{i}^{2} f_{\theta}(x_{i})(y_{i,\lambda_{i}} - f_{\theta}(x_{i})) + (b_{i} + a_{i}^{2})(y_{i,\lambda_{i}} - f_{\theta}(x_{i})) + (a_{i}(f_{\theta}(x_{i}) + \frac{1}{2}))^{2} - \frac{5a_{i}^{2}}{4})\lambda_{i}^{2}$$

and for  $y_{i,\lambda_i} = 1$ ,

$$-2a_i^2 \le -2a_i^2 f_{\theta}(x_i)(y_{i,\lambda_i} - f_{\theta}(x_i)) \le 0 \text{ and } a_i^2 \ge a_i^2(y_{i,\lambda_i} - f_{\theta}(x_i)) \ge 0$$

for  $y_{i,\lambda_i} = 0$ 

$$2a_i^2 \ge -2a_i^2 f_{\theta}(x_i)(y_{i,\lambda_i} - f_{\theta}(x_i)) \ge 0 \text{ and } -a_i^2 \le a_i^2(y_{i,\lambda_i} - f_{\theta}(x_i)) \le 0,$$

we have

$$L_{\theta}(\vec{0}) - \sum_{i=1}^{T} (|a_i| + \frac{17}{8}a_i^2 + \frac{1}{2}|b_i|) \le L_{\theta}(\vec{\lambda}) \le L_{\theta}(\vec{0}) + \sum_{i=1}^{T} (|a_i| + \frac{5}{2}a_i^2 + \frac{1}{2}|b_i|).$$

Thus the conclusion holds.

### A.4 Additional Theoretical Results: Classification Task

Classification. In this part, we address classification problems, that is, (input, label) pairs are given to us. We adopt a slightly different augmentation approach for classification problems. Instead of using two pairs (input<sub>1</sub>, label<sub>1</sub>) and (input<sub>2</sub>, label<sub>2</sub>) to generate a new sample, one sample (input, label) will be used. Since each input  $\in \mathbb{R}^{d \times T}$  is time series data, we generate a new sample in a similar way by equations (22) and (23). The only difference is the form of input. Let s = (x, y) such that  $x = (x_0, \dots, x_T)$  where  $x_i \in \mathbb{R}^d$  and  $y \in \mathbb{R}$ . For each  $\vec{\lambda} = (\lambda_1, \dots, \lambda_T) \in [0, 1]^T$ , the augmented sample is defined as follows;  $s_{\vec{\lambda}} = (x_{\vec{\lambda}}, y)$  such that  $x_{\vec{\lambda}} = (x_0, x_{1,\lambda_1}, \dots, x_{T,\lambda_T})$  where

$$x_{i,\lambda_i} = (1 - \lambda_i)x_i + \lambda_i x_{i-1,\lambda_{i-1}}.$$

In this case, we view  $s_{\vec{\lambda}}$  itself as one of the possible realizations. So  $s_{\vec{\lambda}}$  depends on all of  $\{\lambda_i\}_{i\in[1:T]}$ .

**Lemma A.4** Let  $x=(x_0,\ldots,x_T)$  be one sample drawn from time series and  $\vec{\lambda}\in[0,1]^T$ . Then

$$\frac{\partial x_{i,\lambda_i}}{\partial \lambda_j} = \begin{cases}
0 & \text{if } i < j \\
(x_{i-1,\lambda_{i-1}} - x_i) & \text{if } i = j \\
(\prod_{k=j+1}^{i} \lambda_k)(x_{i-1,\lambda_{i-1}} - x_i) & \text{if } i > j
\end{cases}$$
(30)

*Proof.* If i < j, then  $x_{i,\lambda_i}$  does not depend on  $\lambda_j$ . Hence  $\frac{\partial x_{i,\lambda_i}}{\partial \lambda_j} = 0$ .

If i = j, then

$$x_{i,\lambda_i} = (1 - \lambda_i)x_i + \lambda_i x_{i-1,\lambda_{i-1}} = x_i + \lambda_i (x_{i-1,\lambda_{i-1}} - x_i).$$

Hence 
$$\frac{\partial x_{i,\lambda_i}}{\partial \lambda_j} = (x_{i-1,\lambda_{i-1}} - x_i)$$
.

If i > j, we prove this by induction on i. The case i = j + 1 shows that

$$x_{j+1,\lambda_{j+1}} = (1 - \lambda_{j+1})x_{j+1} + \lambda_{j+1}x_{j,\lambda_j} = x_{j+1} + \lambda_{j+1}(x_{j,\lambda_j} - x_{j+1}).$$

Since  $x_{j+1}$  is not depedent on  $\lambda_j$ ,  $\frac{\partial x_{j,\lambda_j}}{\partial \lambda_j} = (x_{j-1,\lambda_{j-1}} - x_j)$ . We now assume that the inequality holds for i and prove it for i+1. Since

$$x_{i+1,\lambda_{i+1}} = (1 - \lambda_{i+1})x_{i+1} + \lambda_{i+1}x_{i,\lambda_i} = x_{i+1} + \lambda_{i+1}(x_{i,\lambda_i} - x_{i+1}) \text{ and}$$

$$\frac{\partial x_{i,\lambda_i}}{\partial \lambda_j} = (\prod_{k=j+1}^{i} \lambda_k)(x_{i-1,\lambda_{i-1}} - x_i),$$

$$\frac{\partial x_{i+1,\lambda_{i+1}}}{\partial \lambda_j} = \lambda_{i+1} (\prod_{k=j+1}^{i} \lambda_k) (x_{i-1,\lambda_{i-1}} - x_i) = (\prod_{k=j+1}^{i+1} \lambda_k) (x_{i-1,\lambda_{i-1}} - x_i).$$

As a consequence of mathematical induction, the conclusion holds

We use neural networks with ReLU activations and sigmoid function as the activation function for the last layer. So, a neural network  $f_{\theta}(x) = \sigma(g_{\theta}(x))$  where  $g_{\theta}(x) = \nabla g_{\theta}(x)^T x + b$ .

**Theorem A.4.** The loss function  $l_{\theta}(\vec{\lambda})$  has error bound with original loss function as follows

$$||l_{\theta}(\vec{\lambda}) - l_{\theta}(\vec{0})|| \le \sum_{u=1}^{T} \sqrt{n+1} (\frac{||A||_{F}}{T} + ||B_{u}||_{F}) ||\nabla g_{\theta}(x)||$$
(31)

where  $\|\cdot\|_F$  is a Frobenius norm.

*Proof.* Let  $x_{\vec{\lambda}} = (x_0, x_{1,\lambda_1}, \dots, x_{T,\lambda_T})$  and (x,y) be one sample. Define the matrices

$$A = \begin{pmatrix} 0 & \vec{0}^T & \\ & \frac{\partial x_{1,\lambda_1}}{\partial \lambda_1} |_{\vec{\lambda} = \vec{0}} & \frac{\partial x_{2,\lambda_2}}{\partial \lambda_1} |_{\vec{\lambda} = \vec{0}} & \cdots & \frac{\partial x_{T,\lambda_T}}{\partial \lambda_1} |_{\vec{\lambda} = \vec{0}} \\ & 0 & \frac{\partial x_{2,\lambda_2}}{\partial \lambda_2} |_{\vec{\lambda} = \vec{0}} & \cdots & \frac{\partial x_{T,\lambda_T}}{\partial \lambda_2} |_{\vec{\lambda} = \vec{0}} \\ & \vdots & \vdots & \vdots & \vdots \\ & 0 & 0 & \cdots & \frac{\partial x_{T,\lambda_T}}{\partial \lambda_T} |_{\vec{\lambda} = \vec{0}} \end{pmatrix}$$
(32)

$$B_{u} = \begin{pmatrix} 0 & \vec{0}^{T} & \\ & \frac{\partial^{2}x_{1,\lambda_{1}}}{\partial\lambda_{u}\partial\lambda_{1}} |_{\vec{\lambda}=\vec{0}} & \frac{\partial^{2}x_{2,\lambda_{2}}}{\partial\lambda_{u}\partial\lambda_{1}} |_{\vec{\lambda}=\vec{0}} & \cdots & \frac{\partial^{2}x_{T,\lambda_{T}}}{\partial\lambda_{u}\partial\lambda_{1}} |_{\vec{\lambda}=\vec{0}} \\ \vec{0} & 0 & \frac{\partial^{2}x_{2,\lambda_{2}}}{\partial\lambda_{u}\partial\lambda_{2}} |_{\vec{\lambda}=\vec{0}} & \cdots & \frac{\partial^{2}x_{T,\lambda_{T}}}{\partial\lambda_{u}\partial\lambda_{2}} |_{\vec{\lambda}=\vec{0}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^{2}x_{T,\lambda_{T}}}{\partial\lambda_{u}\partial\lambda_{T}} |_{\vec{\lambda}=\vec{0}} \end{pmatrix}$$
(33)

where  $\vec{0} \in \mathbb{R}^{T \times 1}$  denoted by column vector and  $e_i$  is a vector whose i-th component is 1 and 0 otherwise. Let  $l(f_{\theta}(x_{\vec{\lambda}}), y) = y \log(f_{\theta}(x_{\vec{\lambda}})) + (1 - y) \log(1 - f_{\theta}(x_{\vec{\lambda}}))$ . Denote  $l_{\theta}(\vec{\lambda}) = l(f_{\theta}(x_{\vec{\lambda}}), y)$ . Using Taylor expansion of l over  $\vec{\lambda}$ , we have

$$l_{\theta}(\vec{\lambda}) = l_{\theta}(\vec{0}) + \sum_{i=1}^{T} \frac{\partial l_{\theta}(\vec{\lambda})}{\partial \lambda_{i}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i} + \frac{1}{2} \sum_{i,j} \frac{\partial^{2} l_{\theta}(\vec{\lambda})}{\partial \lambda_{i} \partial \lambda_{j}} \Big|_{\vec{\lambda} = \vec{0}} \lambda_{i} \lambda_{j} + r(\vec{\lambda}) ||\vec{\lambda}||^{2}$$
(34)

Note that

$$\frac{\partial l_{\theta}(\vec{\lambda})}{\partial x_{i,\lambda_{i}}} = y \frac{\partial \log(f_{\theta}(x_{\vec{\lambda}}))}{\partial x_{i,\lambda_{i}}} + (1 - y) \frac{\partial \log(1 - f_{\theta}(x_{\vec{\lambda}}))}{\partial x_{i,\lambda_{i}}}$$

$$= y \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} - (1 - y) \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}}$$

$$= \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} \frac{y(1 - f_{\theta}(x_{\vec{\lambda}})) + (y - 1)f_{\theta}(x_{\vec{\lambda}})}{(1 - f_{\theta}(x_{\vec{\lambda}}))f_{\theta}(x_{\vec{\lambda}})}$$

$$= \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} \frac{y - f_{\theta}(x_{\vec{\lambda}})}{(1 - f_{\theta}(x_{\vec{\lambda}}))f_{\theta}(x_{\vec{\lambda}})}$$
(35)

and

$$\frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} = \frac{\partial \sigma(g_{\theta}(x_{\vec{\lambda}}))}{\partial x_{i,\lambda_{i}}} 
= \frac{\partial g_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} \sigma(g_{\theta}(x_{\vec{\lambda}}))(1 - \sigma(g_{\theta}(x_{\vec{\lambda}}))) 
= \frac{\partial (\nabla g_{\theta}(x_{\vec{\lambda}})^{T} x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} \sigma(g_{\theta}(x_{\vec{\lambda}}))(1 - \sigma(g_{\theta}(x_{\vec{\lambda}}))) 
= (\frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}}) \sigma(g_{\theta}(x_{\vec{\lambda}}))(1 - \sigma(g_{\theta}(x_{\vec{\lambda}}))) 
= \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \sigma(g_{\theta}(x_{\vec{\lambda}}))(1 - \sigma(g_{\theta}(x_{\vec{\lambda}}))).$$
(36)

Since the *i*-th feature  $x_{i,\lambda_i}$  of  $x_{\vec{\lambda}}$  depends on  $\{\lambda_1,\ldots,\lambda_i\}$ , we have

$$\frac{\partial l_{\theta}(\vec{\lambda})}{\partial \lambda_{j}} = \sum_{i=1}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \frac{\partial l_{\theta}(\vec{\lambda})}{\partial x_{i,\lambda_{i}}} 
= \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \frac{\partial l_{\theta}(\vec{\lambda})}{\partial x_{i,\lambda_{i}}} 
= \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{i,\lambda_{i}}} \frac{y - f_{\theta}(x_{\vec{\lambda}})}{(1 - f_{\theta}(x_{\vec{\lambda}}))f_{\theta}(x_{\vec{\lambda}})} 
= \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} (\frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}}) \cdot$$

$$\sigma(g_{\theta}(x_{\vec{\lambda}}))(1 - \sigma(g_{\theta}(x_{\vec{\lambda}}))) \frac{y - f_{\theta}(x_{\vec{\lambda}})}{(1 - f_{\theta}(x_{\vec{\lambda}}))f_{\theta}(x_{\vec{\lambda}})} 
= \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} (\frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}})(y - f_{\theta}(x_{\vec{\lambda}})) 
= \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i}(y - f_{\theta}(x_{\vec{\lambda}}))$$

Thus we have

$$\sum_{j=1}^{T} \frac{\partial l_{\theta}(\vec{\lambda})}{\partial \lambda_{j}} \bigg|_{\vec{\lambda} = \vec{0}} \lambda_{j} = (y - f_{\theta}(x))(0, \vec{\lambda})^{T} A((\nabla^{2} g_{\theta}(x))x + \nabla g_{\theta}(x))$$

$$= (y - f_{\theta}(x))(0, \vec{\lambda})^{T} A \nabla g_{\theta}(x)$$
(38)

Hence we have

$$\left|\sum_{j=1}^{T} \frac{\partial l_{\theta}(\vec{\lambda})}{\partial \lambda_{j}}\right|_{\vec{\lambda} = \vec{0}} \leq \sqrt{n+1} \|A\|_{F} \|\nabla g_{\theta}(x)\|$$
(39)

Now we consider the second partial derivative of the loss function  $l_{\theta}(\vec{\lambda})$ .

$$\frac{\partial^{2} l_{\theta}(\vec{\lambda})}{\partial \lambda_{u} \partial \lambda_{j}} = \sum_{i=j}^{T} \frac{\partial^{2} x_{i,\lambda_{i}}}{\partial \lambda_{u} \partial \lambda_{j}} \left( \frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}} \right) (y - f_{\theta}(x_{\vec{\lambda}})) + \\
(y - f_{\theta}(x_{\vec{\lambda}})) \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \sum_{k=u}^{T} \frac{\partial x_{k,\lambda_{k}}}{\partial \lambda_{u}} \frac{\partial}{\partial x_{k,\lambda_{k}}} \left( \frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}} \right) + \\
\sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \left( \frac{\partial \nabla g_{\theta}(x_{\vec{\lambda}})^{T}}{\partial x_{i,\lambda_{i}}} x_{\vec{\lambda}} + \nabla g_{\theta}(x_{\vec{\lambda}})^{T} \frac{\partial x_{\vec{\lambda}}}{\partial x_{i,\lambda_{i}}} \right) \sum_{k=u}^{T} \frac{\partial x_{k,\lambda_{k}}}{\partial \lambda_{u}} \frac{\partial}{\partial x_{k,\lambda_{k}}} (y - f_{\theta}(x_{\vec{\lambda}})) \\
= \sum_{i=j}^{T} \frac{\partial^{2} x_{i,\lambda_{i}}}{\partial \lambda_{u} \partial \lambda_{j}} (\nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i}) (y - f_{\theta}(x_{\vec{\lambda}})) + \sum_{i=j}^{T} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} (\nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i}) \frac{\partial}{\partial \lambda_{u}} (y - f_{\theta}(x_{\vec{\lambda}})) \\
= \sum_{i=j}^{T} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \left( \frac{\partial^{2} x_{i,\lambda_{i}}}{\partial \lambda_{u} \partial \lambda_{j}} (y - f_{\theta}(x_{\vec{\lambda}})) - \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \sum_{k=u}^{T} \frac{\partial x_{k,\lambda_{k}}}{\partial \lambda_{u}} \frac{\partial f_{\theta}(x_{\vec{\lambda}})}{\partial x_{k,\lambda_{k}}} \right) \\
= \sum_{i=j}^{T} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \left( \frac{\partial^{2} x_{i,\lambda_{i}}}{\partial \lambda_{u} \partial \lambda_{j}} (y - f_{\theta}(x_{\vec{\lambda}})) - \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{u}} (y - f_{\theta}(x_{\vec{\lambda}})) - \frac{\partial x_{i,\lambda_{k}}}{\partial \lambda_{u} \partial \lambda_{j}} \sum_{k=u}^{T} \frac{\partial x_{k,\lambda_{k}}}{\partial \lambda_{u} \partial \lambda_{j}} (y - f_{\theta}(x_{\vec{\lambda}})) \right)$$

Since we have

$$\sum_{u=1}^{T} \lambda_{u} \sum_{j=1}^{T} \lambda_{j} \sum_{i=j}^{T} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \left( \frac{\partial^{2} x_{i,\lambda_{i}}}{\partial \lambda_{u} \partial \lambda_{j}} (y - f_{\theta}(x_{\vec{\lambda}})) \right) \Big|_{\vec{\lambda} = \vec{0}} = \sum_{u=1}^{T} \lambda_{u} (y - f_{\theta}(x)) (0, \vec{\lambda})^{T} B_{u} \nabla g_{\theta}(x)$$

$$(41)$$

and

$$\sum_{j=1}^{T} \lambda_{j} \sum_{i=j}^{T} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} \sum_{u=1}^{T} \lambda_{u} \sum_{k=u}^{T} \frac{\partial x_{k,\lambda_{k}}}{\partial \lambda_{u}} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{k} f_{\theta}(x_{\vec{\lambda}}) (1 - f_{\theta}(x_{\vec{\lambda}})) \Big|_{\vec{\lambda} = \vec{0}}$$

$$= \sum_{j=1}^{T} \lambda_{j} \sum_{i=j}^{T} \nabla g_{\theta}(x_{\vec{\lambda}})^{T} e_{i} \frac{\partial x_{i,\lambda_{i}}}{\partial \lambda_{j}} f_{\theta}(x) (1 - f_{\theta}(x)) (0, \vec{\lambda})^{T} A \nabla g_{\theta}(x)$$

$$= f_{\theta}(x) (1 - f_{\theta}(x)) (0, \vec{\lambda})^{T} A \nabla g_{\theta}(x) \nabla g_{\theta}(x)^{T} A^{T}(0, \vec{\lambda}),$$
(42)

the following equation hold

$$\left| \sum_{u=1}^{T} \sum_{j=1}^{T} \lambda_{u} \lambda_{j} \frac{\partial^{2} l_{\theta}(\vec{\lambda})}{\partial \lambda_{u} \partial \lambda_{j}} \right|_{\vec{\lambda} = \vec{0}} =$$

$$\sum_{u=1}^{T} \lambda_{u} (y - f_{\theta}(x)) (0, \vec{\lambda})^{T} B_{u} \nabla g_{\theta}(x) - f_{\theta}(x) (1 - f_{\theta}(x)) (0, \vec{\lambda})^{T} A \nabla g_{\theta}(x) \nabla g_{\theta}(x)^{T} A^{T}(0, \vec{\lambda})$$

$$\leq \sum_{u=1}^{T} \sqrt{n+1} \|B_{u}\|_{F} \|\nabla g_{\theta}(x)\|.$$

$$(43)$$

The last inequality holds due to

$$f_{\theta}(x)(1 - f_{\theta}(x))(0, \vec{\lambda})^T A \nabla g_{\theta}(x) \nabla g_{\theta}(x)^T A^T(0, \vec{\lambda}) \ge 0 \tag{44}$$

By equations (34), (39) and (43), the conclusion holds.

### A.5 Proof of Theorem 4.4

**Description** Denote by  $S_{True}$  the true sample space. There exists all possible realizations  $\{S_i\}_{i\in I}$  for some index set I with  $|I|=\infty$ . We observe only one possible realization  $S=\{s_0,s_1,\ldots,s_{n_i}\}$  where  $s_t=(x_t,y_t)$  and S is sampled from  $S_{True}$ . For each sample  $s_t=(x_t,y_t)$ , we generate augmented samples and let  $S_{(x_t,y_t)}$  be the set of such augmented samples.

We start to construct an augmented sample space in order to approximate the true sample space. For each sample  $(x_t, y_t) \in S$ , set up  $S_{(x_t, y_t)}$  as follows.

In the case for Binary Trend Prediction,

$$S_{x_t,y_t,0} = \{ ((1-\lambda)x_t + \lambda x_{t-1}, y_{t-1}) \mid 1 > \lambda \ge \frac{1}{2} \},$$

$$S_{x_t,y_t,1} = \{ ((1-\lambda)x_t + \lambda x_{t-1}, y_t) \mid 0 \le \lambda < \frac{1}{2} \} \text{ and } S_{(x_t,y_t)} = S_{x_t,y_t,0} \cup S_{x_t,y_t,1}.$$

In the case for Regression, let  $S_{(x_t,y_t)} = \{((1-\lambda)x_t + \lambda x_{t-1}, (1-\lambda)y_t + \lambda y_{t-1}) \mid \lambda \in [0,1)\}$ 

In the case for Classification, let  $S_{(x_t,y_t)}=\{(x_{t,\vec{\lambda}},y_t)\,|\,\vec{\lambda}\in[0,1)^T\}$  such that  $x_{t,\vec{\lambda}}=(x_{t,0},x_{t,1,\lambda_1},\ldots,x_{t,T,\lambda_T})$  where

$$x_{t,i,\lambda_i} = (1 - \lambda_i)x_{t,i} + \lambda_i x_{t,i-1,\lambda_{i-1}}.$$

**Lemma A.5.1.**  $S_{(x_t,y_t)}$  is the disjoint union of  $S_{x_t,y_t,0}$  and  $S_{x_t,y_t,1}$ .

*Proof.* For  $t \in [1:T]$ , if  $y_t \neq y_{t-1}$ , it is obvious that  $S_{x_t,y_t,0} \cap S_{x_t,y_t,1} = \emptyset$ . Otherwise, not to be  $S_{x_t,y_t,0} \cap S_{x_t,y_t,1} = \emptyset$ , the following inequality holds  $(1-\lambda_1)x_t + \lambda_1x_{t-1} = (1-\lambda_2)x_t + \lambda_2x_{t-1}$ . This implies that  $(\lambda_1 - \lambda_2)(x_t - x_{t-1}) = \vec{0}$  which is absurd by  $x_t - x_{t-1} \neq \vec{0}$  and  $\lambda_1 - \lambda_2 \neq 0$ .

**Lemma A.5.2.** (Augmented space is closed) Let  $(x, y), (x', y') \in S_{(x_t, y_t)}$ .

- 1. (Binary Trend Prediction) If y = y', then  $((1 \lambda)x + \lambda x', y) \in S_{(x_t, y_t)}$  for  $\lambda \in [0, 1)$ .
- 2. (Regression)  $((1-\lambda)x + \lambda x', (1-\lambda)y + \lambda y') \in S_{(x_t,y_t)}$  for  $\lambda \in [0,1)$ .

*Proof.* (1) Note that y=y' implies  $y_{\lambda}=y$ . Since  $(x,y),(x',y)\in S_{(x_t,y_t)}, x=(1-\lambda_1)x_t+\lambda_1x_{t-1}$  and  $x'=(1-\lambda_2)x_t+\lambda_2x_{t-1}$ . Hence we have

$$(1 - \lambda)x + \lambda x' = (1 - \lambda)(1 - \lambda_1)x_t + (1 - \lambda)\lambda_1x_{t-1} + \lambda(1 - \lambda_2)x_t + \lambda\lambda_2x_{t-1}$$
$$= (1 - \lambda - \lambda_1 + \lambda\lambda_1 + \lambda - \lambda\lambda_2)x_t + (\lambda_1 - \lambda\lambda_1 + \lambda\lambda_2)x_{t-1}$$
$$= (1 - (\lambda_1 - \lambda\lambda_1 + \lambda\lambda_2)x_t + (\lambda_1 - \lambda\lambda_1 + \lambda\lambda_2)x_{t-1}$$

For the conclusion, it suffices to show that  $\lambda_1 - \lambda \lambda_1 + \lambda \lambda_2 \in [0,1)$ . Indeed,  $\lambda_1 - \lambda \lambda_1 + \lambda \lambda_2 = (1-\lambda)\lambda_1 + \lambda \lambda_2$  and  $\lambda_1, \lambda_2 \in [0,1)$ . Thus  $\lambda_1 - \lambda \lambda_1 + \lambda \lambda_2 \in [0,1)$ .

(2) In order to prove this, it is enough to replace x, x' with y, y' respectively in the proof of (1).

**Assumption 1.** (Disjointness) the true sample space  $S_{True}$  is the disjoint countable union of all possible augmented sample spaces.

Consider the probability space is  $(S_{True}, \mathcal{F}, \mathcal{P})$ . Let f be a measurable function from  $S_{True}$  to  $\mathbb{R}^p$  for some  $p \in \mathbb{N}$ . For each sample  $s = (x, y) \in S_{True}$ , define  $\overline{f}(s) = \mathbb{E}[f(s') \mid s' \in S_{(x,y)}]$ . This is well-defined by Assumption 1, and  $\overline{f}$  is a measurable function.

**Lemma A.5.3.** (Effects of the average function) With notation as above, the following holds.

- 1. The law of total expectation:  $\mathbb{E}_{s \sim \mathcal{P}}[f(s)] = \mathbb{E}_{s \sim \mathcal{P}}[\overline{f}(s)].$
- 2. The law of total covariance:  $Cov_{s\sim \mathcal{P}}f(s) = Cov_{s\sim \mathcal{P}}\overline{f}(s) + \mathbb{E}_{s\sim \mathcal{P}}[Cov(f(s)|\overline{f}(s))].$

*Proof.* From the disjointedness of the sample space, we have

$$\mathbb{E}[\overline{f}(s)] = \mathbb{E}[\mathbb{E}[f(s') \mid s' \in S_{(x,y)}]] = \mathbb{E}[\mathbb{E}[f(s') \mid \overline{f}(s)]]. \tag{45}$$

Thus the law of total expectation and the law of total covariance naturally follow.

**Lemma A.5.4.** (Sup inequ) Let f and g be functions which have the same domain and range. Then

$$\sup_{\theta} |f(\theta)| - \sup_{\theta} |g(\theta)| \le \sup_{\theta} |f(\theta) - g(\theta)| \tag{46}$$

Proof.

$$\begin{aligned} \sup_{\theta} & |f(\theta)| \leq \sup_{\theta} |f(\theta) - g(\theta) + g(\theta)| \\ & \leq \sup_{\theta} (|f(\theta) - g(\theta)| + |g(\theta)|) \\ & \leq \sup_{\theta} |f(\theta) - g(\theta)| + \sup_{\theta} |g(\theta)| \end{aligned}$$

Thus the conclusion holds.

We assume that the true distribution  $\mathcal{P}$  is parametrized by  $\theta_*$  and denote by  $\mathcal{P}_{\theta_*}$  on the true sample space  $S_{True}$ . We have one realization from the true distribution  $\mathcal{P}_{\theta_*}$  and this is  $\{(x_i,y_i)\}_{i\in[0:n]}$ . We set:

$$\theta_{*} = \underset{\theta}{\operatorname{argmin}} \mathbb{E}[l(f_{\theta}(x), y)]$$

$$\theta_{n} = \underset{\theta}{\operatorname{argmin}} \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_{i}), y_{i})$$

$$\theta_{\operatorname{aug}} = \underset{\theta}{\operatorname{argmin}} \mathbb{E}[\mathbb{E}[l(f_{\theta}(x'), y') | (x', y') \in S_{(x,y)}]]$$

$$\theta_{\operatorname{aug},n} = \underset{\theta}{\operatorname{argmin}} \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta}(x'), y') | (x', y') \in S_{(x_{i}, y_{i})}].$$

$$\mathcal{R}_{n}(l \circ \Theta) = \mathbb{E}\left[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} l(f_{\theta}(x_{i}), y_{i}) \right| \right]$$

$$(47)$$

We will compare  $\theta_*, \theta_0, \theta_{\text{aug}}$  and  $\theta_{\text{aug},n}$ . Define  $\bar{l}(f_{\theta}(x), y) = \mathbb{E}[l(f_{\theta}(x'), y') | (x', y') \in S_{(x,y)}]$  as an average loss function.

**Assumption 2.** The parameter space  $\Theta$  is big enough to contain a parameter  $\theta$  which makes  $\left|\frac{1}{n+1}\sum_{i=0}^{n}\epsilon_{i}l(f_{\theta}(x_{i}),y_{i})\right|=\frac{m_{i}}{n+1}$  where  $m_{i}$  is the number of  $\epsilon_{i}$  whose value is 1 (Note that  $l(\cdot,\cdot)\in[0,1]$ ).

**Proof of Theorem 4.4.** We follow notation of Theorem 4.4.

$$\mathbb{E}[l(f_{\theta_{\text{aug},n}}(x),y)] - \mathbb{E}[l(f_{\theta_*}(x),y)] = u_1 + u_2 + u_3 + u_4 + u_5$$

where

$$u_{1} = \mathbb{E}[l(f_{\theta_{\text{aug},n}}(x), y)] - \mathbb{E}[\mathbb{E}[l(f_{\theta_{\text{aug},n}}(x'), y') | (x', y') \in S_{(x,y)}]]$$

$$u_{2} = \mathbb{E}[\mathbb{E}[l(f_{\theta_{\text{aug},n}}(x'), y') | (x', y') \in S_{(x,y)}]] - \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta_{\text{aug},n}}(x'), y') | (x', y') \in S_{(x_{i},y_{i})}]$$

$$u_{3} = \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta_{\text{aug},n}}(x'), y') | (x', y') \in S_{(x_{i},y_{i})}] - \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta_{*}}(x'), y') | (x', y') \in S_{(x_{i},y_{i})}]$$

$$u_{4} = \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta_{*}}(x'), y') | (x', y') \in S_{(x_{i},y_{i})}] - \mathbb{E}[\mathbb{E}[l(f_{\theta_{*}}(x'), y') | (x', y') \in S_{(x,y)}]]$$

$$u_{5} = \mathbb{E}[\mathbb{E}[l(f_{\theta_{*}}(x'), y') | (x', y') \in S_{(x,y)}]] - \mathbb{E}[l(f_{\theta_{*}}(x), y)]$$

We get

$$u_1 + u_5 \le 2 \sup_{\theta \in \Theta} \left| \mathbb{E}[l(f_{\theta}(x), y)] - \mathbb{E}[\mathbb{E}[l(f_{\theta}(x'), y') | (x', y') \in S_{(x,y)}]] \right|$$
(49)

By Lemma A.5.3,

$$\mathbb{E}[l(f_{\theta}(x), y)] - \mathbb{E}[\mathbb{E}[l(f_{\theta}(x'), y') | (x', y') \in S_{(x,y)}]] = 0$$

By Hoeffding's inequality,  $u_4$  has the following bound

$$u_4 < \sqrt{\frac{\log(2/\delta)}{2(n+1)}}\tag{50}$$

with probability at least  $1 - \delta/2$ . Moreover, Rademacher complexity holds for  $u_2$ , so we have

$$u_2 \le 2\mathcal{R}_n(\bar{l} \circ \Theta) + \sqrt{\frac{\log(2/\delta)}{2(n+1)}} \tag{51}$$

with probability at least  $1 - \delta/2$ . By equations (50) and (51), we get the following inequality

$$u_2 + u_4 \le 2\mathcal{R}_n(\bar{l} \circ \Theta) + \sqrt{\frac{2\log(2/\delta)}{n+1}} \tag{52}$$

with probability at least  $1 - \delta/2$  where

$$\mathcal{R}_n(\bar{l} \circ \Theta) = \mathbb{E}[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n \epsilon_i \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}] \right|]$$

Since  $\theta_{\text{aug},n}$  is an optimal parameter for  $\frac{1}{n+1}\sum_{i=0}^{n}\mathbb{E}[l(f_{\theta}(x'),y')\,|\,(x',y')\in S_{(x_i,y_i)}],\,u_3\leq 0$ . With all together, we conclude that

$$\mathbb{E}[l(f_{\theta_{\text{aug},n}}(x), y)] - \mathbb{E}[l(f_{\theta_*}(x), y)] < 2\mathcal{R}_n(\bar{l} \circ \Theta) + \sqrt{\frac{2\log(2/\delta)}{n+1}}.$$
 (53)

Now we will prove that

$$\mathcal{R}_n(l \circ \Theta) \le \mathcal{R}_n(l \circ \Theta) \tag{54}$$

 $\mathcal{R}_n(\bar{l}\circ\Theta) \leq \mathcal{R}_n(l\circ\Theta) \tag{54}$  Since  $\bar{l}(f_{\theta}(x_i),y_i) = \mathbb{E}[l(f_{\theta}(x'),y')\,|\,(x',y')\in S_{(x,y)}]$  and  $l(\cdot,\cdot)\in[0,1]$ , we have  $\bar{l}(f_{\theta}(x_i),y_i)\leq 1$ . Hence the

$$\left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} \bar{l}(f_{\theta}(x_{i}), y_{i}) \right| = \left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x_{i}, y_{i})}] \right| \le \frac{m_{i}}{n+1}$$
 (55)

where  $m_i$  is the number of  $\epsilon_i$  whose value is 1. Thus

$$\mathbb{E}\left[\sup_{\theta\in\Theta}\left|\frac{1}{n+1}\sum_{i=0}^{n}\epsilon_{i}\bar{l}(f_{\theta}(x_{i}),y_{i})\right| - \sup_{\theta\in\Theta}\left|\frac{1}{n+1}\sum_{i=0}^{n}\epsilon_{i}l(f_{\theta}(x_{i}),y_{i})\right|\right] \leq 0$$
(56)

which means

$$\mathcal{R}_n(\bar{l}\circ\Theta) - \mathcal{R}_n(l\circ\Theta) \le 0 \tag{57}$$

**Proposition A.5.** (Compare  $\theta_{\text{aug},n}$  with  $\theta_n$ ) With notation of Theorem 4.4 and without any assumption, we have

$$\mathcal{R}_n(\bar{l} \circ \Theta) - \mathcal{R}_n(l \circ \Theta) \le \sup_{\theta \in \Theta} |\bar{l}(f_{\theta}(x), y) - l(f_{\theta}(x), y)|$$
(58)

Proof. Since

$$\mathcal{R}_n(\bar{l} \circ \Theta) = \mathbb{E}[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n \epsilon_i \bar{l}(f_{\theta}(x_i), y_i) \right|]$$

and

$$\mathcal{R}_n(l \circ \Theta) = \mathbb{E}[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n \epsilon_i l(f_{\theta}(x_i), y_i) \right|]$$

Hence

$$\mathcal{R}_{n}(\bar{l} \circ \Theta) - \mathcal{R}_{n}(l \circ \Theta) = \mathbb{E}\left[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} \bar{l}(f_{\theta}(x_{i}), y_{i}) \right| - \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} l(f_{\theta}(x_{i}), y_{i}) \right| \right] \\
\leq \mathbb{E}\left[\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \epsilon_{i} (l(f_{\theta}(x_{i}), y_{i}) - \bar{l}(f_{\theta}(x_{i}), y_{i})) \right| \right] \\
\leq \sup_{\theta \in \Theta} \left| \bar{l}(f_{\theta}(x), y) - l(f_{\theta}(x), y) \right| \tag{59}$$

The last inequality holds by By Lemma A.5.4.

**Remark.** We use the following well known Rademacher bound for  $\theta_n$ .

$$\mathbb{E}[l(f_{\theta_n}(x), y)] - \mathbb{E}[l(f_{\theta_*}(x), y)] < 2\mathcal{R}_n(l \circ \Theta) + \sqrt{\frac{2\log(2/\delta)}{n+1}}$$
(60)

For completeness, we give a proof. Decompose the generalization error into the following terms

$$\mathbb{E}[l(f_{\theta_n}(x), y)] - \mathbb{E}[l(f_{\theta_*}(x), y)]$$

$$= \mathbb{E}[l(f_{\theta_n}(x), y)] - \frac{1}{n+1} \sum_{i=0}^n l(f_{\theta_n}(x_i), y_i) + \frac{1}{n+1} \sum_{i=0}^n l(f_{\theta_n}(x_i), y_i) - \mathbb{E}[l(f_{\theta_*}(x), y)]$$
(61)

Now we get the bound of the equation

$$\mathbb{E}[l(f_{\theta_{n}}(x), y)] - \mathbb{E}[l(f_{\theta_{*}}(x), y)] \leq \mathbb{E}[l(f_{\theta_{n}}(x), y)] - \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta_{n}}(x_{i}), y_{i}) + \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta_{*}}(x_{i}), y_{i}) - \mathbb{E}[l(f_{\theta_{*}}(x), y)]$$

$$\leq \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_{i}), y_{i}) - \mathbb{E}[l(f_{\theta}(x), y)]] \right| + \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta_{*}}(x_{i}), y_{i}) - \mathbb{E}[l(f_{\theta_{*}}(x), y)]$$

$$(62)$$

By Hoeffding's inequality, we get

$$\mathcal{P}\left[\frac{1}{n+1}\sum_{i=0}^{n}l(f_{\theta_*}(x_i), y_i) - \mathbb{E}[l(f_{\theta_*}(x), y)] \ge t\right] \le \exp(-2(n+1)t^2) \tag{63}$$

So with probability at least  $1 - \delta/2$ , we have

$$\frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta_*}(x_i), y_i) - \mathbb{E}[l(f_{\theta_*}(x), y)] < \sqrt{\frac{\log(2/\delta)}{2(n+1)}}$$
(64)

So as to control the LHS of the equation (62), we will prove that

$$\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_i), y_i) - \mathbb{E}[l(f_{\theta}(x), y)]] \right| < 2\mathcal{R}_n(l \circ \Theta) + \sqrt{\frac{\log(2/\delta)}{2(n+1)}}$$
(65)

Set

$$\phi(s_0, \dots, s_n) = \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n l(f_{\theta}(x_i), y_i) - \mathbb{E}[l(f_{\theta}(x), y)]] \right|$$
 (66)

Then

$$|\phi(s_0,\ldots,s_{i-1},s_i,s_{i+1},\ldots,s_n) - \phi(s_0,\ldots,s_{i-1},s_i',s_{i+1},\ldots,s_n)|$$

$$\leq \left| \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_{i}), y_{i}) - \mathbb{E}[l(f_{\theta}(x), y)]] \right| \\
- \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_{i}), y_{i}) - \frac{1}{n+1} l(f_{\theta}(x_{i}), y_{i}) + \frac{1}{n+1} l(f_{\theta}(x'_{i}), y'_{i}) - \mathbb{E}[l(f_{\theta}(x), y)]] \right| \\
\leq \left| \sup_{\theta} \left| \frac{1}{n+1} (l(f_{\theta}(x_{i}), y_{i}) - l(f_{\theta}(x'_{i}), y'_{i})) \right| \right| \leq \frac{1}{n+1}$$
(67)

The last inequality holds by Lemma A.5 and  $l(f_{\theta}(x), y) \in [0, 1]$ . McDiarmid's inequality holds for the function  $\phi$ , so we have

$$\mathcal{P}[\phi(s_0,\ldots,s_n)] - \mathbb{E}[\phi] \ge t] \ge \exp(-2(n+1)t^2)$$
(68)

This implies that with probability  $1 - \delta/2$ , we get

$$\phi(s_0, \dots, s_n) - \mathbb{E}[\phi] < \sqrt{\frac{\log(2/\delta)}{2(n+1)}}$$
(69)

Moreover, it is well known that

$$\mathbb{E}[\phi] < 2\mathcal{R}_n(l \circ \Theta) \tag{70}$$

By equations (69) and (70), we have

$$\phi(s_0, \dots, s_n) = \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^n l(f_{\theta}(x_i), y_i) - \mathbb{E}[l(f_{\theta}(x), y)]] \right| < 2\mathcal{R}_n(l \circ \Theta) + \sqrt{\frac{\log(2/\delta)}{2(n+1)}}$$
(71)

### A.6 Proof of Theorem 4.5

**Asymptotic Results.** Note that the sample space is contained in  $\mathbb{R}^{d+1}$  and the parameter space  $\Theta \subseteq \mathbb{R}^p$ . Under regularity conditions, it is well known that  $\theta_n$  is asymptotically normal with covariance given by the inverse Fisher information matrix. We will see that  $\theta_{\text{aug},n}$  is also asymptotically normal with covariance.

**Theorem 4.5.** (Asymptotic normality) Assume  $\Theta$  is open, Assumptions 1, 3 and 4 hold. Then  $\theta_n$  and  $\theta_{\text{aug},n}$  admit the following Bahadur representation;

$$\sqrt{n+1}(\theta_n - \theta_*) = \frac{1}{\sqrt{n+1}} V_{\theta_*}^{-1} \sum_{i=0}^n \nabla l(f_{\theta_*}(x_i), y_i) + o_{\mathcal{P}}(1)$$

$$\sqrt{n+1}(\theta_{\text{aug},n} - \theta_*) = \frac{1}{\sqrt{n+1}} V_{\theta_*}^{-1} \sum_{i=0}^n \nabla \bar{l}(f_{\theta_*}(x_i), y_i) + o_{\mathcal{P}}(1)$$
(72)

Therefore, both  $\theta_n$  and  $\theta_{\text{aug},n}$  are asymptotically normal

$$\sqrt{n+1}(\theta_n - \theta_*) \to N(0, \Sigma_0) \text{ and } \sqrt{n+1}(\theta_{\text{aug},n} - \theta_*) \to N(0, \Sigma_{\text{aug}})$$
 (73)

where the covariance is given by

$$\Sigma_0 = V_{\theta_*}^{-1} \mathbb{E}[\nabla l(f_{\theta_*}(x), y) \nabla l(f_{\theta_*}(x), y)^T] V_{\theta_*}^{-1}$$

$$\Sigma_{\text{aug}} = \Sigma_0 - V_{\theta_*}^{-1} \mathbb{E}[XX^T] V_{\theta_*}^{-1}$$
(74)

where  $X = \nabla l(f_{\theta_*}(x), y) - \nabla \bar{l}(f_{\theta_*}(x), y)$ . As a consequence, the asymptotic relative efficiency of  $\theta_{\text{aug},n}$  compared to  $\theta_n$  is  $\text{RE} = \frac{tr(\Sigma_0)}{tr(\Sigma_{\text{aug}})} \geq 1$ 

**Proof of Theorem 4.5**. The results for  $\theta_n$  have already proven in [24, Theorem 5.23]. Lemma A.6 and the assumption  $S_{True} = S$  guarantee that the pair  $(\theta_* = \theta_{\text{aug}}, \bar{l})$  is applied to [24, Theorem 5.23]. Therefore  $\theta_{\text{aug},n}$  is asymptotically normal and satisfies equation (16). Let

$$X = \nabla l(f_{\theta_*}(x), y) - \nabla \bar{l}(f_{\theta_*}(x), y).$$

Then by Lemma A.5.3, we have

$$\Sigma_{\text{aug}} = \Sigma_0 - V_{\theta_n}^{-1} \mathbb{E}[XX^T] V_{\theta_n}^{-1}. \tag{75}$$

Since 
$$tr(V_{\theta_*}^{-1}\mathbb{E}[XX^T]V_{\theta_*}^{-1}) \geq 0$$
, we have  $RE = \frac{tr(\Sigma_0)}{tr(\Sigma_{ang})} \geq 1$ 

**Assumption 3.** (Regularity of the population risk minimizer) For the minimizer  $\theta_*$  of the population risk and any  $\epsilon > 0$ , we have

$$\sup_{\{\|\theta - \theta_*\| \ge \epsilon \mid \theta \in \Theta\}} \mathbb{E}[l(f_{\theta}(x), y)] > \mathbb{E}[l(f_{\theta_*}(x), y)]$$
(76)

**Assumption 4.** (Regularity of the loss function) For the loss function  $l(\theta, \cdot)$ , we assume that

1. uniform weak law of large number holds

$$\sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} l(f_{\theta}(x_i), y_i) - \mathbb{E}[l(f_{\theta}(x), y)] \right| \to 0$$

- 2. for each  $\theta$  in  $\Theta$ , the map  $(x,y) \to l(f_{\theta}(x),y)$  is measurable
- 3. the map  $\theta \to l(f_{\theta}(x), y)$  is differentiable at  $\theta_*$  for almost every (x, y)
- 4. for every  $\epsilon > 0$ , there exists a function  $l' \in L^2(\mathcal{P})$  such that for almost every (x, y) and for every  $\theta_1, \theta_2 \in N(\theta_0, \epsilon)$ , we have

$$|l(f_{\theta_1}(x), y) - l(f_{\theta_2}(x), y)| \le l'(x, y) \|\theta_1 - \theta_2\|$$
(77)

5. the map  $\theta \to \mathbb{E}[l(f_{\theta}(x), y)]$  admits a second-order Taylor expansion at  $\theta_*$  with non-singular second derivatives matrix  $V_{\theta_*}$ 

**Lemma A.6.** (Regularity of the augmented loss) Assume that  $S_{True} = S$ . If the pair  $(\theta_*, l)$  satisfies Assumptions 3 and 4, then the Assumptions hold for the pair  $(\theta_{\text{aug}}, \bar{l})$ .

*Proof.* By Lemma A.5, we have  $\mathbb{E}[l(f_{\theta}(x), y)] = \mathbb{E}[\bar{l}(f_{\theta}(x), y)]$  for any  $\theta \in \Theta$ . By the assumption  $S = S_{True}$ , we get  $\theta_* = \theta_{\text{aug}}$ . This gives rise to Assumption 3 for  $(\theta_{\text{aug}}, \bar{l})$ . We prove Assumption 4 for  $(\theta_{\text{aug}}, \bar{l})$  step by step. For part 1, we have

$$\begin{split} \sup_{\theta \in \Theta} & \left| \frac{1}{n+1} \sum_{i=0}^{n} \bar{l}(f_{\theta}(x_{i}), y_{i}) - \mathbb{E}[\bar{l}(f_{\theta}(x), y)]] \right| \\ &= \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}] - \mathbb{E}[\mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}]] \right| \\ &= \sup_{\theta \in \Theta} \left| \frac{1}{n+1} \sum_{i=0}^{n} \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)})] - \mathbb{E}[l(f_{\theta}(x), y)] \right| \\ &= \sup_{\theta \in \Theta} \left| \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S \setminus \{S_{(x_{1}, y_{1})}, \dots, S_{(x_{n}, y_{n})}\}] \right| \\ &= o_{\mathcal{P}}(1) \end{split}$$

where inequalities hold by Lemma A.5.3 and Assumption 1. For part 2,  $\bar{l}$  is measurable since l is measurable and  $\bar{l}(f_{\theta}(x), y) = \mathbb{E}[l(f_{\theta}(x'), y') \mid (x', y') \in S_{(x,y)}]$ . For part 3, we have

$$\begin{split} &\lim_{\delta \to 0} \frac{\left| \overline{l}(f_{\theta_* + \delta}(x), y) - \overline{l}(f_{\theta_*}(x), y) - \delta^T \nabla \overline{l}(f_{\theta_*}(x), y) \right|}{\|\delta\|} \\ &= \lim_{\delta \to 0} \frac{\left| \mathbb{E}[l(f_{\theta_* + \delta}(x'), y') - l(f_{\theta_*}(x'), y') \mid (x', y') \in S_{(x, y)}] - \delta^T \mathbb{E}[\nabla l(f_{\theta_*}(x'), y') \mid (x', y') \in S_{(x, y)}] \right|}{\|\delta\|} \\ &= \lim_{\delta \to 0} \frac{\left| \mathbb{E}[l(f_{\theta_* + \delta}(x'), y')] - l(f_{\theta_*}(x'), y') - \delta^T \nabla l(f_{\theta_*}(x'), y') \mid (x', y') \in S_{(x, y)}] \right|}{\|\delta\|} \\ &\leq \lim_{\delta \to 0} \frac{\mathbb{E}[\left| l(f_{\theta_* + \delta}(x'), y') - l(f_{\theta_*}(x'), y') - \delta^T \nabla l(f_{\theta_*}(x'), y') \right| \mid (x', y') \in S_{(x, y)}]}{\|\delta\|} \\ &\leq \mathbb{E}[\lim_{\delta \to 0} \left| \frac{l(f_{\theta_* + \delta}(x'), y') - l(f_{\theta_*}(x'), y') - \delta^T \nabla l(f_{\theta_*}(x'), y')}{\|\delta\|} \right| \mid (x', y') \in S_{(x, y)}] = 0 \end{split}$$

where the last inequality holds by Lebesgue's dominated convergence theorem. For part 4, we have

$$|\bar{l}(f_{\theta_{1}}(x), y) - \bar{l}(f_{\theta_{2}}(x), y)| = |\mathbb{E}[l(f_{\theta_{1}}(x'), y') | (x', y') \in S_{(x,y)}] - \mathbb{E}[l(f_{\theta_{2}}(x'), y') | (x', y') \in S_{(x,y)}]|$$

$$= |\mathbb{E}[l(f_{\theta_{1}}(x'), y') - l(f_{\theta_{2}}(x'), y') | (x', y') \in S_{(x,y)}]|$$

$$\leq \mathbb{E}[|l(f_{\theta_{1}}(x'), y') - l(f_{\theta_{2}}(x'), y')| | (x', y') \in S_{(x,y)}]$$

$$\leq \mathbb{E}[l'(f(x'), y') | (x', y') \in S_{(x,y)}](\|\theta_{1} - \theta_{2}\|)$$
(78)

where the last inequality holds by the assumption for the function l. If  $l' \in L^2(\mathcal{P})$ , then l'' defined by  $l''(f_{\theta}(x), y) = \mathbb{E}[l'(f_{\theta}(x), y) \mid (x', y') \in S_{(x,y)}]$  is also in  $L^2(\mathcal{P})$ .

For part 5, by Lemma A.5,  $\mathbb{E}[l(f_{\theta}(x), y)] = \mathbb{E}[\bar{l}(f_{\theta}(x), y)]$  which implies the conclusion.

# **B** Augmentation Training Procedures

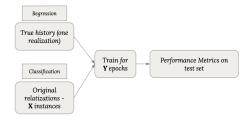


Figure 7: Training procedures for non Augmented case.

Figure 7 shows the non-augmented training for classification and regression problems, which follows the standard ML training procedure where we first train model using the training data and then evaluate model performance on test data.

Figure 8 shows the RIM training methodology for classification and for regression problems. For classification where we have limited but labeled time series, we first use augmented sample to learn the extrinsic features of the underlying system and then train it on the original data to learn the intrinsic features. For regression, we could have first used augmented sample for training as in classification. However, since in regression we only have one realization, it is

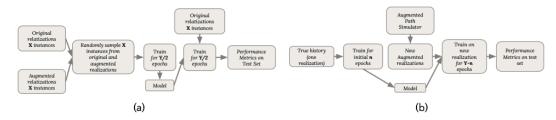


Figure 8: Training procedure for Classification (left) and Regression (right) with the Augmented case using the Recursive Interpolation Method (RIM).

not sufficient to learn the intrinsic features. As a result, in regression, we first train the ML model using the historical original time series to learn the intrinsic features and then use augmented sample to learn the extrinsic features for a better generalization.

# C Dataset, Setup, and RL Policy Deployment

#### C.1 Dataset

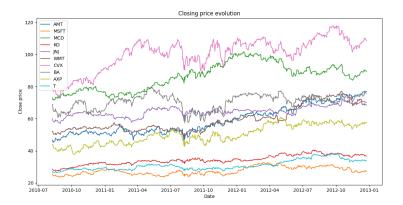


Figure 9: Evolution of stock prices.

For RL, the data comes from the Quandl finance database that has daily data. Our RL models are trained over 600 trading days from 2010-08-09 to 2012-12-25 and tested over 200 trading days from 2012-12-26 to 2013-10-11 on a portfolio consisting of ten selected stocks: American Tower Corp. (AMT), American Express Company (AXP), Boeing Company (BA), Chevron Corporation (CVX), Johnson & Johnson (JNJ), Coca-Cola Co (KO), McDonald's Corp. (MCD), Microsoft Corporation (MSFT), AT&T Inc. (T) and Walmart Inc (WMT). To promote the diversification of the portfolio, these stocks were selected from different sectors of the S&P 500, so that they are uncorrelated as much as possible as shown in Fig. 9.

# C.2 Setup

Experiments were run on a 24-core machine with 33GB of memory. All algorithms were implemented in Python using Keras and Tensorflow libraries. Each method is executed in an asynchronously parallel set up of 2 NVIDIA Quadro P5000 (16GB) GPUs, that is, it can evaluate multiple models in parallel, with each model on a single GPU. When the evaluation of one model finishes, the methods can incorporate the result and immediately re-deploy the next job without waiting for the others to finish.

### **C.3** Policy Deployment

In our RL experiments, the investment decisions to rebalance the portfolio are made daily and each input signal represents a multidimensional tensor that aggregates historical open, low, high, close prices and volume. It should be noted that our training and testing include the transaction costs (TC). We used the typical cost due to bid-ask spread and market impact that is 0.25%. We believe these are reasonable transaction costs for the portfolio trades.

#### C.4 RL Pseudocode

### **Algorithm 3:** RIM: RL Training

```
input :Observed time series data S = \{s_0, s_1, \dots, s_T\} where s_i = (x_i, y_i) for x_i \in \mathbb{R}^d and y_i \in \mathbb{R} for i \in [0:T].

1 Initialize \theta parameter for the policy network, Y epochs, and a distribution \mathcal{D} with support [0, 1).

1 Initialize \vec{\lambda} = 1 to Y do

2 Initialize \vec{\lambda} = (\lambda_1, \dots, \lambda_T) with \lambda_i \sim \mathcal{D} // Initialize interpolation coefficients vector

Augmented Path Simulator Generate an augmented trajectory S_{\vec{\lambda}} = \{s_0, s_{1,\lambda_1}, \dots, s_{T,\lambda_T}\} where s_{i,\lambda_i} = (x_{i,\lambda_i}, y_{i,\lambda_i})

for t = 1 to T do

3 | a_t \sim \pi(.|s_{t,\lambda_t}) | s_t' \sim p(.|s_{t,\lambda_t}, a_t) | UpdateCritic(S_t) | UpdateActor(S_t) // Data augmentation is applied to the samples for actor training as well end

9 end
```

# **D** Hyperparameters

We note that the primary objective of the conducted experiments is to show that RIM can improve model performance. Therefore, in all experiments, instead of finding optimal set of parameters for augmented and non-augmented trained model, we compare the performance of augmented trained model and non-augmented trained model with the same hyperparameter configuration. We demonstrated in Section 5 that RIM indeed improves model performance with same hyperparameter configuration. However, are these improvements robust to other hyperparameters? To answer this question, we conducted sensitivity analysis for two supervised learning tasks (Indoor user movement classification and Air Quality regression) and RL task (Portfolio Management).

### D.1 Hyperparameter Sensitivity for Supervised Tasks

For Indoor movement classification task, we conducted the same experiment in section 5 with 9 different hyperparameter configurations as shown in table 1 and observed that for all the cases RIM outperforms Non-Augmented case (with smaller mean test loss and higher mean test accuracy) which solidifies our claim of enhancement observed in model performance when we use RIM.

For Air quality regression task, we again conducted the same experiment as in section 5 with 8 different hyperparameter configurations as shown in table 2. Here too, we find that RIM outperforms Non-Augmented case (with smaller mean test MSE and higher mean test accuracy) which confirms our claim of enhancement observed in model performance when we use RIM.

# D.2 Hyperparameter Sensitivity for RL Task

The hyperparameter space is represented by a hypercube: the more values it contains the harder it is to explore all the possible combinations. To efficiently find the optimal set of hyperparameters, we explored the hyperparameter space using Bayesian optimization (BO) [25]. Table 3 shows the range of values for the hyperparameters used during the training and validation phase. The learning rate controls the speed at which neural network parameters are updated. The window is used to allow the deep RL agents to utilize a range of historical data values to relax the Markov assumption. We allow the use of 2 days up to 30 days of historical data. The number of filters and kernel strides are the

Table 1: In the table, the Test Loss and Test Accuracy are the mean(standard deviation) over 10 runs for 50 epochs for varying Filters and Kernel size for Indoor User Movement Classification Task

Filters	Kernel Size	Test Loss – RIM	Test Loss - NoAug	Test Acc – RIM	Test Acc - NoAug
16	3	1.09 (0.43)	2.72 (0.95)	0.71 (0.04)	0.63 (0.02)
16	4	0.90 (0.45)	1.70 (0.69)	0.76 (0.04)	0.68 (0.04)
16	5	0.86 (0.19)	1.00 (0.19)	0.72 (0.01)	0.60 (0.05)
32	3	1.37 (0.31)	3.07 (0.67)	0.72 (0.02)	0.62 (0.05)
32	4	1.75 (0.71)	2.70 (1.20)	0.74 (0.03)	0.68 (0.03)
32	5	1.70 (0.70)	2.57 (0.85)	0.66 (0.08)	0.64 (0.06)
64	3	2.99 (2.17)	3.31 (1.36)	0.68 (0.07)	0.68 (0.06)
64	4	2.60 (0.70)	4.80 (4.30)	0.73 (0.03)	0.68 (0.02)
64	5	3.30 (0.74)	4.90 (4.08)	0.66 (0.03)	0.60 (0.06)

Table 2: In the table, the MSE and Accuracy are the mean(standard deviation) over 10 runs for 50 epochs for varying Epoch Initial, Batch Size, LSTM Layer and Dense Layer for Air Quality Regression Task on Test Data

<b>Epoch Init</b>	<b>Batch Size</b>	LSTM Layer	Dense Layer	MSE – RIM	MSE - NoAug	Acc – RIM	Acc - NoAug
5	16	100	50	5.34 (0.11)	5.43 (0.25)	0.63 (0.03)	0.47 (0.08)
1	32	80	40	5.40 (0.08)	5.44 (0.04)	0.63 (0.05)	0.51 ( 0.02)
5	16	150	50	5.59 (0.01)	5.74 (0.35)	0.66 (0.03)	0.53 (0.10)
6	32	200	100	5.23 (0.10)	5.55 (0.19)	0.64 (0.02)	0.51 (0.06)
10	16	150	100	5.32 (0.05)	5.48 (0.11)	0.57 (0.02)	0.51 (0.35)
1	64	200	100	5.59 (0.07)	5.67 (0.23)	0.65 (0.02)	0.52 (0.03)
10	16	100	50	5.44 (0.24)	5.49 (0.38)	0.59 (0.05)	0.56 (0.07)
15	16	100	50	5.56 (0.08)	5.60 (0.16)	0.65 (0.01)	0.56 (0.02)

hyperparameters for the convolution neural networks. It is important to carefully optimize these parameters in order to capture the best feature representations used by the policy networks. Finally, the training and testing sizes may also impact the RL performance. So, we also consider them as hyperparameters.

Table 3: Hyperparameters used by our RL algorithms.

Table 3. Hyperparameters used by our RE argorithms.						
Parameters	Bounds	Type				
Learning rate (lr)	$10^{-5} - 5.10^{-1}$	Discrete				
Trading cost (tc)	2–30	Discrete				
Number of filters (nf)	2–52	Discrete				
Kernel Strides (ks)	2–10	Discrete				
Window	2-30	Discrete				
Training size (train)	20-500	Discrete				
Testing size (test)	5–100	Discrete				

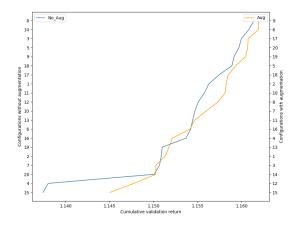


Figure 10: Hyperparameter sensitivity for DPG: the vertical axes list all the models we evaluated by its index. The detailed hyperparameter configurations each index refers to are listed in Tables 4 and 5. The horizontal axis shows the cumulative total validation return. The blue line shows the validation performance for DPG without augmentation. The orange line shows the validation performance for DPG using RIM. The worst to best models are ordered from bottom to top.

Table 4: DPG configurations without augmentation

lr	tc	nf	ks	window	train	test	validation cumulative return
0.001	0.0225	[48, 20, 1710]	9	19	50	10	1.1563
1.0	0.0125	[50, 38, 1340]	3	6	500	100	1.1509
0.0005	0.0125	[14, 6, 1300]	4	16	50	50	1.1600
0.01	0.0125	[28, 28, 210]	4	13	100	10	1.1380
0.01	0.0175	[16, 2, 1080]	3	9	50	5	1.1597
0.001	0.0125	[18, 12, 1290]	8	16	50	50	1.1542
1.0	0.0175	[20, 20, 1090]	6	16	50	10	1.1507
0.0005	0.0075	[18, 2, 1270]	3	16	50	50	1.1614
1,00E-05	0.0225	[10, 10, 1310]	8	9	50	50	1.1591
0.1	0.0225	[10, 10, 770]	2	17	200	20	1.15470
5,00E-05	0.0075	[12, 16, 1300]	7	17	50	50	1.1558
0.05	0.0125	[16, 10, 1090]	3	7	50	5	1.1551
0.0001	0.0075	[24, 12, 1080]	5	15	50	5	1.1544
0.1	0.0175	[44, 2, 1500]	2	11	20	10	1.16082
0.005	0.0025	[2, 42, 270]	8	16	500	50	1.1374
5,00E-05	0.0175	[24, 10, 1090]	2	11	50	5	1.1537
1.0	0.0175	[32, 12, 1490]	2	8	20	20	1.1574
0.5	0.0075	[46, 6, 950]	4	18	20	20	1.1589
0.0001	0.0025	[30, 26, 1190]	7	9	500	100	1.1510
0.01	0.0025	[50, 16, 940]	7	15	20	5	1.1501

Table 5: DPG configurations with augmentation

	Table 5: DPG configurations with augmentation						<u> </u>
lr	tc	nf	ks	window	train	test	validation cumulative return
0.005	0.0175	[36, 36, 1960]	4	9	20	10	1.1513
0.0001	0.0025	[22, 14, 1270]	9	10	50	50	1.1582
0.1	0.0125	[40, 22, 880]	3	6	20	10	1.1503
1,00E-05	0.0075	[8, 16, 760]	2	6	200	20	1.1521
0.005	0.0175	[44, 6, 1500]	5	6	20	20	1.1593
1,00E-05	0.0025	[2, 2, 1310]	2	19	50	50	1.1619
0.001	0.0125	[36, 6, 960]	3	17	20	20	1.1560
0.0001	0.0075	[22, 6, 770]	9	13	200	20	1.1573
1,00E-05	0.0025	[22, 2, 1320]	2	19	50	50	1.1619
0.5	0.0225	[28, 2, 970]	7	12	20	5	1.1581
0.5	0.0125	[40, 14, 960]	9	18	20	10	1.1547
0.01	0.0175	[32, 16, 760]	5	7	200	20	1.1478
0.01	0.0025	[24, 10, 780]	7	16	200	5	1.1517
0.005	0.0025	[46, 12, 770]	9	19	200	20	1.1502
0.01	0.0225	[48, 44, 1190]	4	10	100	50	1.1450
1.0	0.0175	[28, 8, 960]	6	13	20	5	1.1541
0.005	0.0125	[36, 2, 1500]	3	15	20	5	1.1607
0.005	0.0225	[18, 8, 1270]	5	9	50	50	1.1585
1.0	0.0125	[8, 6, 1260]	7	8	50	50	1.1604
1.0	0.0025	[28, 2, 1250]	2	7	50	50	1.1607