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Ecole Nationale Supérieure des Mines SAINT-ETIENNE

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Mathematical Morphology

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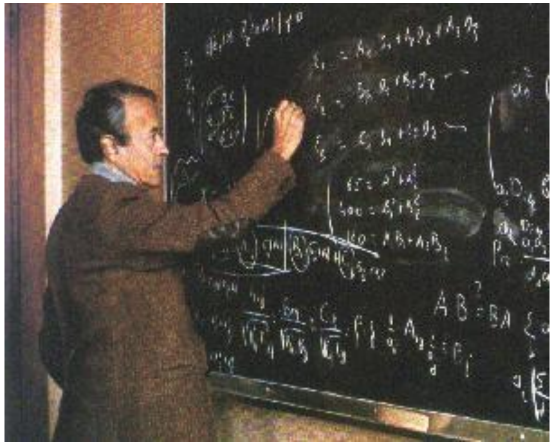
MINISTÈRE DE L'ÉCONOMIE DE L'INDUSTRIE ET DE L'EMPLOI

History

- **Morphology**
  - The word “morphology” stems from the Greek words ‘morphé ‘and ‘logos’, meaning “the study of forms”
  - It is used in a large number of scientific disciplines including biology and geography
- **Origin**
  - Invented in the early 1960s by Georges Matheron and Jean Serra
  - Study of porous media (geostatistics), automatic analysis of images occurring in mineralogy and petrography



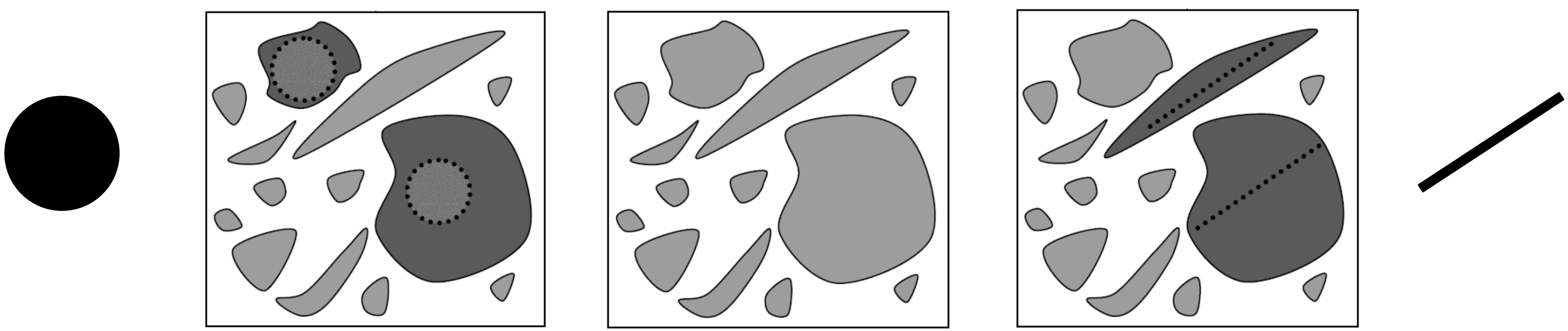
G. Matheron (1930-2000)

J. Serra (1940-~)

Principle of Mathematical Morphology (MM)

- **Study the Shape of Objects (Images)**
  - Geometry
  - Topology
  - Neighborhood information

- **Key Idea**
  - Extracting knowledge from the relation of an image and a simple, small probe (called the structuring element), which is a predefined shape.
  - It is checked in each pixel, how does this shape match or miss local shapes in the image.



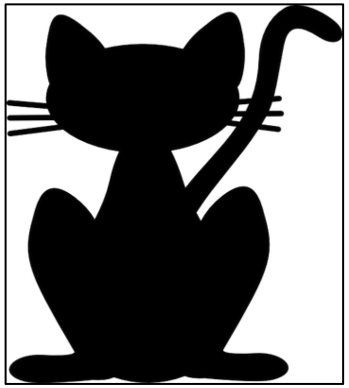
Morphological Image Processing

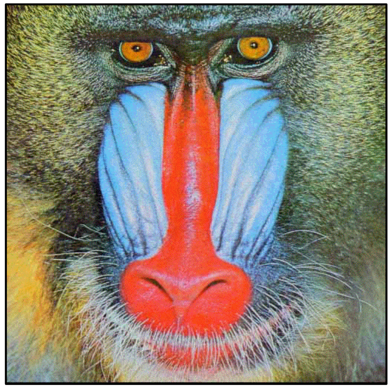
- **Tools for**
  - Filtering
  - Segmentation
  - Measurements
  - Texture analysis
  - Shape recognition
  - Scene interpretation

- **Mathematical Bases**
  - Set theory
  - Lattice algebra

- **Application in Various Domains**
  - Material sciences
  - Process engineering
  - Biomedical imaging
  - …

- **Nature of MM**
  - Set-Valued (binary images)
  - Functional (gray level images)
  - Multivariate (color/multispectral images)
  - Tensorial (tensor images)





Basic Structures

Linear Image Processing

- Vector space
- V: set of vectors
- K: set of scalars

- K: field
- V: commutative group
- Additive law
- Multiplicative law

Mathematical Morphology

- Complete lattice
- L: set of elements
- $\leq$: binary relation

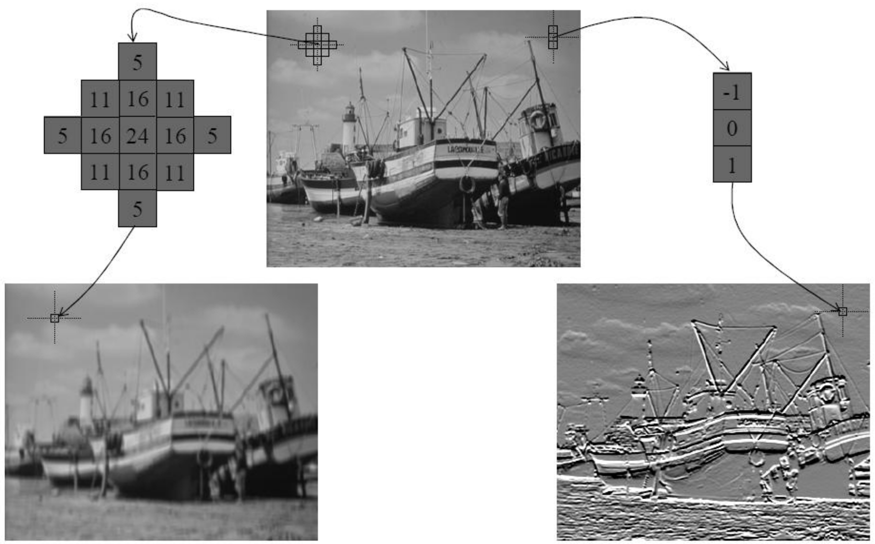
- Partial ordering
  - $A \leq A$ (reflexivity)
  - $A \leq B, B \leq A \Rightarrow A = B$ (antisymmetry)
  - $A \leq B, B \leq C \Rightarrow A \leq C$ (transitivity)
- Existence of an infimum
- Existence of a supremum

completeness

Basic Operations

- **Linear Image Processing**
  - Basic operators: those which preserve these laws

    $\varphi(\sum_i \lambda_i f_i) = \sum_i \lambda_i \varphi(f_i)$ / linearity
  - Resulting operator: convolution



- **Mathematical Morphology**
  - Basic operators: those which preserve these laws

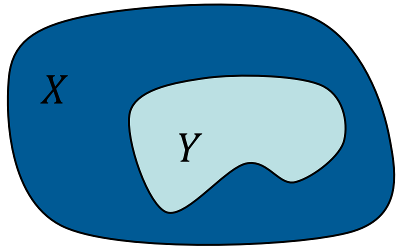
    $X \leq Y \Rightarrow \varphi(X) \leq \varphi(Y)$ / ordering preservation

    $\varphi(\vee X_i) = \vee \varphi(X_i)$ / commutation under sup.

    $\varphi(\wedge X_i) = \wedge \varphi(X_i)$ / commutation under inf.
  - Resulting operators: erosion and dilation

Examples of Complete Lattice

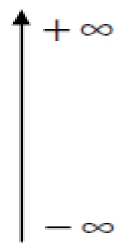
- **Lattice of Subsets**
  - Partial ordering: $\subseteq$
  - Sup: $\cup$
  - Inf: $\cap$



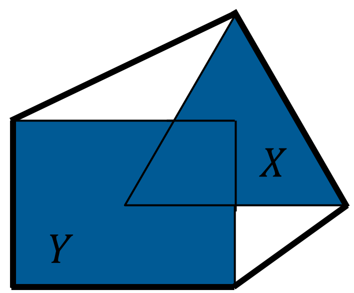
$Y \subseteq X$

Set-Valued Morphology (Binary Images)

- **Lattice of Real Numbers**
  - Total ordering: $\leq$
  - Sup: $\vee$ (usual sense)
  - Inf: $\wedge$ (usual sense)



- **Lattice of Convex Sets**
  - Partial ordering: $\subseteq$
  - Sup: convex hull of the union
  - Inf: $\cap$

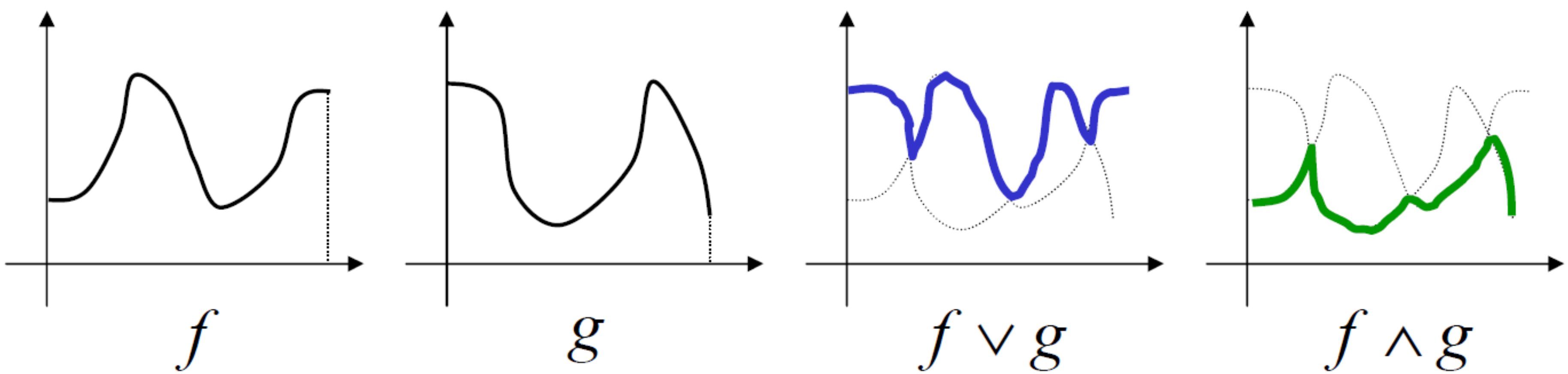


$\mathrm{CH}(X \cup Y)$

Examples of Complete Lattice

- **Lattice of Functions**
  - Partial ordering: ≤ $f \leq g \Leftrightarrow f(x) \leq g(x)$
  - Sup: ∨ $(\vee f_i)(x) = \vee (f_i(x))$
  - Inf: ∧ $(\wedge f_i)(x) = \wedge (f_i(x))$

Functional Morphology (Gray-Level Images)



Duality Principle in Lattices

- **Symmetrical Role of Sup. and Inf.**

- **Involution**
  - $\bar{.}: L \rightarrow L$
  - Function that is its own inverse: $\bar{\bar{X}} = X$
  - Operator that permutes Sup. and Inf.

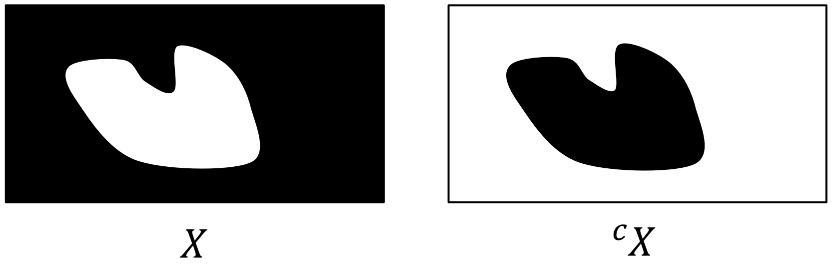
$$\mathrm{Sup}(\bar{X}) = \overline{\mathrm{Inf}\,(X)} \qquad \mathrm{Inf}(\bar{X}) = \overline{\mathrm{Sup}\,(X)}$$

- **Duality**
  - Two operators φ and φ^* are dual with respect to the involution $\bar{.}$ iff:

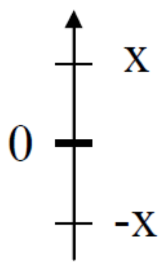
$$\varphi(\bar{X}) = \overline{\varphi^*(X)}$$

Examples of Involution

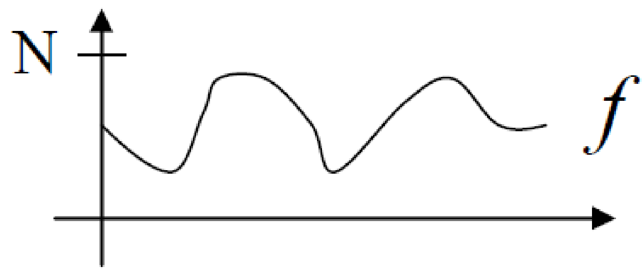
- **Lattice of Subsets**
  - Complement (foreground/background)

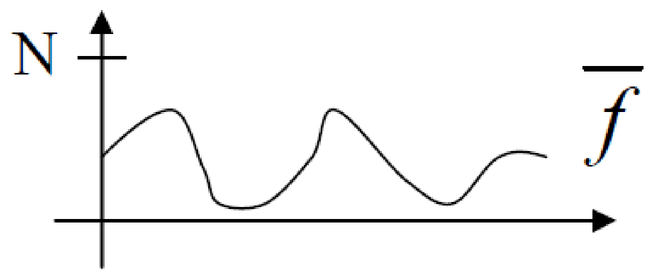


- **Lattice of Real Numbers**
  - Opposite



- **Lattice of Functions in [0,M]**
  - Reflexion with respect to M/2





Self Duality

- **Linear Image Processing**
  - The convolution operator is self-dual, that is dual of itself:

$$f * (-g) = -(f * g)$$

  - This means that positive or negative (bright and dark) components are processed in a symmetrical way

- **Mathematical Morphology**
  - The fundamental duality between Sup and Inf translates to all morphological tools
  - In general, morphological operations go py pair and correspond to each other by duality (erosion/dilation, opening/closing)
  - However, operators may also be
    - Self-dual:

$$\varphi(\bar{X}) = \overline{\varphi(X)}$$ (e.g. morph. centre)

    - Invariant under duality:

$$\varphi(\bar{X}) = \varphi(X)$$ (e.g. boundary set)

Fundamental Properties of Operators

- **Increasing**

$$X \leq Y \Leftrightarrow \varphi(X) \leq \varphi(Y)$$

- **Extensivity**

$$X \leq \varphi(X)$$

- **Anti-Extensivity**

$$\varphi(X) \leq X$$

- **Idempotence**

$$\varphi\big(\varphi(X)\big) = \varphi(X)$$

Lattices of Operators

- **For every lattice *L*, the class *L'* of the operations $\varphi: L \rightarrow L$ is a sub-lattice iff:**

$$\varphi \leq \xi \ (\text{in } L') \Leftrightarrow \varphi(X) \leq \xi(X) \ \text{ for all } X \in L$$

$$(\vee\, \varphi_i)(X) \ (\text{in } L') = \vee\, \varphi_i(X) \ \ (\text{in } L)$$

$$(\wedge\, \varphi_i)(X) \ (\text{in } L') = \wedge\, \varphi_i(X) \ \ (\text{in } L)$$

- **Examples**
  - Mappings which are increasing and extensive
  - Mappings which are increasing and anti-extensive

- **More Generally**
  - Openings
  - Filters
  - …

Scaling and Operators

- **It is often neccessary to modify the analyzing scale, i.e. to perform a space similitude $X \to \lambda X$. Two situations can be distinguised:**

- **Commutation under Scaling**
  - Examples: boundary, skeleton, center of gravity

$$\varphi(X) = \frac{1}{\lambda}\varphi(\lambda X)$$

- **Compatibility with Scaling**
  - Examples: granulometries, alternate sequantial filters

$$\varphi(X) = \frac{1}{\lambda}\varphi_\lambda(\lambda X)$$

  - Modeling of multiscale decomposition (pyramids, scale-space)

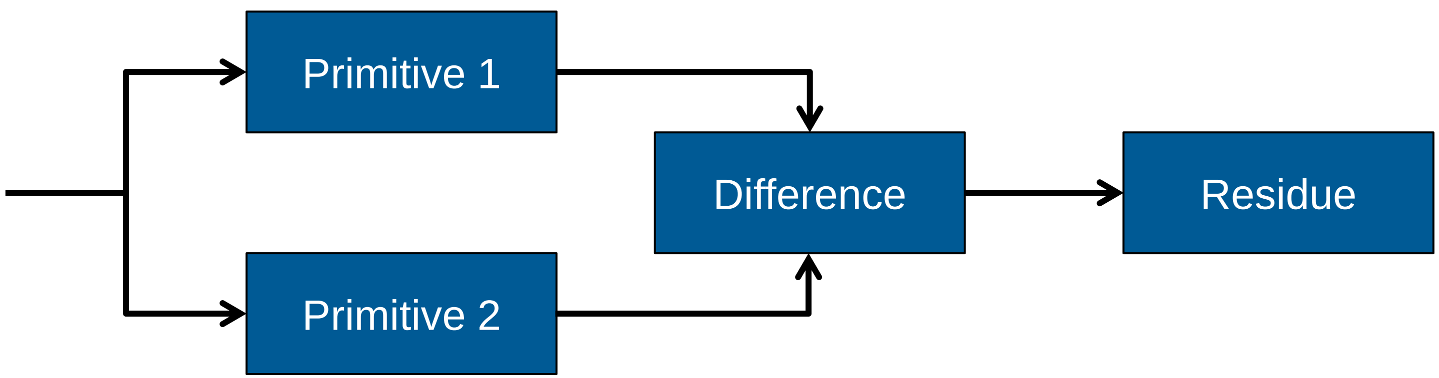
Four Principles of Mathematical Morphology

- **Compatibility with Translation**
  - The morphological operator should not depend on the translation
- **Compatibility with Scaling**
  - The morphological operator should not depend on the scale
- **Local Knowledge**
  - The morphological operator is a local operator (probe/structuring element)
- **Continuity (Semi-Continuity)**
  - The morphological operator should not exhibit abrupt changes of its behavior

Notion of Residues

- **The theory of morphological filters has highlighted**
  - The increasing property
  - The idempotence property
  - The ordering rules between transformations

- **There is a family of transformations which study**
  - The difference between two (or many) basic transformations



Classification of Residues

- **Main Classification in 3 Groups**

Residues

Two primitives
φ, ξ

Examples
- Gradient
- Top hat

Two families of primitives
$\{\varphi_i\}, \{\xi_i\}$

Examples
- Ultimate erosion
- Skeleton

Residue with 'hit or miss' transformations

Examples
- Thinning
- Thickenning
- SKIZ

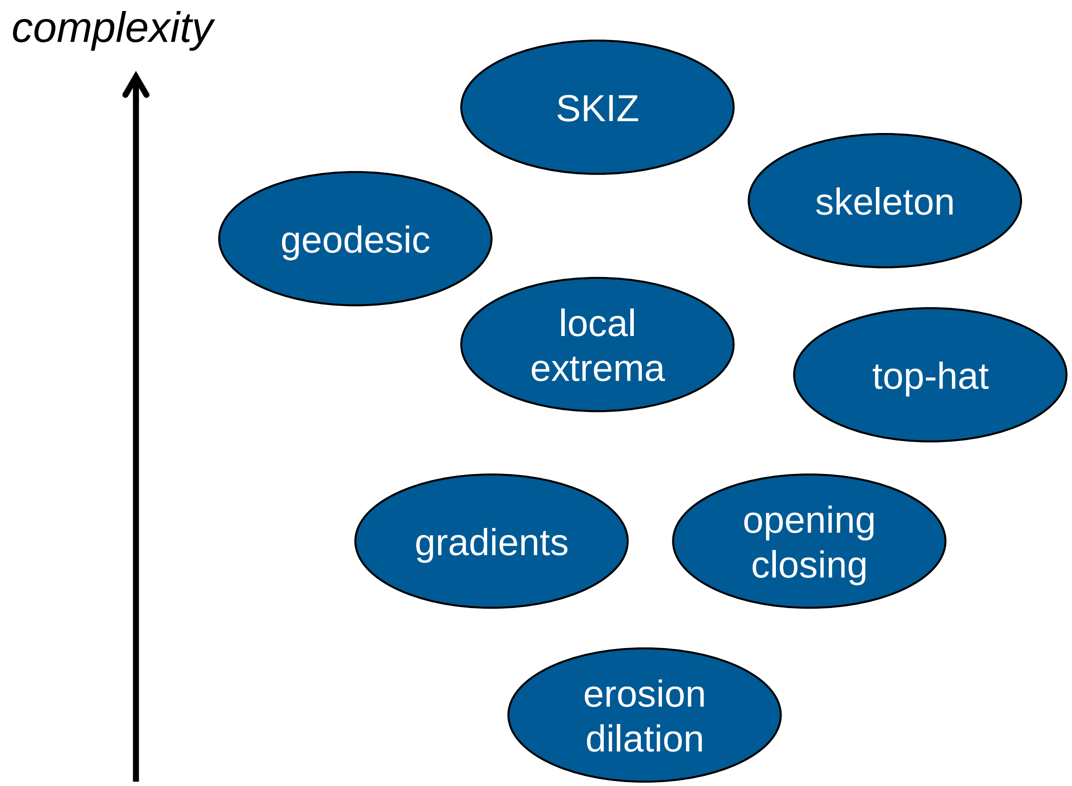
Construction of Morphological Operators

- **Using Composition**

$$\varphi(x) = \xi(\psi(x))$$

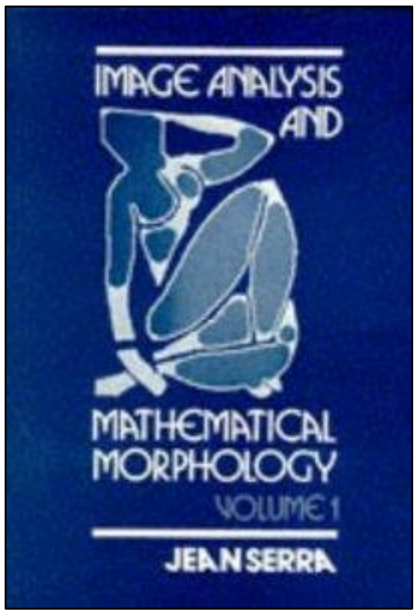
- **Using Residue**

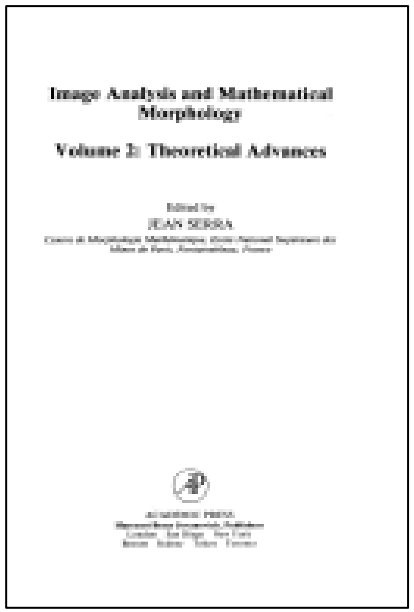
$$\varphi(x) = \xi(x) - \psi(x)$$

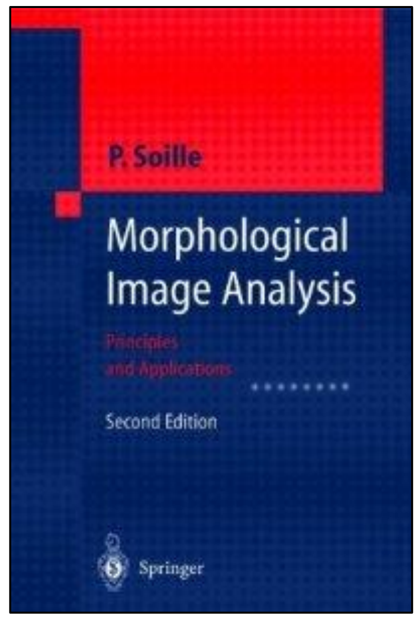


A Few References

- **J. Serra**, *Image Analysis and Mathematical Morphology, Volume I*, Academic Press, New-York, 1982.
- **J. Serra** (Ed.), *Image Analysis and Mathematical Morphology, Part II: Theoretical Advances*, Academic Press, London, 1988.
- **P. Soille**, *Morphological Image Analysis, 2^{nd} Edition*, Springer-Verlag, Berlin, 2004.







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Binary Mathematical Morphology

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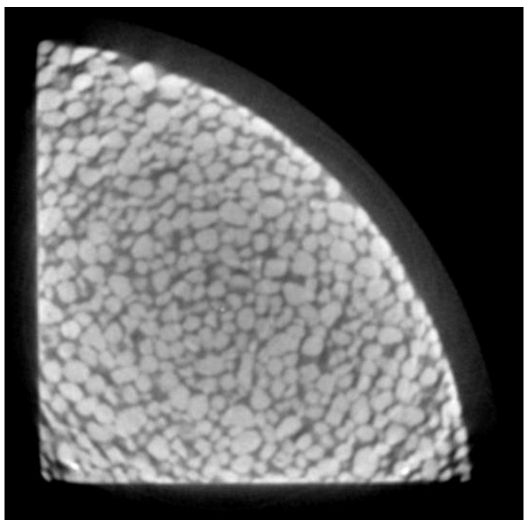
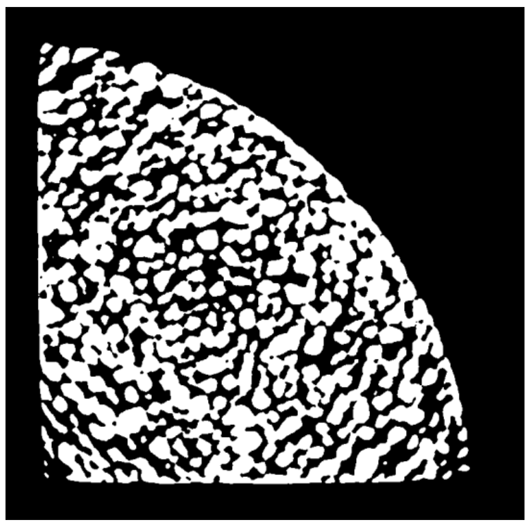
Binary Morphology

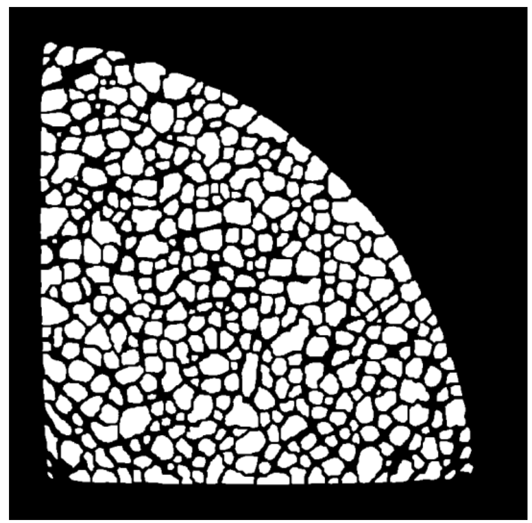
- **Mainly used for**
  - Object segmentation/recognition
    - Binary images often suffer from noise
    - Binary regions also suffer from noise (holes, cracks, protusions…)
  - Object measurements



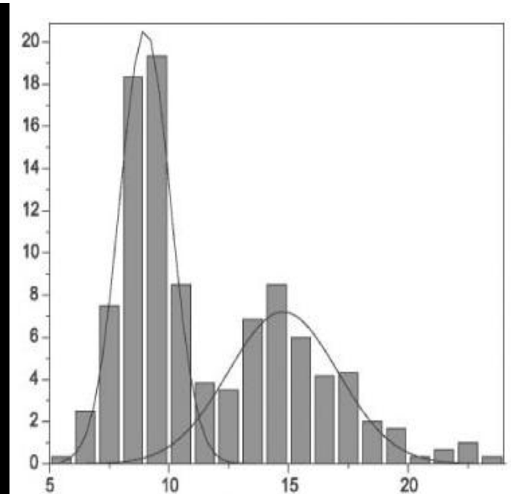
Original image
of sand grains
(X-ray tomography)



Thresholded image

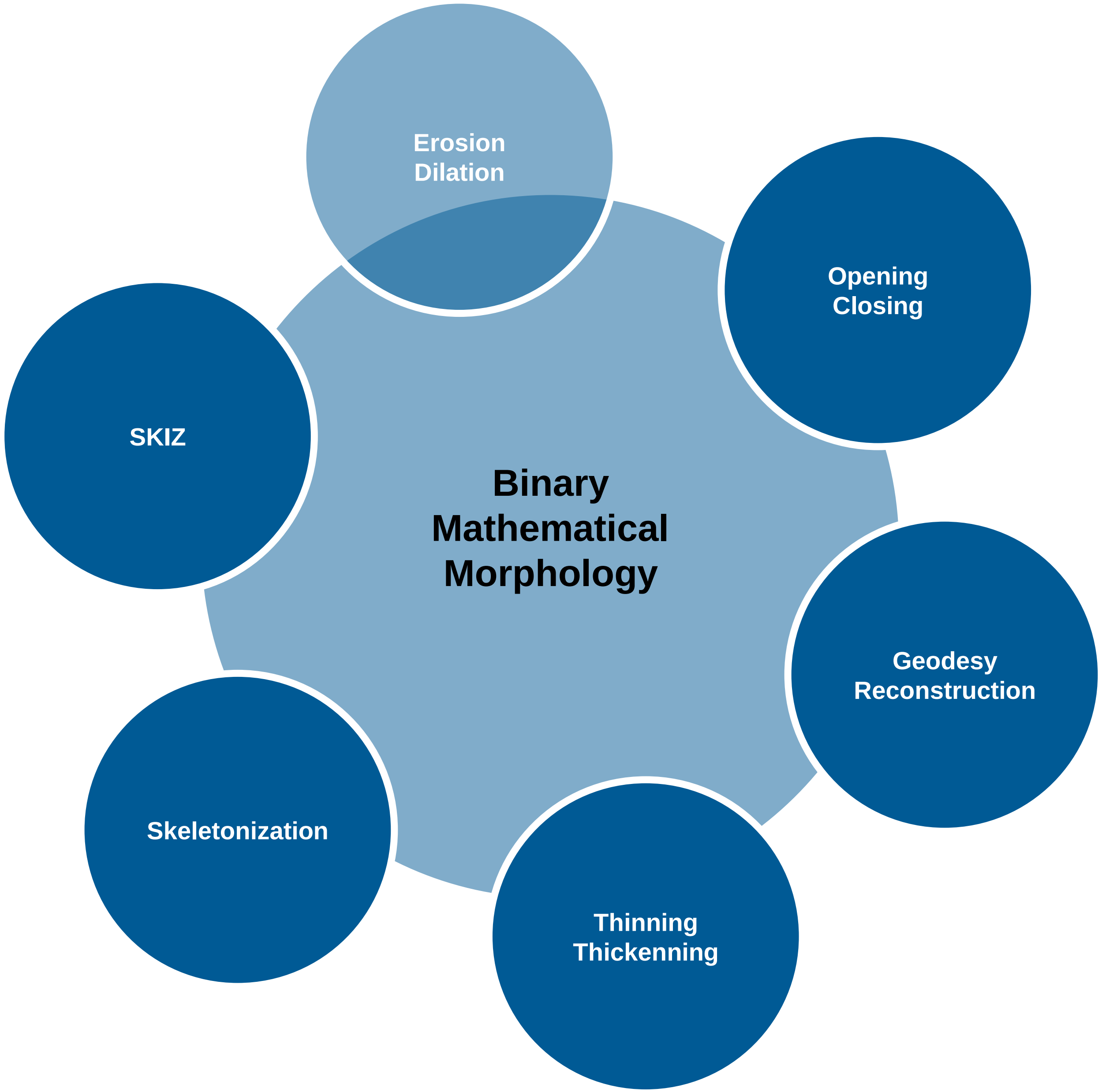


Segmented image



Quantification

Binary Mathematical Morphology

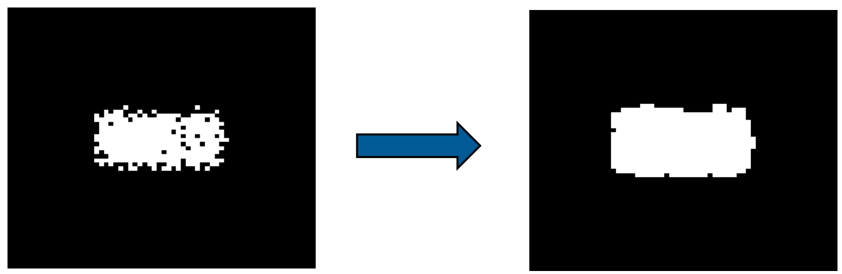


Erosion and Dilation

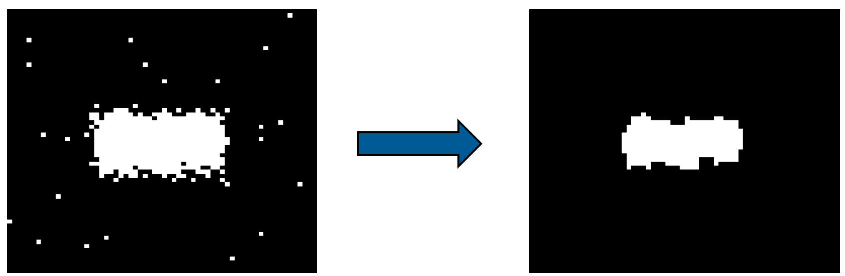
- **Primary Operations of MM**

- **More Complicated Operators can be designed by means of combining Dilations and Erosions**

- **Dilation**
  - Fills holes
  - Smoothes object boundaries
  - Objects become slightly larger



- **Erosion**
  - Removes isolated noisy pixels
  - Smoothes object boundaries
  - Objects become slightly smaller



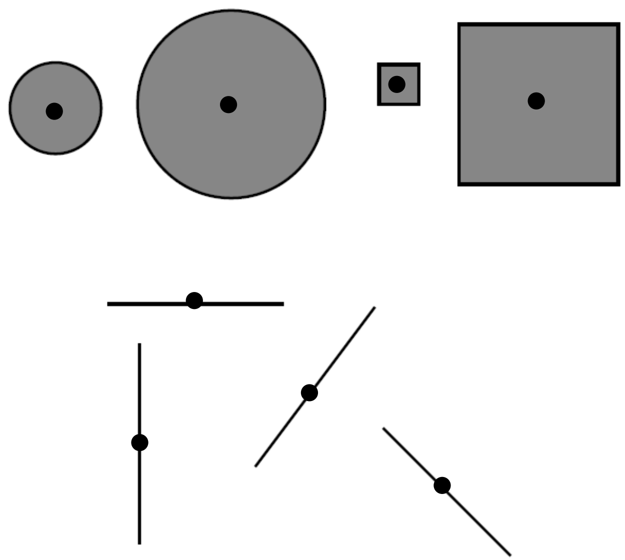
- **Use of a Structuring Element (SE)**

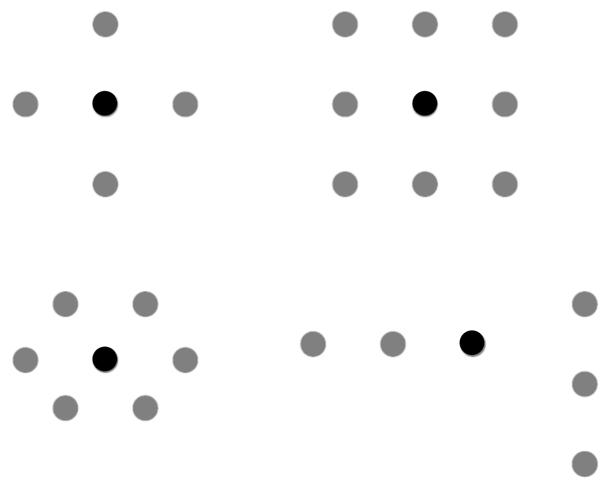
Structuring Element (SE)

- **Various Shape, Size**

- **Origin**
  - Not necessarily inside the SE

- **Continuous and Discrete**





Erosion

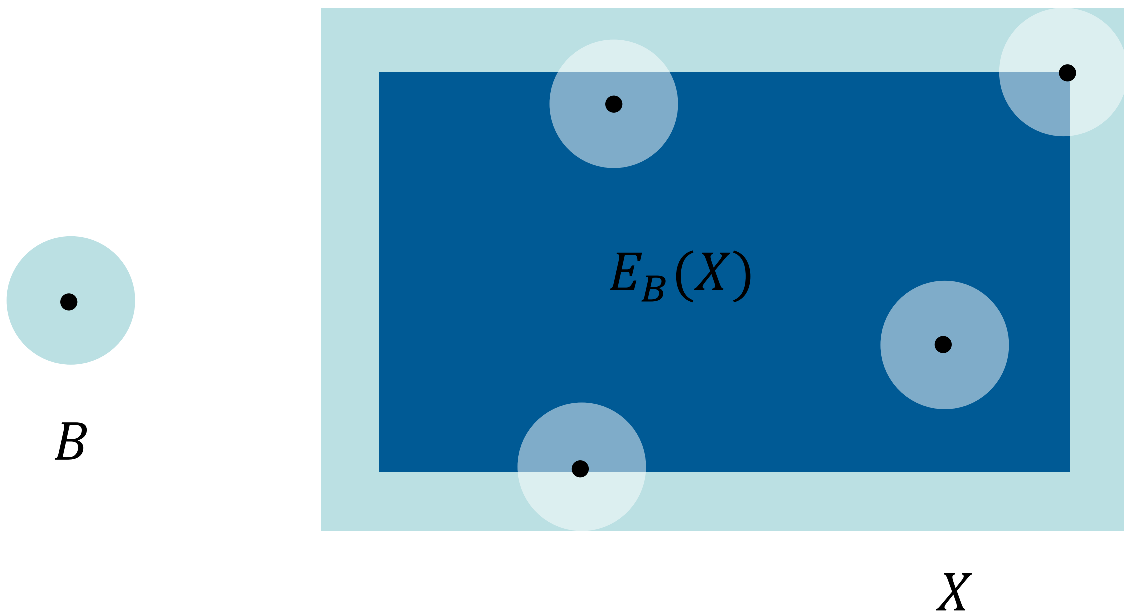
- **Does the SE Match the Set?**

- **Definition**

$$E_B(X) = \{x;\ B_x \subseteq X\}$$

where $B_x = \{b + x; x \in X\}$

- **Illustration**

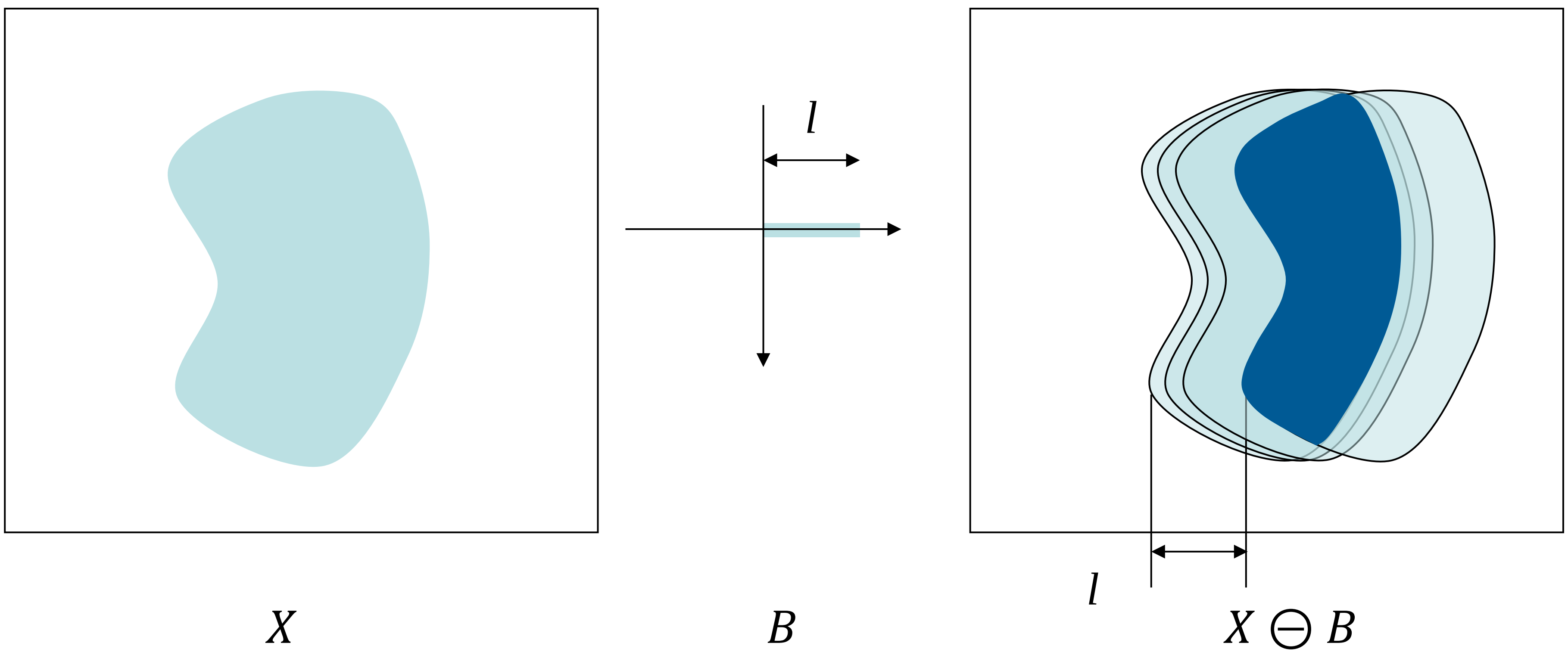


Erosion

- **Link with the Minkowski Subtraction**

$$S \ominus B = \bigcap_{b \in B} S_b$$

where $S_b = \{s + b; s \in S\}$



Erosion

- **Link with the Minkowski Subtraction**

$$E_B(X) = \{x;\ B_x \subseteq X\}$$

$$= \{x;\ \forall b \in B, x + b \in X\}$$

$$= \{x; \forall b \in B, x \in X_{-b}\}$$

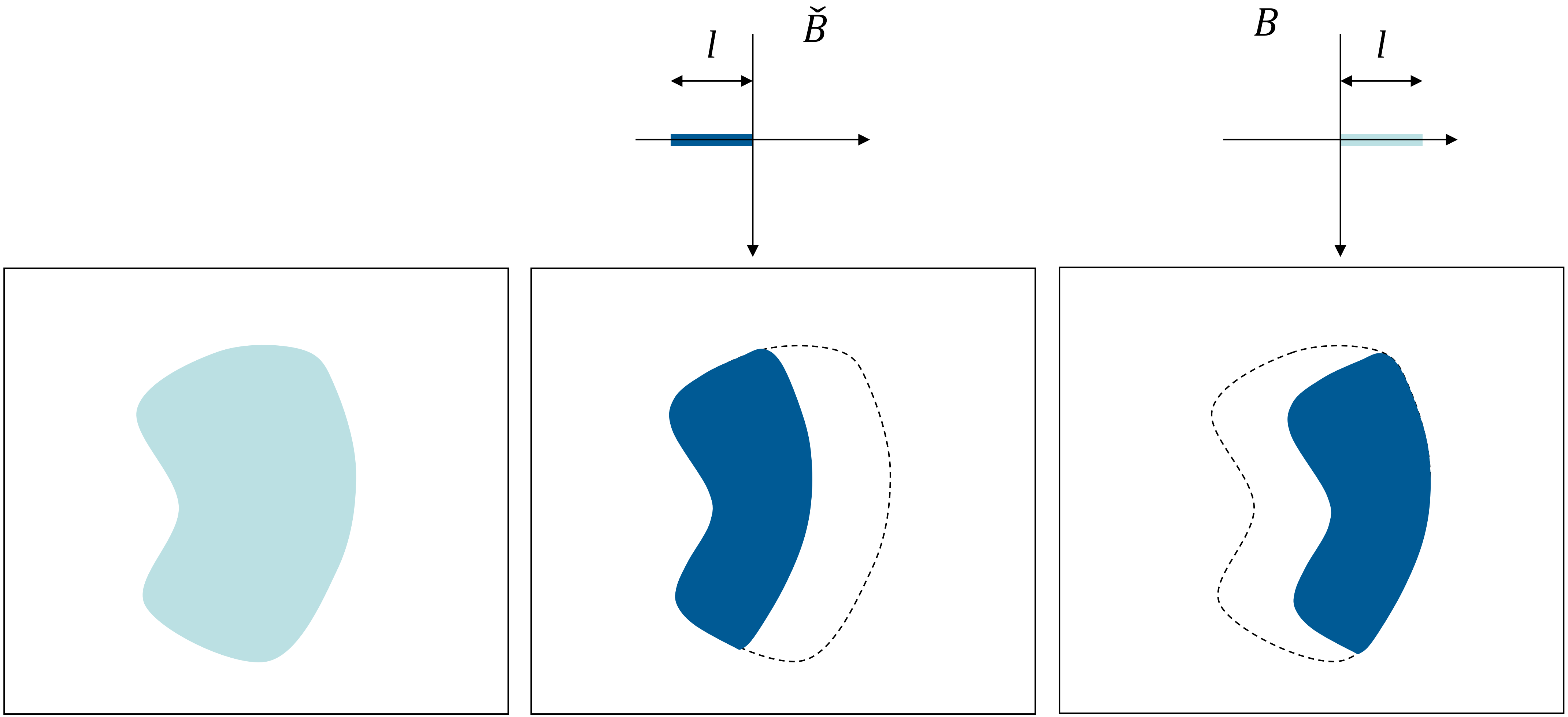
$$= \left\{x; \forall b \in \check{B}, x \in X_b\right\}$$ where $\check{B} = \{-b; b \in B\}$ (transposed/reflected SE)

$$= \bigcap_{b \in \check{B}} X_b$$

$$= \mathrm{X} \ominus \check{B}$$

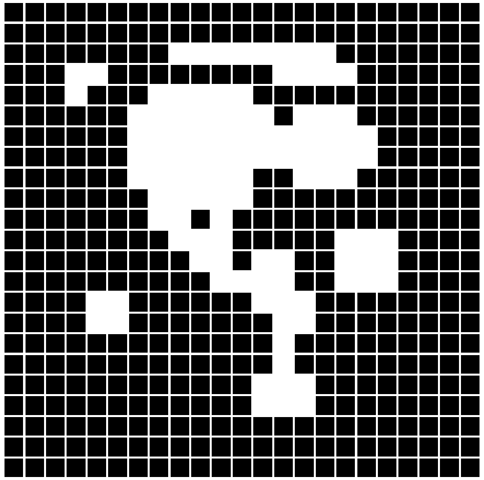
Erosion

- **Erosion is not the Minkowski Subtraction!**



X $\quad$ $E_B(X) = X \ominus \check{B}$ $\quad$ $X \ominus B$

Erosion with Various Structuring Elements



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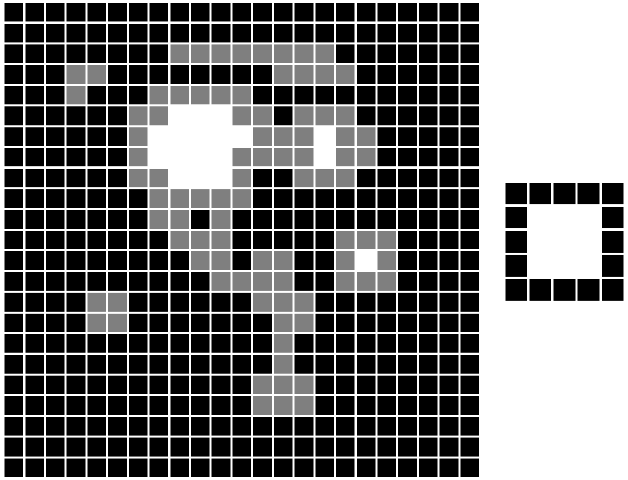
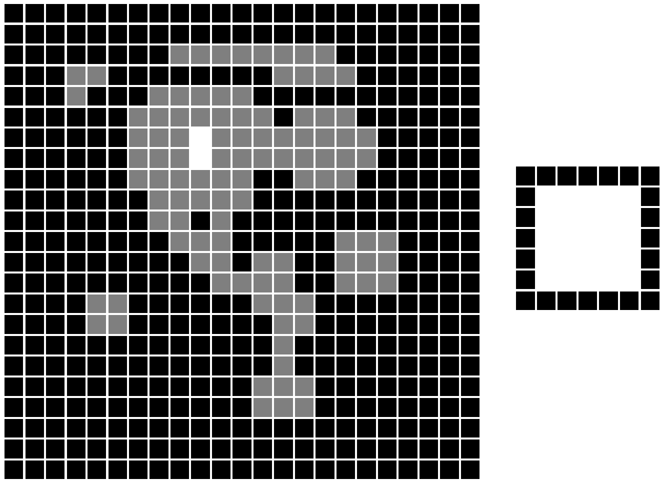
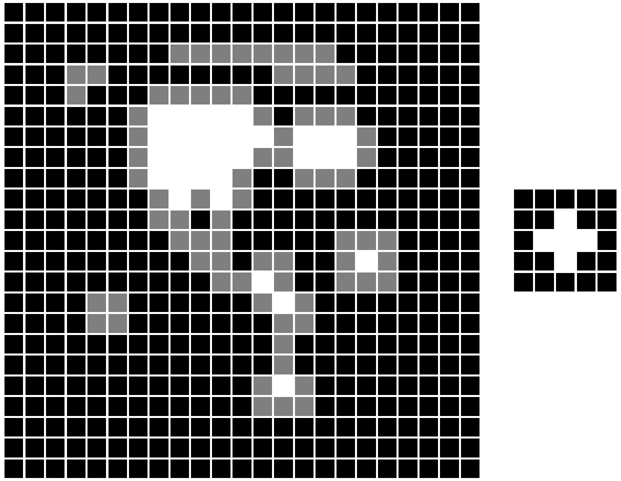
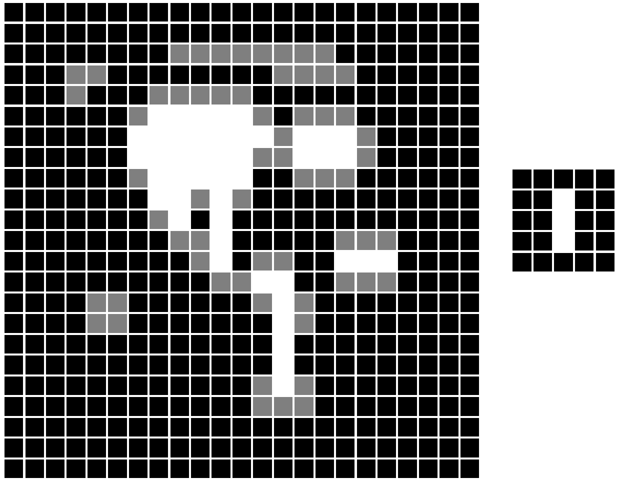
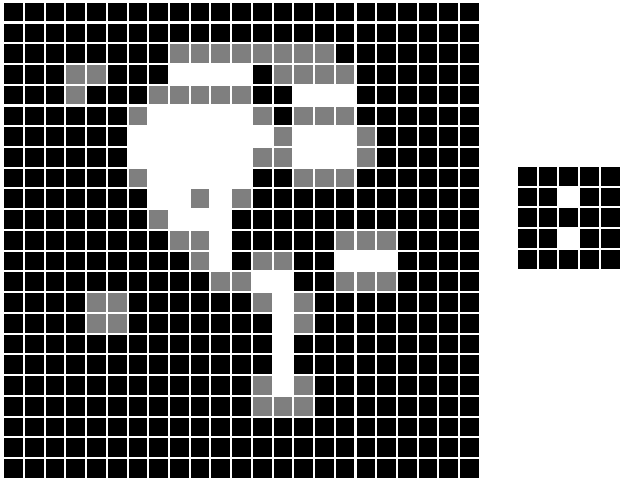






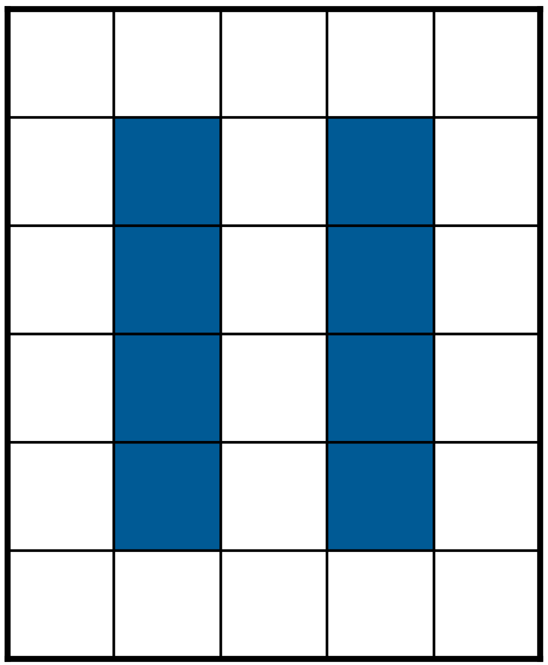




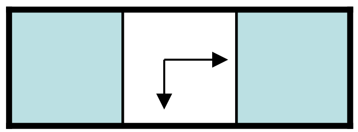
Properties of Erosion

- **Anti-Extensivity**

  $E_B(X) \subseteq \mathrm{X} \;\text{ if } 0 \in B$

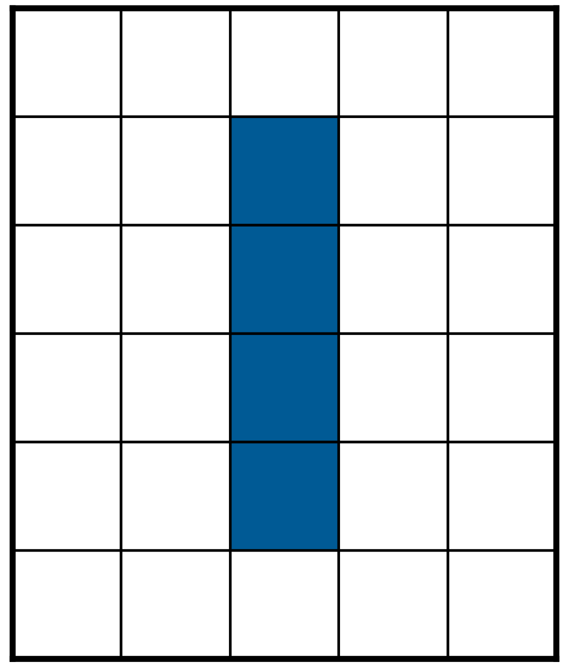


X



B

$0 \notin B \text{ and } E_B(X) \nsubseteq \mathrm{X}$

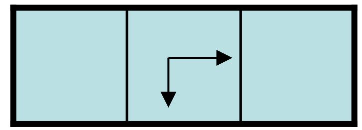


$E_B(X)$

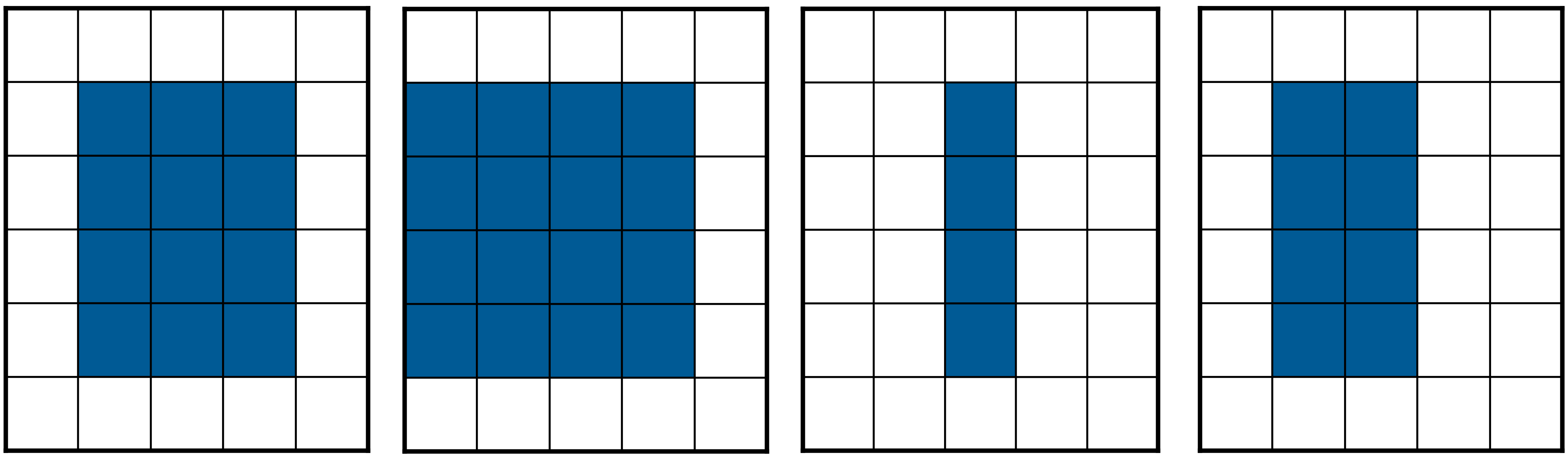
Properties of Erosion

- **Increasing**

$$X \subseteq Y \Rightarrow E_B(X) \subseteq E_B(Y)$$



B

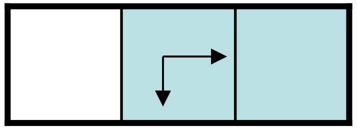


X Y $E_B(X)$ $E_B(Y)$

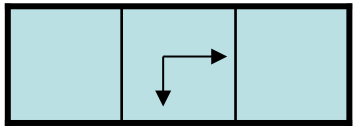
Properties of Erosion

- **Decreasing with respect to the SE**

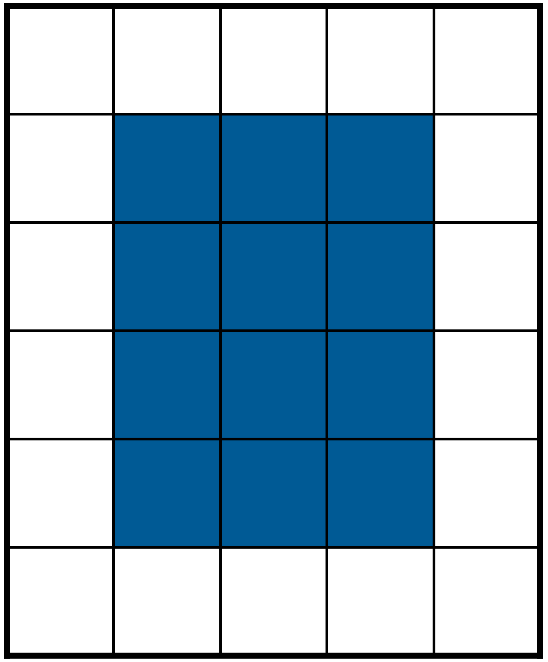
$$B \subseteq B' \Rightarrow E_{B'}(X) \subseteq E_B(X)$$



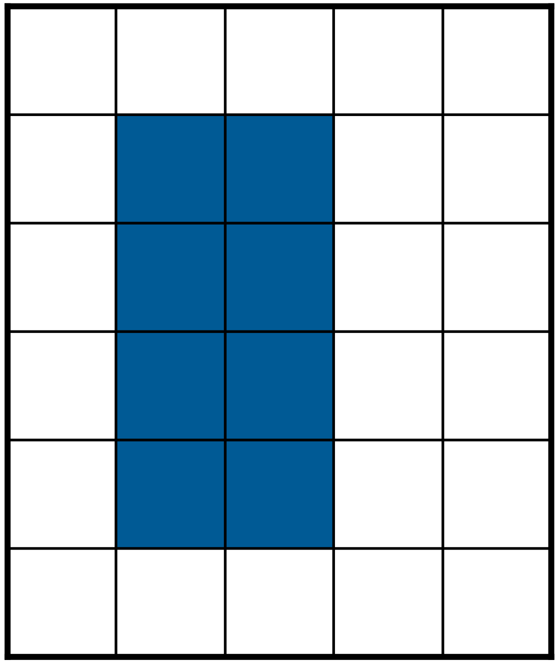
B



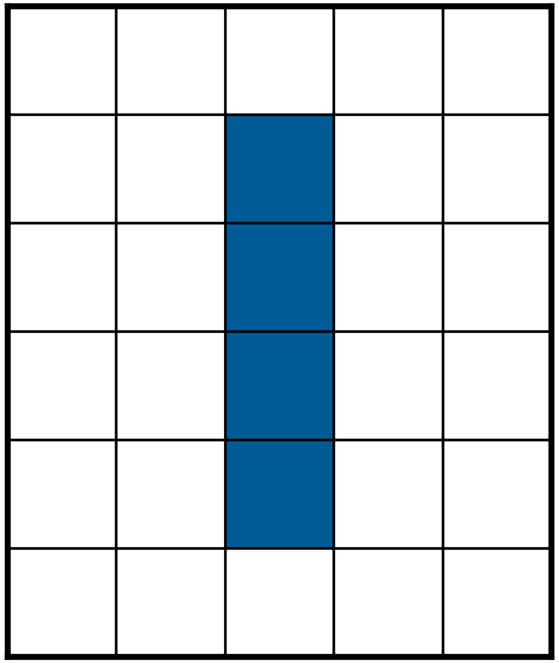
B'



X



$E_B(X)$

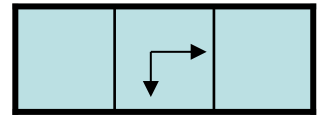


$E_{B'}(X)$

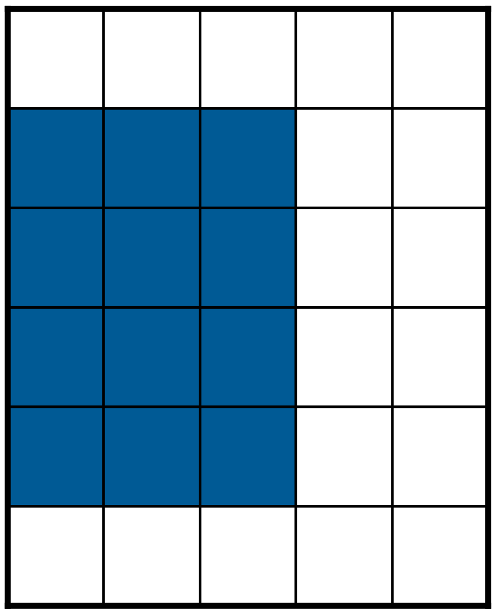
Properties of Erosion

- **Commutativity with Intersection, not with Union**

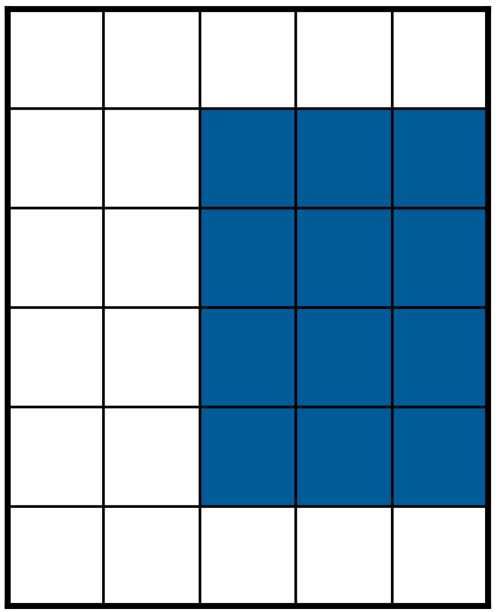
$$E_B(X \cap Y) = E_B(X) \cap E_B(Y)$$
$$E_B(X) \cup E_B(Y) \subseteq E_B(X \cup Y)$$



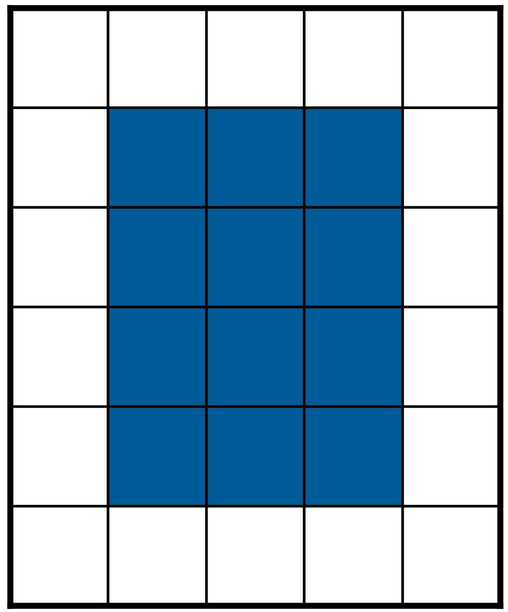
B



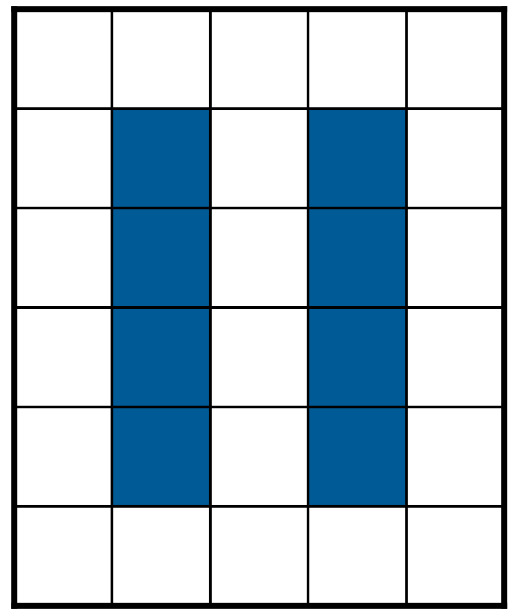
X



Y



$E_B(X \cup Y)$

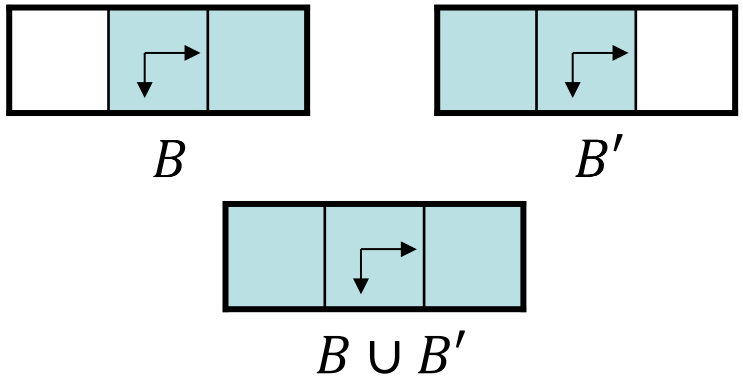


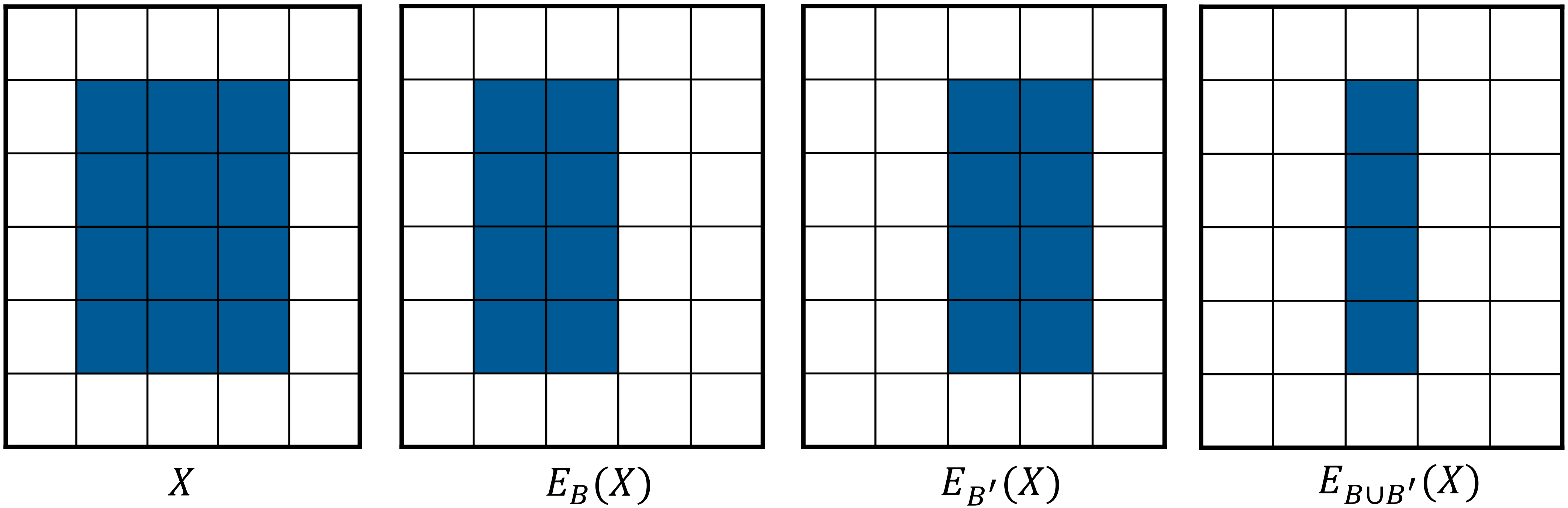
$E_B(X) \cup E_B(Y)$

Properties of Erosion

- **Structuring Element Decomposition**

$$E_{B\cup B'}(X) = E_B(X) \cap E_{B'}(X)$$

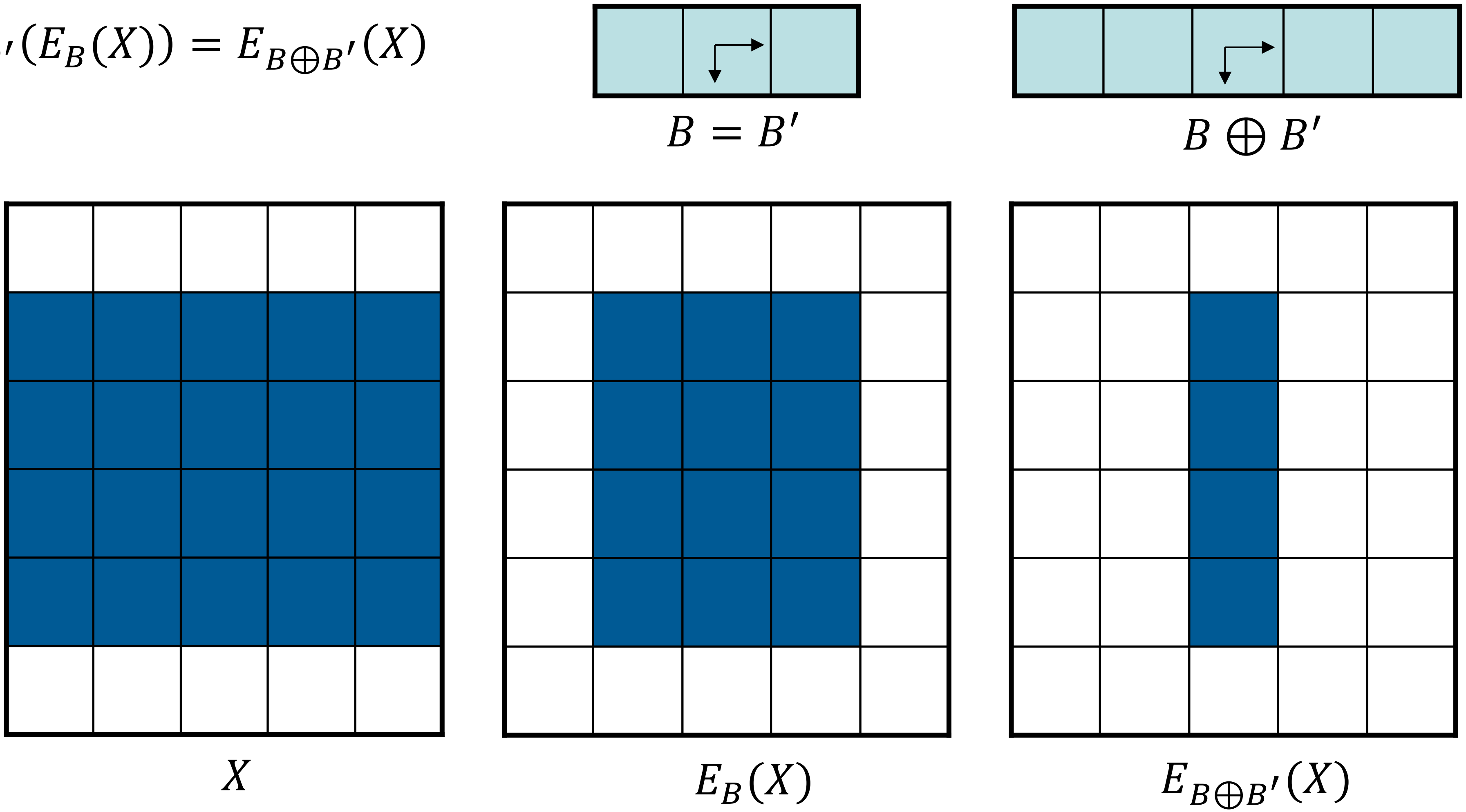




Properties of Erosion

- **Iterativity**

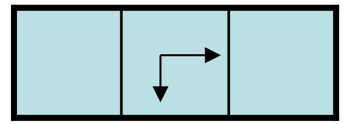
$$E_{B'}(E_B(X)) = E_{B \oplus B'}(X)$$



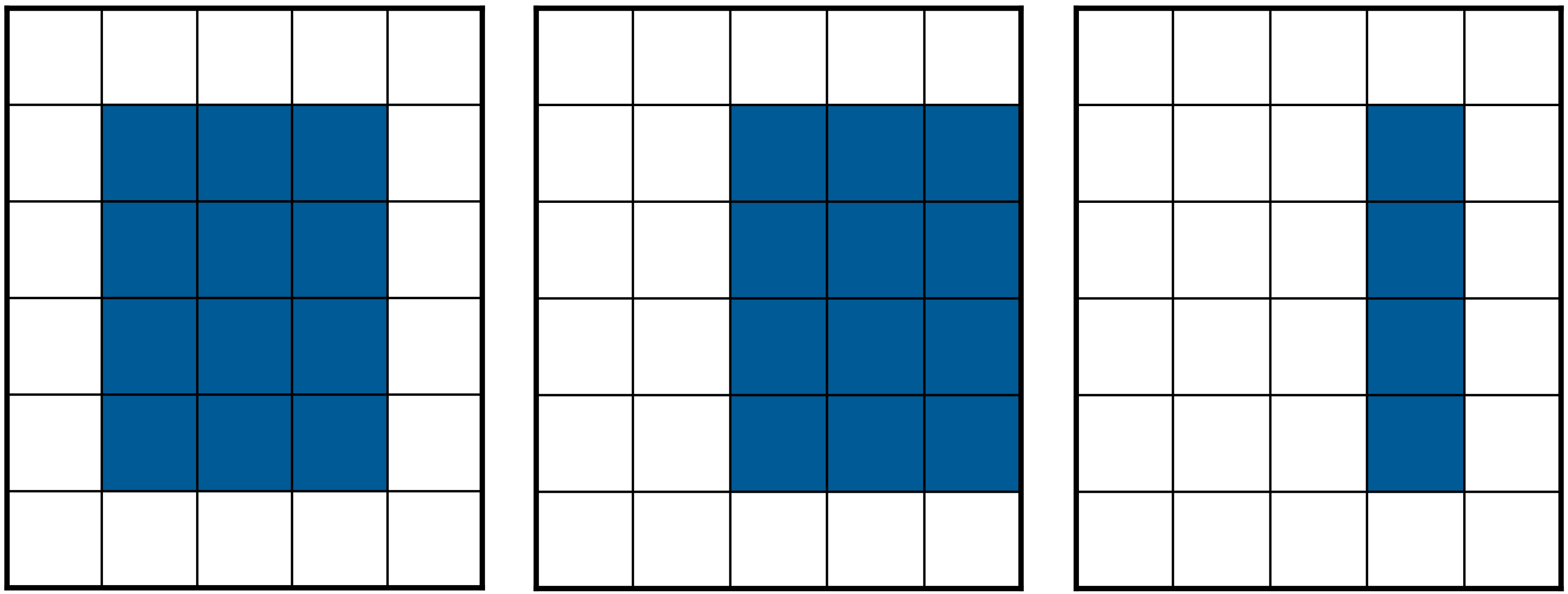
Properties of Erosion

- **Translation Invariance**

$$E_B(X_z) = (E_B(X))_z$$



B

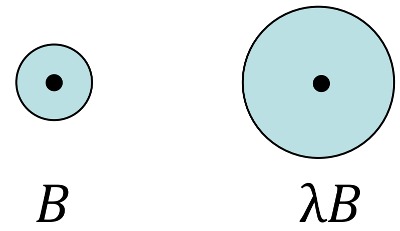


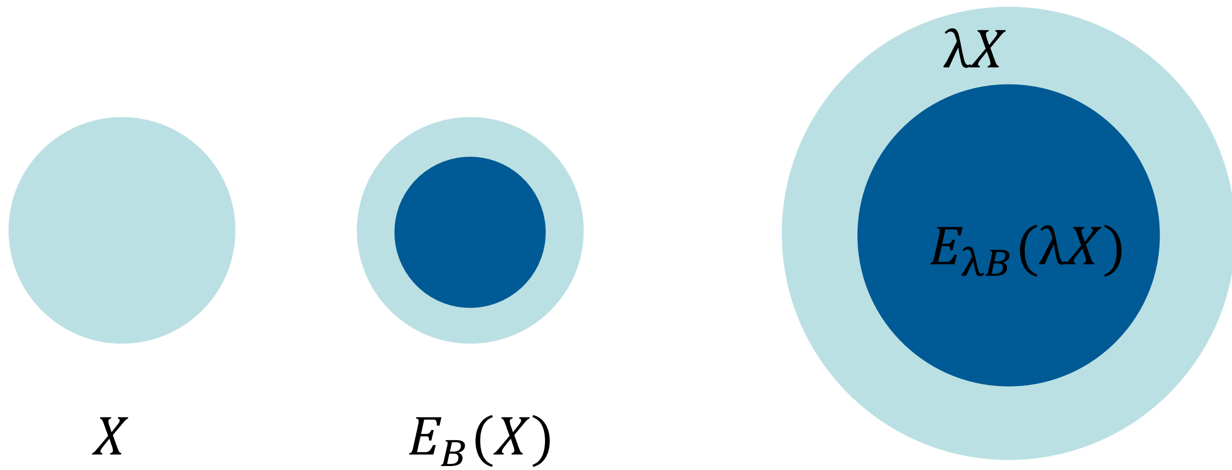
X X_z $E_B(X_z)$

Properties of Erosion

- **Compatibility with Scales**

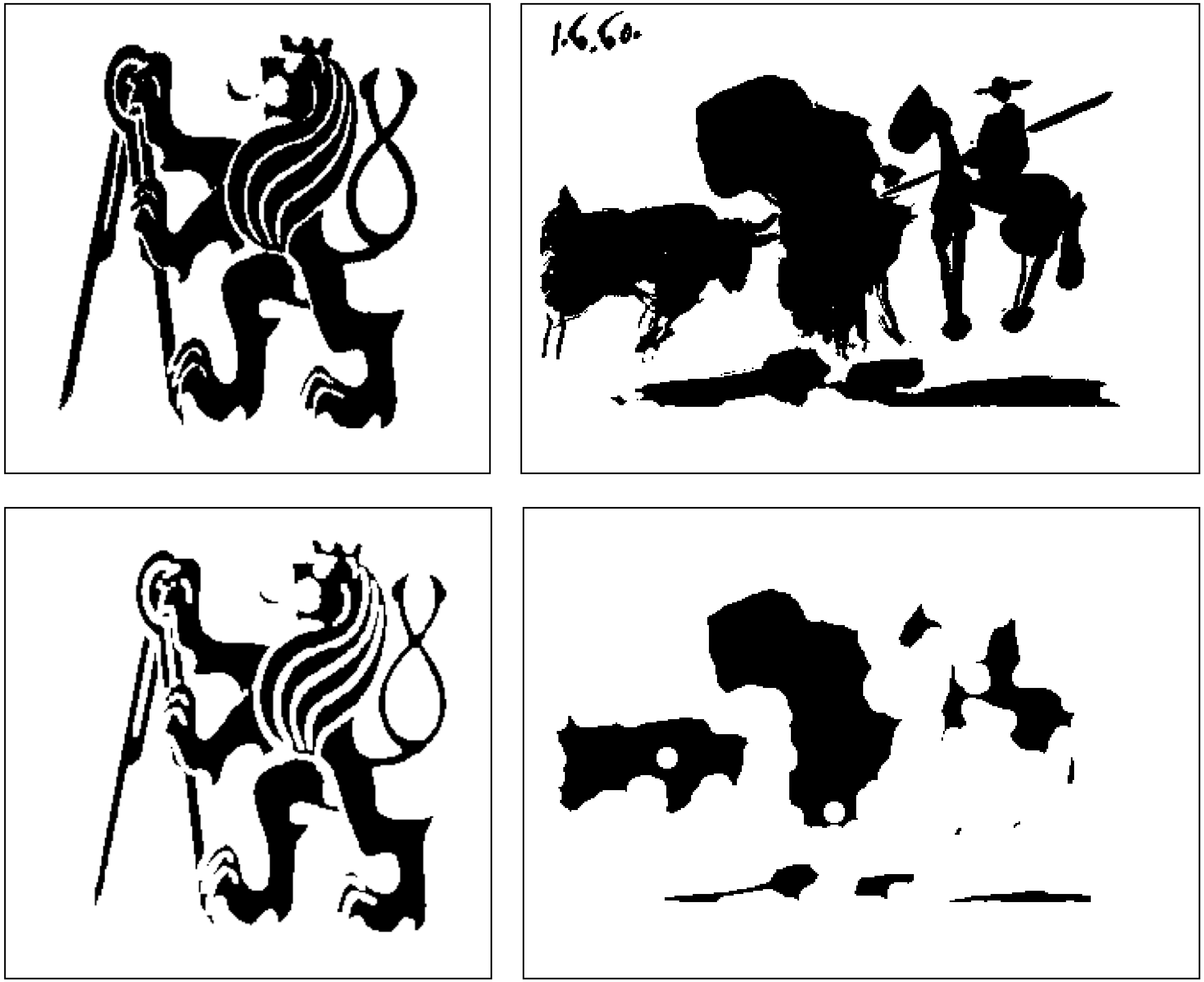
$$\lambda E_B(X) = E_{\lambda B}(\lambda X)$$





Erosion

- **Illustration**
  - Foreground is black
  - Background is white



Dilation

- **Dilation is the Dual by Complementation of Erosion**

$$D_B(X) = \overline{E_B(\bar{X})}$$

$$= \overline{\bar{X} \ominus \check{B}}$$ (transposed/reflected SE)

$$= \overline{\bigcap_{b \in \check{B}} \overline{X_b}}$$

$$= \bigcup_{b \in \check{B}} X_b$$

$$= \{x;\ \exists z \in X, \exists b \in \check{B} \mid x = z + b\}$$

$$= \{x;\ \exists z \in X, \exists b \in B \mid x = z - b\}$$

$$= \{x;\ \exists z \in X, \exists b \in B \mid b + x = z\}$$

$$= \{x;\ B_x \cap X \neq \emptyset\}$$

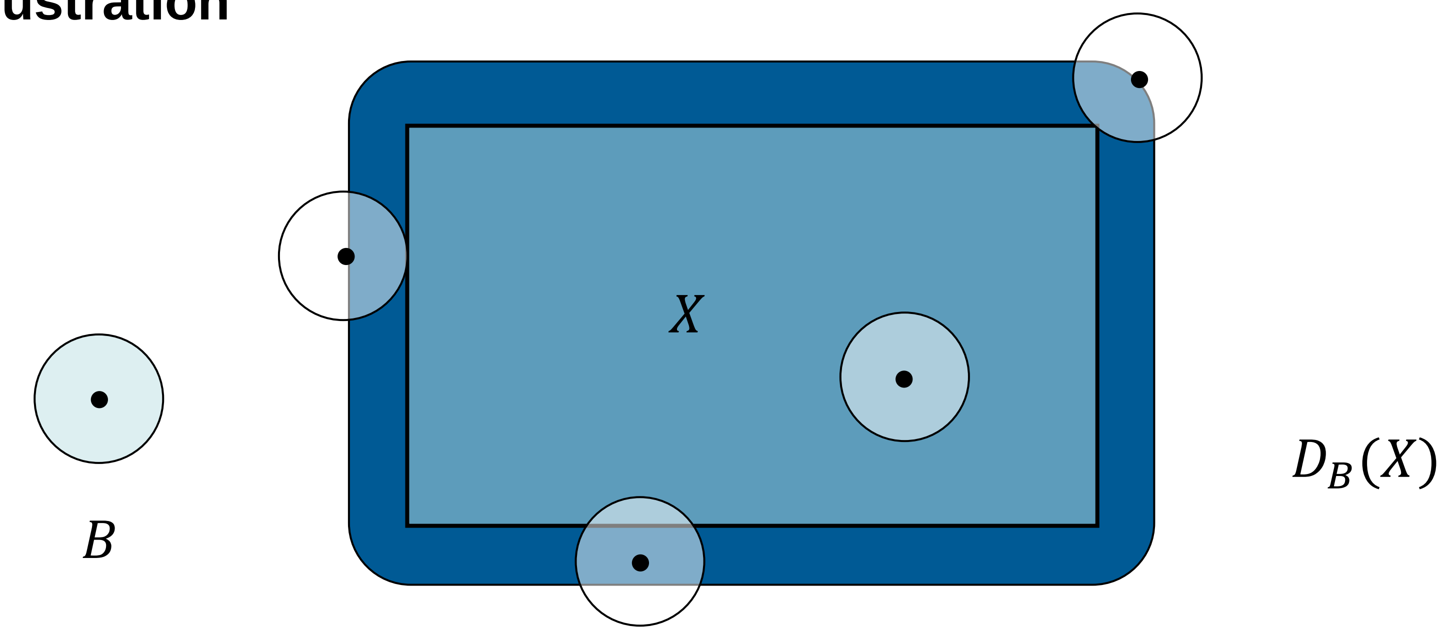
Dilation

- **Definition**

$$D_B(X) = \bigcup_{b \in \check{B}} X_b = \{x;\ B_x \cap X \neq \emptyset\}$$

where $B_x = \{b + x; x \in X\}$

- **Illustration**

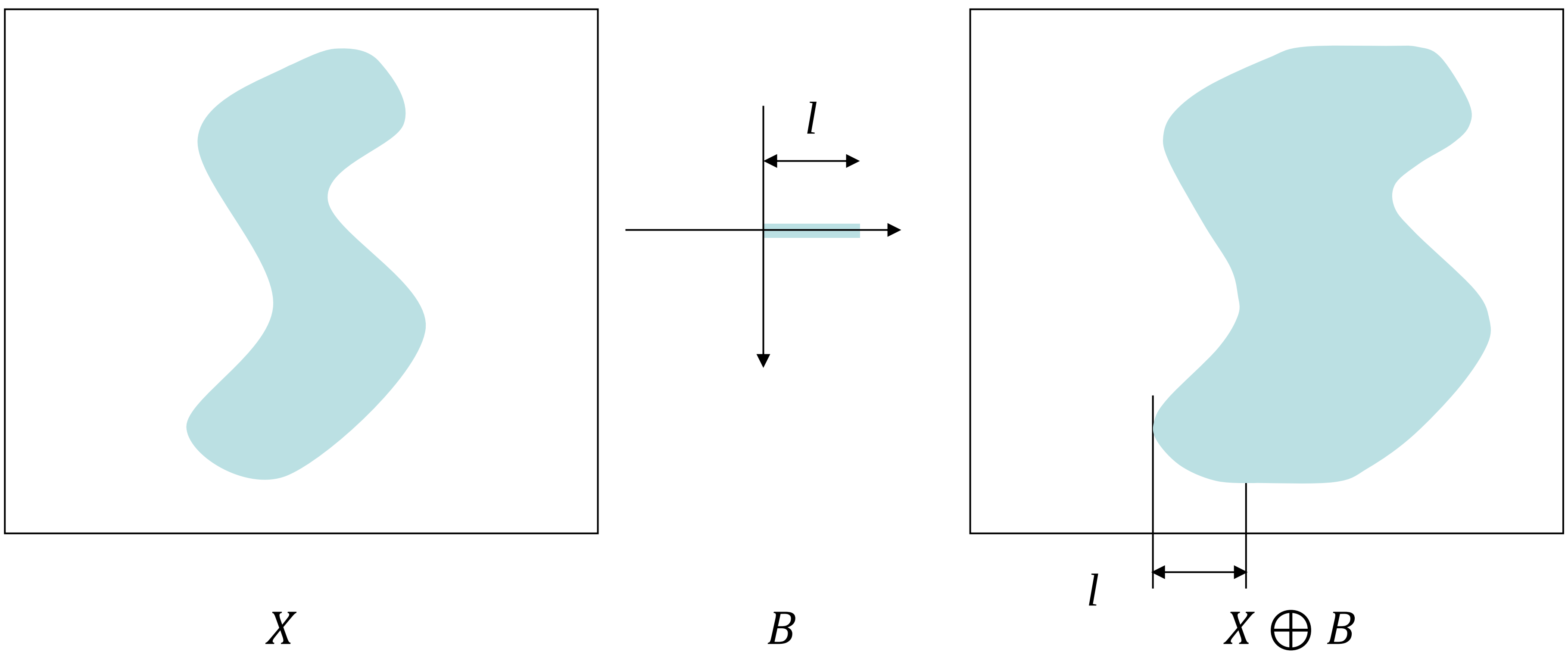


Dilation

- **Link with the Minkowski Sum**

$$S \oplus B = \{s + b; s \in S, b \in B\} = \bigcup_{b \in B} S_b$$

where $S_b = \{s + b; s \in S\}$



Dilation

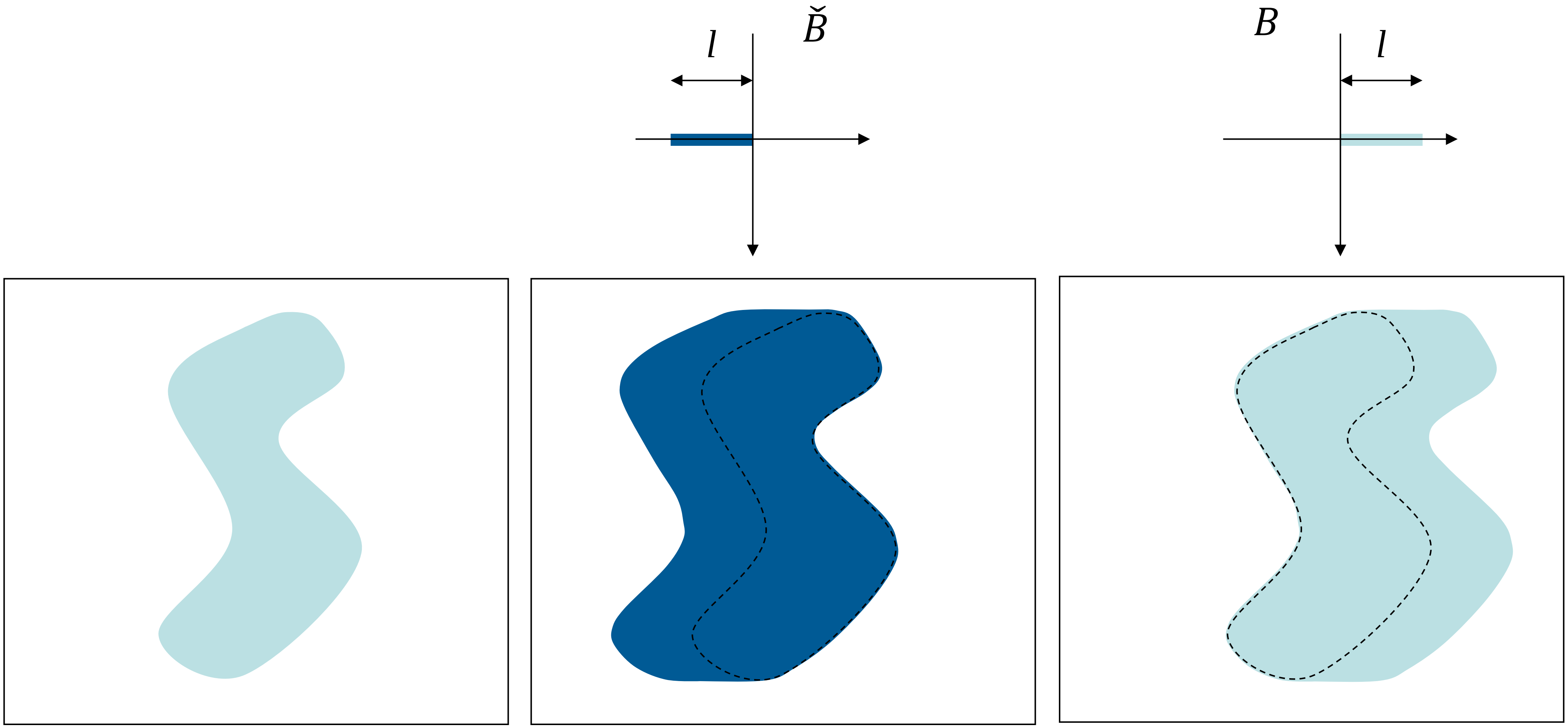
- **Link with the Minkowski Sum**

$$D_B(X) = \bigcup_{b \in \check{B}} X_b \quad \text{where} \quad \check{B} = \{-b; b \in B\} \quad \text{(transposed/reflected SE)}$$

$$= \mathrm{X} \oplus \check{B}$$

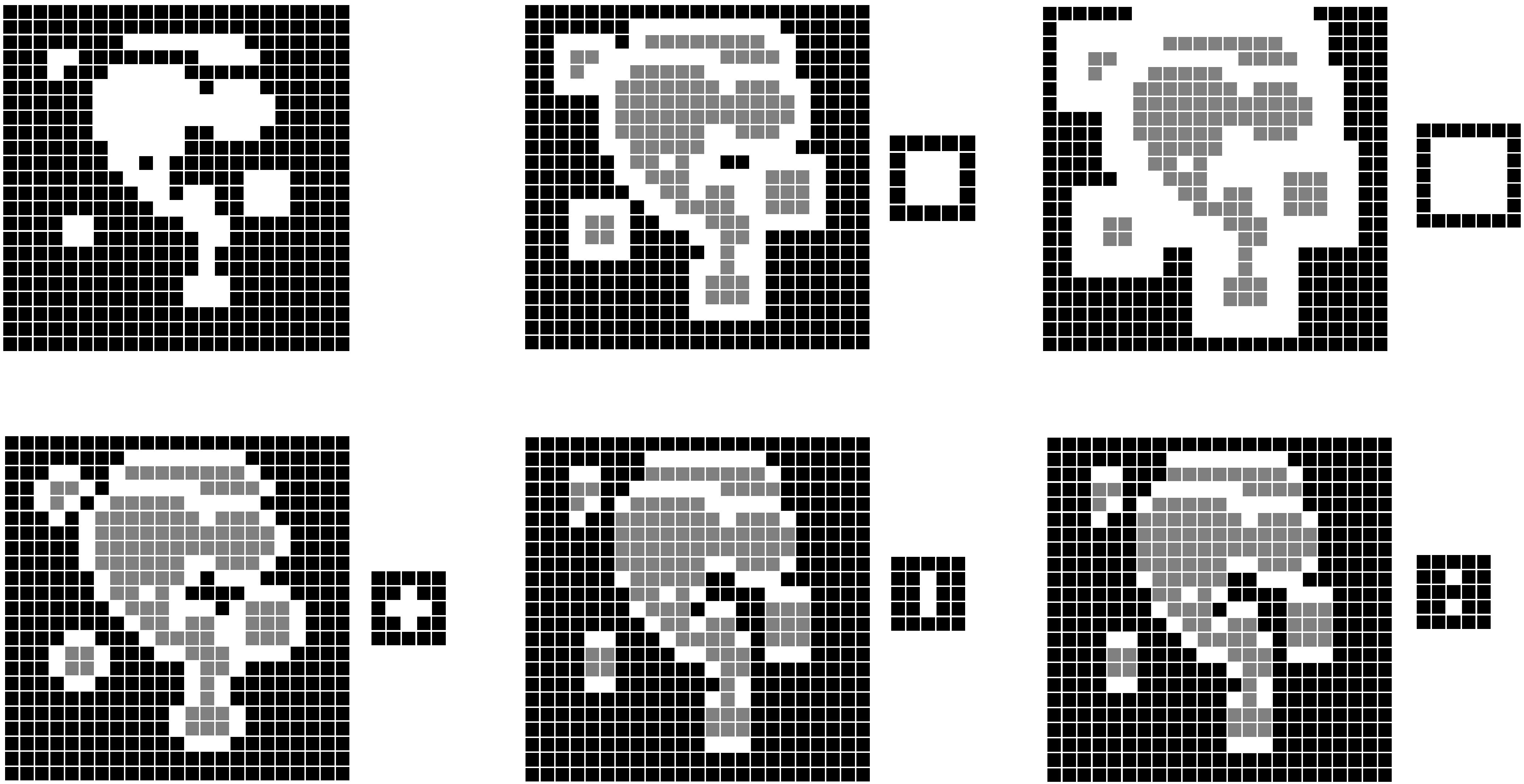
Dilation

- **Dilation is not the Minkowski Sum!**



X $\quad$ $D_B(X) = X \oplus \check{B}$ $\quad$ $X \oplus B$

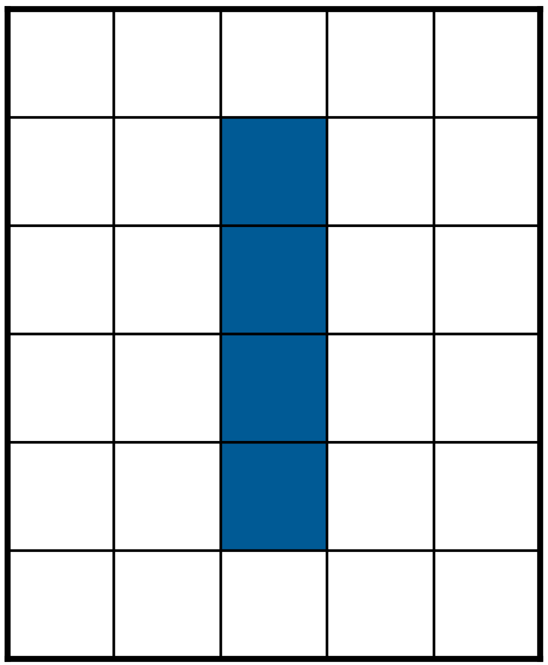
Dilation with Various Structuring Elements

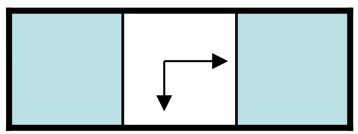


Properties of Dilation

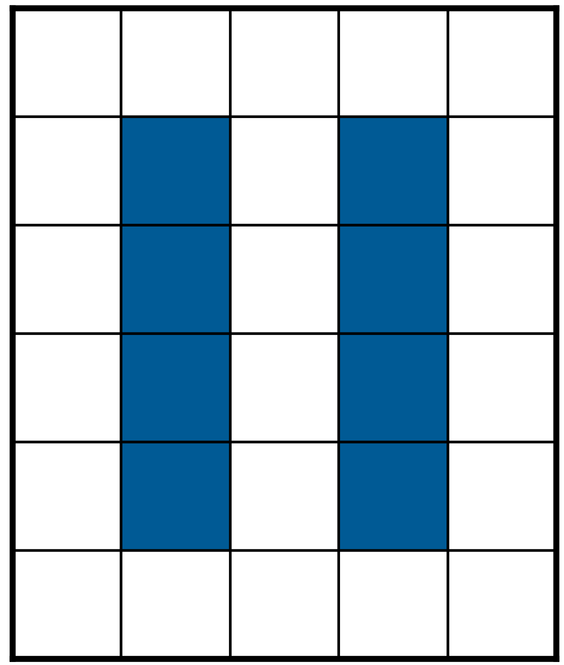
- **Extensivity**

$$X \subseteq D_B(X) \text{ if } 0 \in B$$





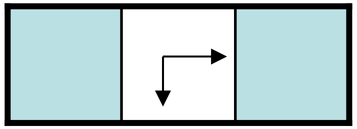
$0 \notin B$ and $X \nsubseteq D_B(X)$



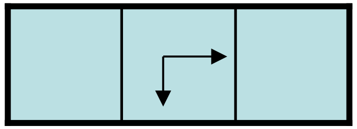
Properties of Dilation

- **Increasing**

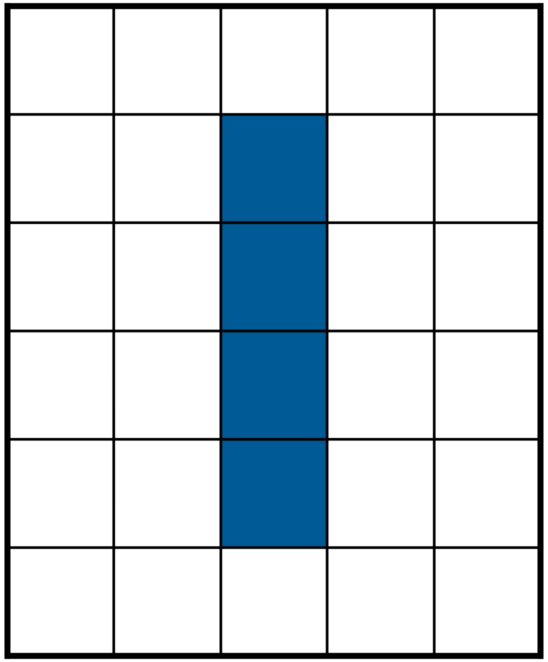
$$X \subseteq Y \Rightarrow D_B(X) \subseteq D_B(Y)$$
$$B \subseteq B' \Rightarrow D_B(X) \subseteq D_{B'}(X)$$



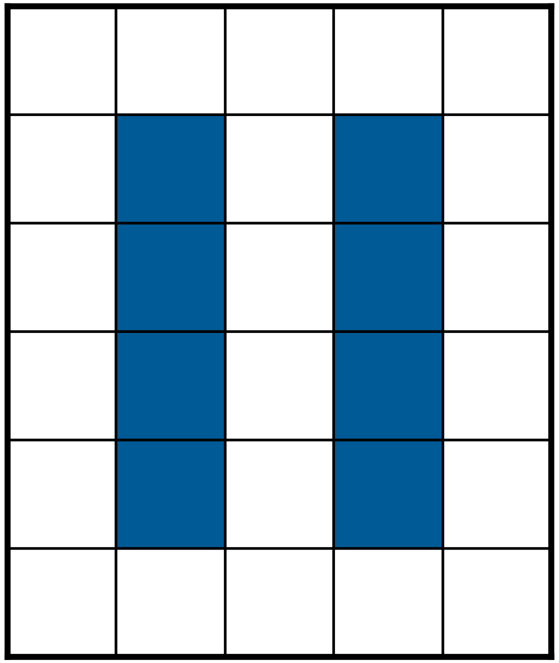
B



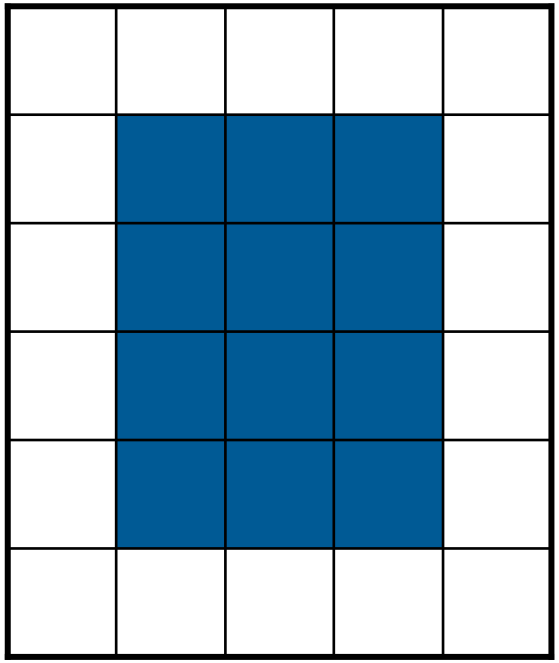
B'



X



$D_B(X)$

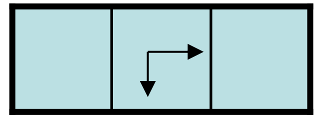


$D_{B'}(X)$

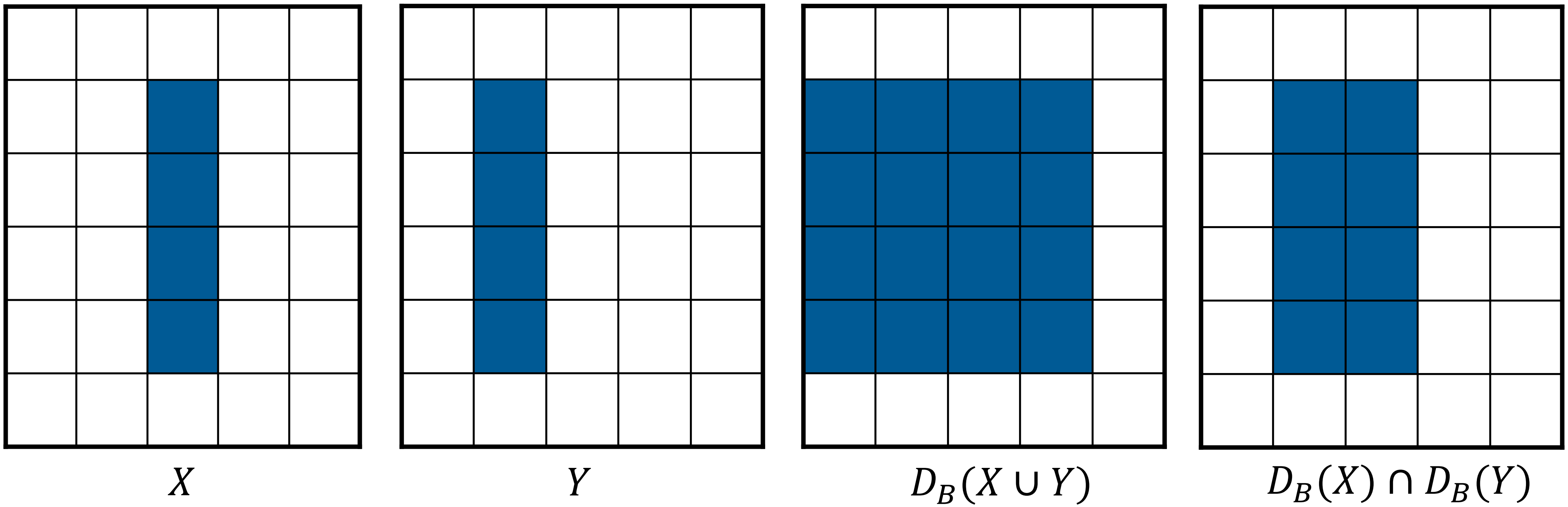
Properties of Dilation

- **Commutativity with Union, not with Intersection**

$$D_B(X \cup Y) = D_B(X) \cup D_B(Y)$$
$$D_B(X \cap Y) \subseteq D_B(X) \cap D_B(Y)$$



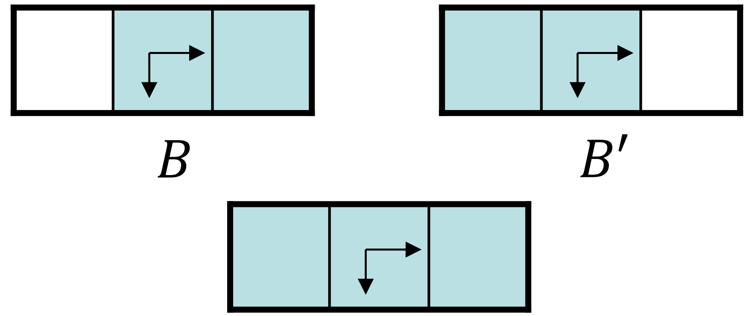
B

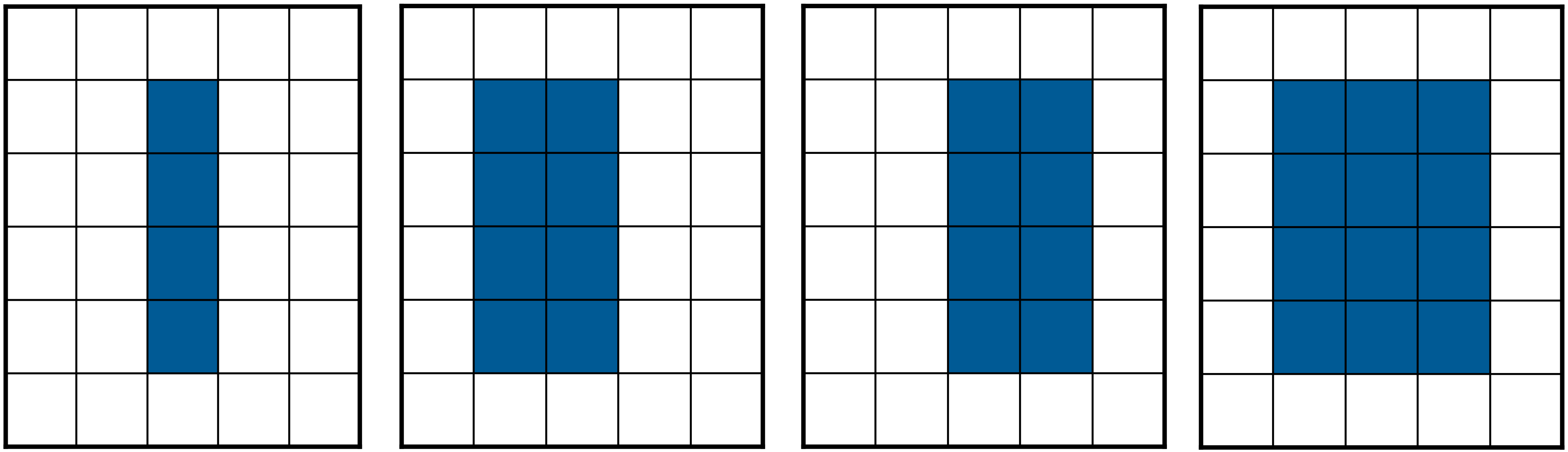


Properties of Dilation

- **Structuring Element Decomposition**

$$D_{B\cup B'}(X) = D_B(X) \cup D_{B'}(X)$$
$$D_{B\cap B'}(X) \subseteq D_B(X) \cap D_{B'}(X)$$

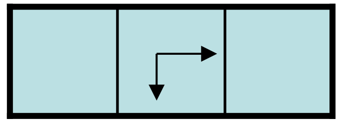




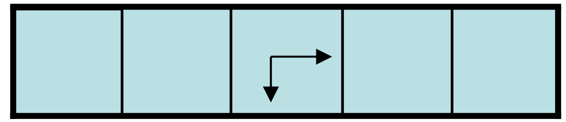
Properties of Dilation

- **Iterativity**

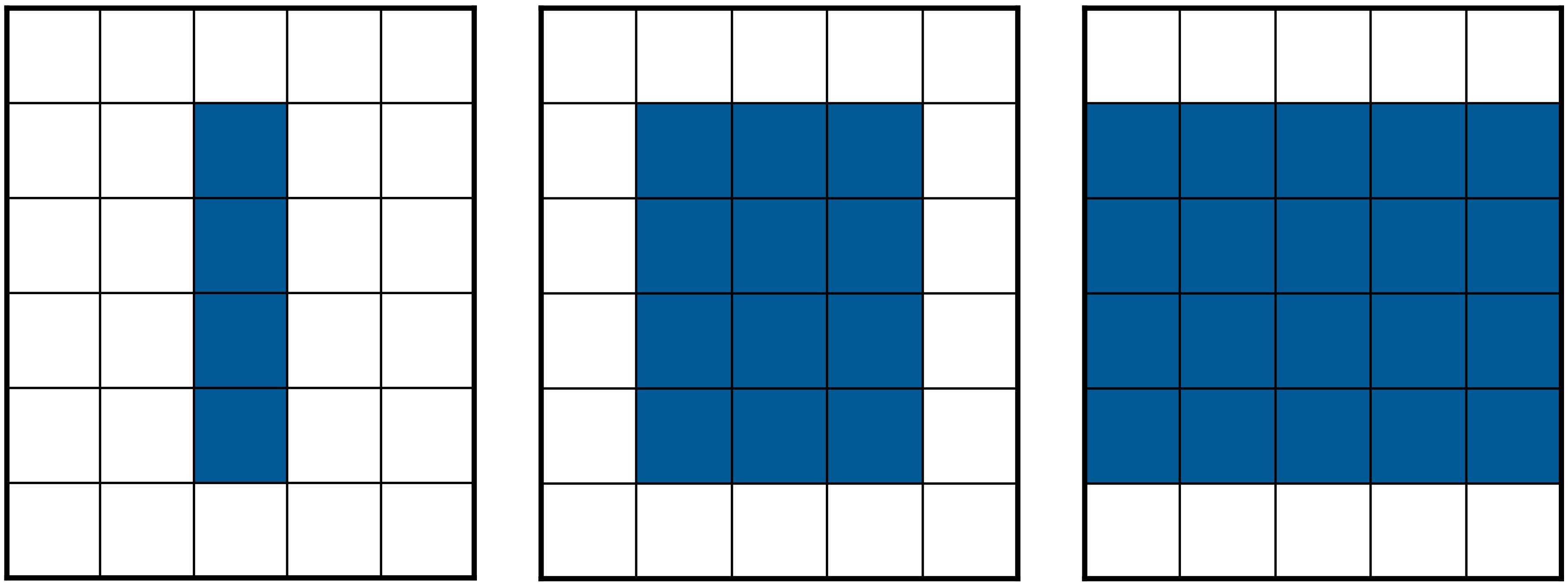
$$D_{B'}(D_B(X)) = D_{B \oplus B'}(X)$$



$B = B'$



$B \oplus B'$

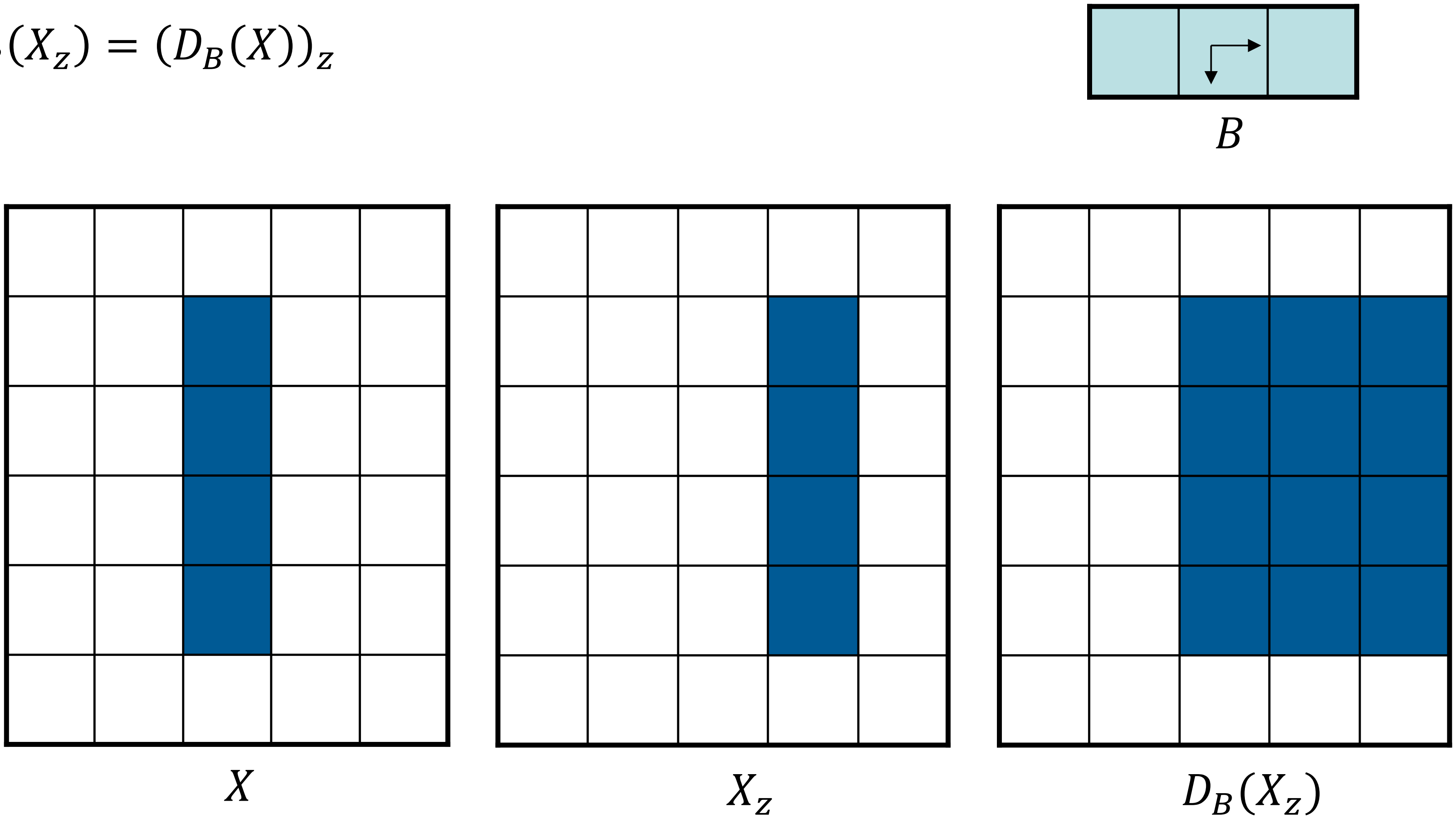


X $\qquad$ $D_B(X)$ $\qquad$ $D_{B \oplus B'}(X)$

Properties of Dilation

- **Translation Invariance**

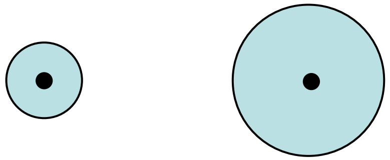
$$D_B(X_z) = (D_B(X))_z$$



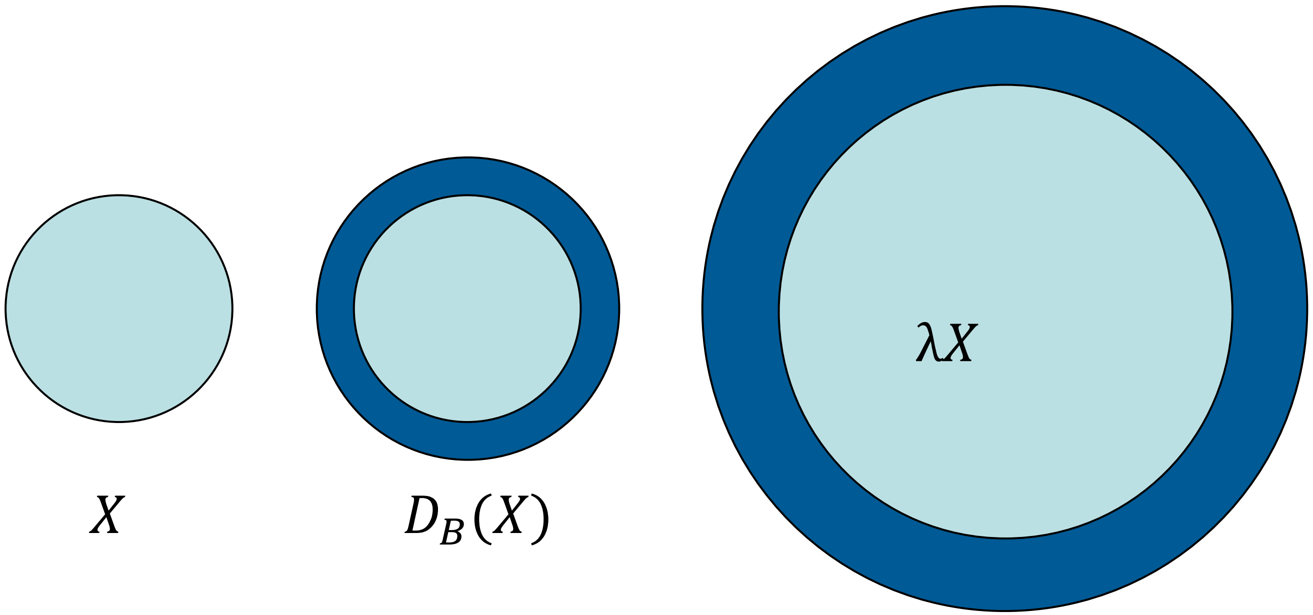
Properties of Dilation

- **Compatibility with Scales**

$$\lambda D_B(X) = D_{\lambda B}(\lambda X)$$



B λB

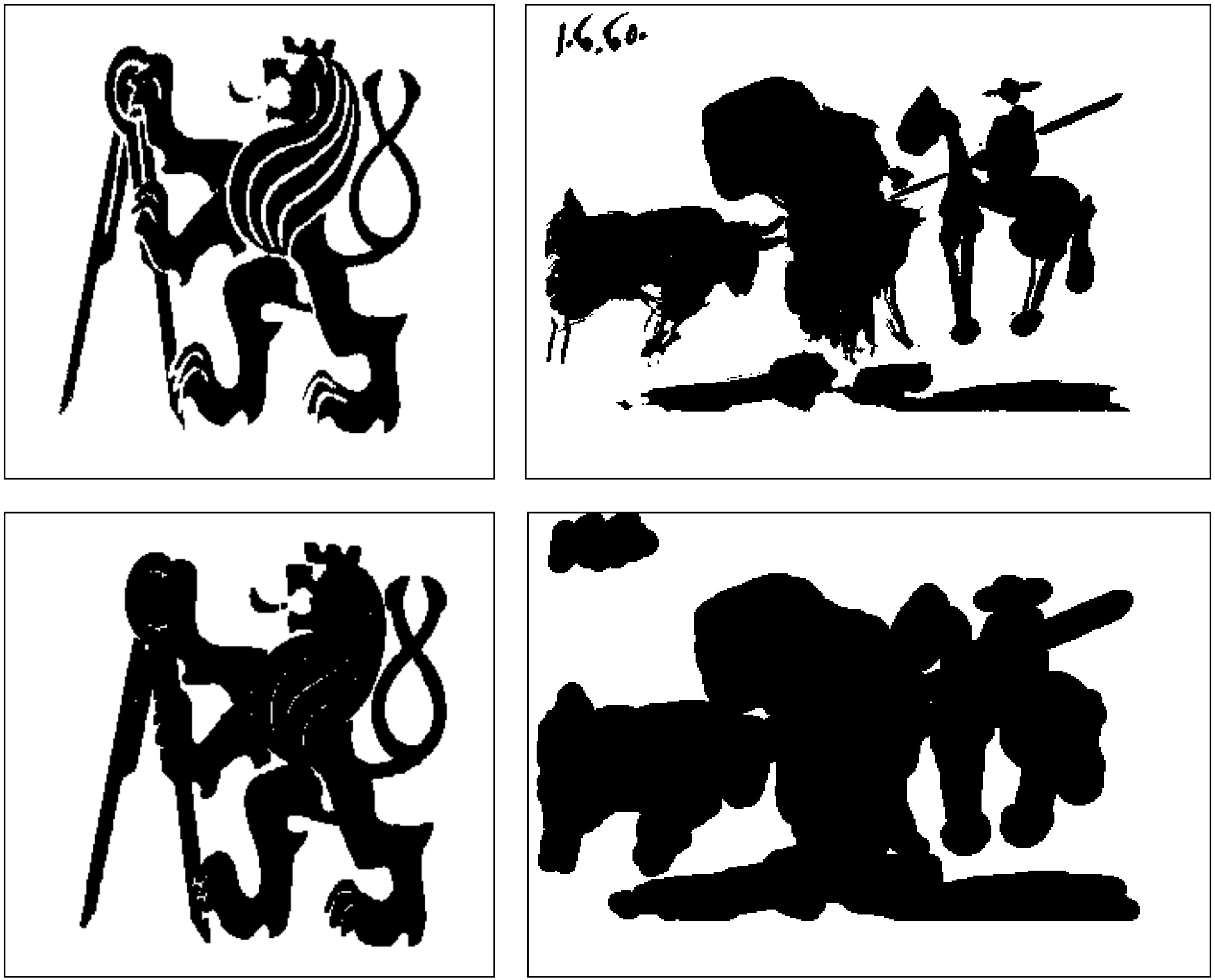


X $D_B(X)$ $D_{\lambda B}(\lambda X)$

Erosion, Dilation

Dilation

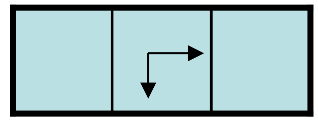
- **Illustration**
  - Foreground is black
  - Background is white



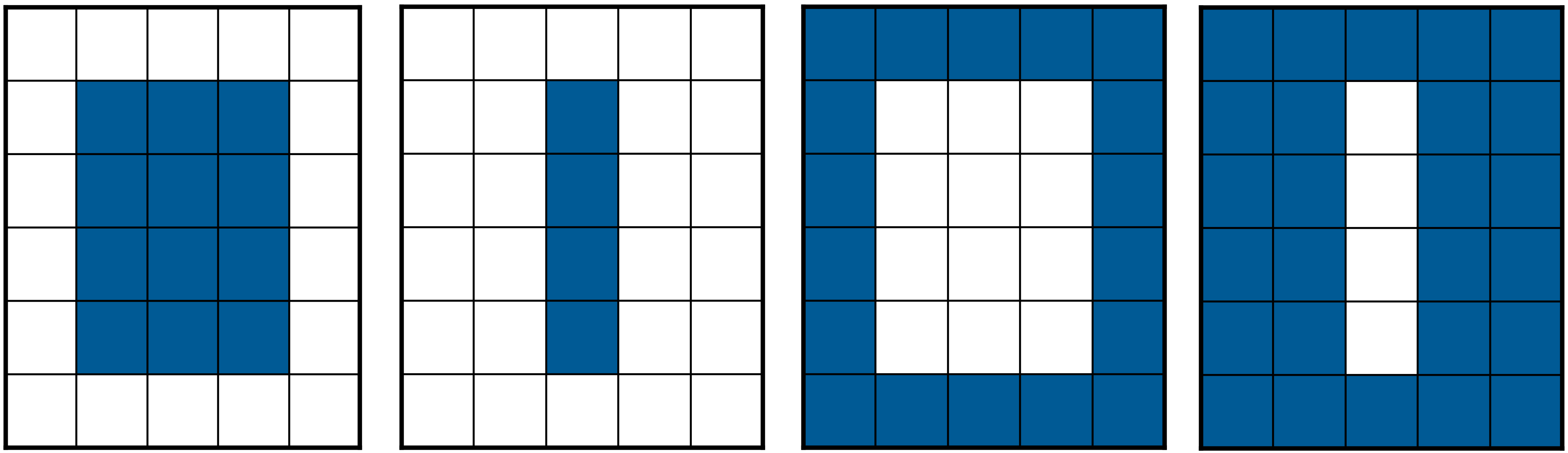
Duality Relationship between Erosion and Dilation

- **Duality with respect to the Complementation**

$$(E_B(X))^c = D_B(X^c)$$



B

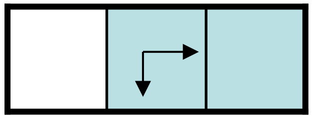


X $E_B(X)$ X^c $D_B(X^c)$

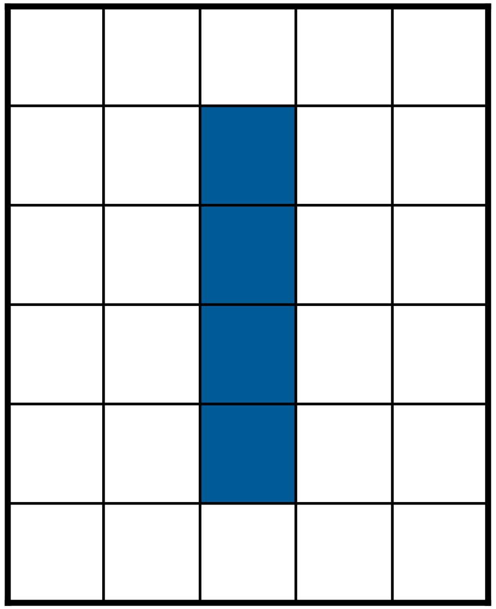
Duality Relationship between Erosion and Dilation

- **Duality with respect to the Adjunction**

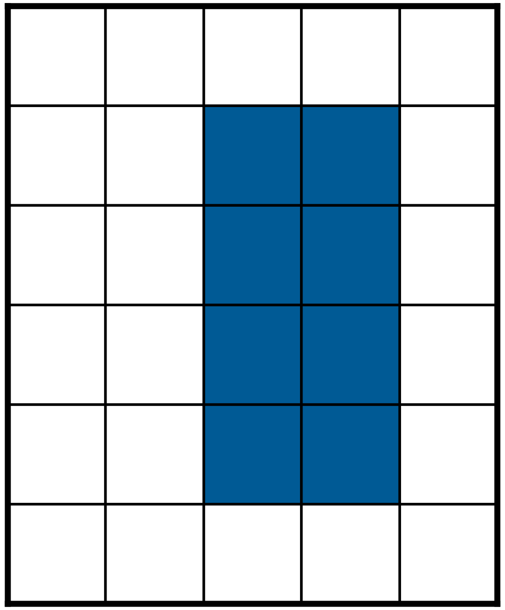
$$D_{\check{B}}(X) \subseteq Y \Leftrightarrow X \subseteq E_B(Y)$$



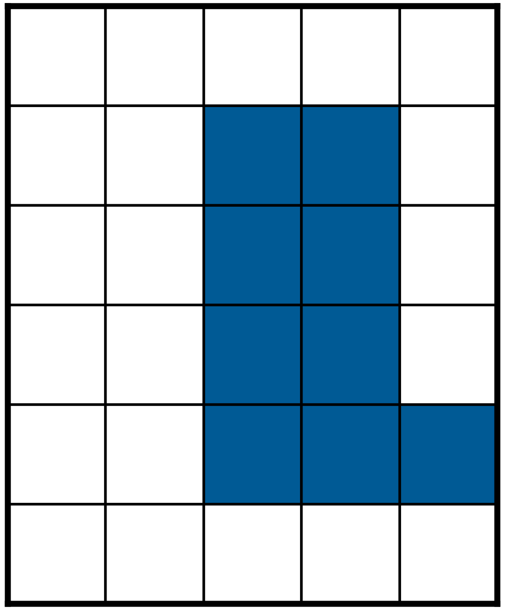
B



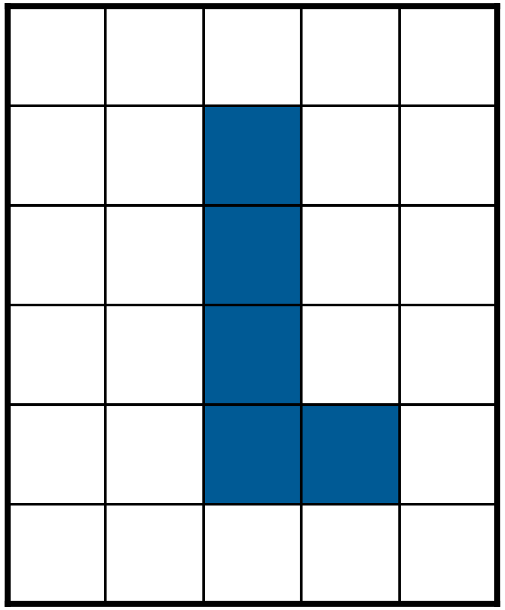
X



$D_{\check{B}}(X)$



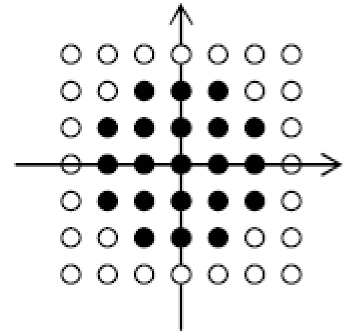
Y

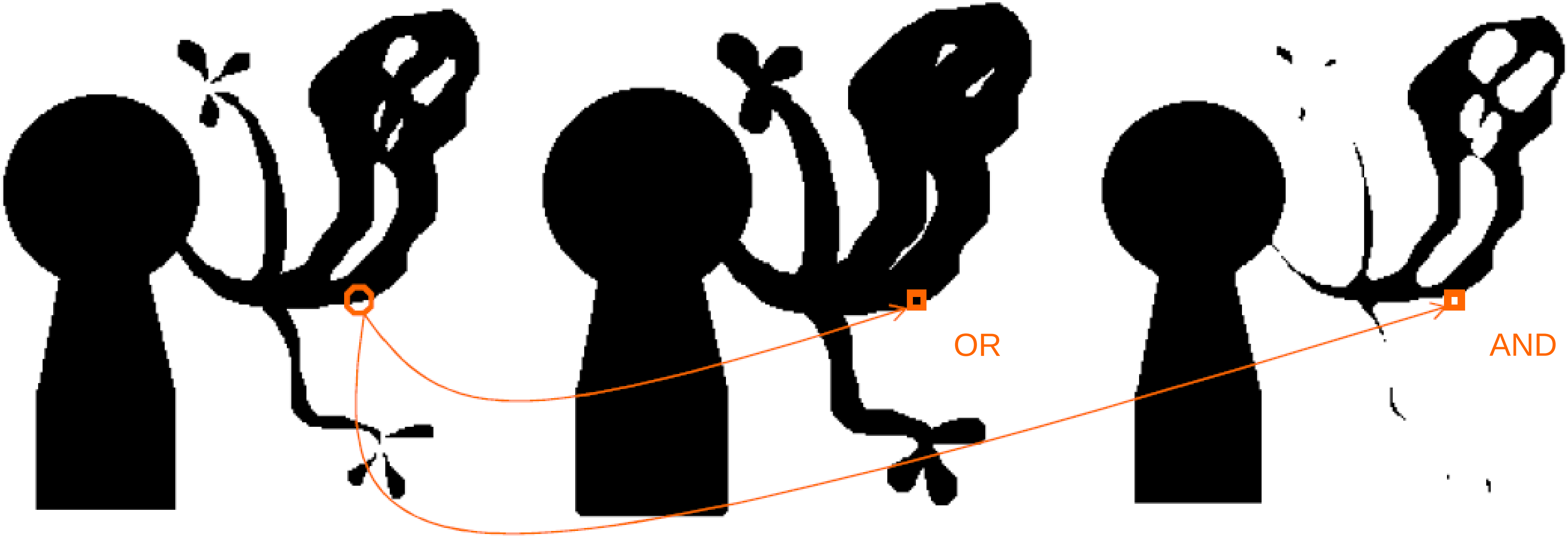


$E_B(Y)$

Application to Digital Images

- **Logical Operations**

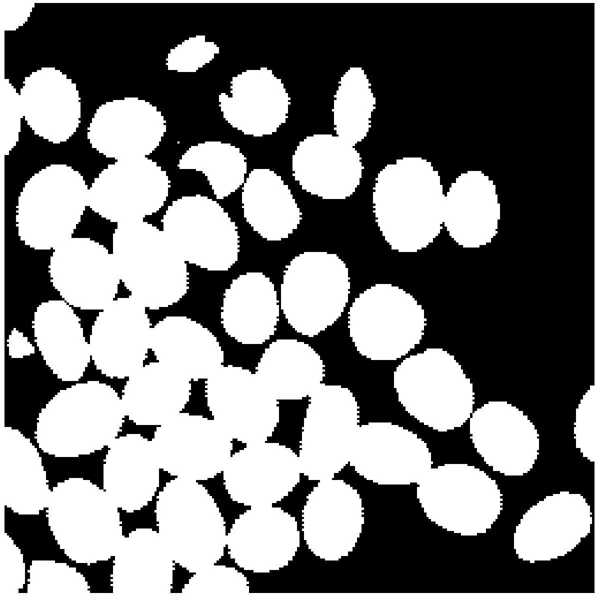




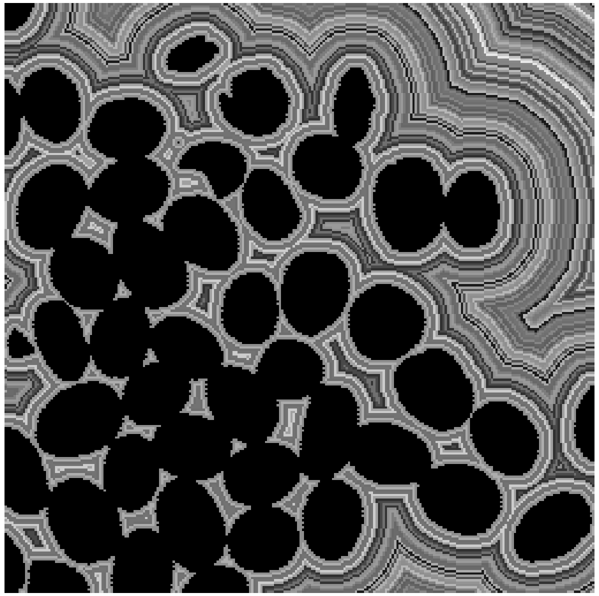
original image X dilated image $D_B(X)$ eroded image $E_B(X)$

Link with Distances

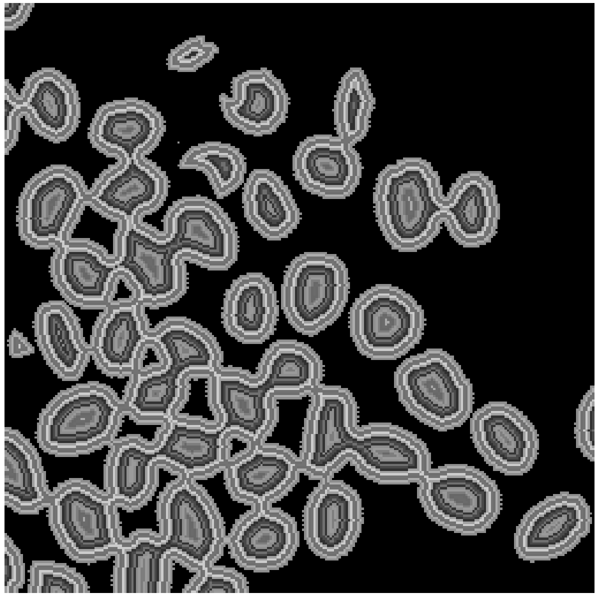
- **Distance maps and Thresholding**
  - Dilation: from background to foreground
  - Erosion: from foreground to background



original image



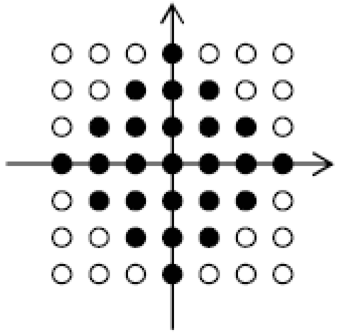
distance map for dilation

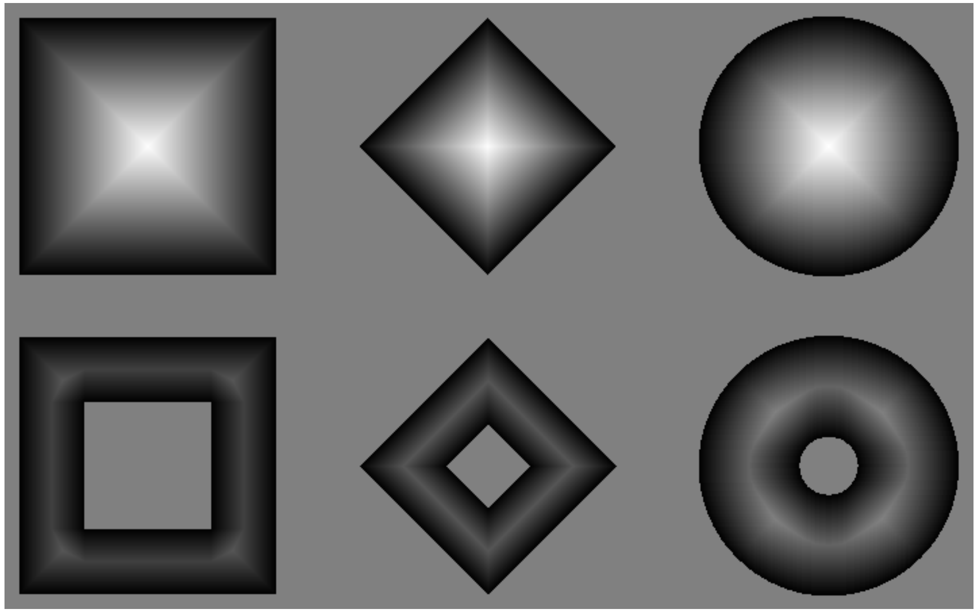


distance map for erosion

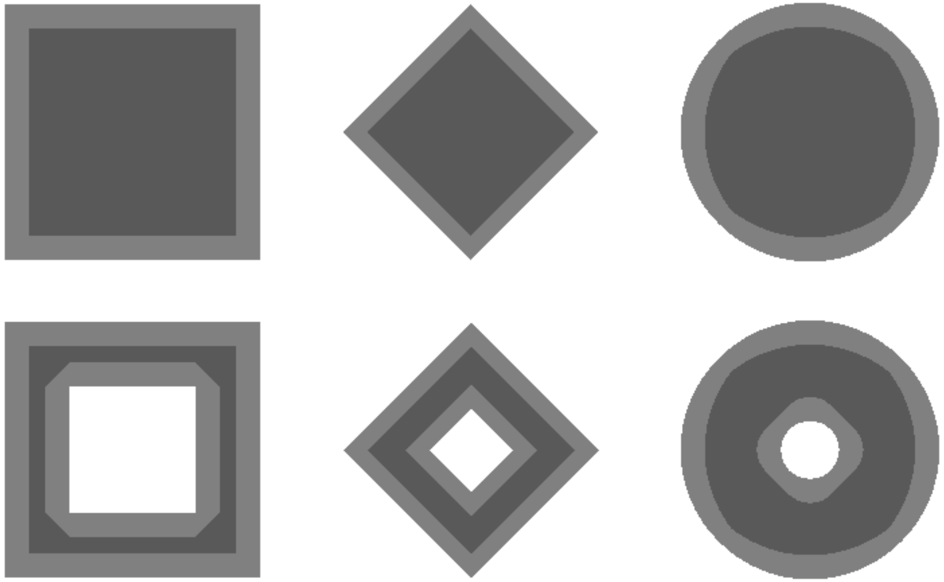
Link with Distances

- **Each distance is associated to a specific SE**
  - Manhattan Distance
  - $d_4(a,b) = |x_a - x_b| + |y_a - y_b|$





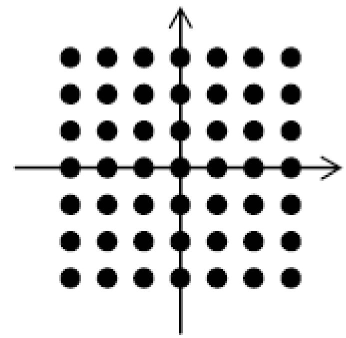
distance map

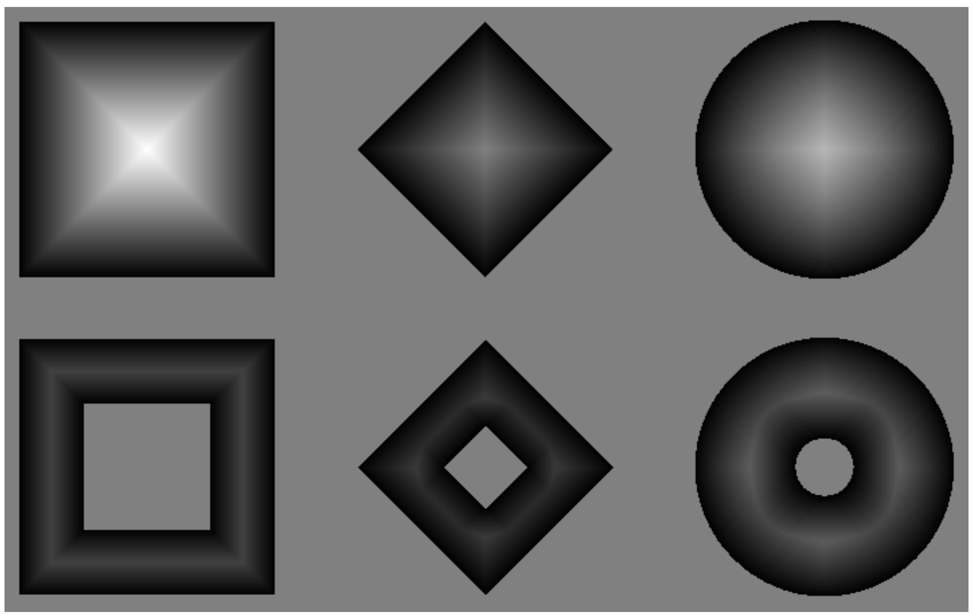


erosion

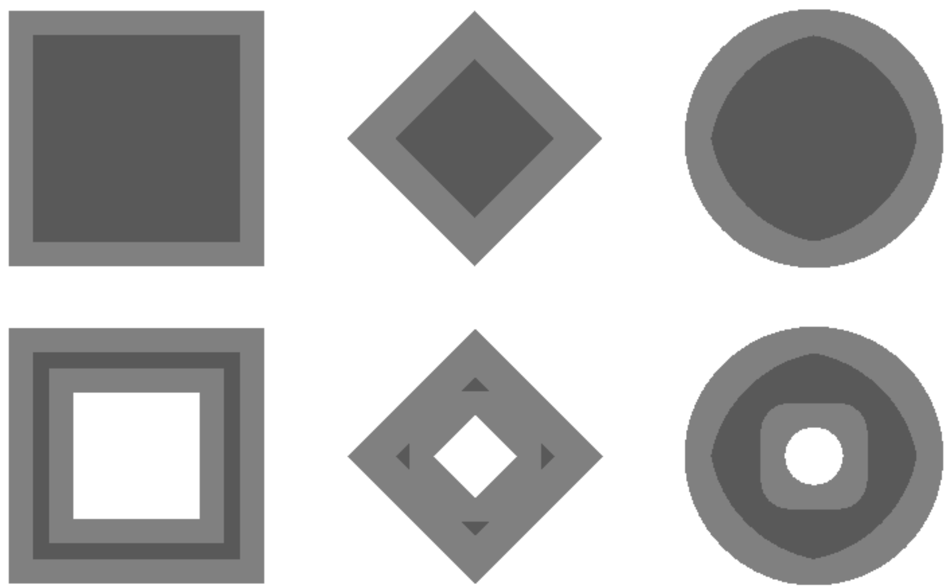
Link with Distances

- **Each distance is associated to a specific SE**
  - Tchebychev Distance
  - $d_8(a, b) = \max(| x_a - x_b |, | y_a - y_b |)$





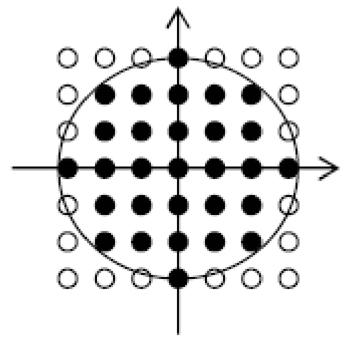
distance map

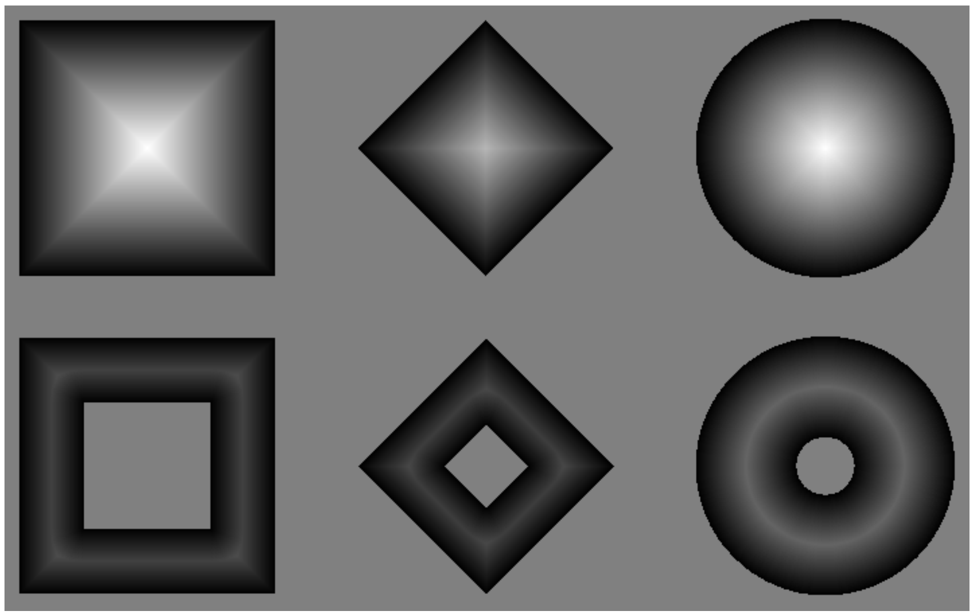


erosion

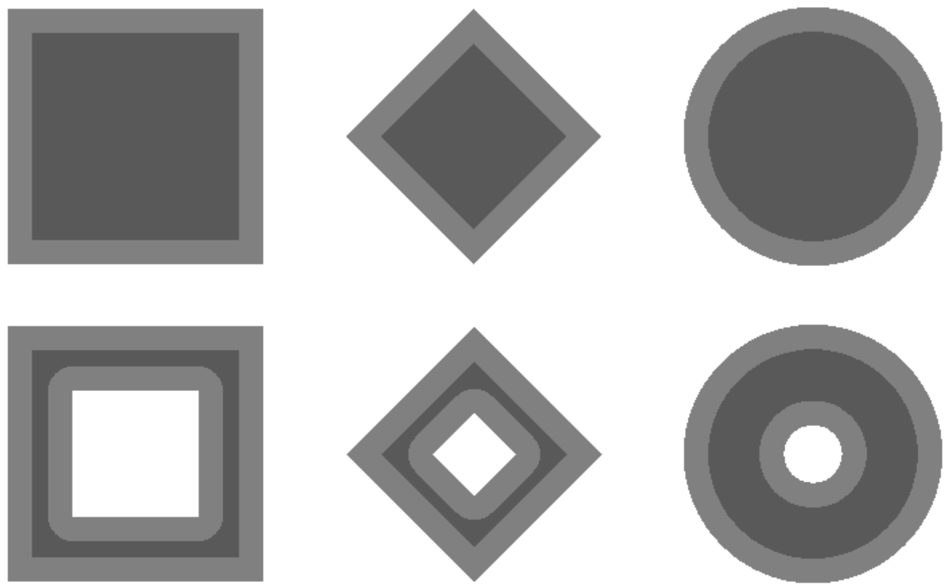
Link with Distances

- **Each distance is associated to a specific SE**
  - (Pseudo-)Euclidean Distance
  - $d_e(a,b) = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$





distance map

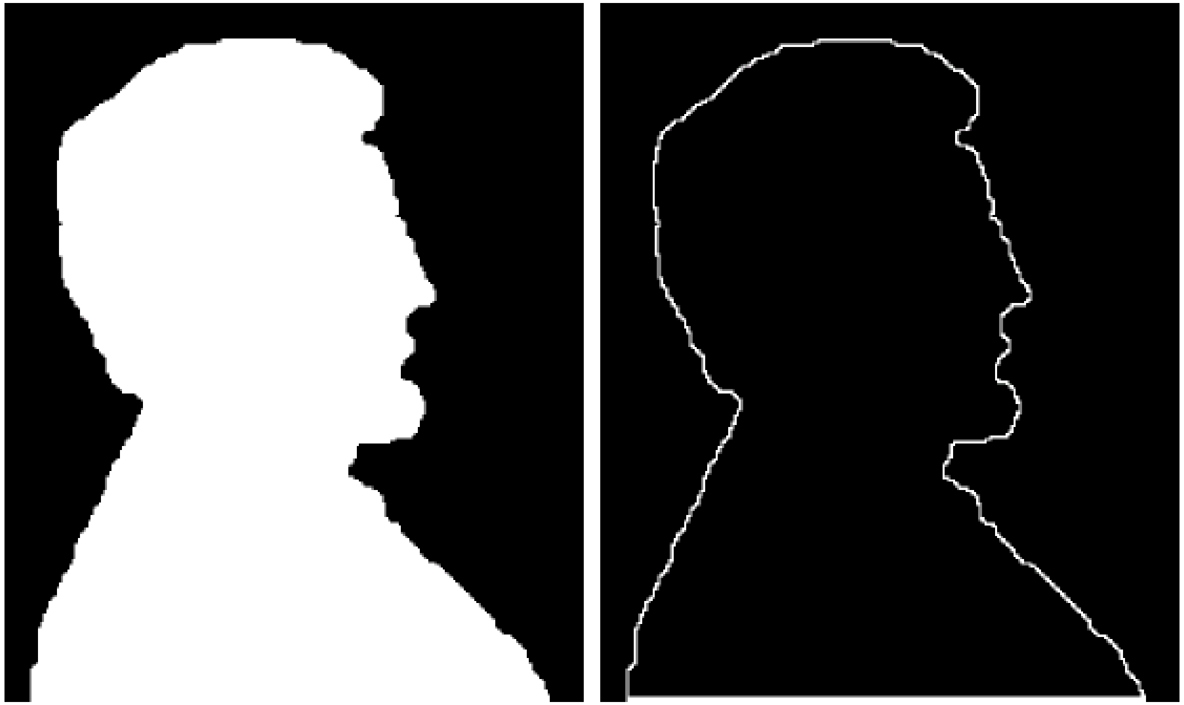


erosion

Application to Boundary Extraction

- **Morphological Gradient**
  - For a small SE size

$$\partial_B(X) = D_B(X) \backslash E_B(X)$$



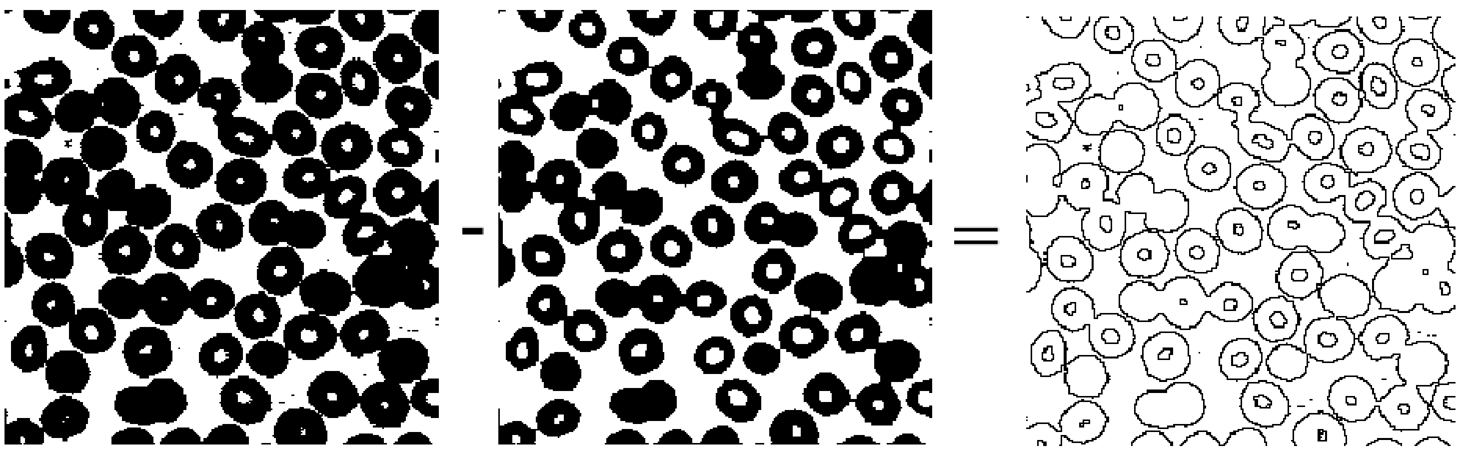
Other Definition of Gradients

- **External Boundary**

$$\partial_B^{ext}(X) = D_B(X) \backslash X$$

- **Internal Boundary**

$$\partial_B^{int}(X) = X \backslash \mathrm{E}_B(X)$$

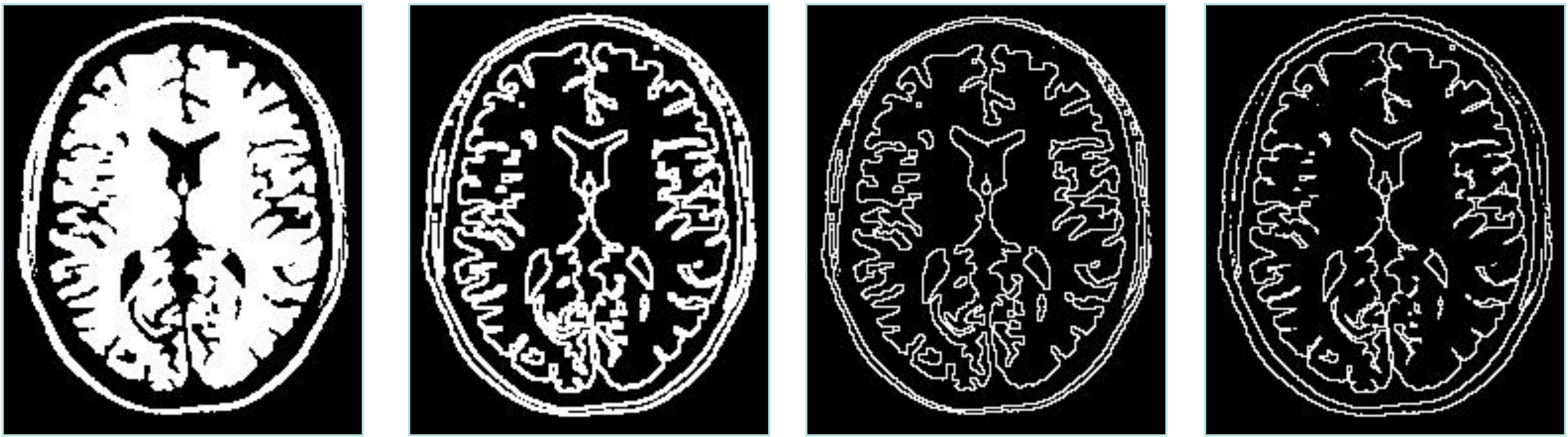


- **Boundary**

$$\partial_B(X) = \partial_B^{int}(X) \cup \partial_B^{ext}(X)$$

Boundary Extraction

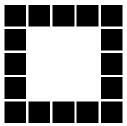
- **Comparison**
  - Brain image



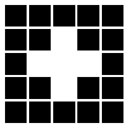
X $\partial_B(X)$ $\partial_B^{int}(X)$ $\partial_B^{ext}(X)$

Effect of the SE Topology

- Example at the Fine Scale

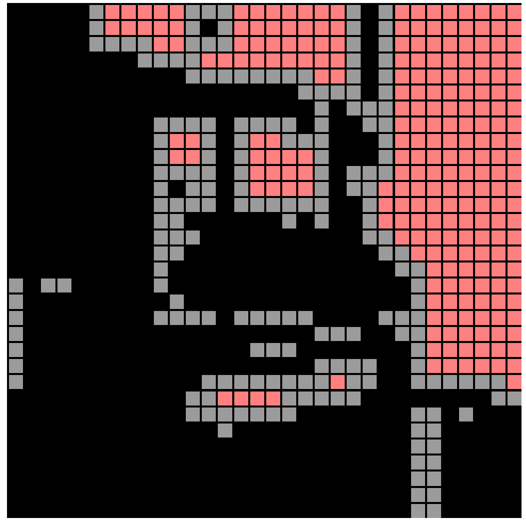


8-connected SE



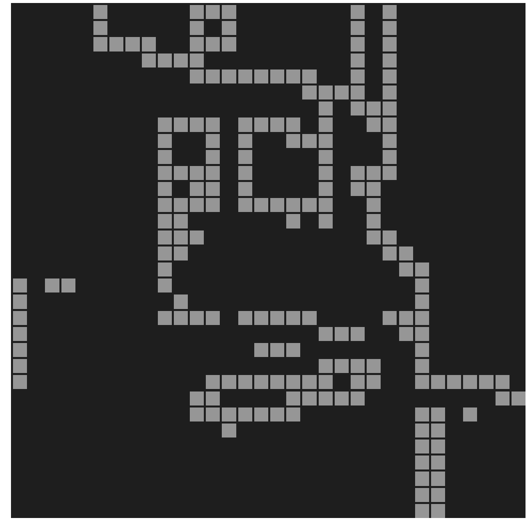
4-connected SE

8-connected SE

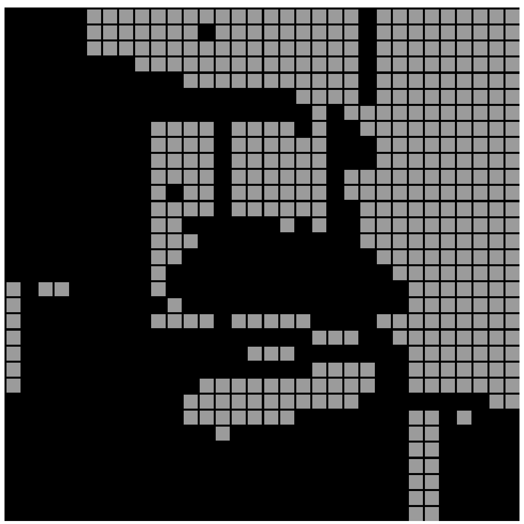


$E_B(X)$

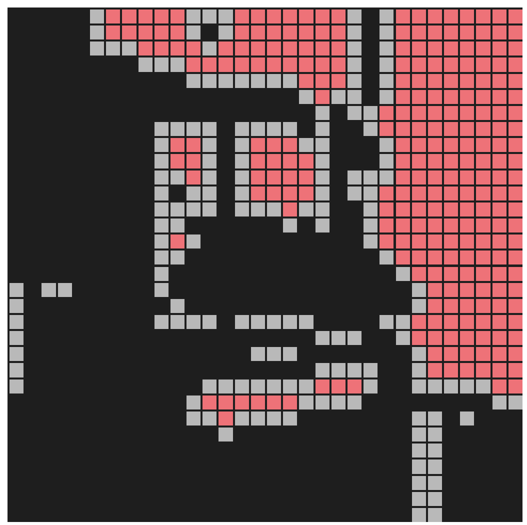
4-connected boundary



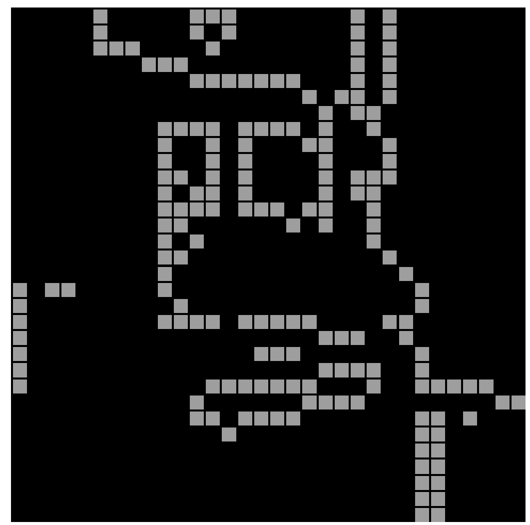
$\partial_B^{int}(X)$



X



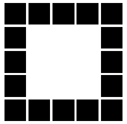
4-connected SE



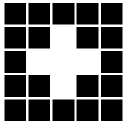
8-connected boundary

Effect of the SE Topology

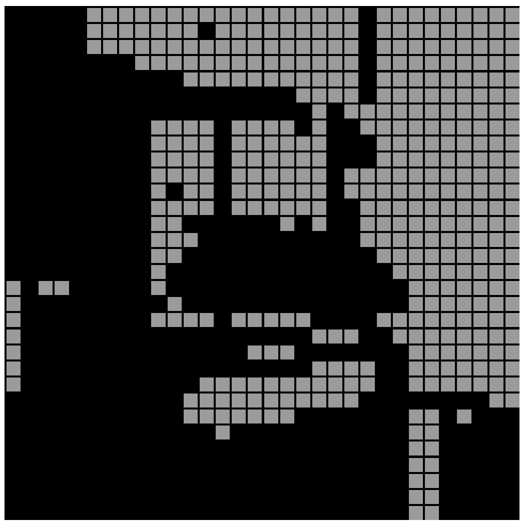
- **Example at the Fine Scale**



8-connected SE

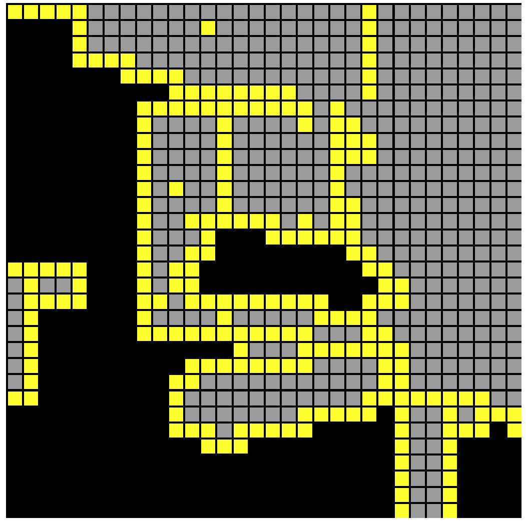


4-connected SE



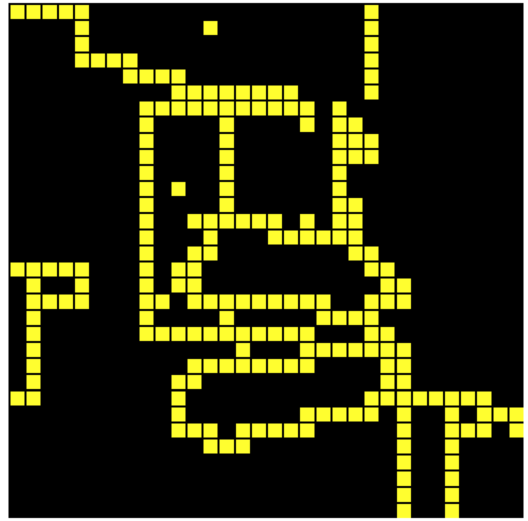
X

8-connected SE

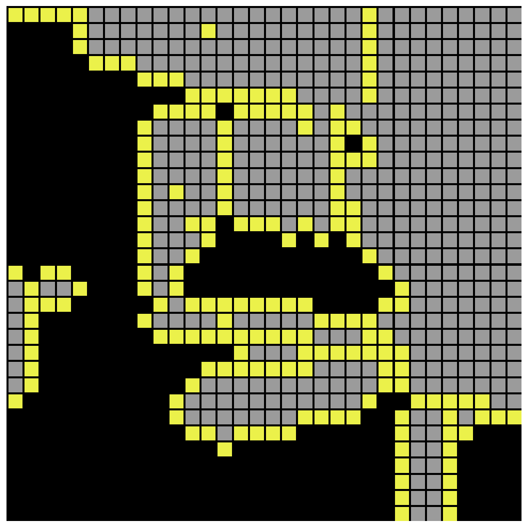


$D_B(X)$

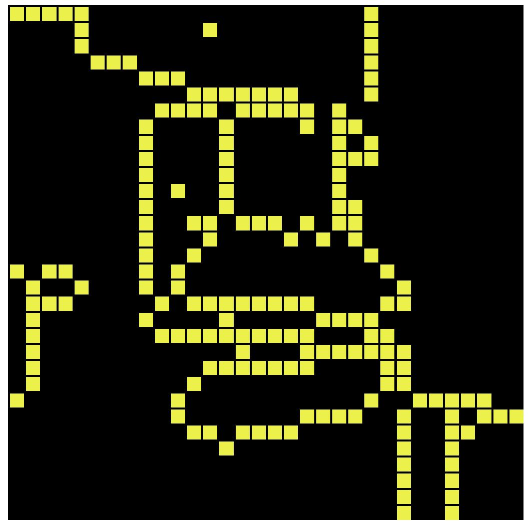
4-connected boundary



$\partial_B^{ext}(X)$



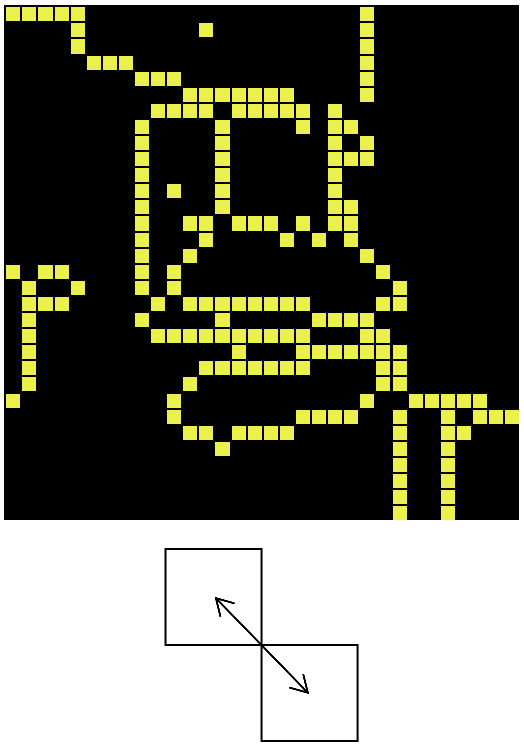
4-connected SE



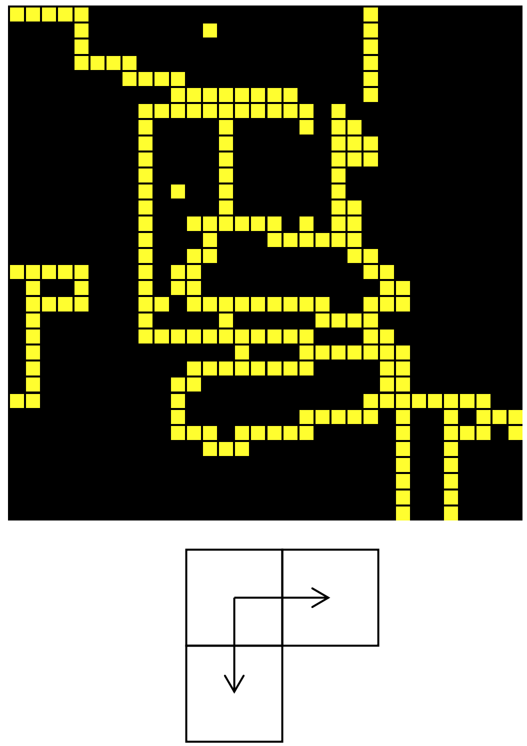
8-connected boundary

Boundary Measurement

- Interior Boundaries are disjoint from Exterior Boundaries
- Topology is important for the Perimeter Computation



$\sqrt{2}$

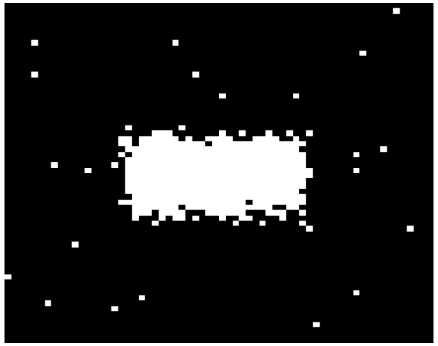


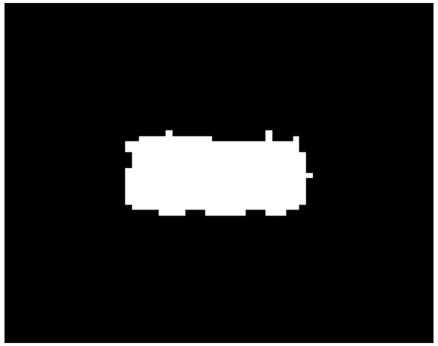
2

Binary Mathematical Morphology

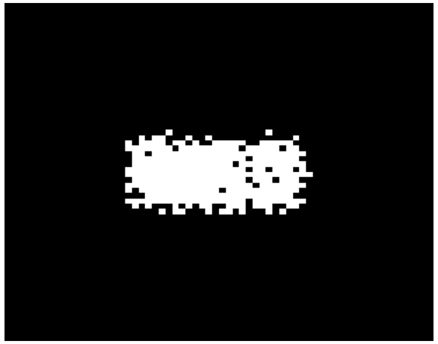
Opening and Closing

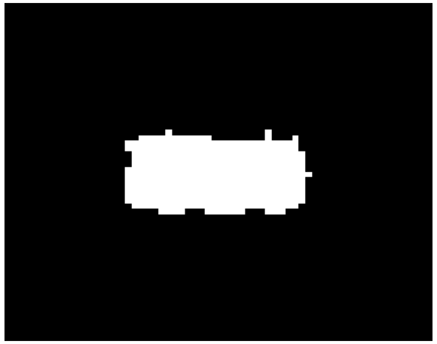
- **Opening**
  - Removes isolated noisy pixels
  - Smoothes object boundaries
  - Eliminates narrow protrusions
  - Does not significantly change the object's size





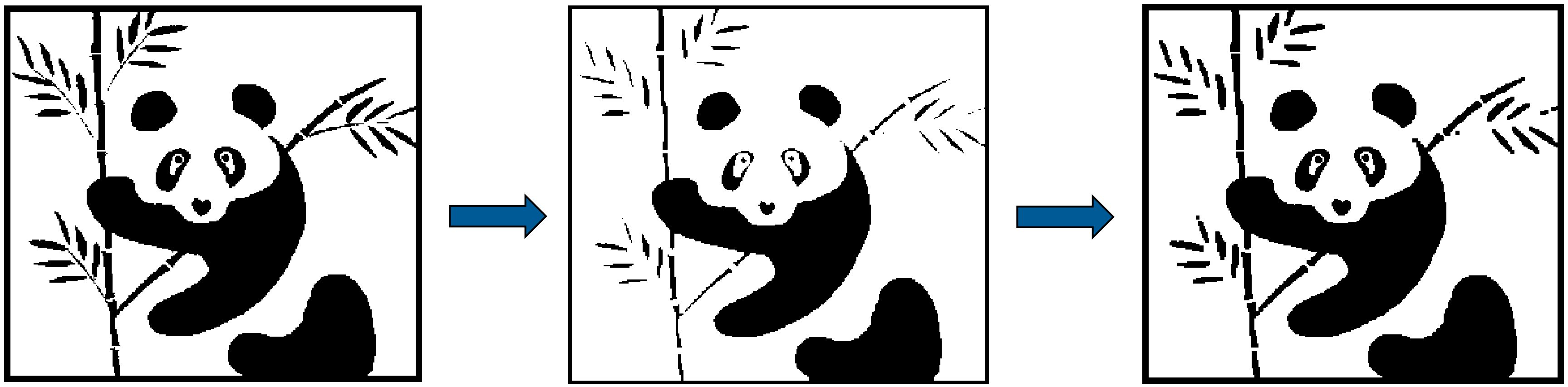
- **Closing**
  - Eliminates small holes
  - Smoothes object boundaries
  - Fills narrow gaps in the contour
  - Does not significantly change the object's size





Opening and Closing

- Composition of Dilation and Erosion



original → erosion → dilation

original → dilation → erosion

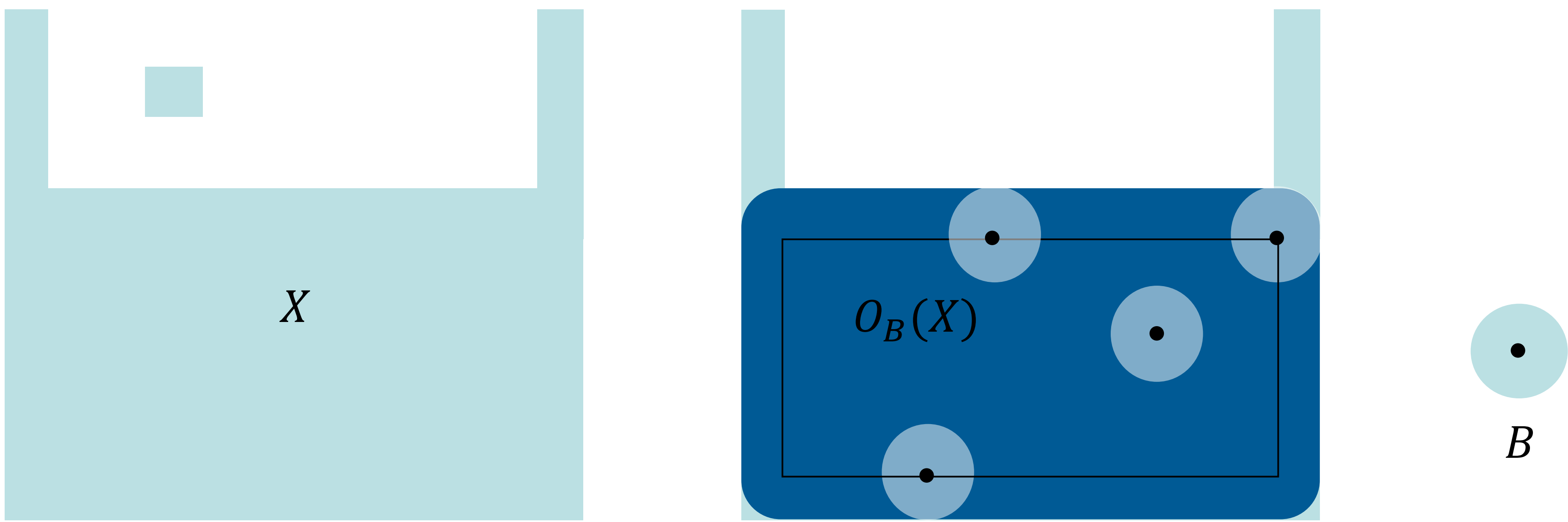
Opening, Closing

Opening

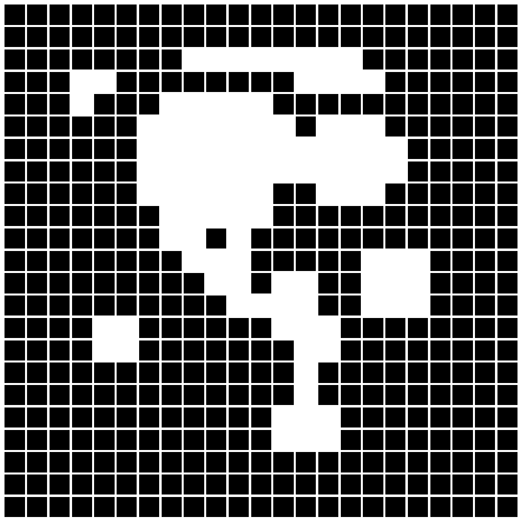
- **Definition**

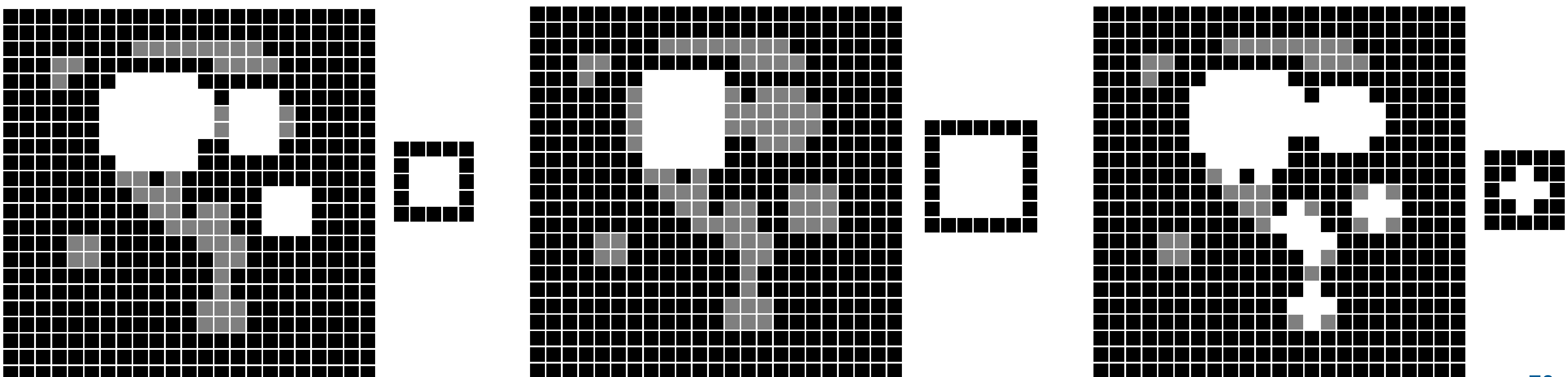
$$O_B(X) = D_{\check{B}}(E_B(X)) = \left(X \ominus \check{B}\right) \oplus B = \bigcup_{B_x \subseteq X} B_x$$

- **Illustration**



Opening with Other Structuring Elements



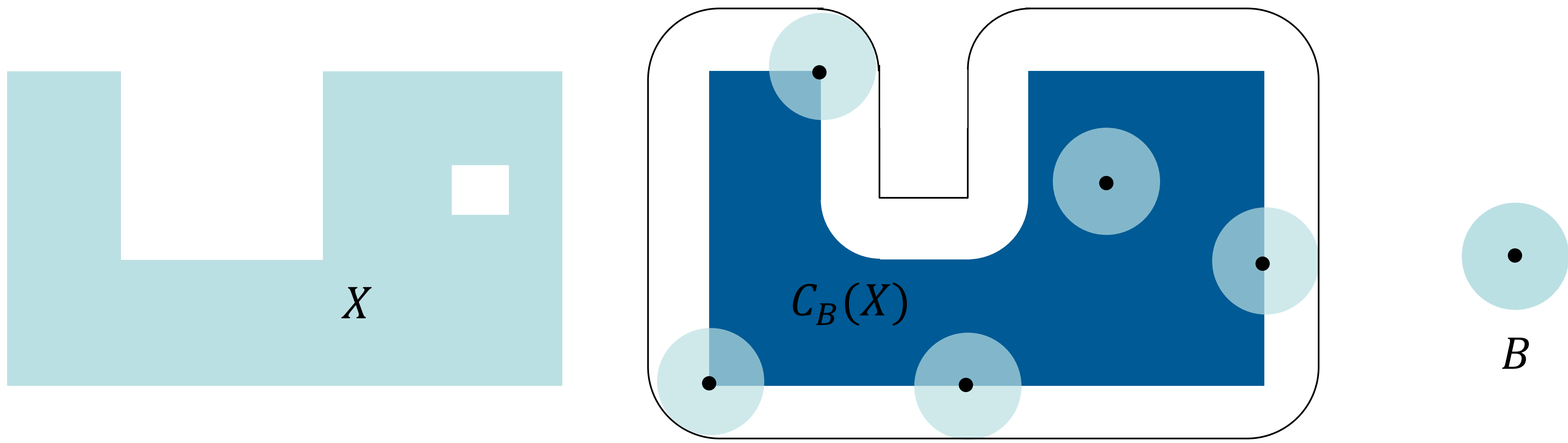


Closing

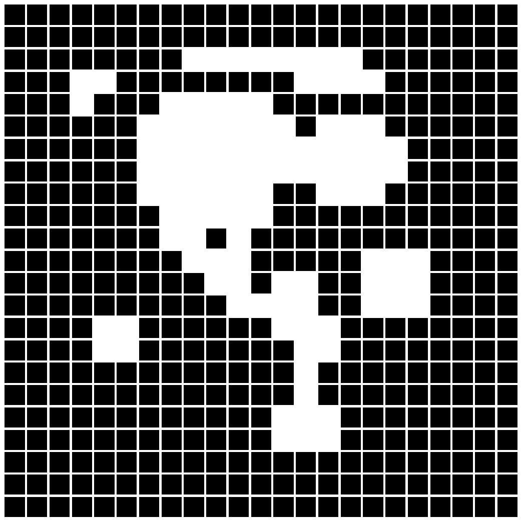
- **Definition**

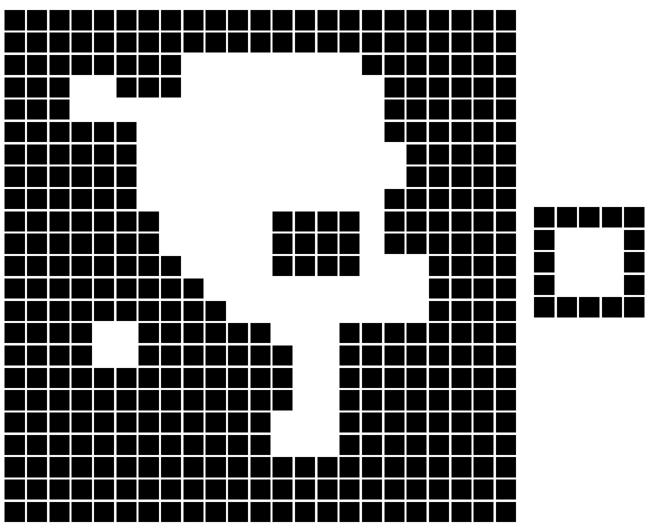
$$C_B(X) = E_{\check{B}}(D_B(X)) = (X \oplus \check{B}) \ominus B = \left(\bigcup_{B_x \subseteq X^c} B_x\right)^c$$

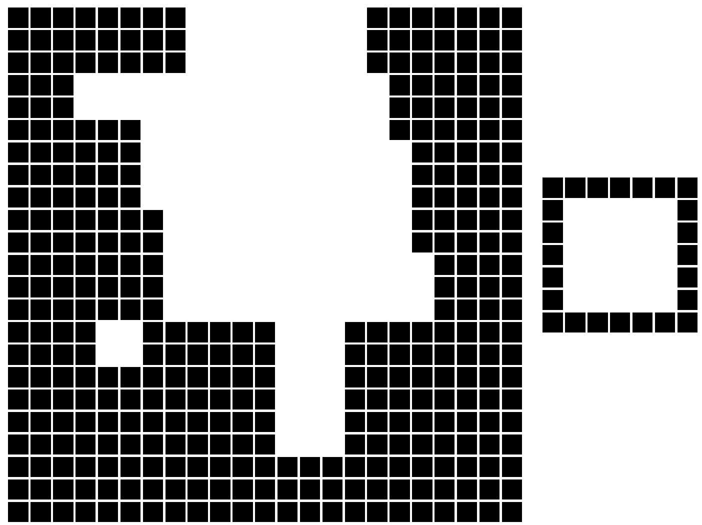
- **Illustration**

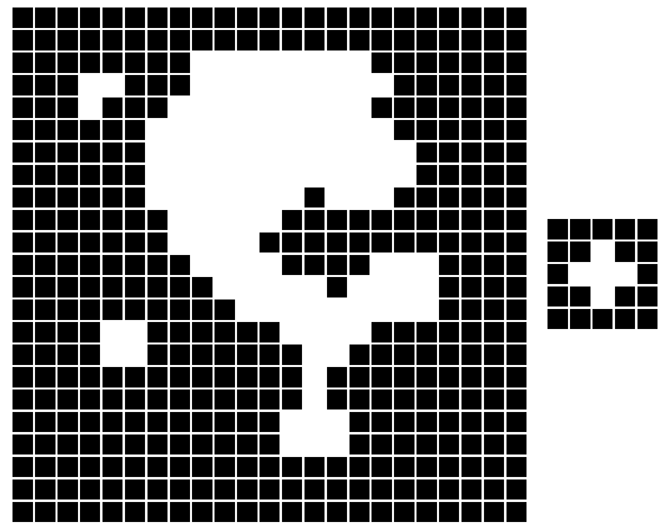


Closing with Other Structuring Elements









Main Properties

- **Extensivity, Anti-Extensivity**

$$O_B(X) \subseteq X \subseteq C_B(X)$$

- **Increasing**

$$X \subseteq Y \Rightarrow O_B(X) \subseteq O_B(Y)$$
$$X \subseteq Y \Rightarrow C_B(X) \subseteq C_B(Y)$$

- **Idempotence**

$$O_B(O_B(X)) = O_B(X)$$
$$C_B(C_B(X)) = C_B(X)$$

- **Duality with respect to the Complementation**

$$O_B(X) = (C_B(X^c))^c$$

- **Duality with respect to the Adjunction**

$$C_{\check{B}}(X) \subseteq Y \Leftrightarrow X \subseteq O_B(Y)$$

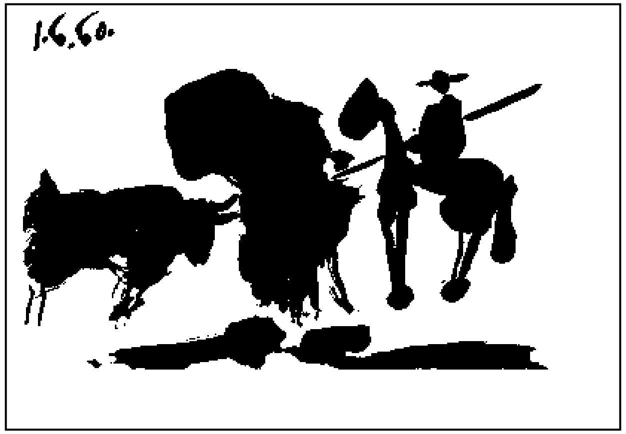
- **Translation Invariance**
- **Compatibility with Scales**

- **Decreasing, Increasing with respect to the SE**

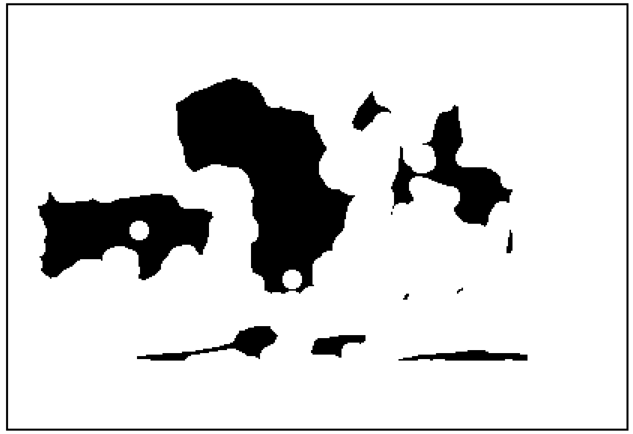
$$B \subseteq B' \Rightarrow O_{B'}(X) \subseteq O_B(X)$$
$$B \subseteq B' \Rightarrow C_B(X) \subseteq C_{B'}(X)$$

Erosion, Dilation, Opening, Closing

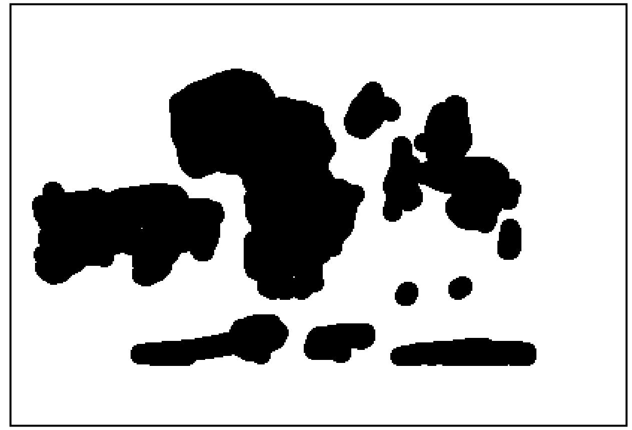
- **Comparison**
  - Foreground is black, background is white



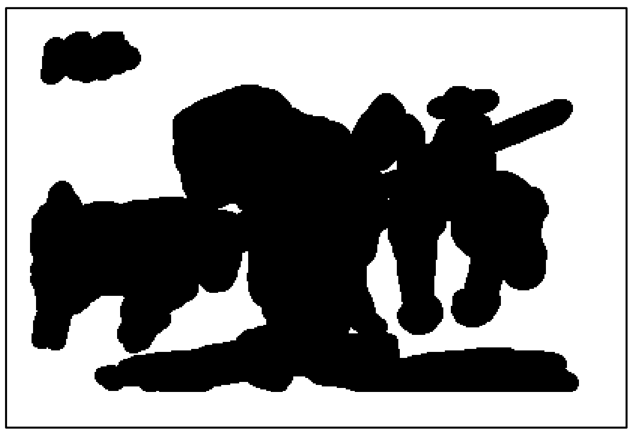
original



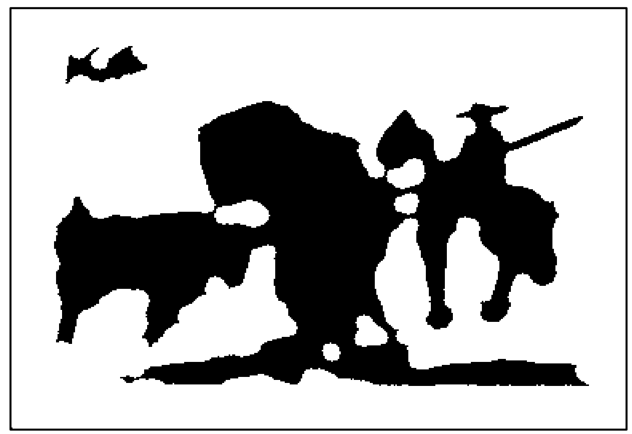
erosion



opening



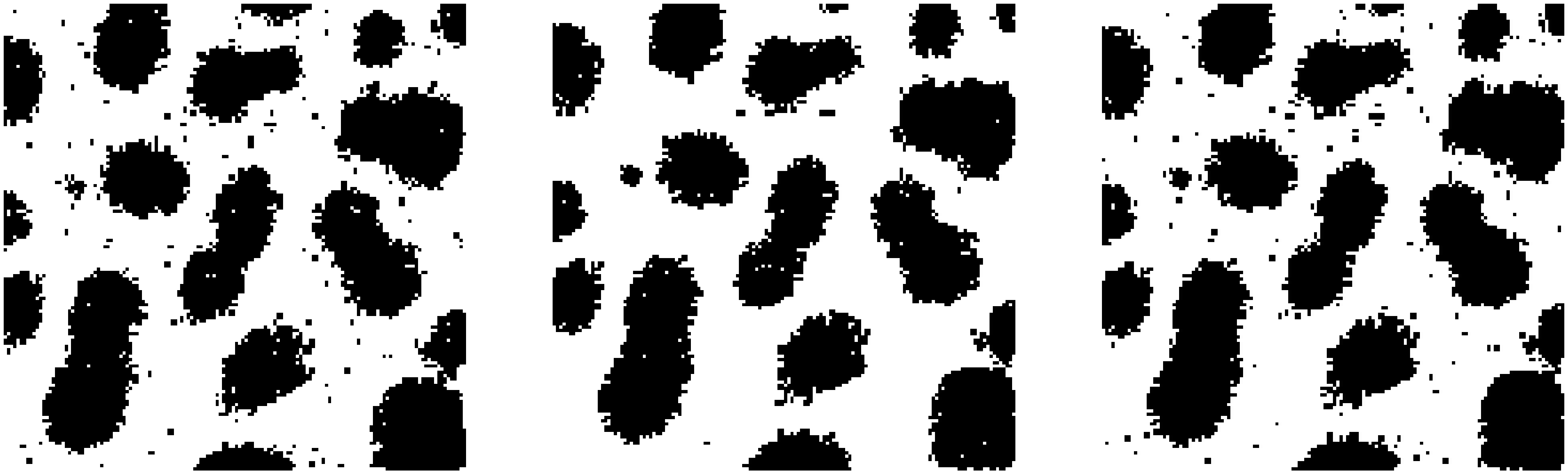
dilation



closing

Application to Image Denoising

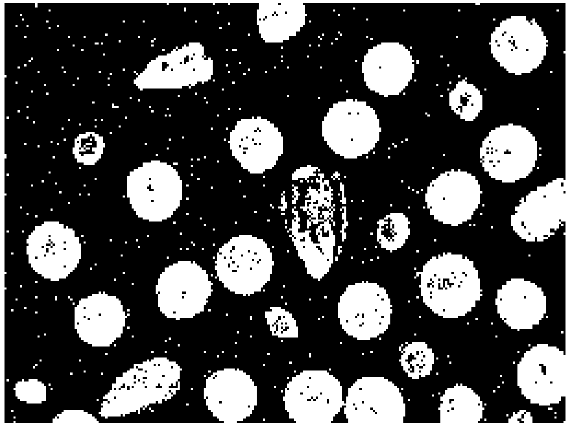
- **Salt and Pepper Noise Reduction**
  - Using closing or opening



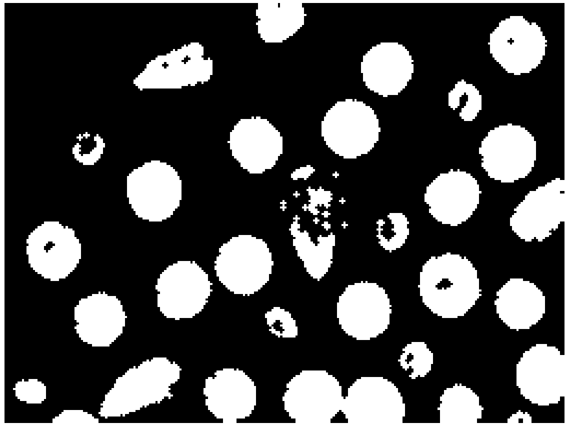
original — closing (pepper removed) — opening (salt removed)

Application to Image Denoising

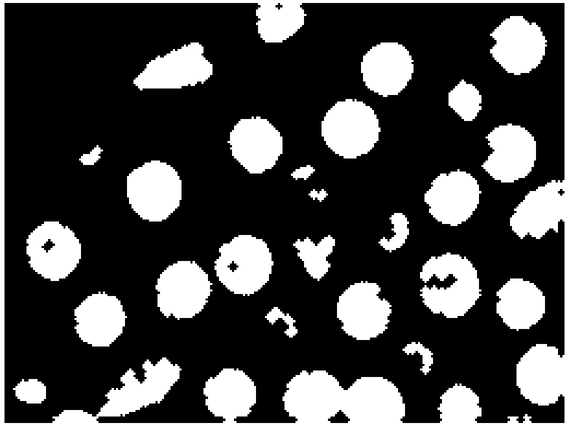
- **Salt and Pepper Noise Reduction**
  - Using alternate filters



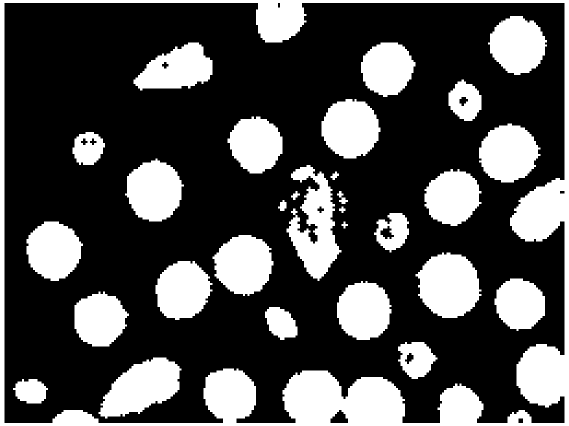
original



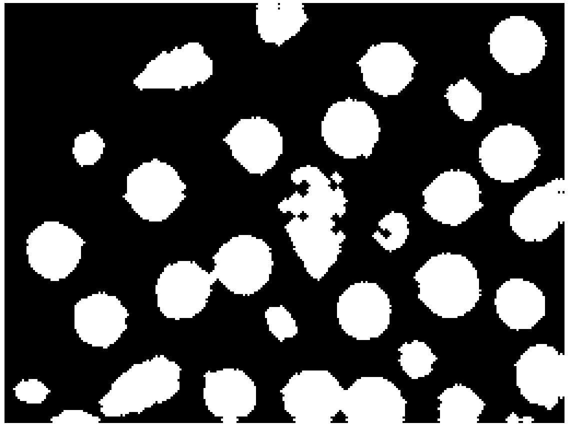
opening + closing (3x3 SE)



opening + closing (5x5 SE)



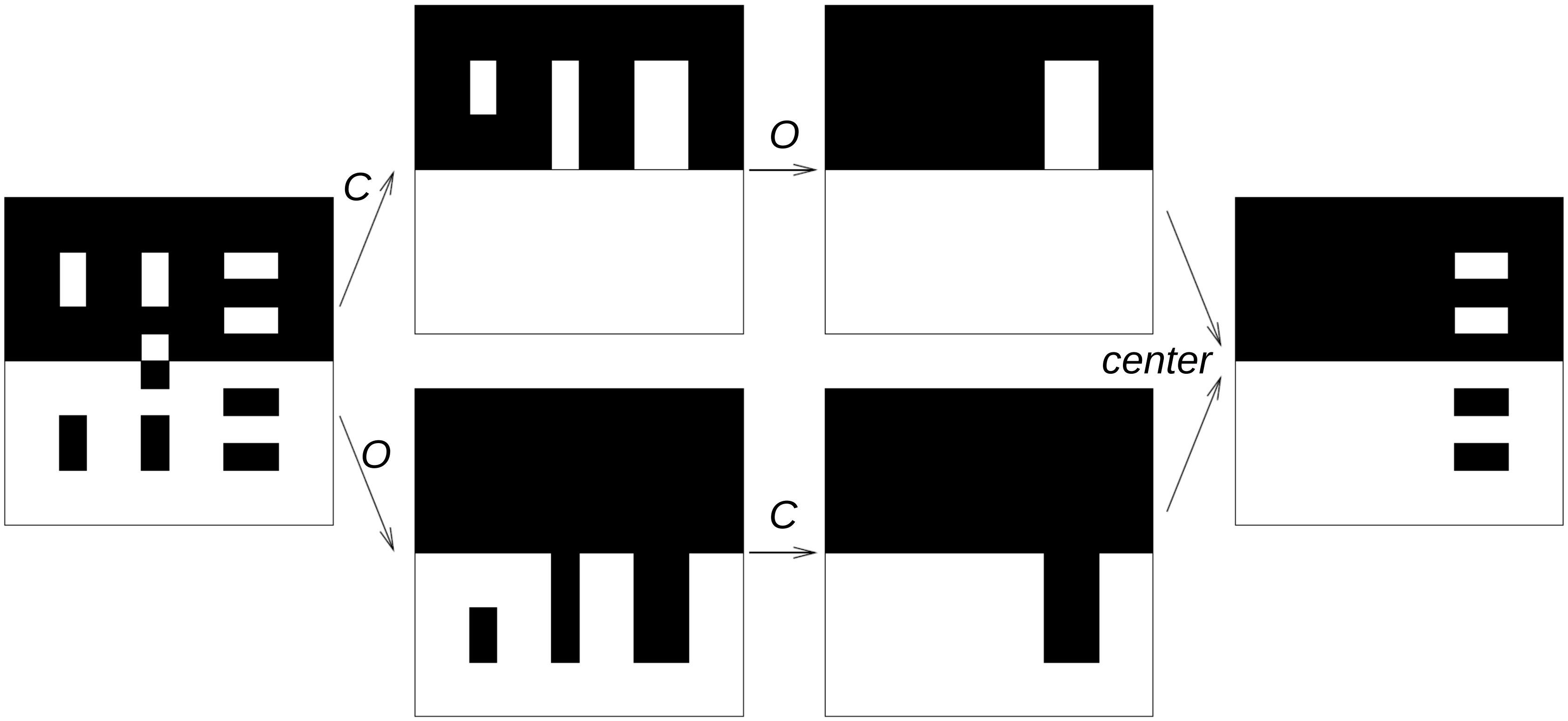
closing + opening (3x3 SE)



closing + opening (5x5 SE)

Morphological Center

- **Acting on Black and White Areas**
  - Auto-dual operator
  - Kind of median morphological operator
- **Illustration**
  - $Median[f, O_B(C_B(f)), C_B(O_B(f))]$



Granulometry

Aim

- Study of the size characteristics of sets
- In physics: use of sieves ψ_λ of increasing meshes $\lambda > 0$

Granulometry by Openings

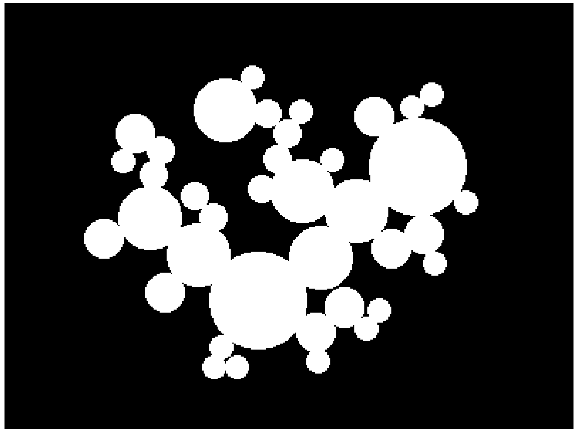
- Physical sieving refers to openings
- The size of the SE plays the role of the sieve meshes
- Decreasing with respect to the parameter: $\lambda \geq \mu > 0 \Rightarrow O_{\lambda B} \leq O_{\mu B}$
- Semi-group property: $O_{\lambda B} O_{\mu B} = O_{\mu B} O_{\lambda B} = O_{\sup(\lambda,\mu) B}$

Anti-Granulometry by Closings

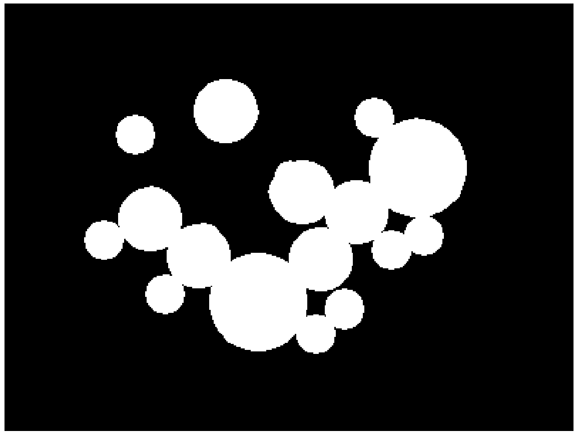
- Increasing with respect to the parameter: $\lambda \geq \mu > 0 \Rightarrow C_{\lambda B} \geq C_{\mu B}$
- Semi-group property: $C_{\lambda B} C_{\mu B} = C_{\mu B} C_{\lambda B} = C_{\sup(\lambda,\mu) B}$

Granulometry

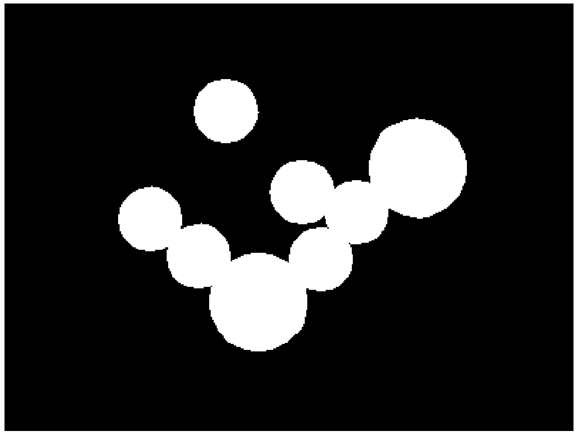
- **Illustration**



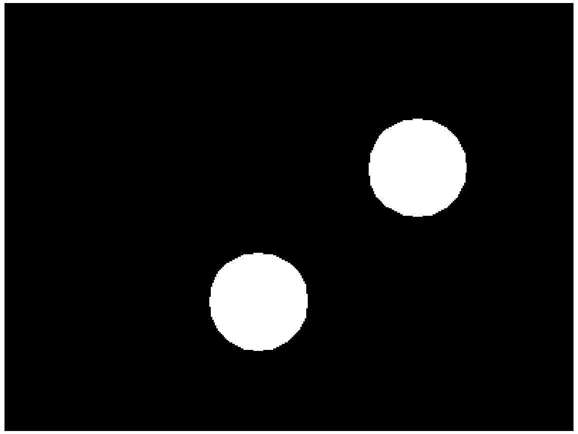
original



opening of size 10



opening of size 15



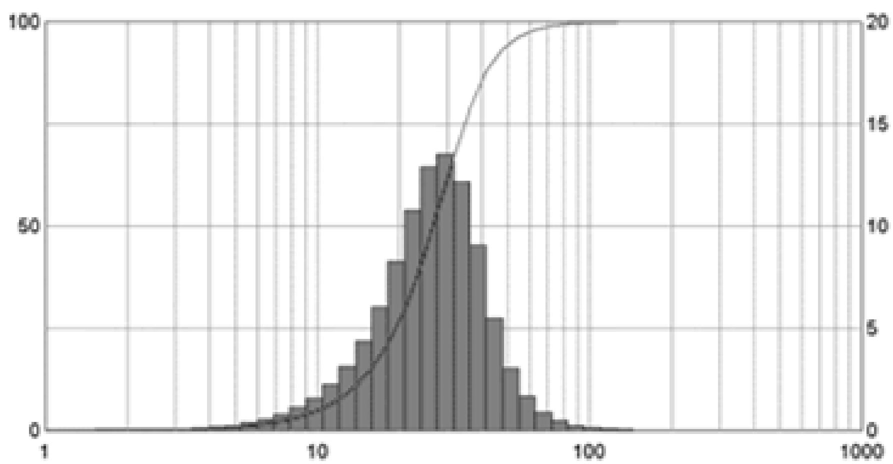
opening of size 25

Application to Particle Size Distribution

Size Measurement

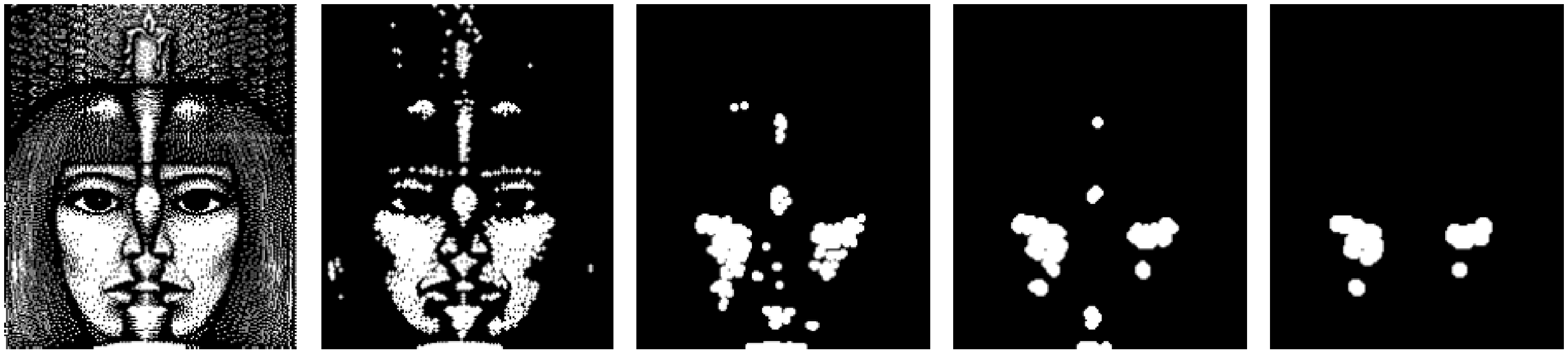
- Cumulative Distribution function: $F_X(\lambda) = 1 - \frac{\mu(O_{\lambda B}(X))}{\mu(X)}$

- Density function / Pattern spectrum: $f_X(\lambda) = F'_X(\lambda)$

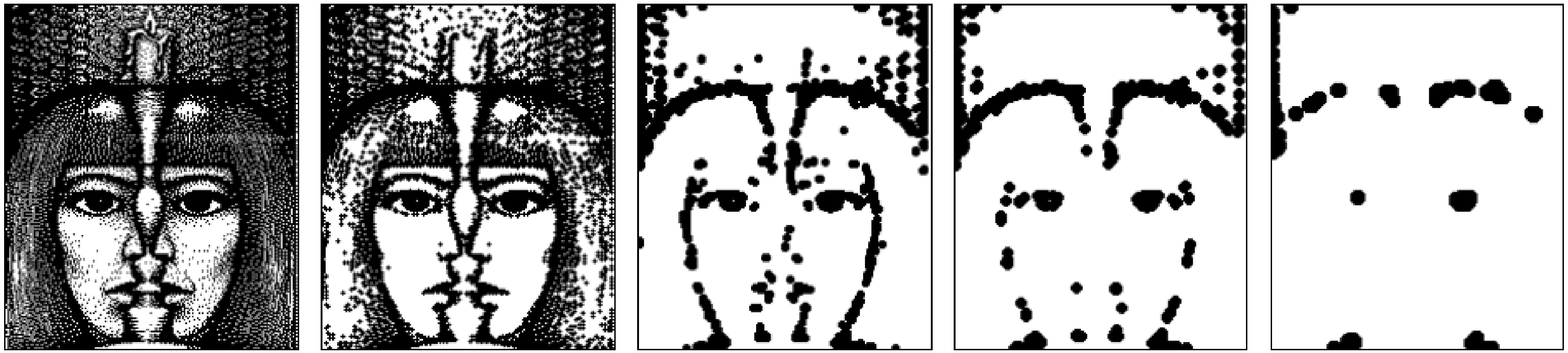


Application to Particle Size Distribution

- Granulometry

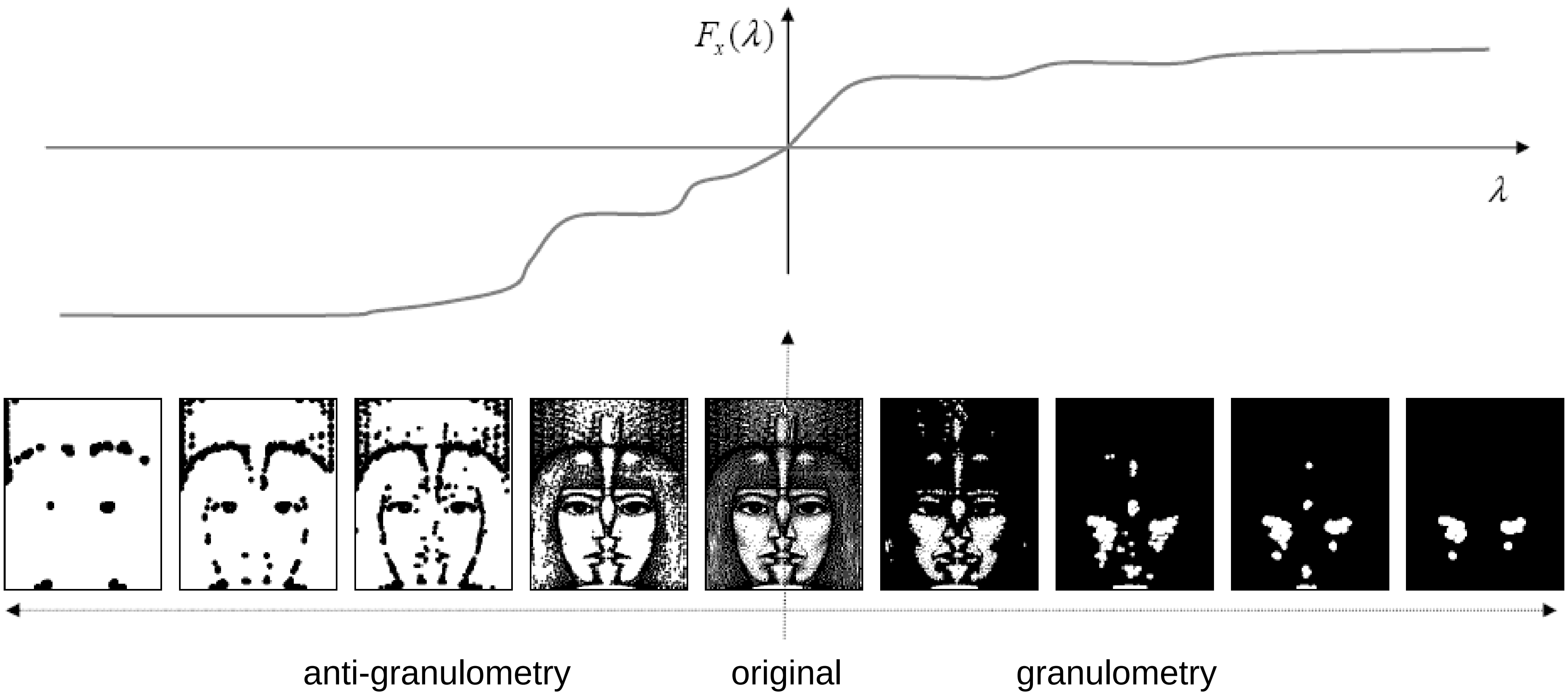


- Anti-Granulometry



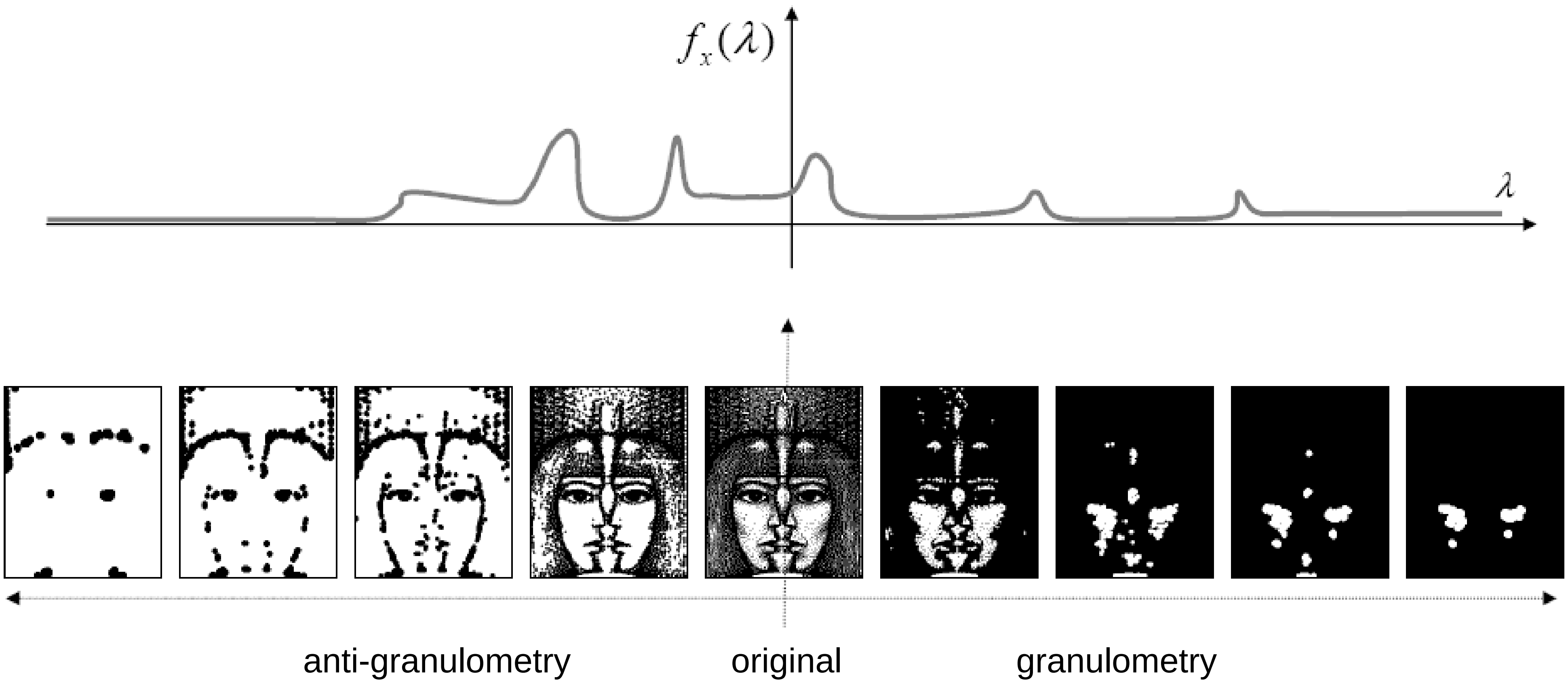
Application to Particle Size Distribution

- Cumulative Distribution Function



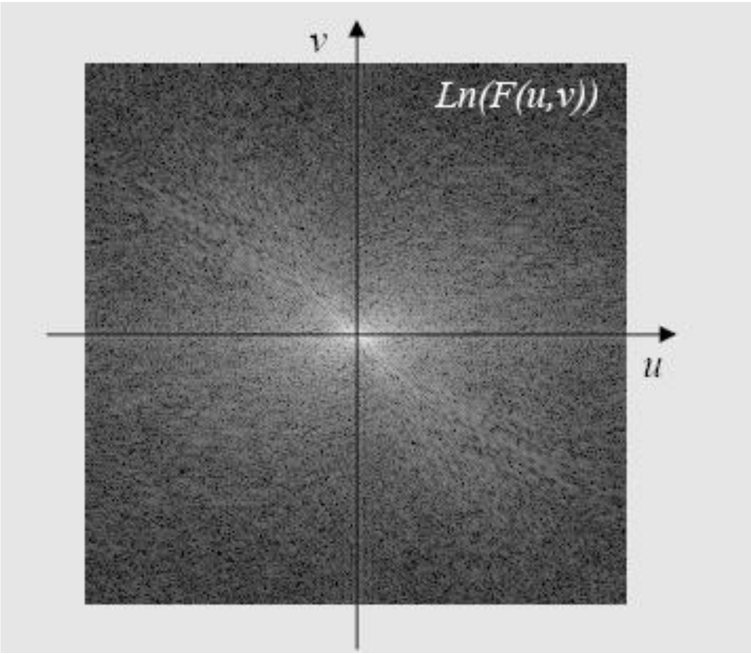
Application to Particle Size Distribution

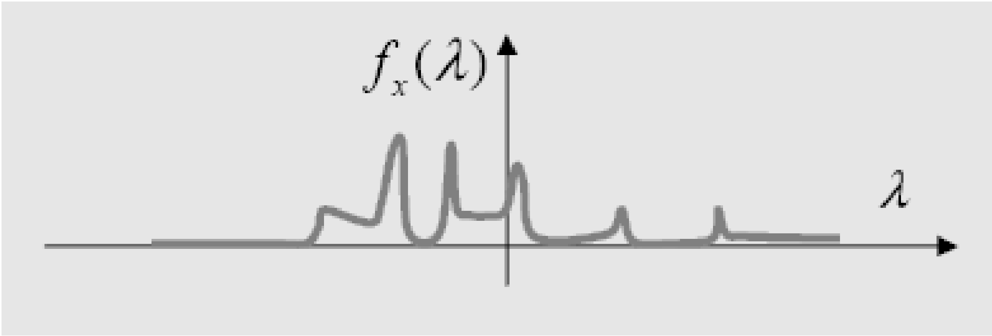
- **Pattern Spectrum**



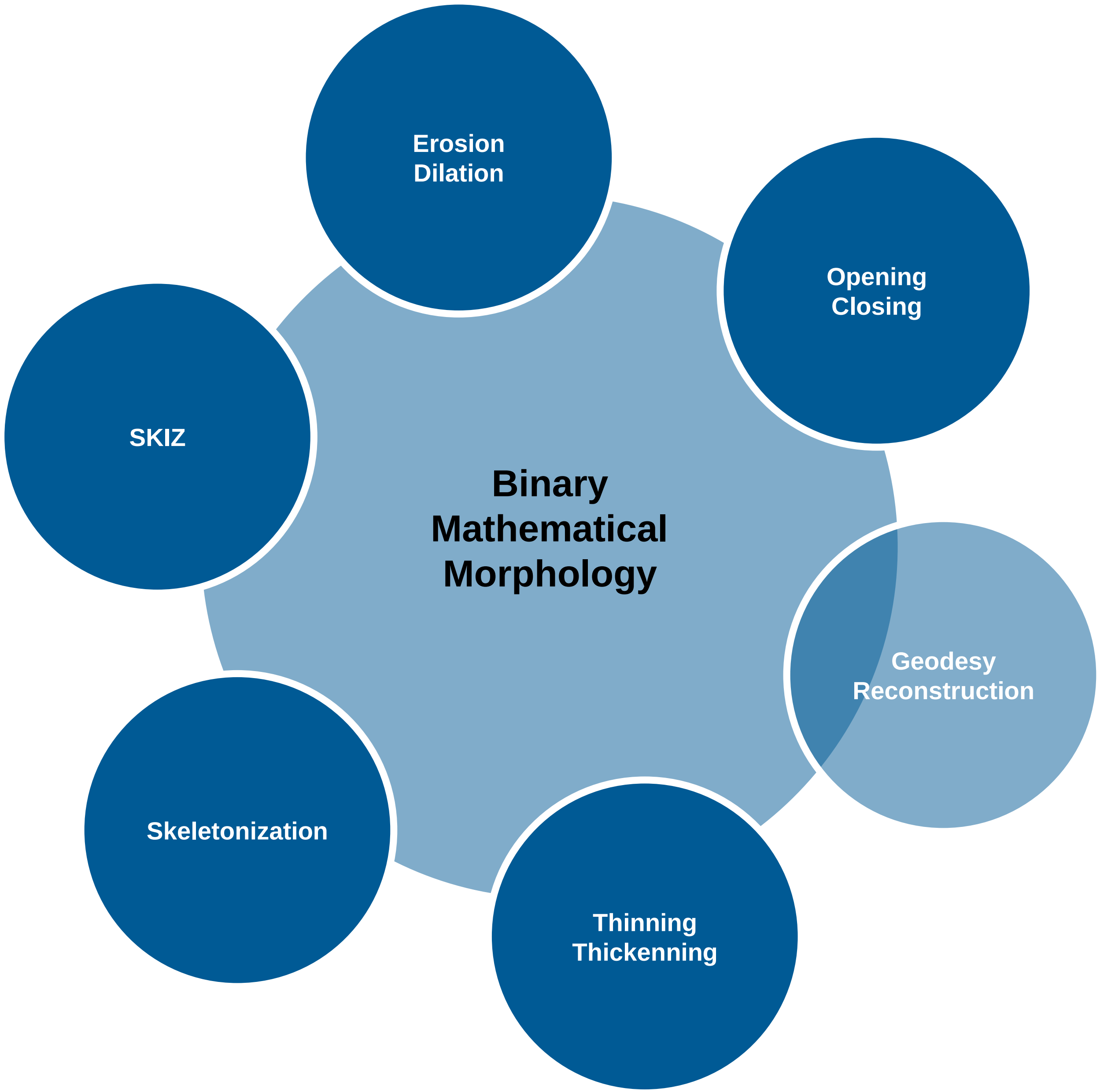
Fourier Analysis vs. Granulometric Analysis

- **Fourier Analysis**
  - Components = complex sinusoids
- **Granulometric Analysis**
  - Components = black and white discs



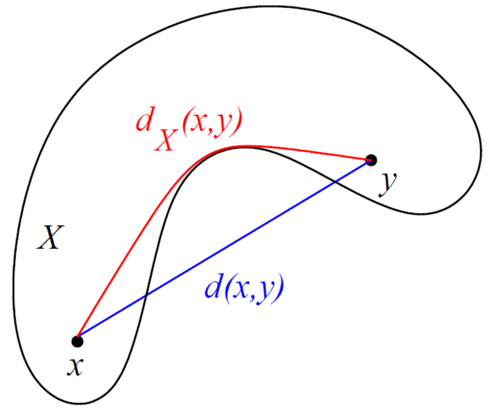


Binary Mathematical Morphology

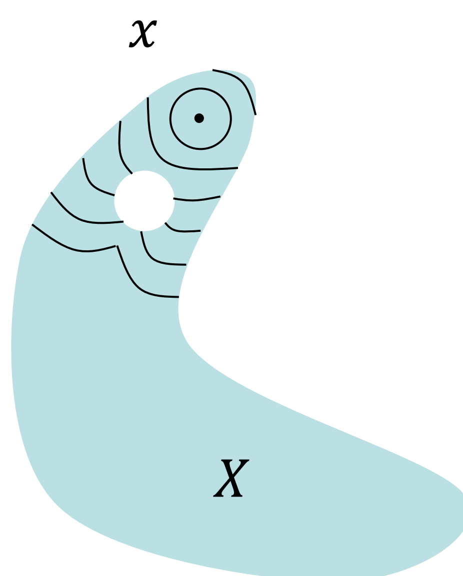


Geodesic Operators

- **Geodesic Distance**
  - Conditional to the marker X: d_X
  - $d_X(x, y)$ = length of the shortest path going from *x* to *y* and included in *X*
  - $X \text{ convex} \Leftrightarrow d_X = d$



- **Geodesic Ball**
  - $B_x^X(r) = \{y \in X; d_X(x, y) \leq r\}$
  - Remark: $B_x^X(r) \subseteq B_x(r)$



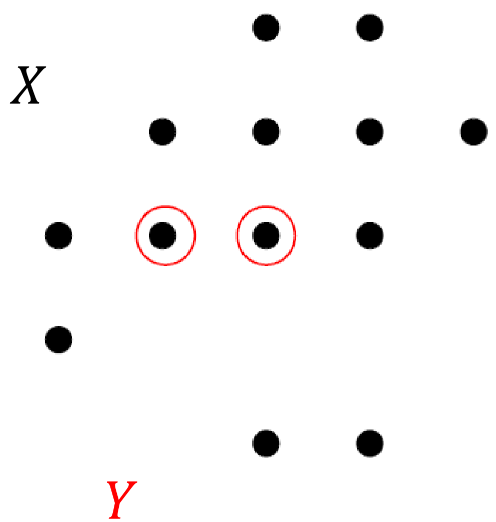
- **Geodesic Dilation and Erosion**
  - $D_{B,X}(Y) = \bigcup_{y \in Y} B_y^X$
  - $E_{B,X}(Y) = \left\{y; B_y^X \subseteq X\right\}$

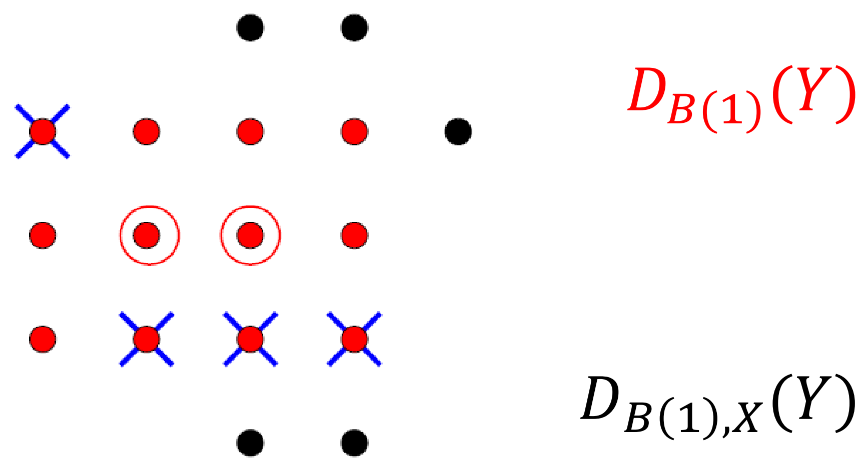
- **Geodesic Opening and Closing**

Geodesic Operators

Illustration in Digital Case

- $D_{\lambda B(1),X}(Y) = \left[D_{B(1)}(Y) \cap X\right]^{\lambda}$

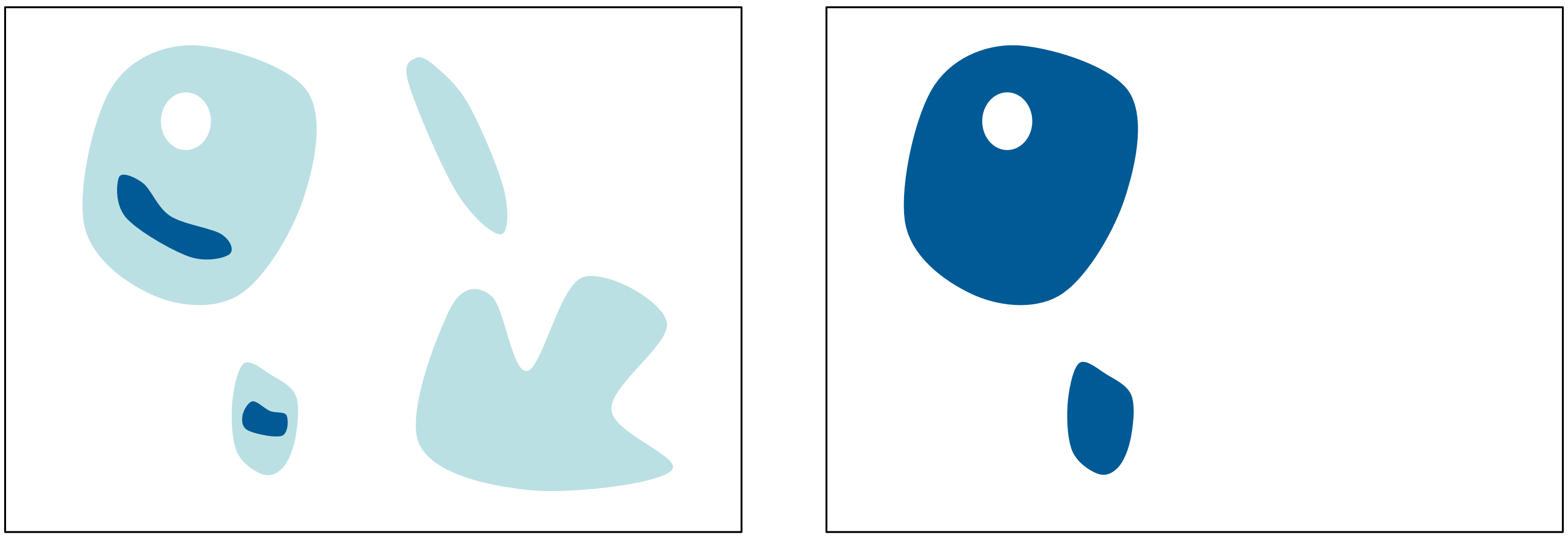




Reconstruction

- **Definition**
  - $D_X^{\infty}(Y) = \left[D_{B(1)}(Y) \cap X\right]^{\infty}$
  - Connected components of X which intersect Y

- **Illustration**

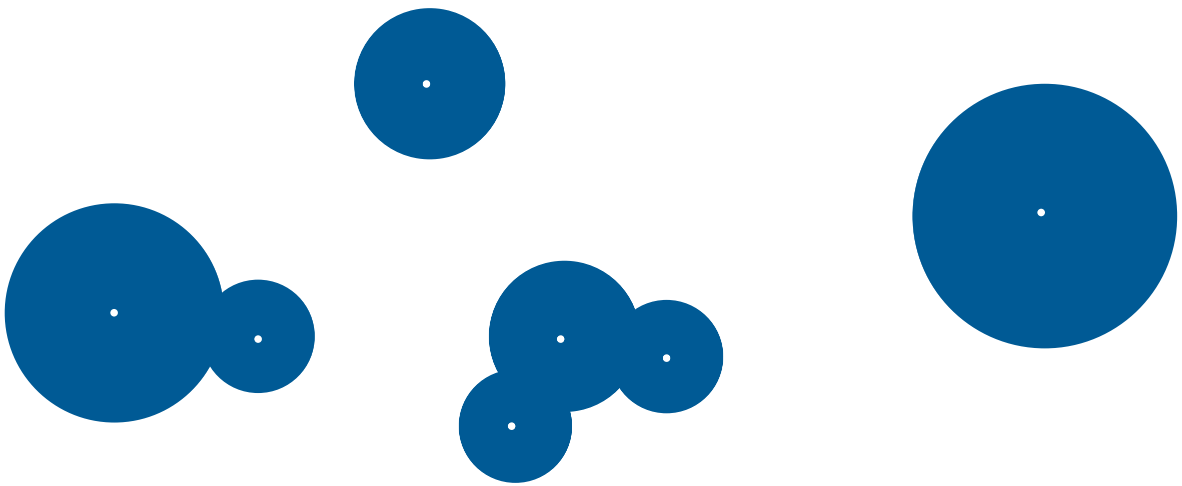


Application to Particle Counting

- **Ultimate Erosion**

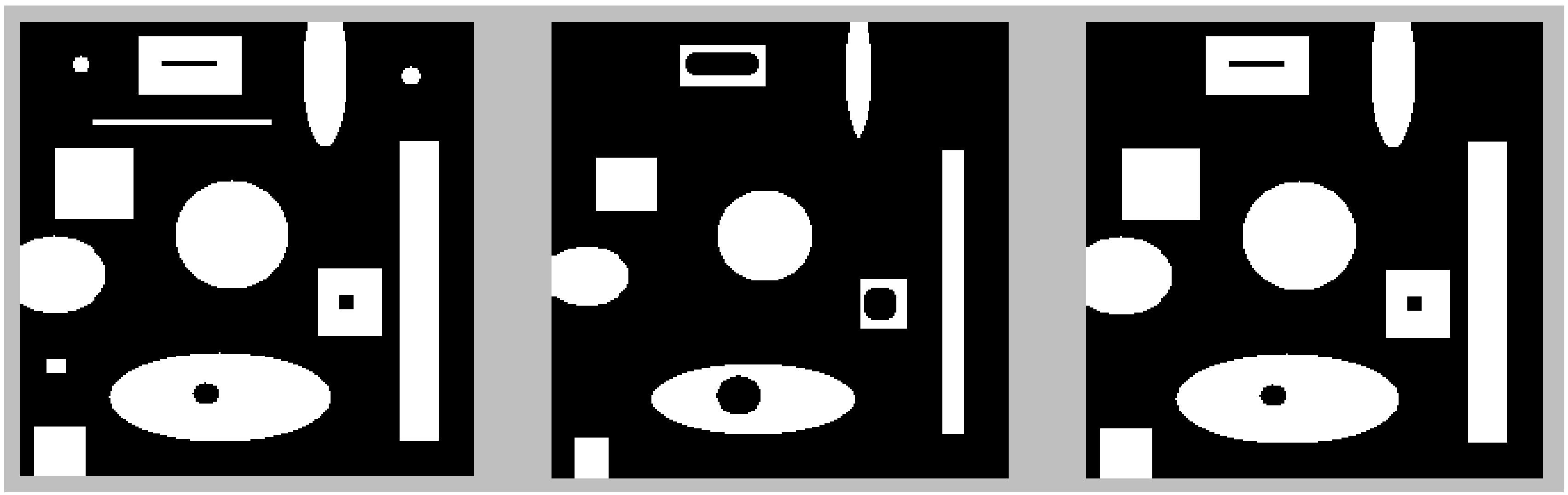
$$E_B^{ult}(X) = \bigcup_n E_{B_n}(X) \setminus D^{\infty}_{E_{B_n}(X)}\left(E_{B_{n+1}}(X)\right)$$

where Y = set of regional maxima of the distance function $\mathrm{d}(x, X^c)$



Application of Reconstruction

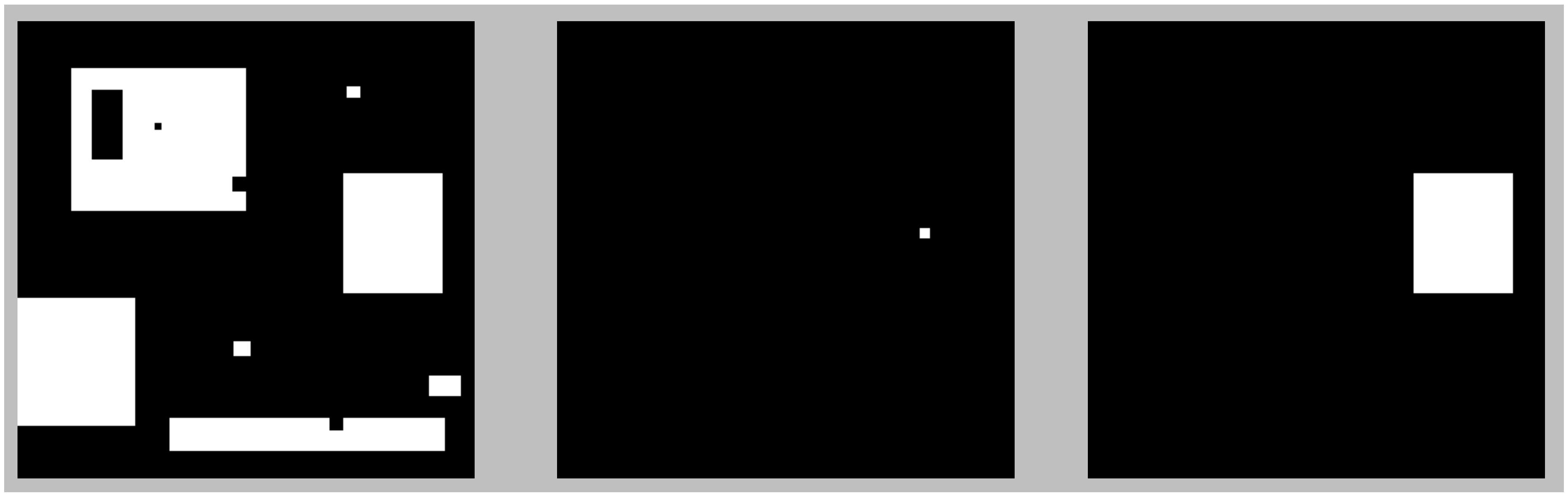
- **Removing small objects**
  - The other objects remain intact



initial — marker = erosion — reconstruction

Application of Reconstruction

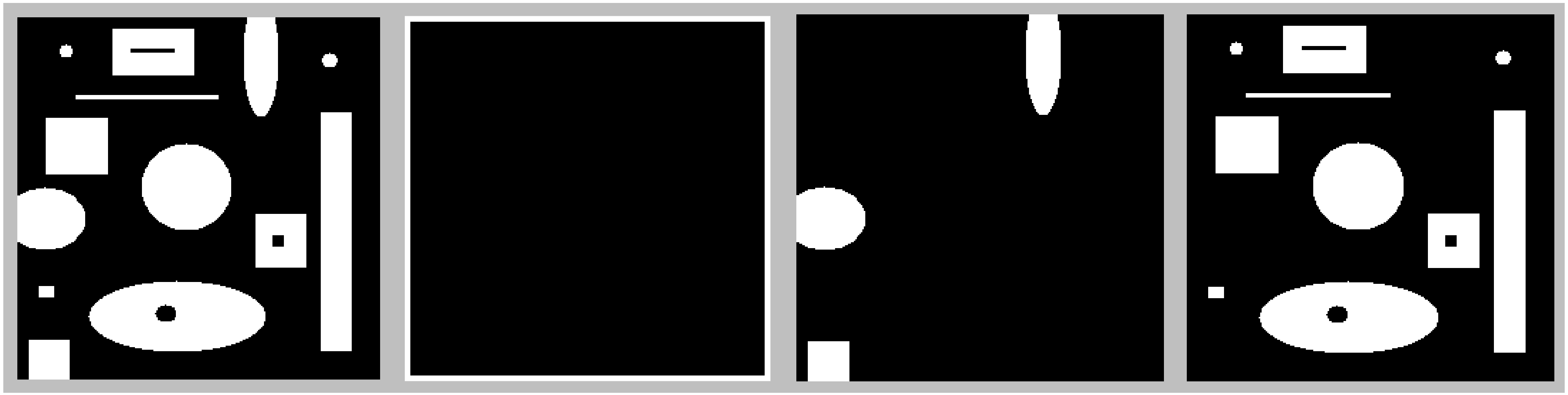
- **Object Individualization**



initial marker reconstruction

Application of Reconstruction

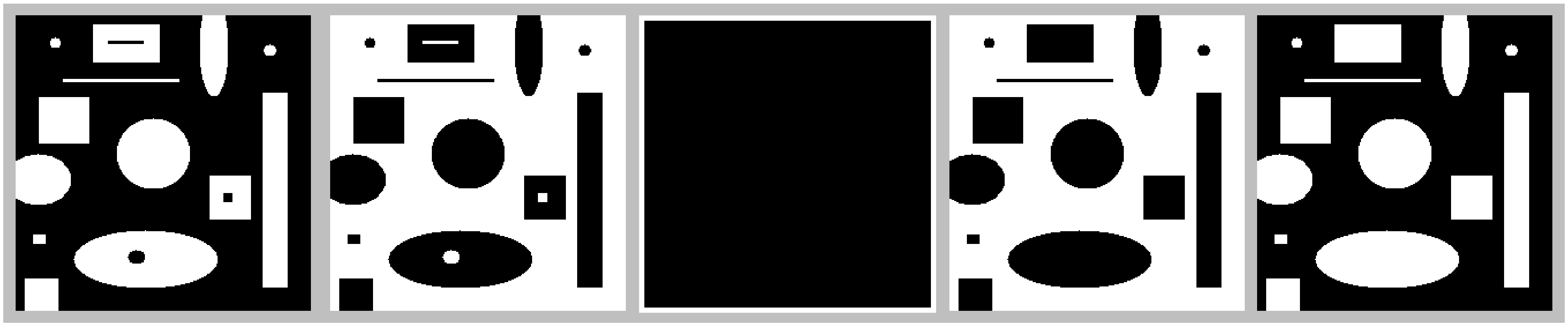
- **Removing Objects located on the Border**



initial marker reconstruction difference

Application of Reconstruction

- **Closing Holes**



initial | inversion | marker | reconstruction | inversion

Application of Reconstruction

- **Hysteresis Thresholding**
  - Using low and high thresholds


initial


high thresholding


low thresholding

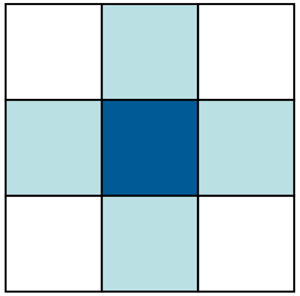

hysteresis thresholding

Binary Mathematical Morphology

Hit-or-Miss Transform

Bi-Phase Structuring Element

- $B = (B_1, B_2)$ with $B_1 \cap B_2 = \emptyset$



B

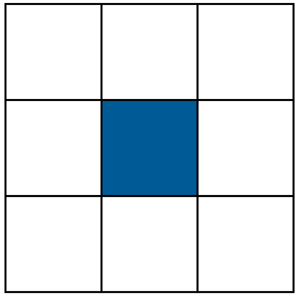
Hit or Miss Principle

- Does B_1 fit the object?

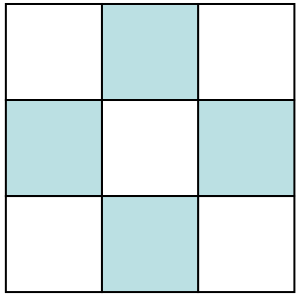
while simultaneously

- Does B_2 miss the object, i.e. fit the background?

$$\{x;\ (B_1)_x \subseteq X, (B_2)_x \subseteq X^c\}$$



B_1: ‘hit’ part



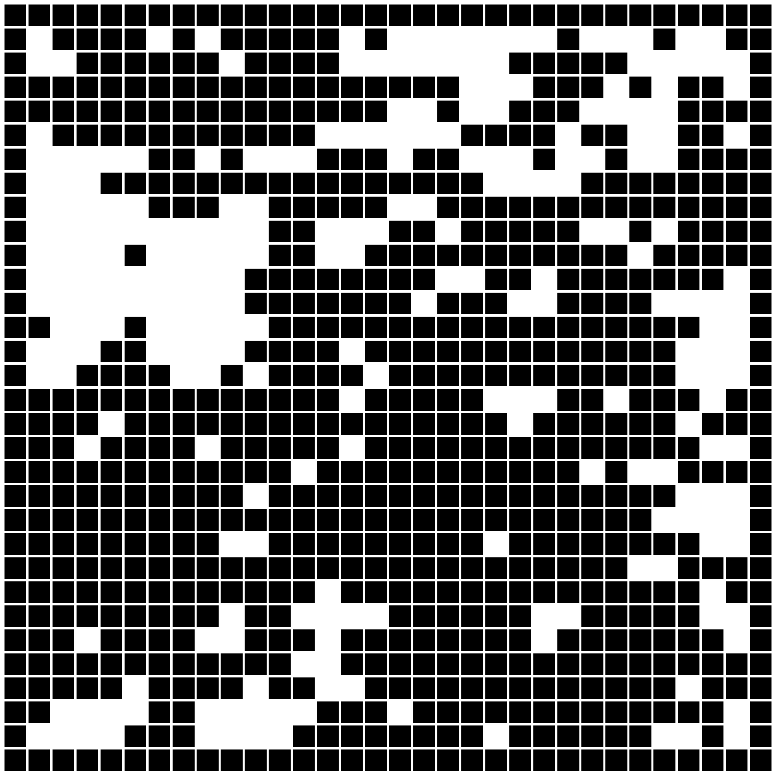
B_2: ‘miss’ part

Definition

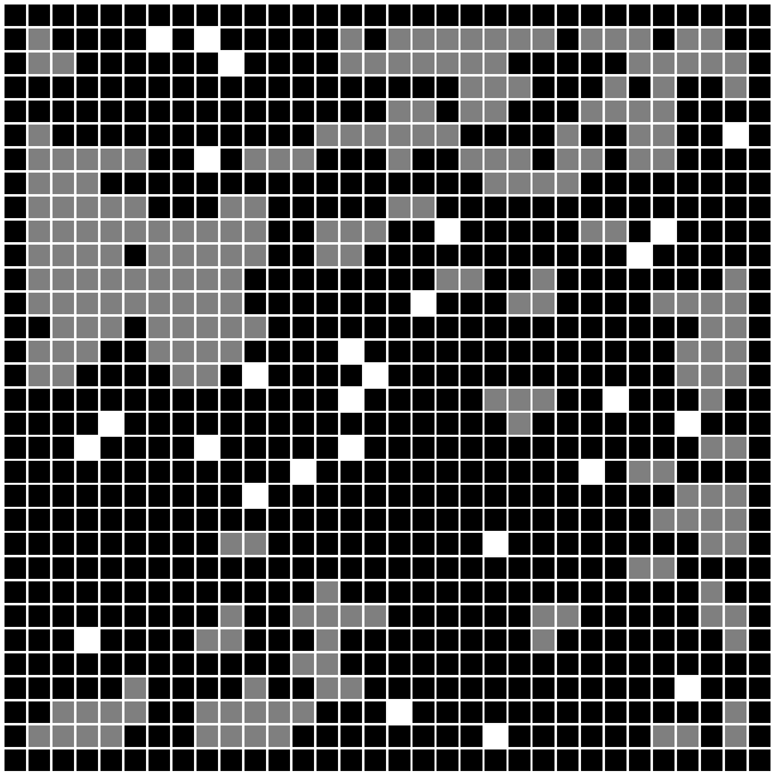
$$X \otimes B = E_{B_1}(X) \cap E_{B_2}(X^c)$$

Hit-or-Miss Transform

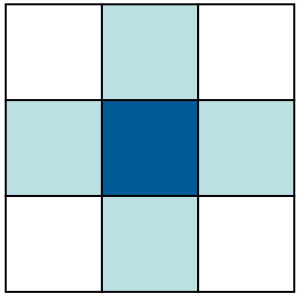
- **Illustration**
  - Isolated points in 4-connectivity



original



hit-or-miss



B

Definitions

- **Thinning**

$$X \circ B = X \setminus (X \otimes B)$$

- **Thickening**

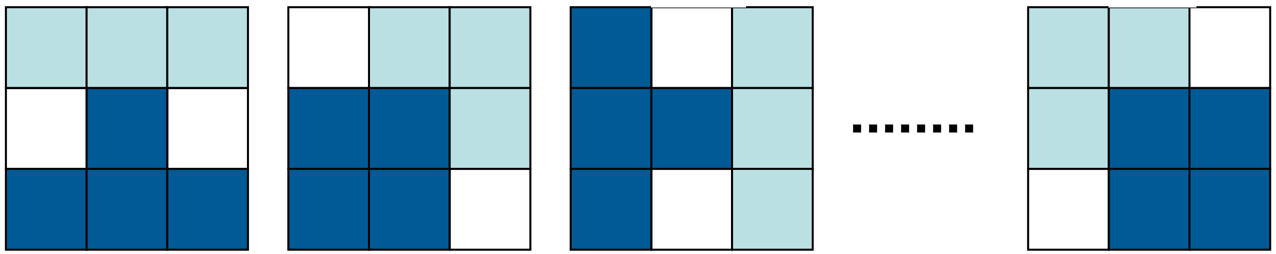
$$X \odot B = X \cup (X \otimes B)$$

- **Depending on the SE (or series of SE), very different results can be achieved**
  - Pruning
  - Skeletonization
  - Convex hull
  - …

Illustration

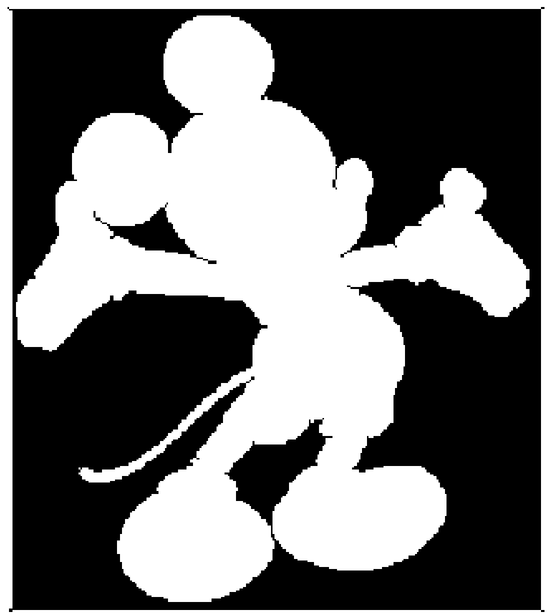
- **Using Specific Series of SE**

B^1 B^2 B^3 .... B^8



original

thinning

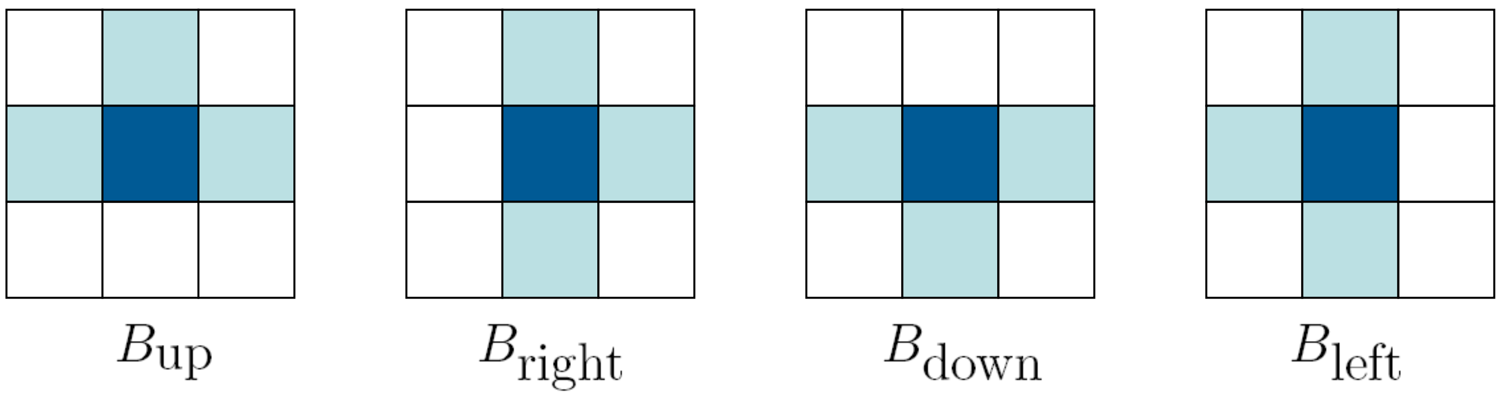


thickening

Pruning

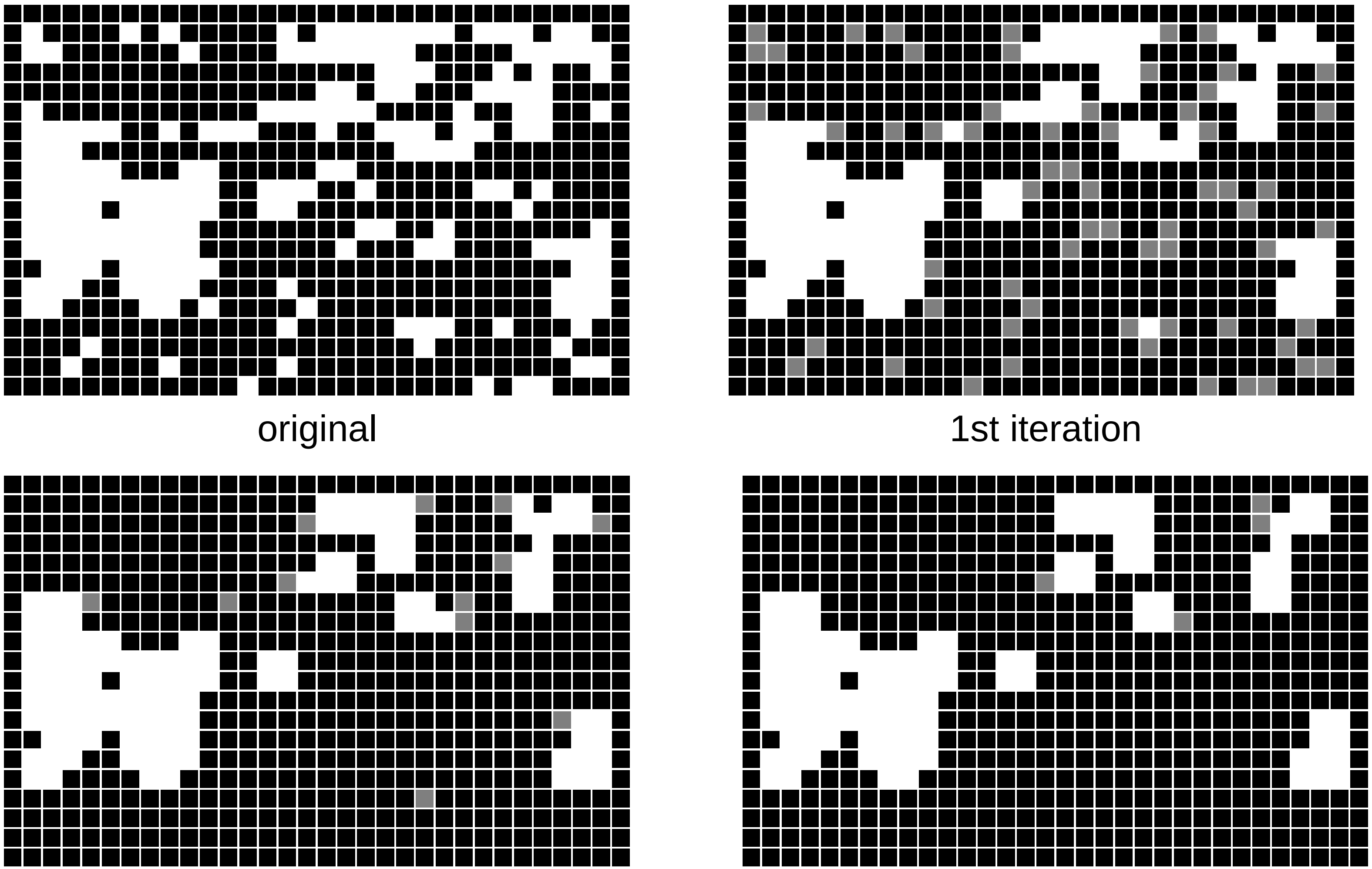
- **Remove End Points by a Sequence of Thinnings**

$$1 \text{ iteration} = (((X \circ B_{\text{up}}) \circ B_{\text{right}}) \circ B_{\text{down}}) \circ B_{\text{left}}$$



Pruning

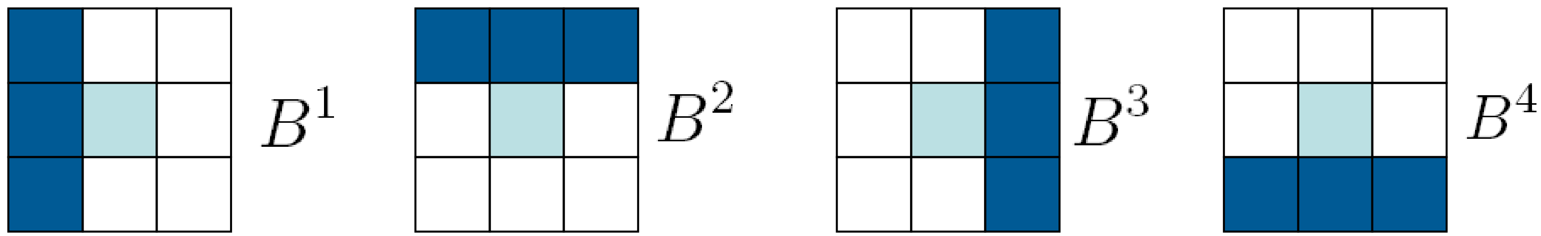
- **Illustration**

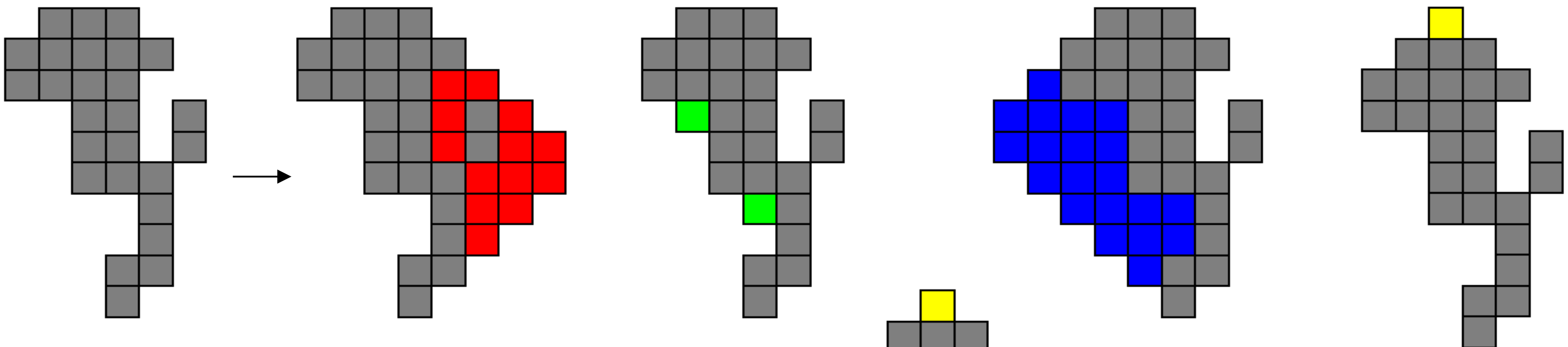


original · 1st iteration · 2nd iteration · 3rd iteration: idempotence

Convex Hull

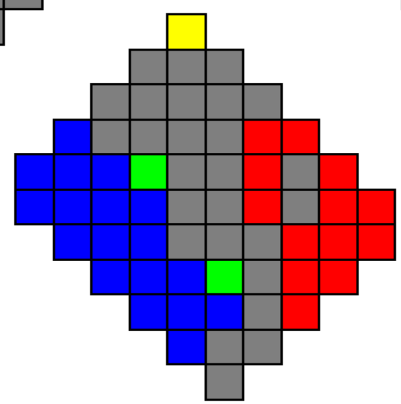
- **Smallest Convex Set Containing the Object**
  - Union of thickenings, each up to idempotence





Original shaper

Thickening with the first mask

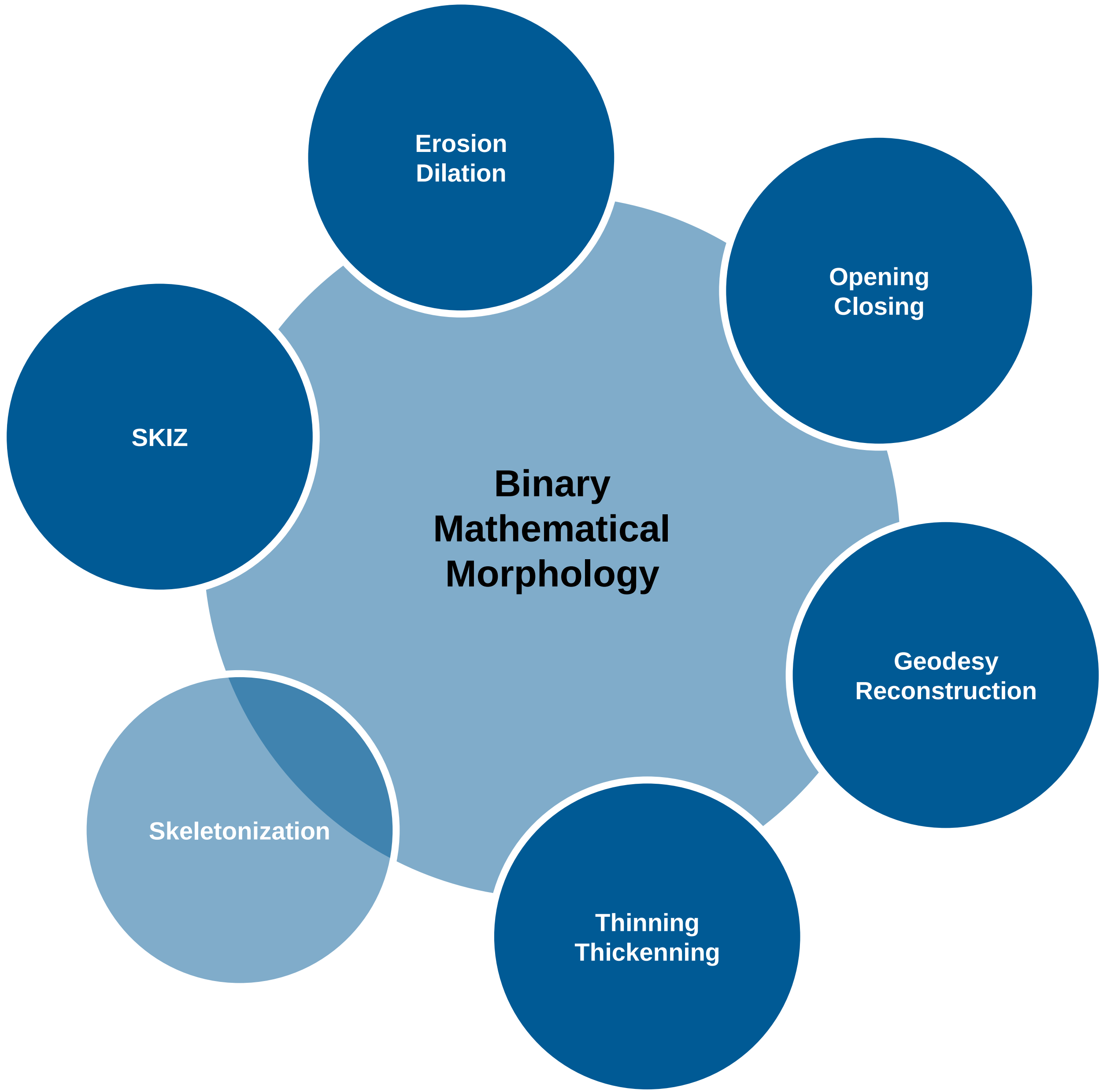


Union of four thickenings

Convex Hull

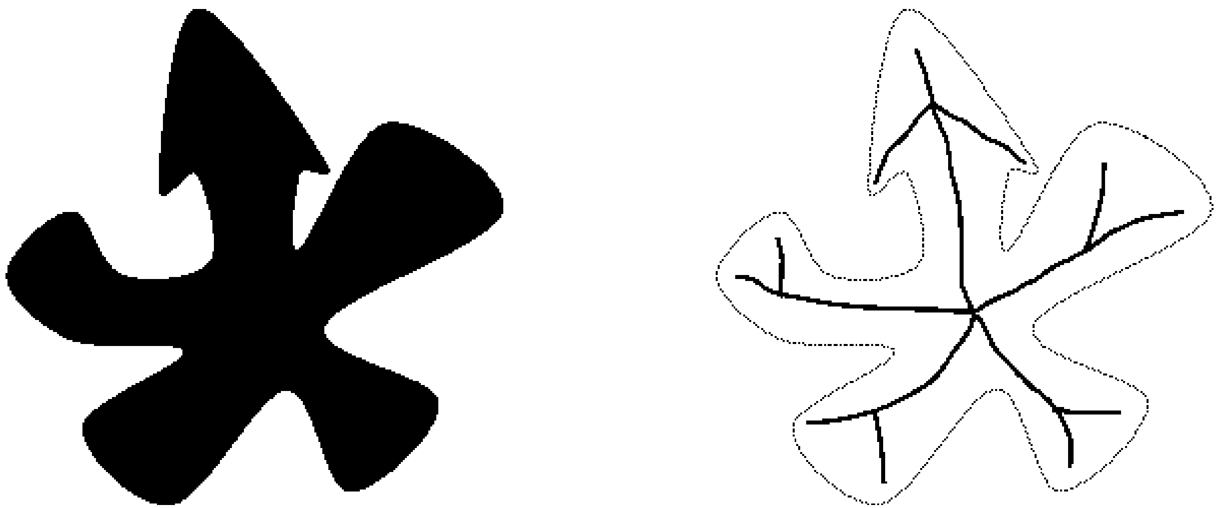
- **Illustration**

Binary Mathematical Morphology



Aim

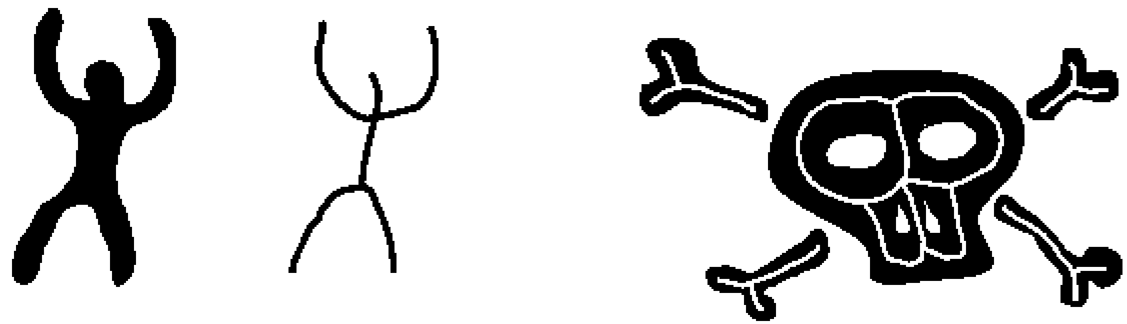
- **Compact Representation of Objects by Minimal Geometrical Information**

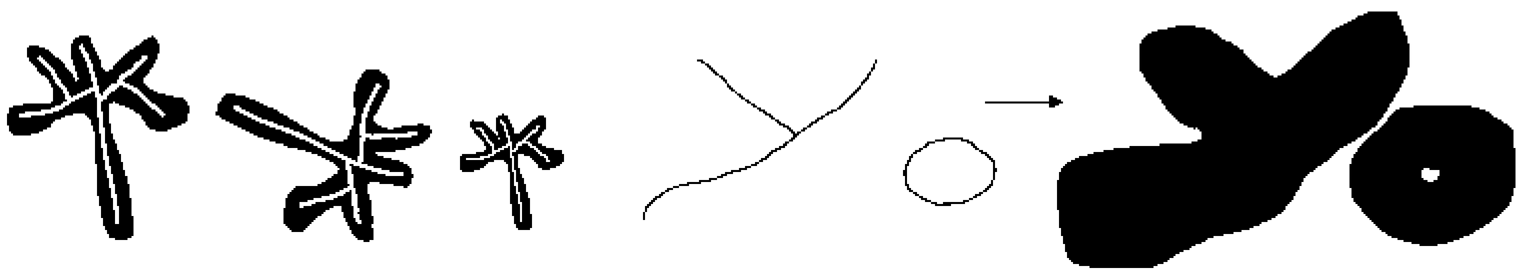


- **Different Approximations/Definition**
  - Skeleton by iterative thinnings
  - Skeleton by maximal balls

Requirements

- Thin Lines
- Geometry Preservation
- Topology Preservation
- Affine Invariance
- Invertibility
- Continuity

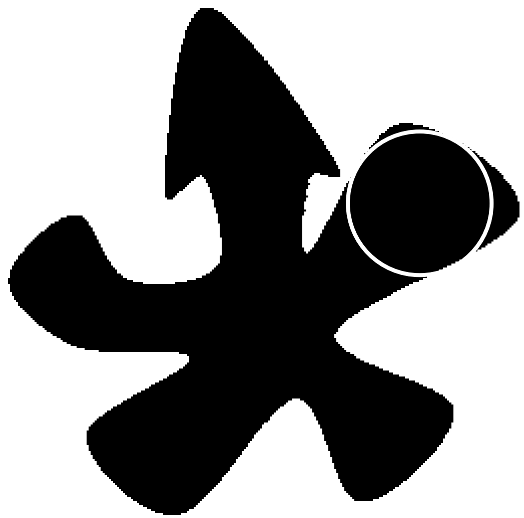




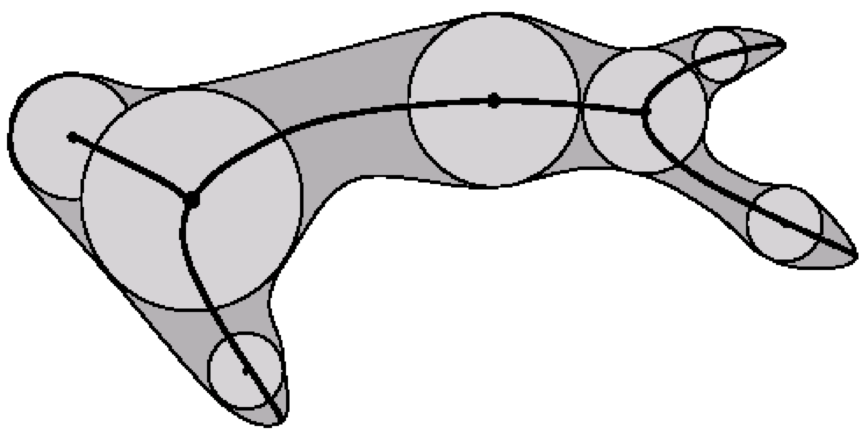
Skeleton by Maximal Balls

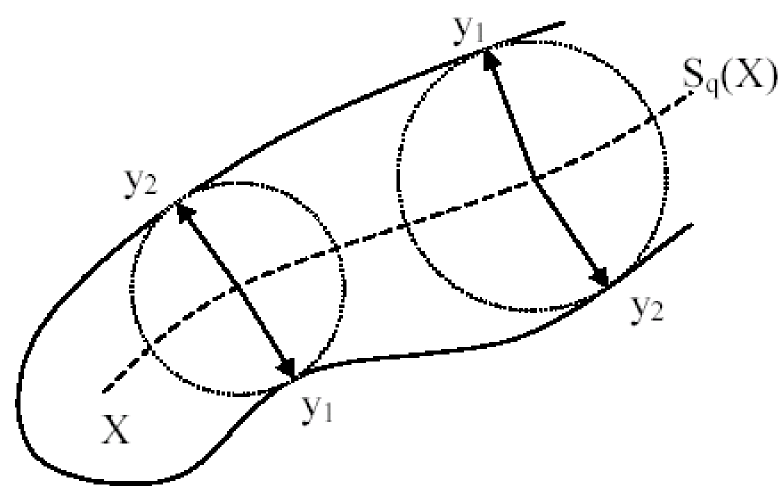
- **Maximal Ball *B***

$$B \subseteq B' \subseteq X \Rightarrow B = B'$$



- **Skeleton**
  - Set of all centers of maximal balls
  - $s \in S(X) \Leftrightarrow (\exists y_1, y_2 \in \partial X, y_1 \neq y_2 \mid d(s, \partial X) = d(s, y_1) = d(s, y_2))$



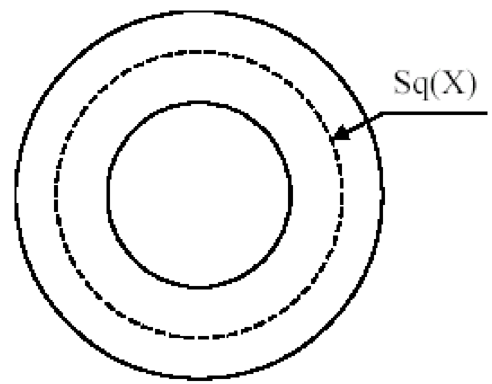


Skeleton by Maximal Balls

- **Definition in the Continuous Case**

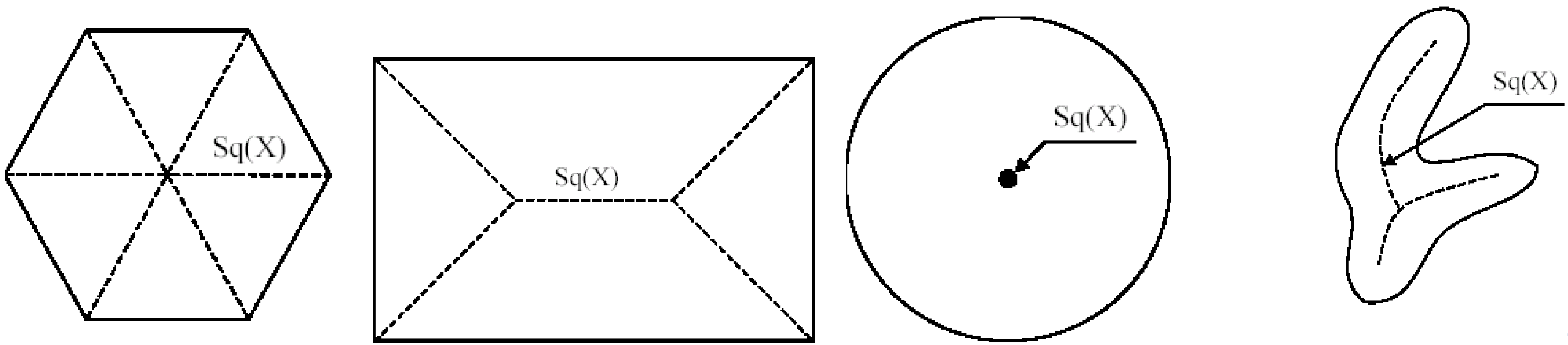
$$S(X) = \bigcup_{\lambda>0} \bigcap_{\mu>0} \{E_{B(\lambda)}(X) \setminus O_{B(\mu)}(E_{B(\lambda)}(X))\}$$

- **Definition in the Digital Case**



$$S(X) = \bigcup_{i \in \mathbb{N}} S_i(X) = \bigcup_{i \in \mathbb{N}} \{E_{B(i)}(X) \setminus O_{B(1)}(E_{B(i)}(X))\}$$

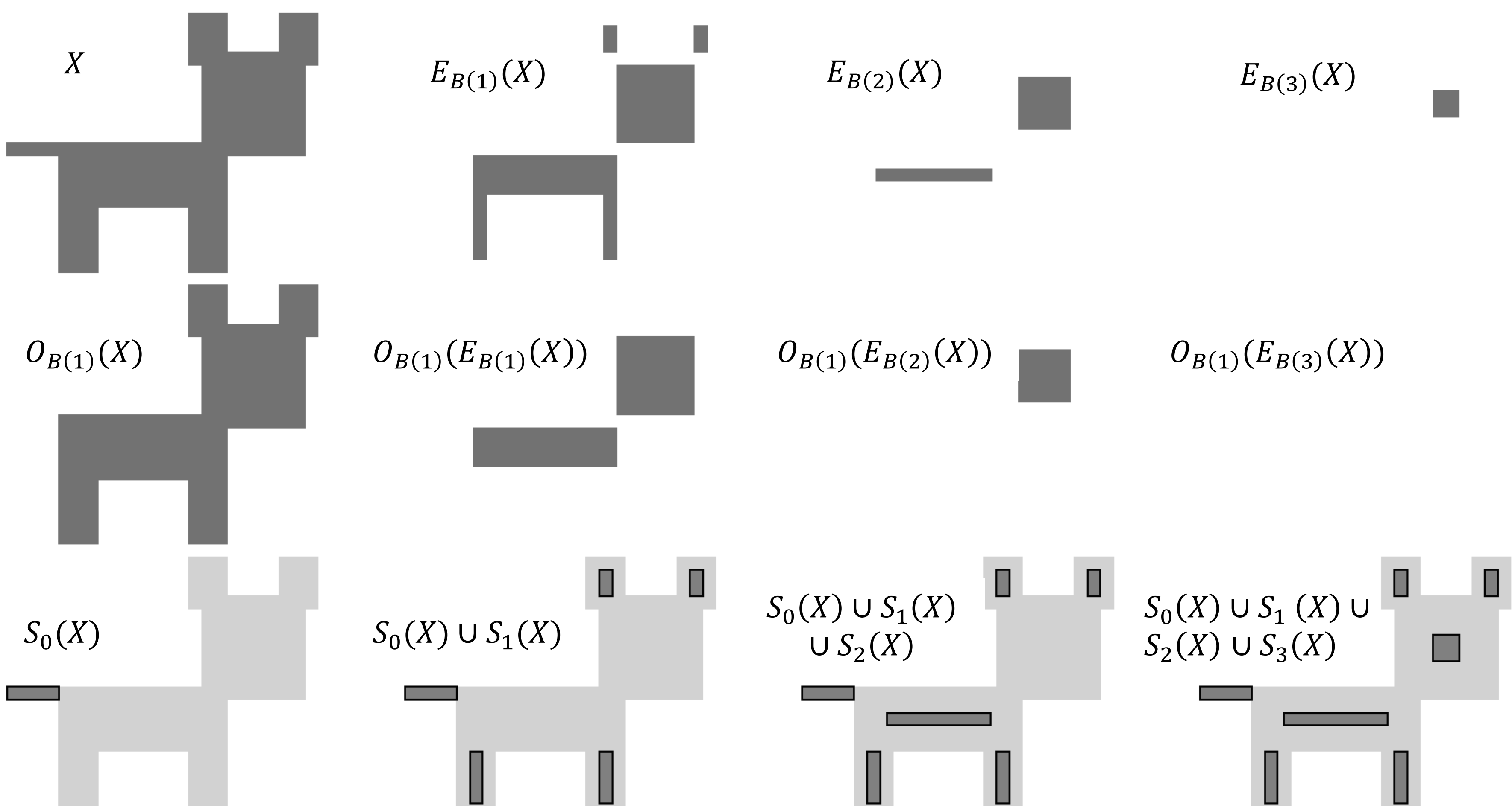
- **Illustration**



Skeleton by Maximal Balls

$$S(X) = \bigcup_{i\in\mathbb{N}} S_i(X) = \bigcup_{i\in\mathbb{N}} \{E_{B(i)}(X) \setminus O_{B(1)}(E_{B(i)}(X))\}$$

- Illustration in the Digital Case



Properties

- **Anti-Extensivity**

  $$S(X) \subseteq X$$

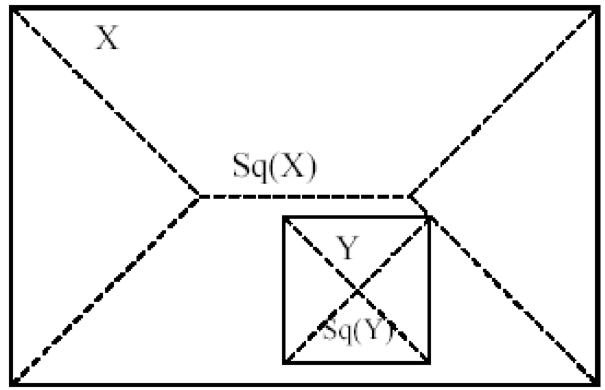
- **Idempotence**

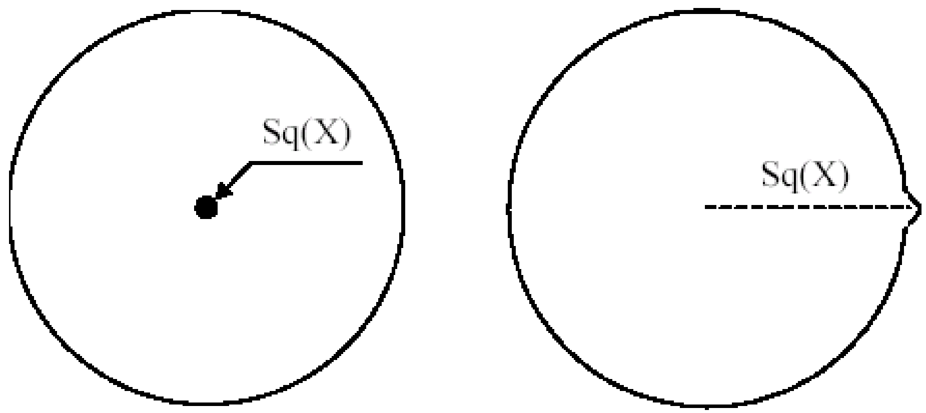
  $$S\big(S(X)\big) = S(X)$$

- **Invertibility**

  $$X = \bigcup_i D_{B(i)}(S_i(X))$$

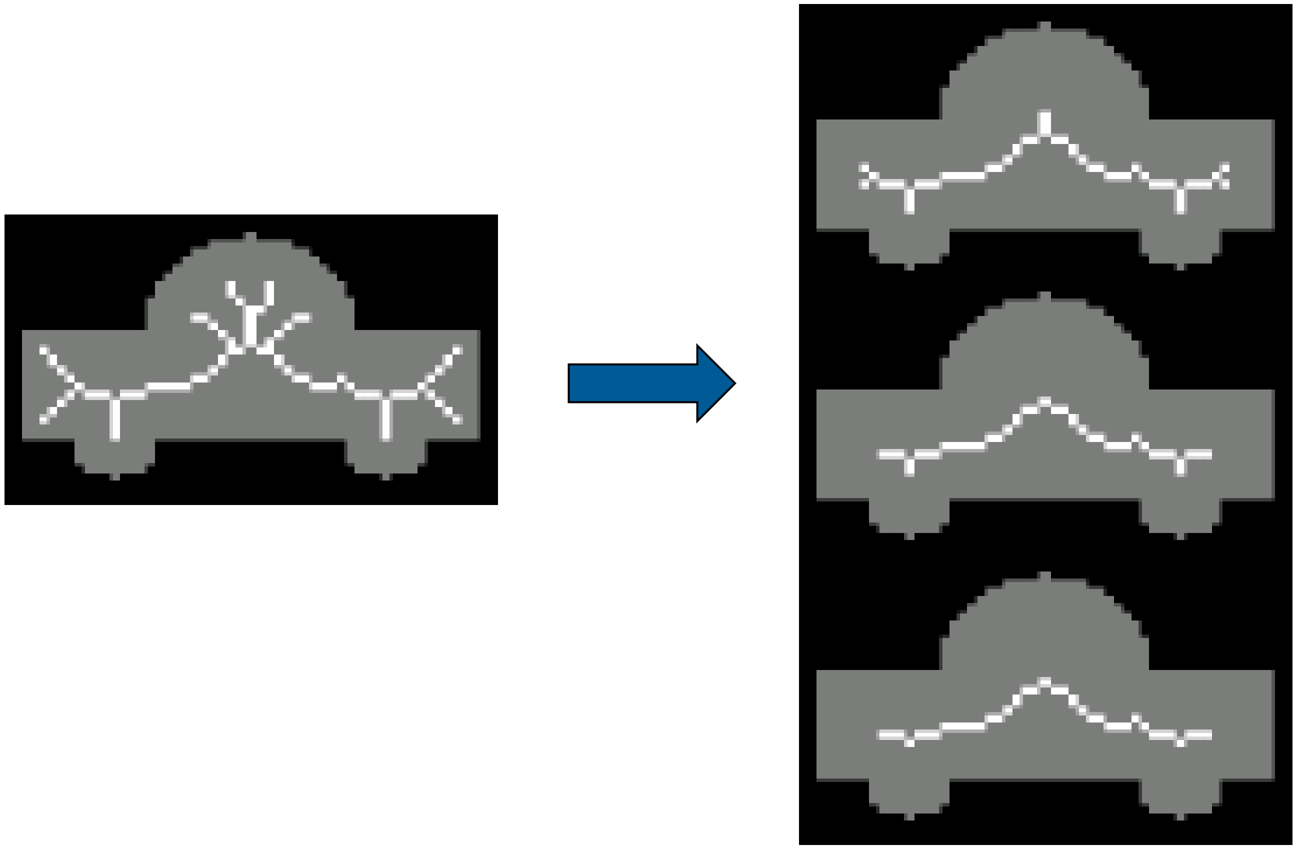
- **No Increasing, No Decreasing**
- **No Continuity**
- **Connectivity only in Continuous Case**





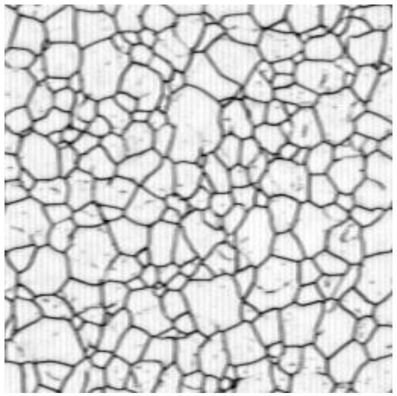
Needs of Pruning

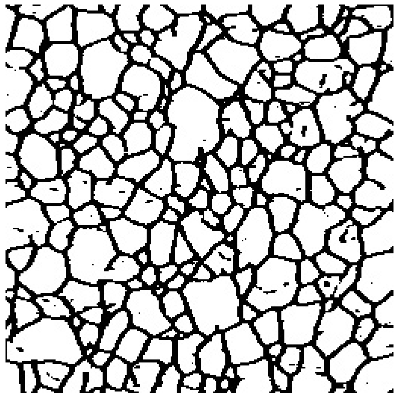
- Iterative Removing of Extremal Points

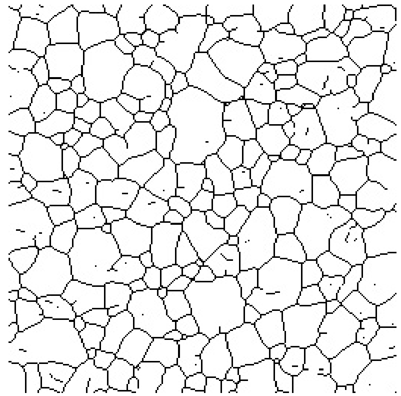


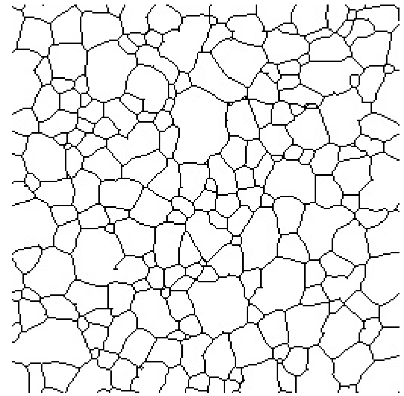
Illustration

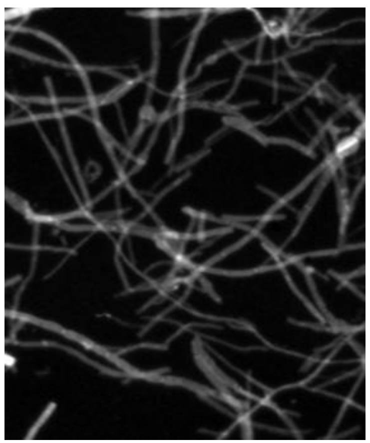
- **Metallurgic Grains and Textile Fibers**


original

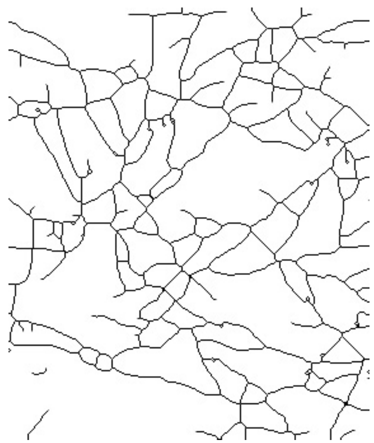

thresholding

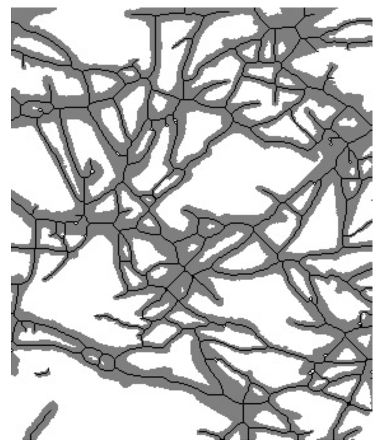

skeletonization


pruning


original


thresholding


skeletonization

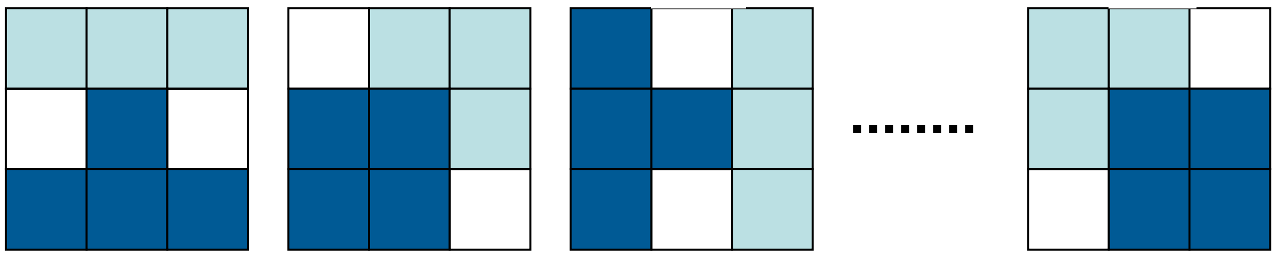

superimposition

Homotopic Skeleton

- **If Connectivity is of Importance!**

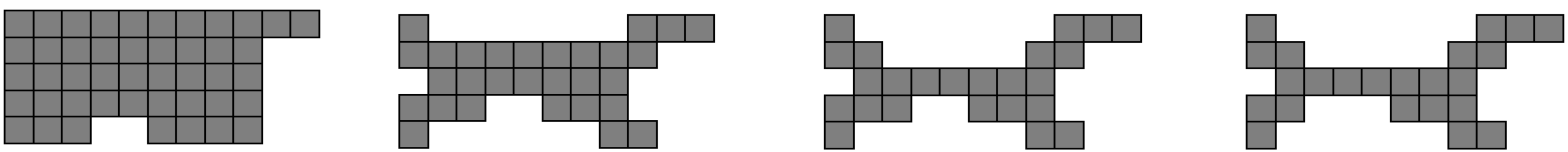
- **Using Specific Series of SE**

B^1 B^2 B^3 B^8



- **Iterative Thinnings**

X $\quad (\ldots(X \bigcirc B^1)\ldots \bigcirc B^8)$



original first iteration second iteration reaches idempotence

Properties

- **Anti-Extensivity**

  $$S(X) \subseteq X$$

- **Idempotence**

  $$S\big(S(X)\big) = S(X)$$

- **No Invertibility**

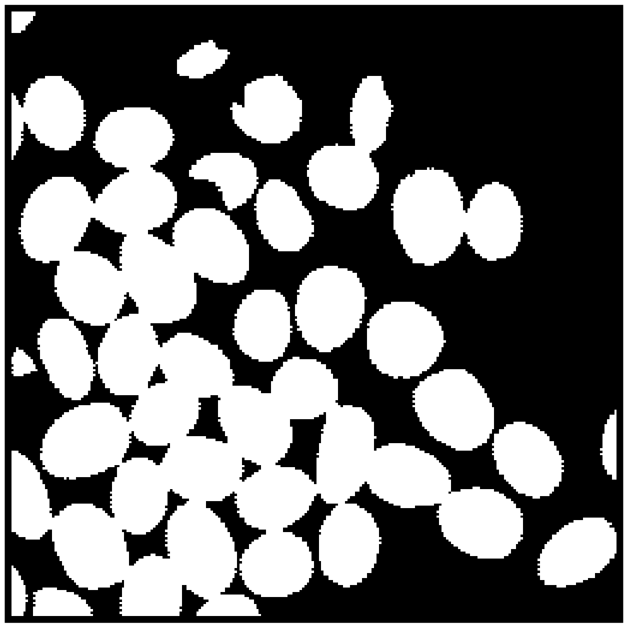
- **No Increasing, No Decreasing**

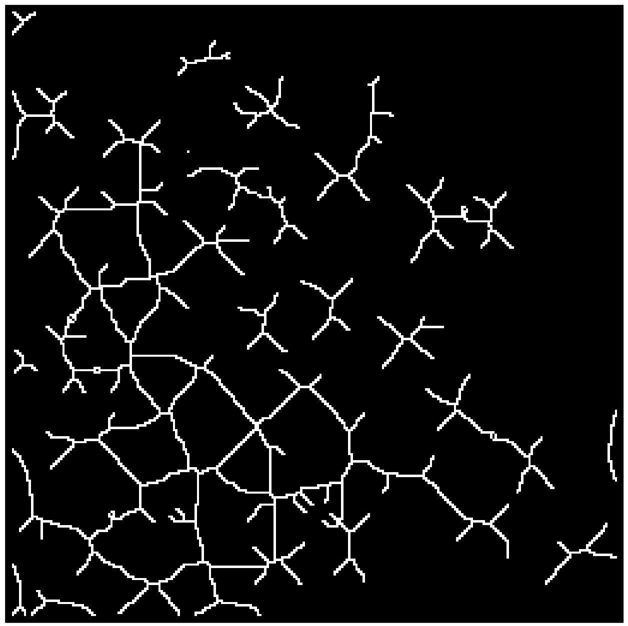
- **No Continuity**

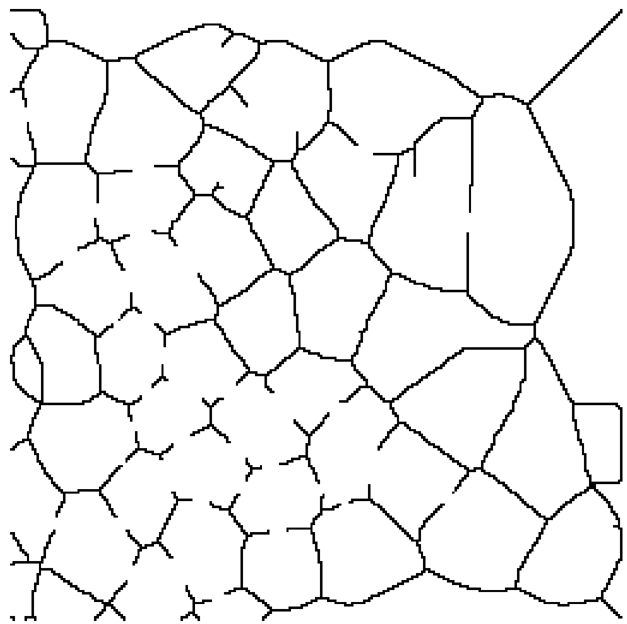
- **Connectivity**

Homotopic Skeleton

- **Illustration**

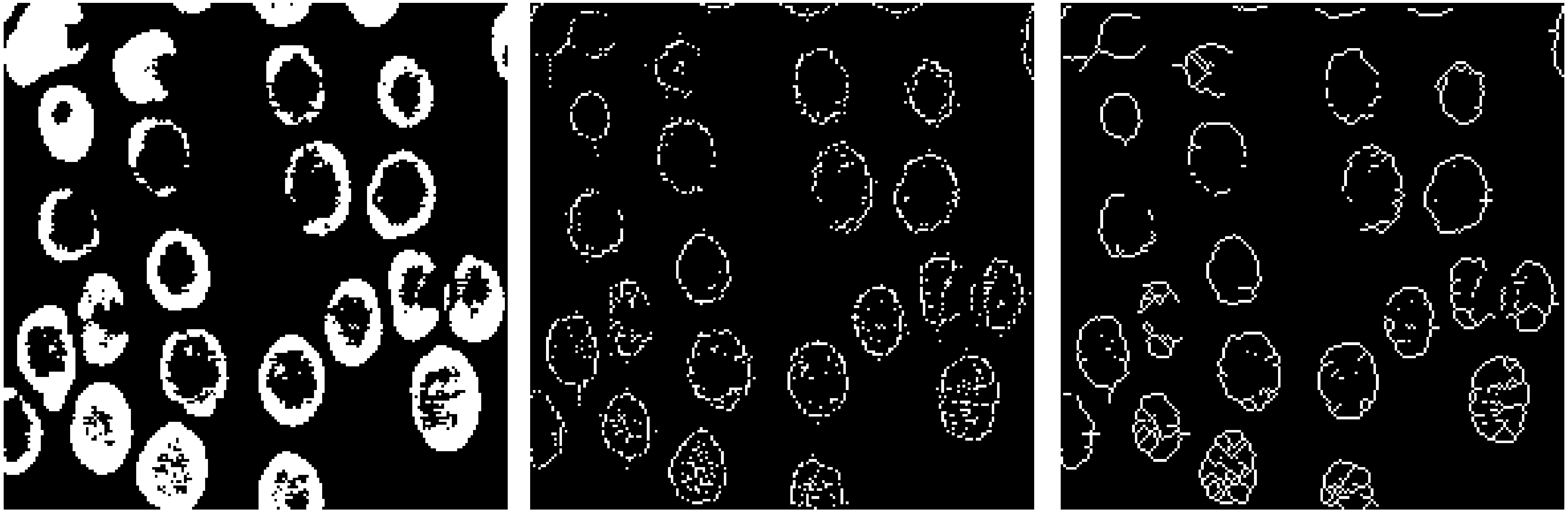

original


skeleton of the foreground


skeleton of the background

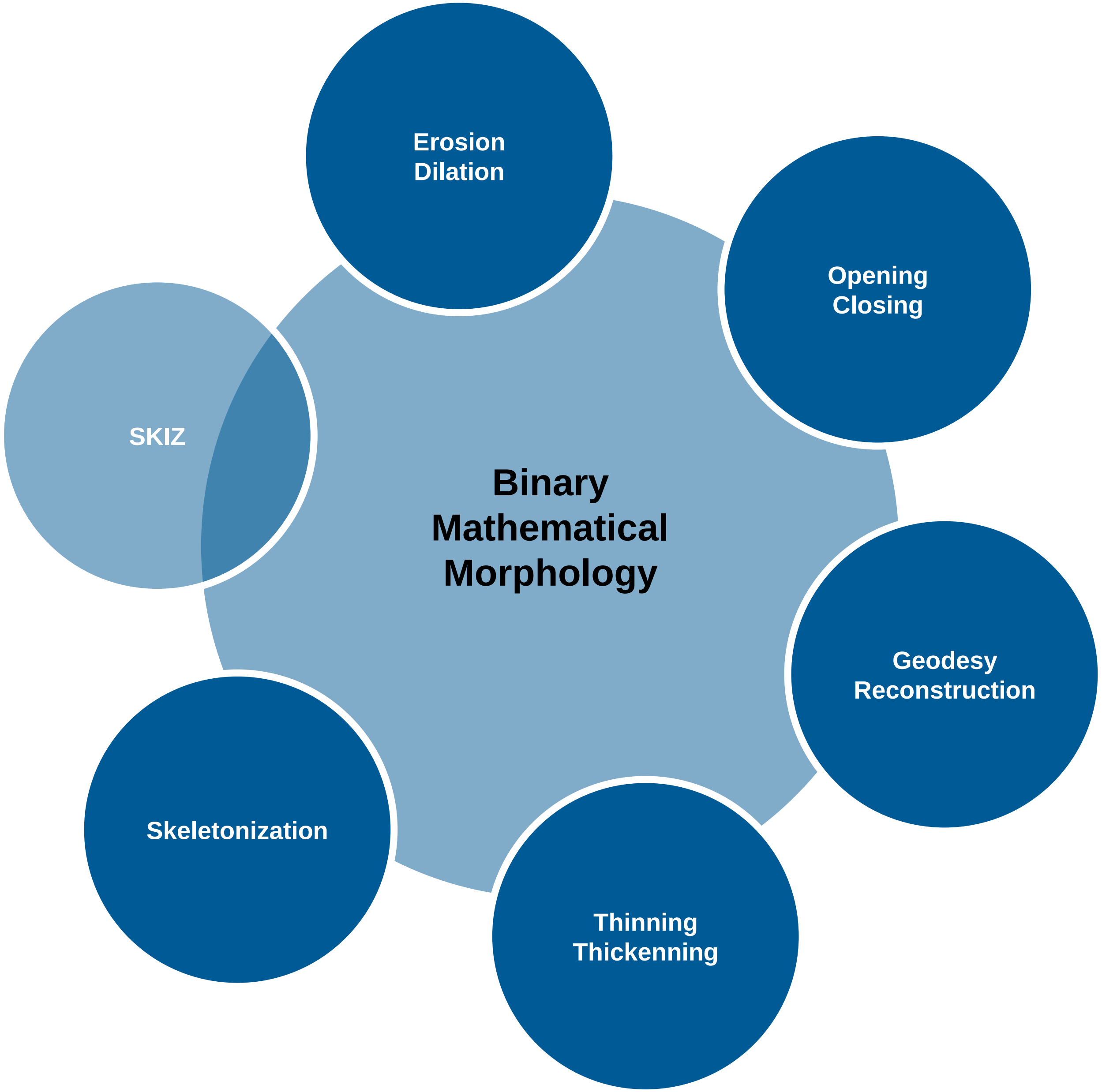
Homotopic vs. Maximal Ball Skeleton

- Comparison



original   skeleton by maximal balls   homotopic skeleton

Binary Mathematical Morphology



Definition

- $\boldsymbol{X} = \bigcup_{\boldsymbol{i}} \boldsymbol{X_i}$

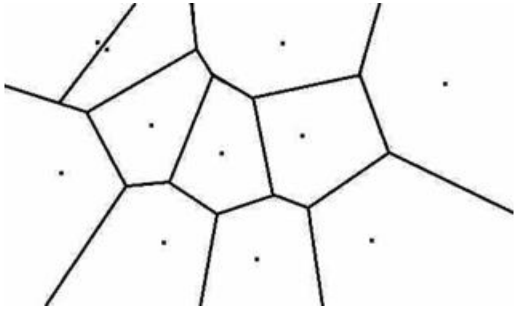
- **Influence Zone**

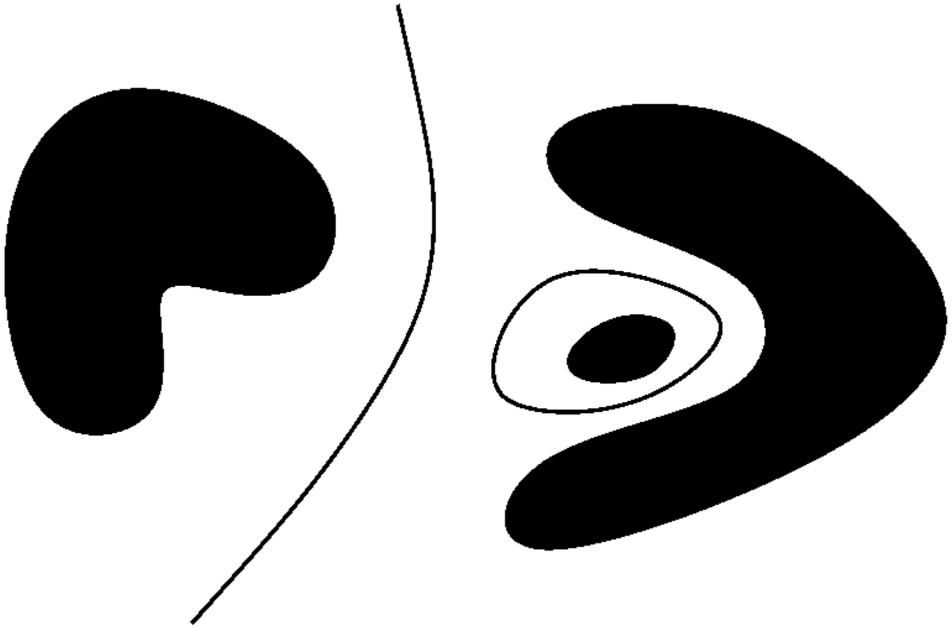
  $$ZI(X_i) = \{x \in X^c; d(x, X_i) < d(x, X \backslash X_i)\}$$

- **Skeleton by Influence Zones**

  $$SKIZ(X) = \left(\bigcup_i ZI(X_i)\right)^c$$

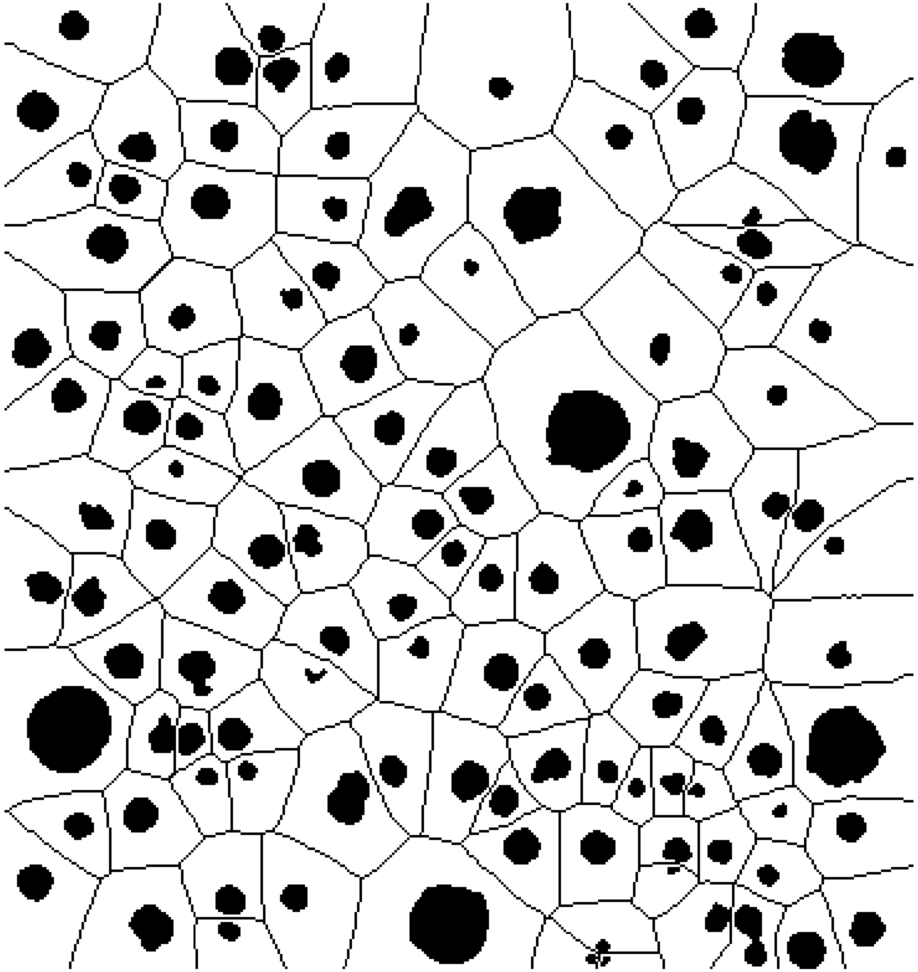
  - Generalized Voronoï diagram





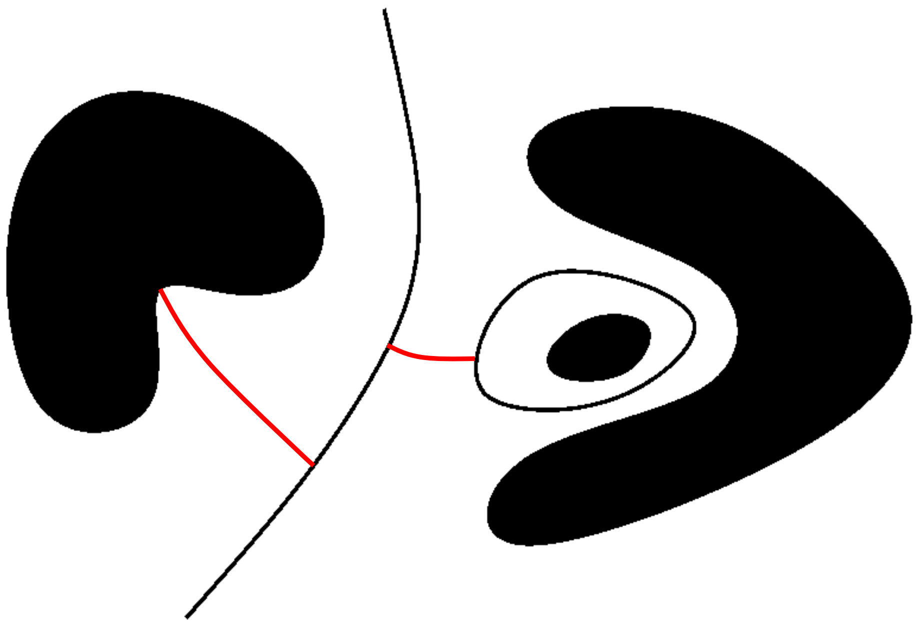
Illustration

- Graphite Nodules



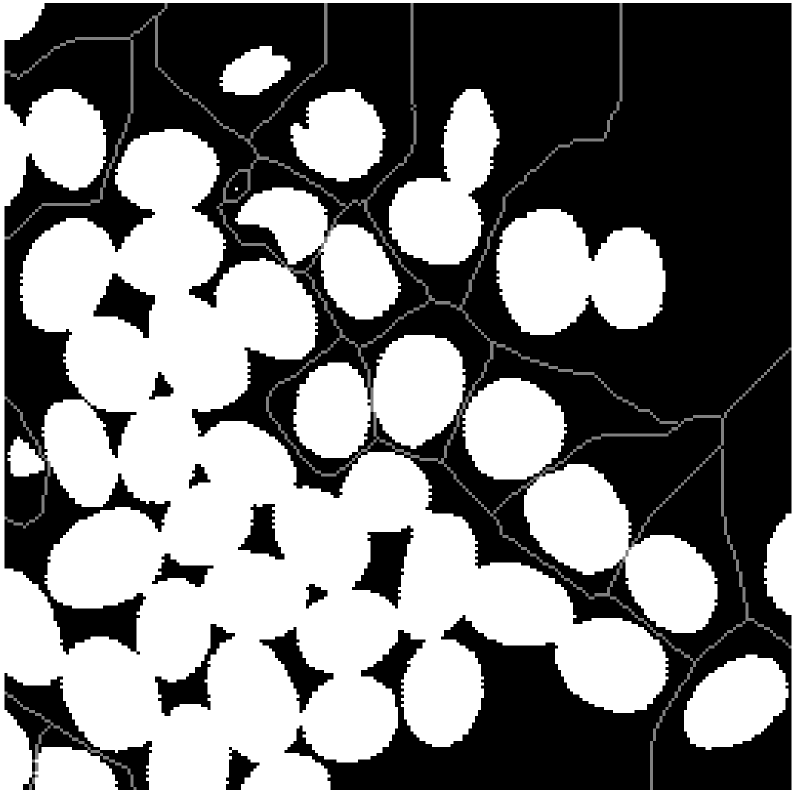
Properties

- **Anti-Extensivity vs. X^c**
- **Idempotence**
- **No Connectivity**
- $SKIZ(X) \subseteq S(X^c)$

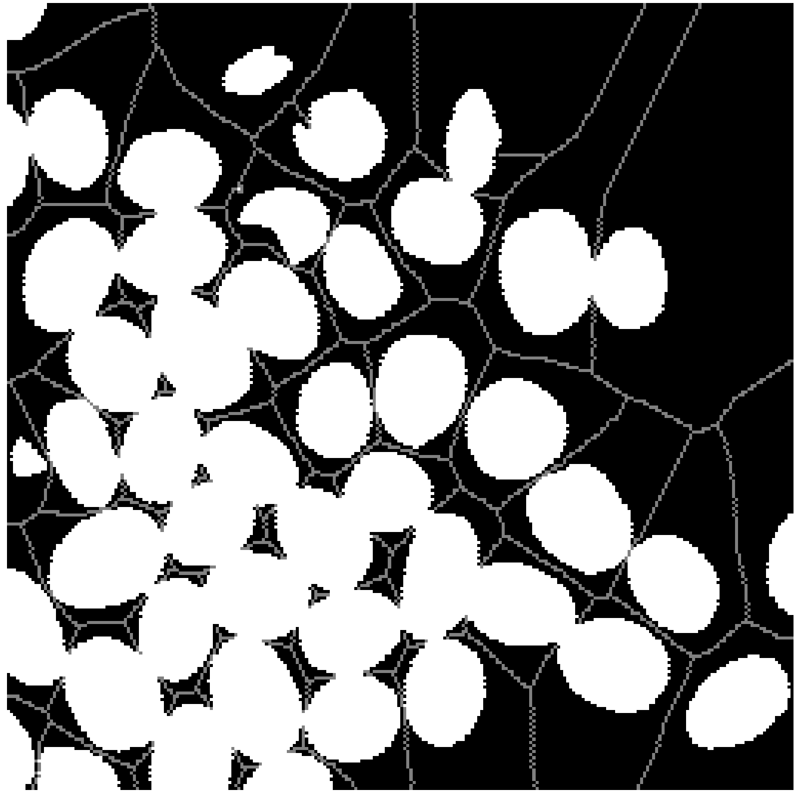


Skeleton and SKIZ

- **Comparison**



skeleton



SKIZ

Geodesic SKIZ

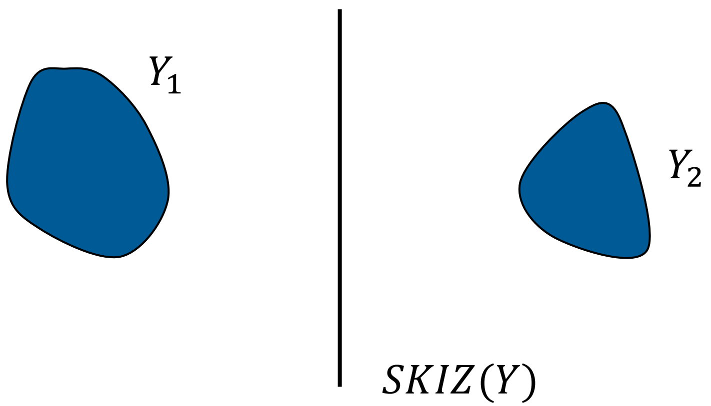
- $\boldsymbol{Y = \bigcup_i Y_i}$

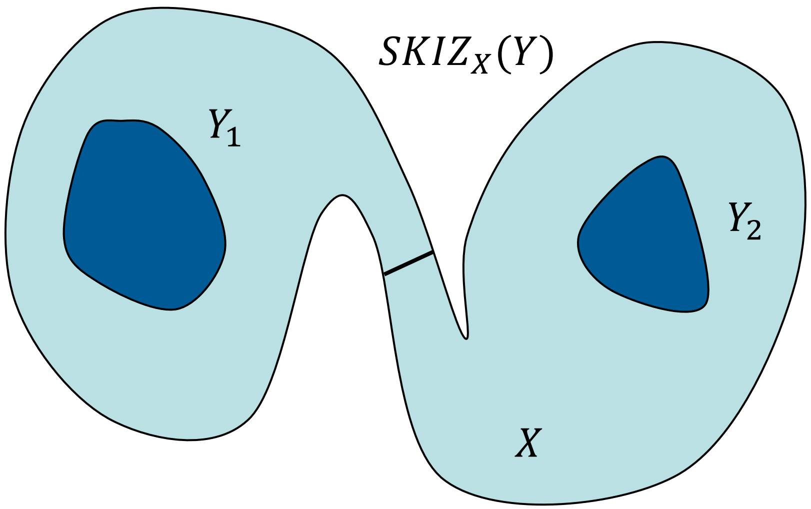
- **Influence Zone**

$$ZI_X(Y_i) = \{x \in X; d_X(x, Y_i) < d_X(x, Y \backslash Y_i)\}$$

- **Skeleton by Influence Zones**

$$SKIZ_X(Y) = X \setminus \left(\bigcup_i ZI_X(Y_i) \right)^c$$





INSPIRING INNOVATION | INNOVANTE PAR TRADITION

Ecole Nationale Supérieure des Mines SAINT-ETIENNE

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Gray-Level Mathematical Morphology

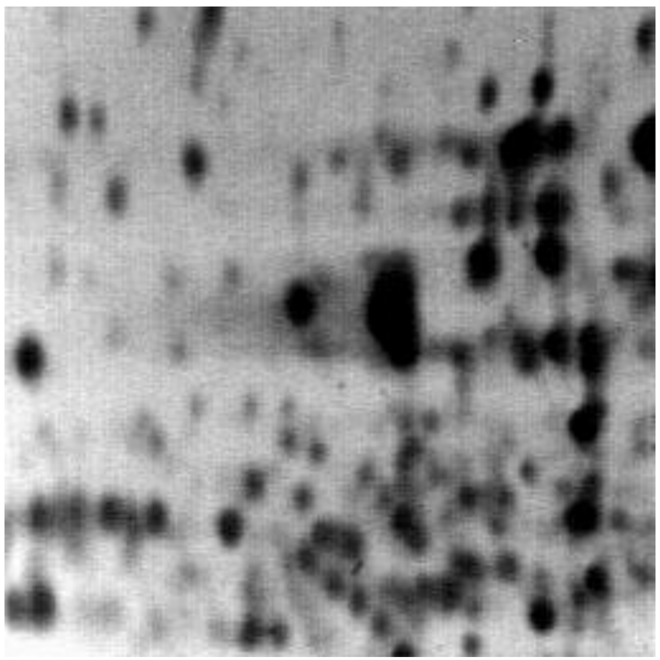
Johan DEBAYLE

Ecole Nationale Supérieure des Mines, Saint-Etienne, FRANCE

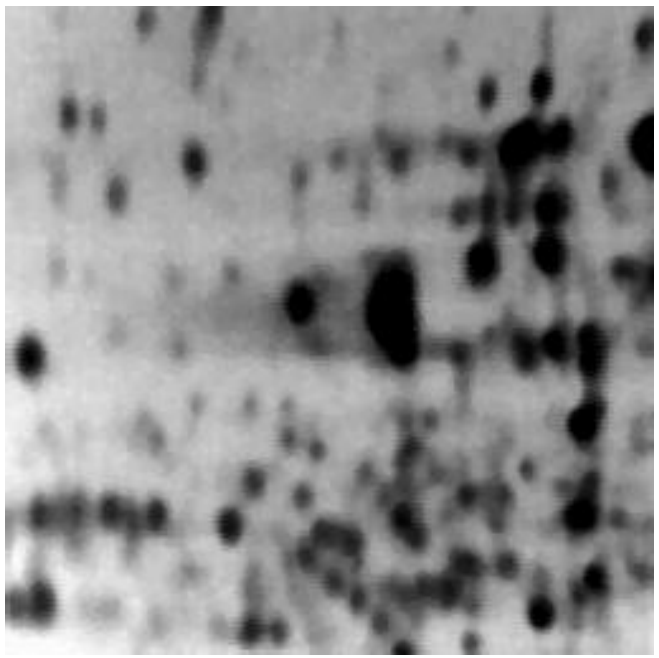
MINISTÈRE DE L'ÉCONOMIE DES FINANCES ET DE L'INDUSTRIE

Gray-Level Morphology

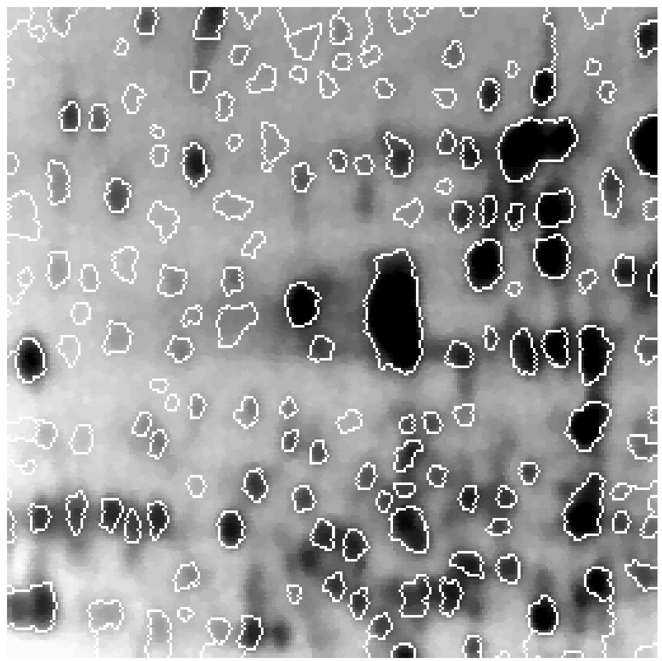
- **Mainly used for**
  - Image filtering
    - Smoothing
    - Enhancement
  - Image Segmentation



Electrophoresis gel image



Filtered image

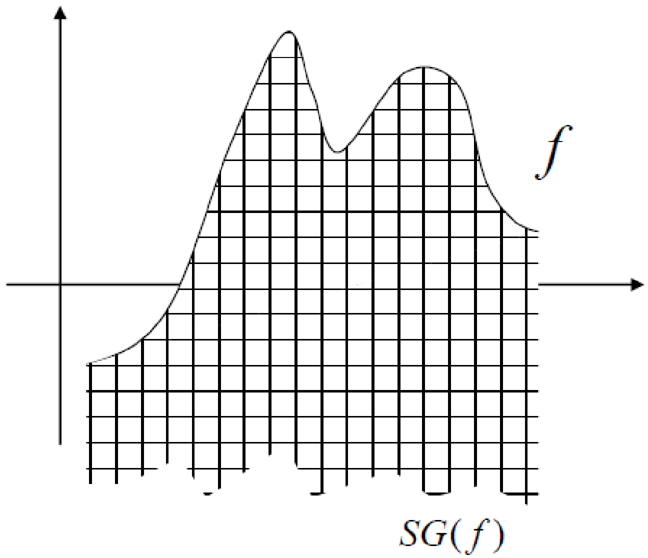


Segmented image

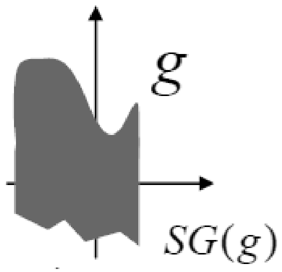
From Sets to Functions

- **Sub-Graph**

  $$SG(f) = \{(x,t); f(x) \geq t\}$$

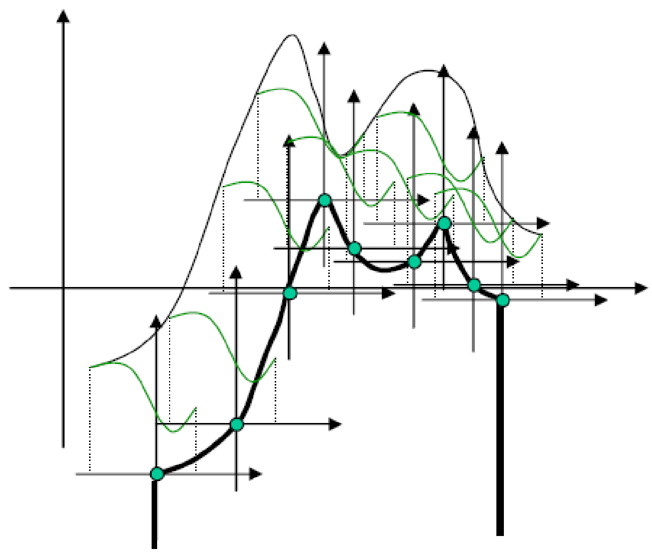


- **Structuring Function**



- **Gray-Level Morphology**
  - φ: binary morphological operator (e.g. dilation, erosion, …)

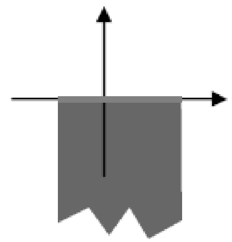
$$SG(\Phi(f)) = \varphi_{SG(g)}(SG(f))$$



From Sets to Functions

- **Using Flat Structuring Functions**
  - i.e. using a structuring element *B*

$$g(x) = \begin{cases} 0 \text{ if } x \in B \\ -\infty \text{ if } x \in B^c \end{cases}$$



$SG(g)$

- **Level Sets**

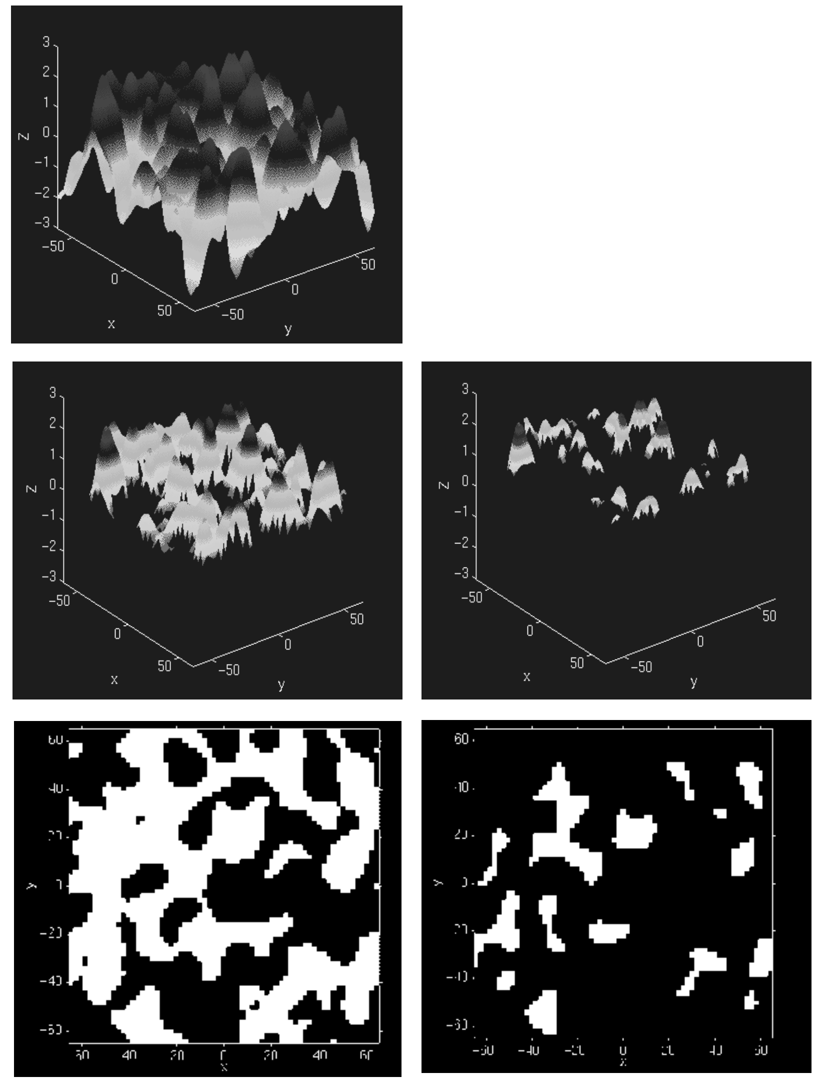
$$X_t(f) = \{x; f(x) \geq t\}$$

- **Reconstruction**

$$f(x) = sup\{t; x \in X_t(f)\}$$

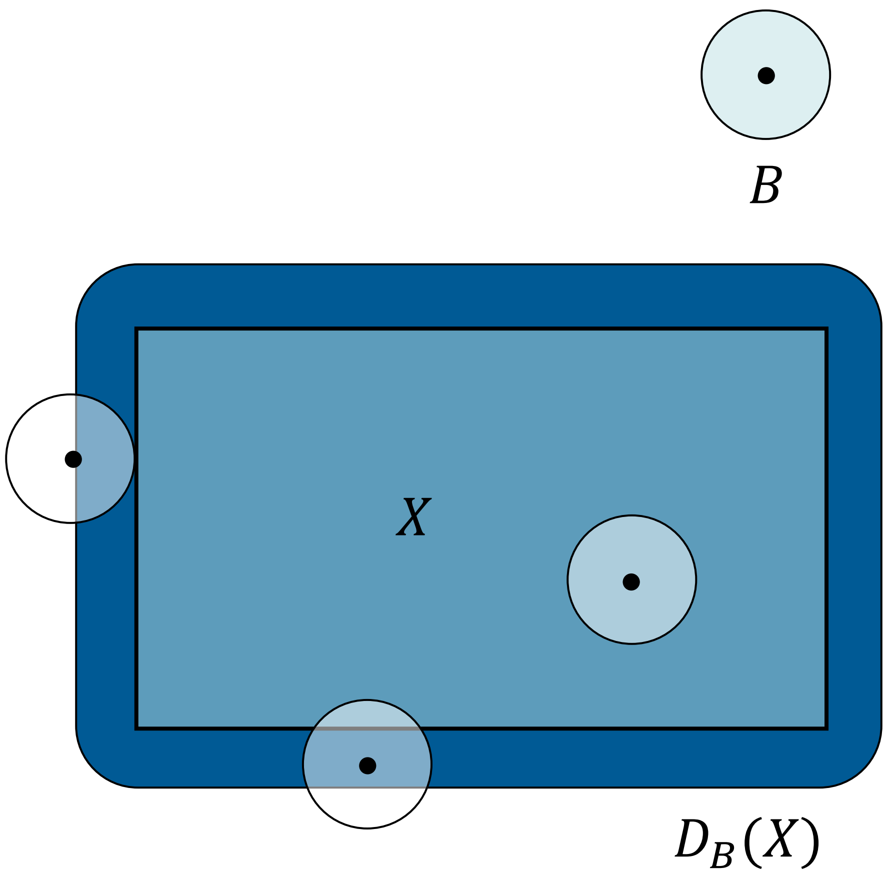
- **Gray-Level Image Transformation**
  - φ_B: binary increasing morphological operator (e.g. dilation, erosion, …)

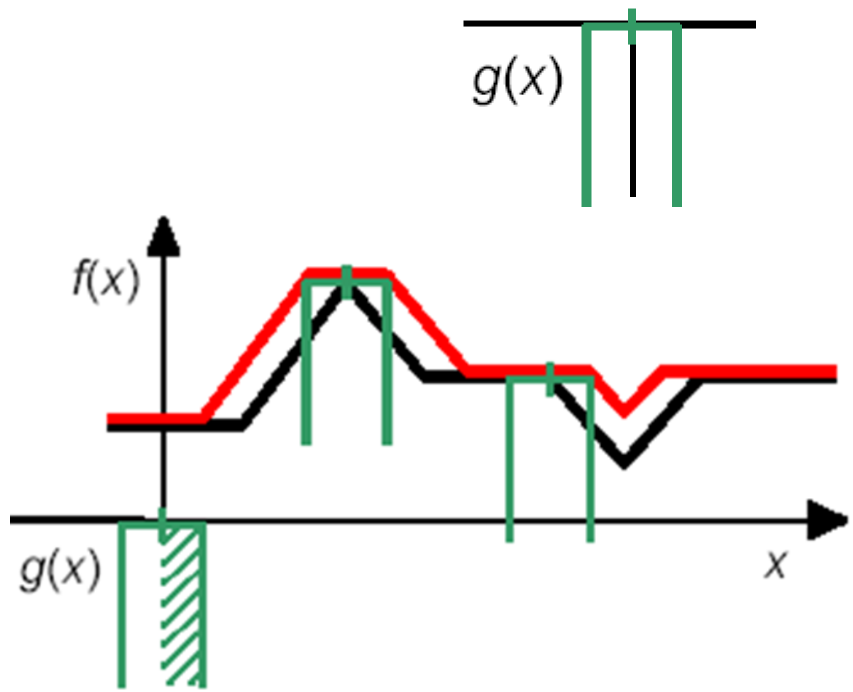
$$\Phi_B(f)(x) = sup\{t; x \in \varphi_B(X_t(f))\}$$



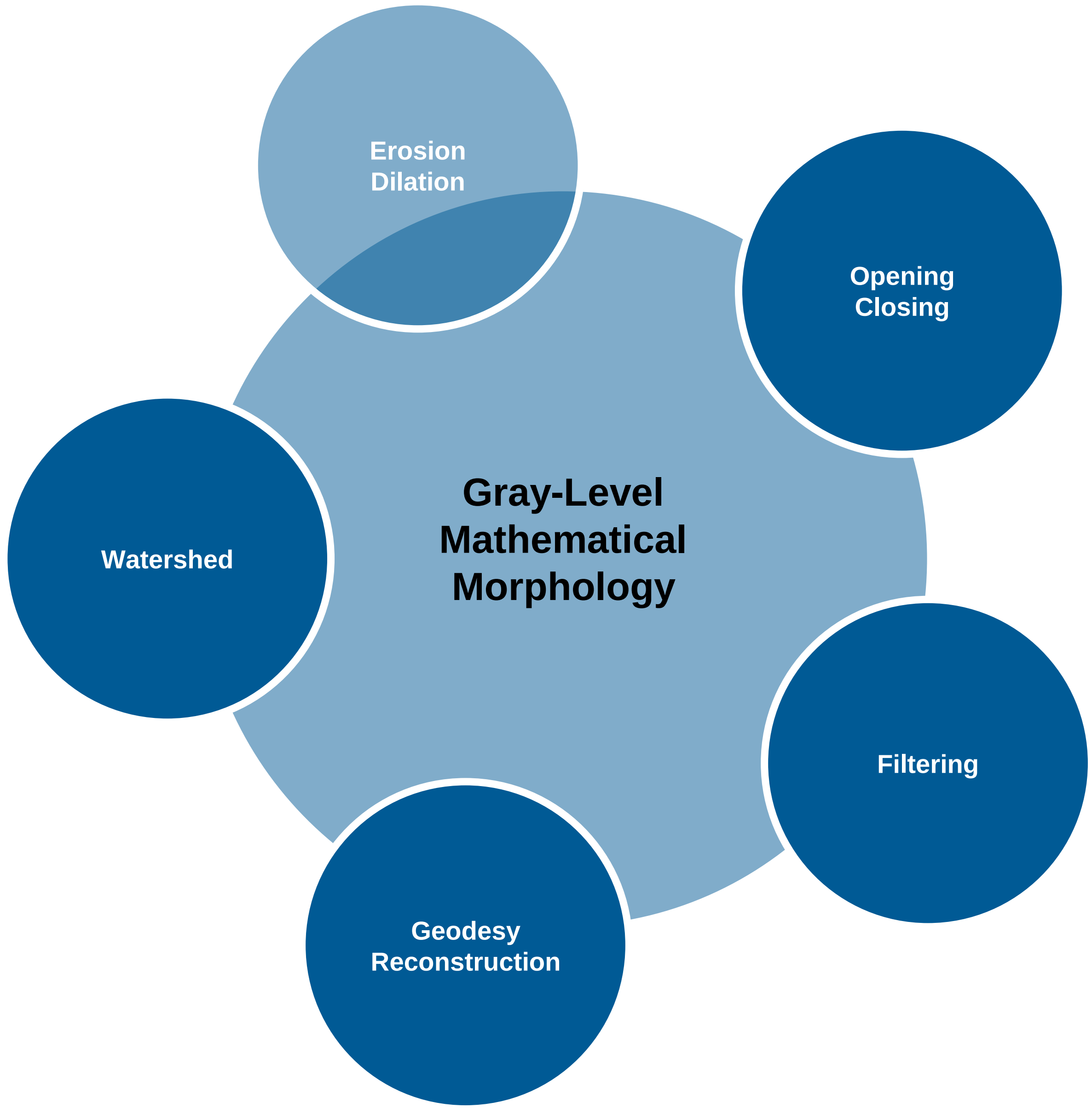
From Sets to Functions

- **Set**
  - $\cup$ and $\cap$
  - $\subseteq$ and $\supseteq$
  - Structuring element *B*
- **Functions**
  - $\vee$ and $\wedge$
  - $\leq$ and $\geq$
  - Flat structuring function *g*





Gray-Level Mathematical Morphology



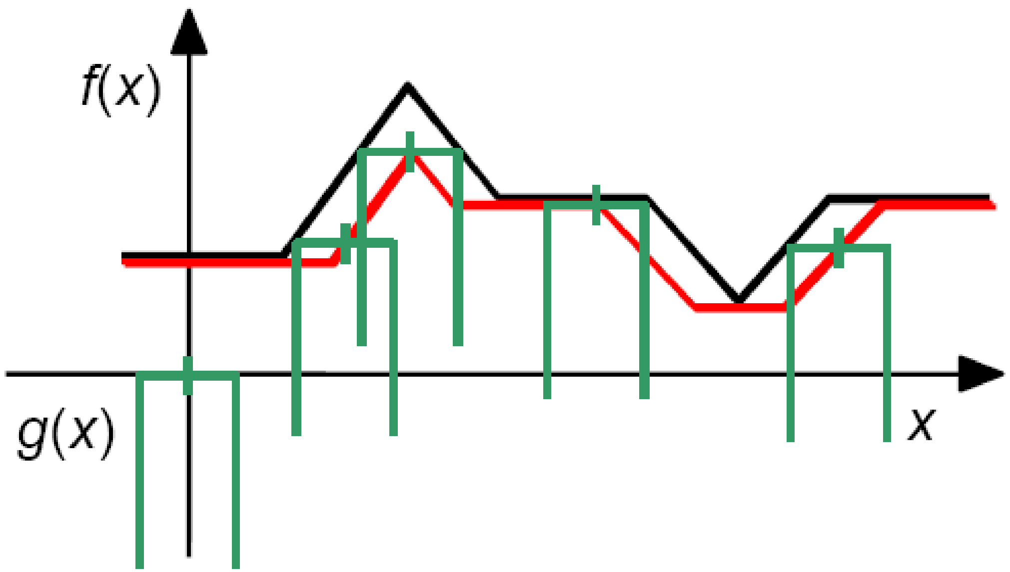
Erosion

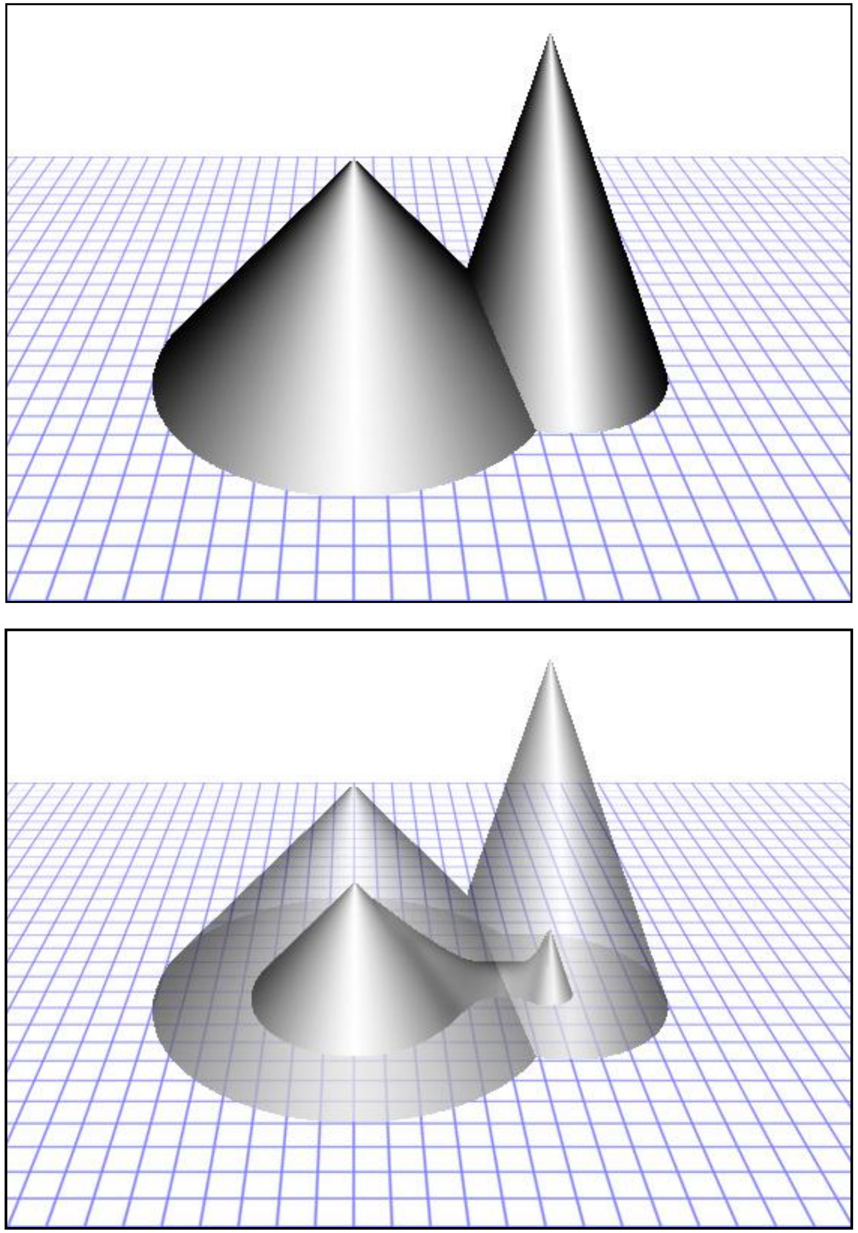
Definition

- For a flat structuring function

$$E_B(f)(x) = \bigwedge_{y \in B_x} f(y)$$

Illustration





Properties of Erosion

- **Anti-Extensivity**
  - If $O \in B$
- **Increasing**
- **Decreasing with respect to the SE**
- **Structuring Element Decomposition**
- **Commutativity with Inf., not with Sup.**
- **Iterativity**
- **Translation Invariance**

Illustration

- **Reducing / Removing Bright Details**


original image


eroded image

Illustration

- **Reducing / Removing Bright Details**

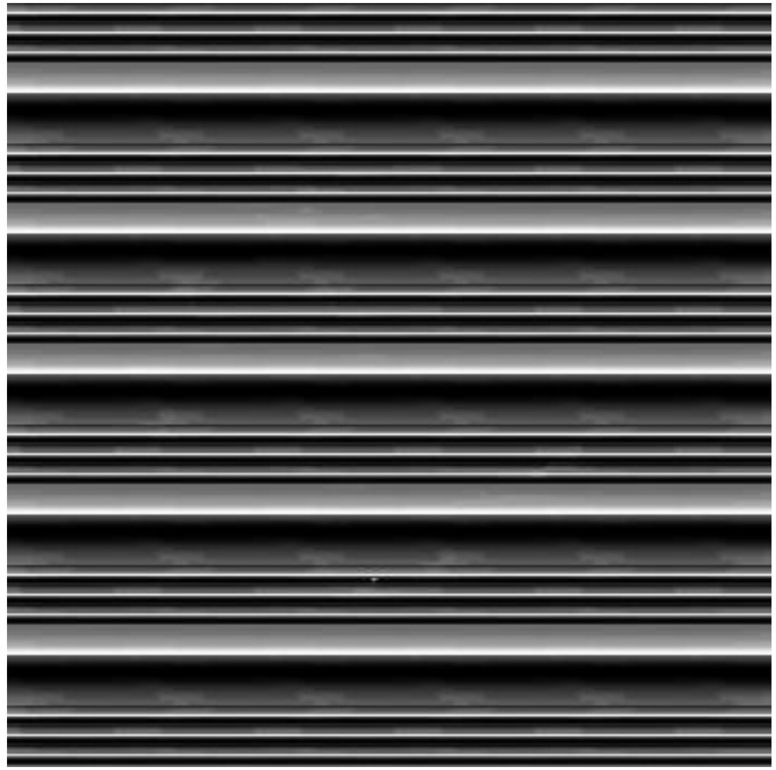
original image


eroded image

Illustration

- **Shape of the SE has an Impact**

original image



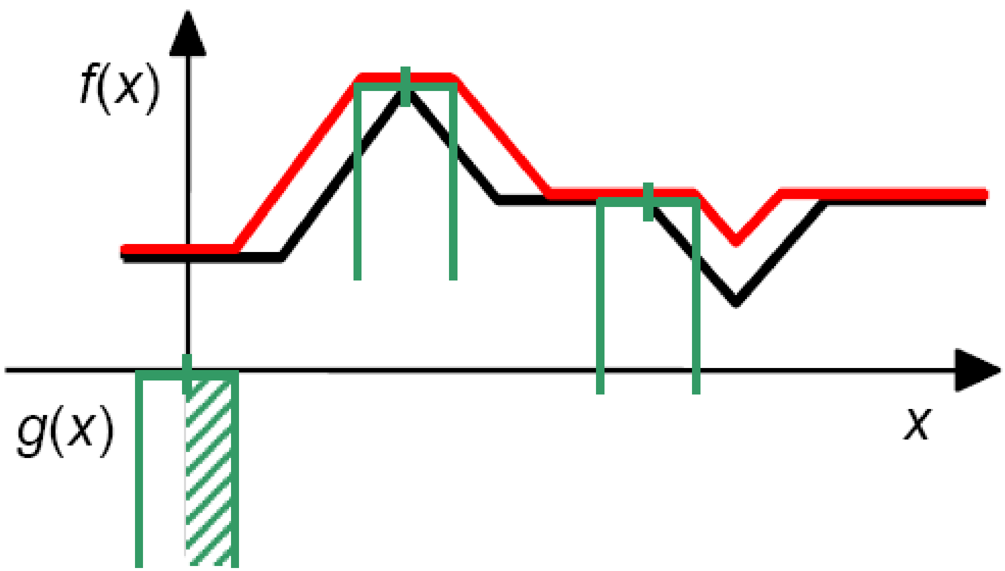
eroded image

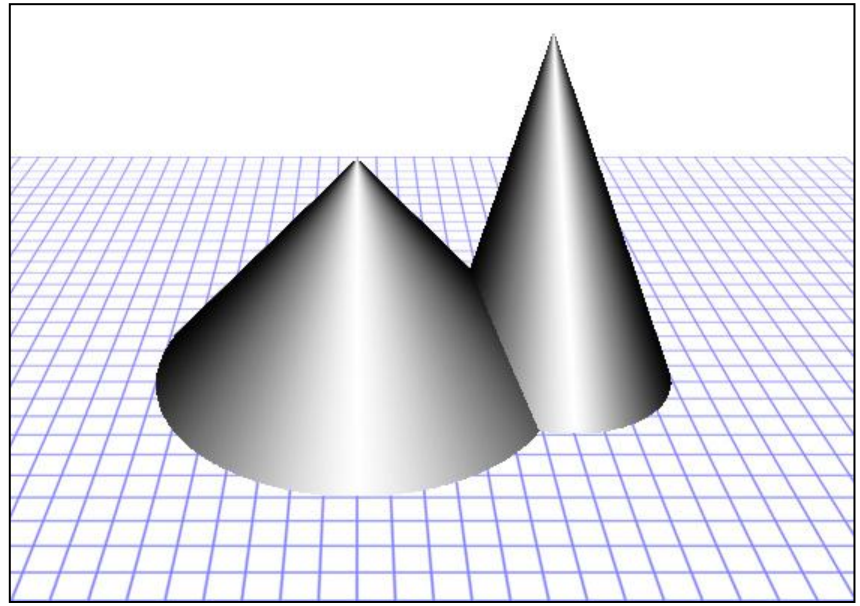
Dilation

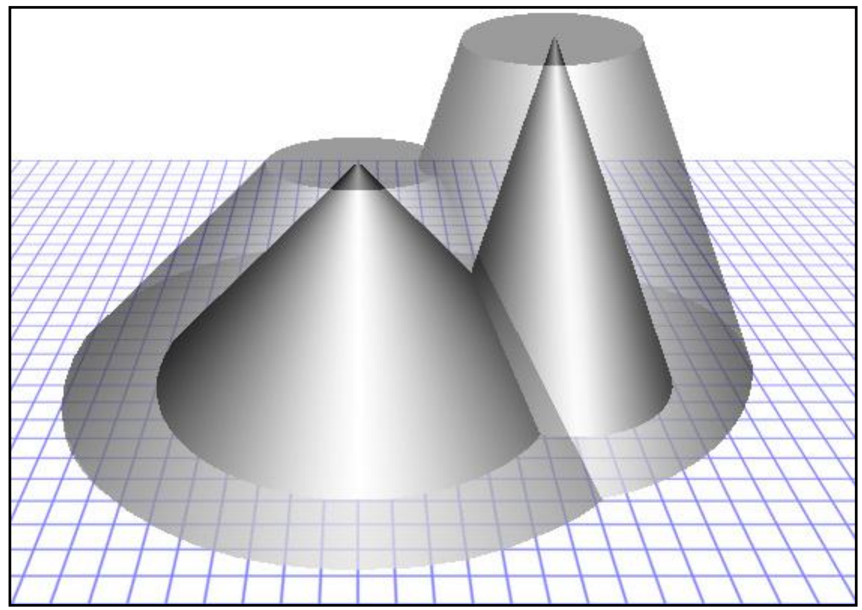
- **Definition**
  - For a flat structuring function

$$D_B(f)(x) = \bigvee_{y \in B_x} f(y)$$

- **Illustration**







Properties of Dilation

- **Extensivity**
  - If $O \in B$
- **Increasing**
- **Increasing with respect to the SE**
- **Structuring Element Decomposition**
- **Commutativity with Sup., not with Inf.**
- **Iterativity**
- **Translation Invariance**

Illustration

- **Reducing / Removing Dark Details**

original image

dilated image

Illustration

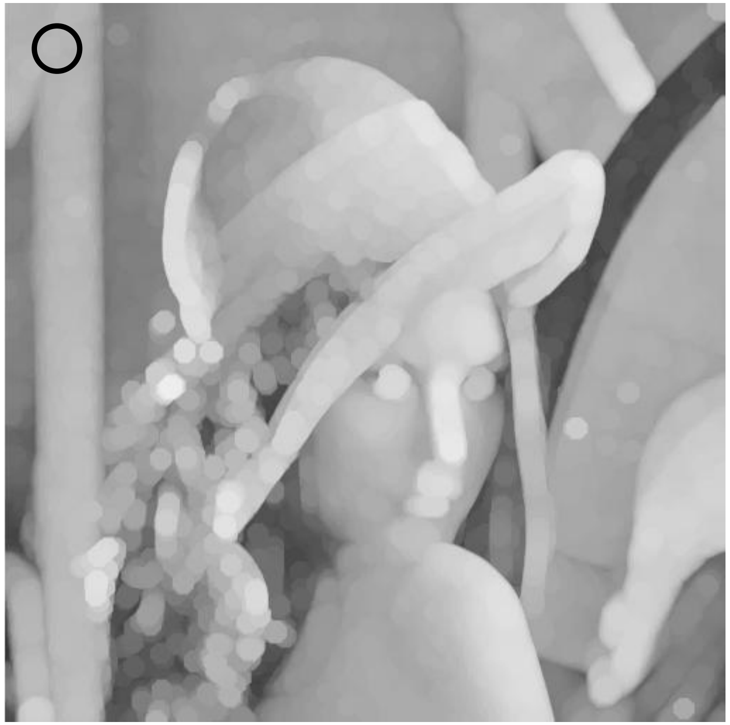
- **Reducing / Removing White Details**

original image

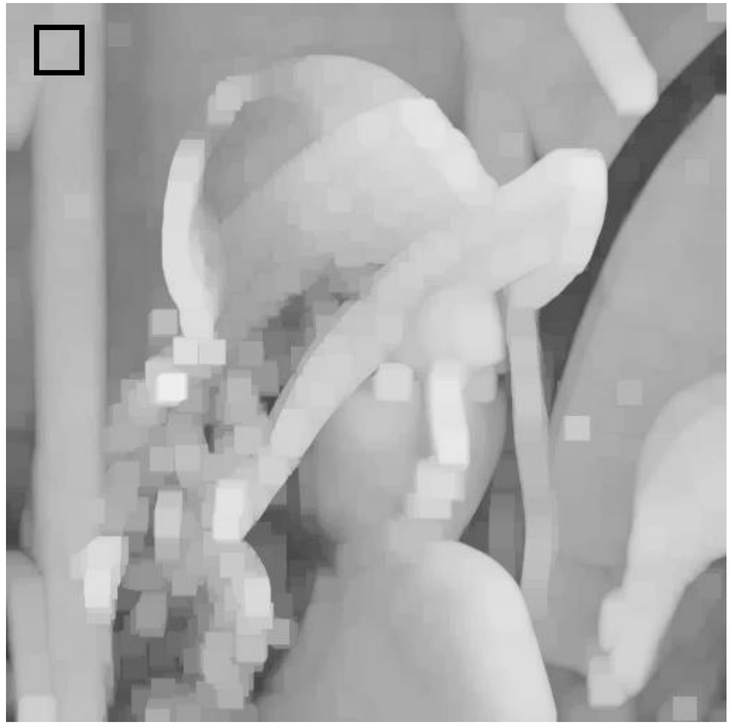
dilated image

Illustration

- **Shape of the SE has an Impact**



dilated image



dilated image

Duality Relationships between Erosion and Dilation

- **Duality with respect to the Reflexion**

$$M - E_B(f) = D_B(M - f)$$

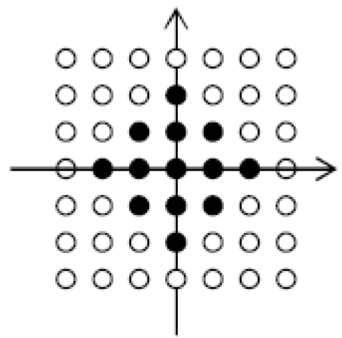
- **Duality with respect to the Adjunction**

$$D_{\check{B}}(f) \leq g \Leftrightarrow f \leq E_B(g)$$

Application to Digital Images

- **Max-Min Operations**

B

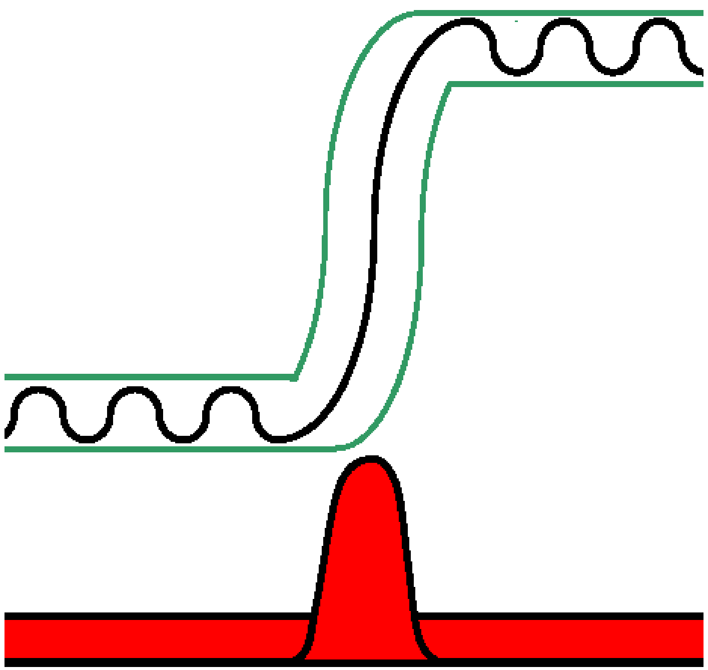


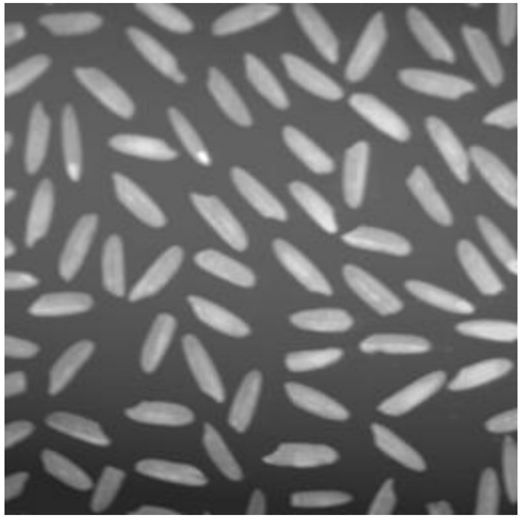
original image f | dilated image $D_B(f)$ | eroded image $E_B(f)$

Application to Boundary Extraction

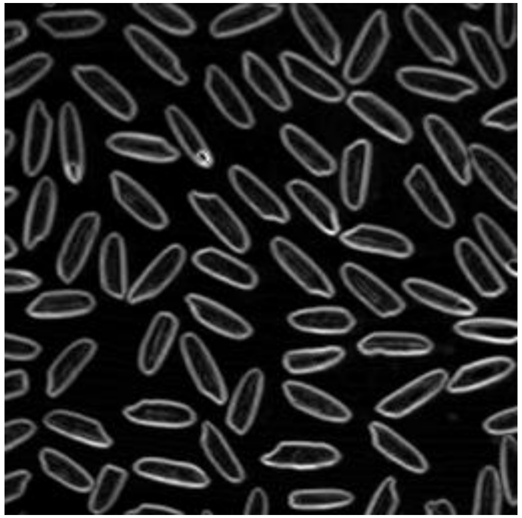
- **Morphological Gradient**
  - For a small SE size

$$\partial_B(f) = D_B(f) \backslash E_B(f)$$





f



$\partial_B(f)$

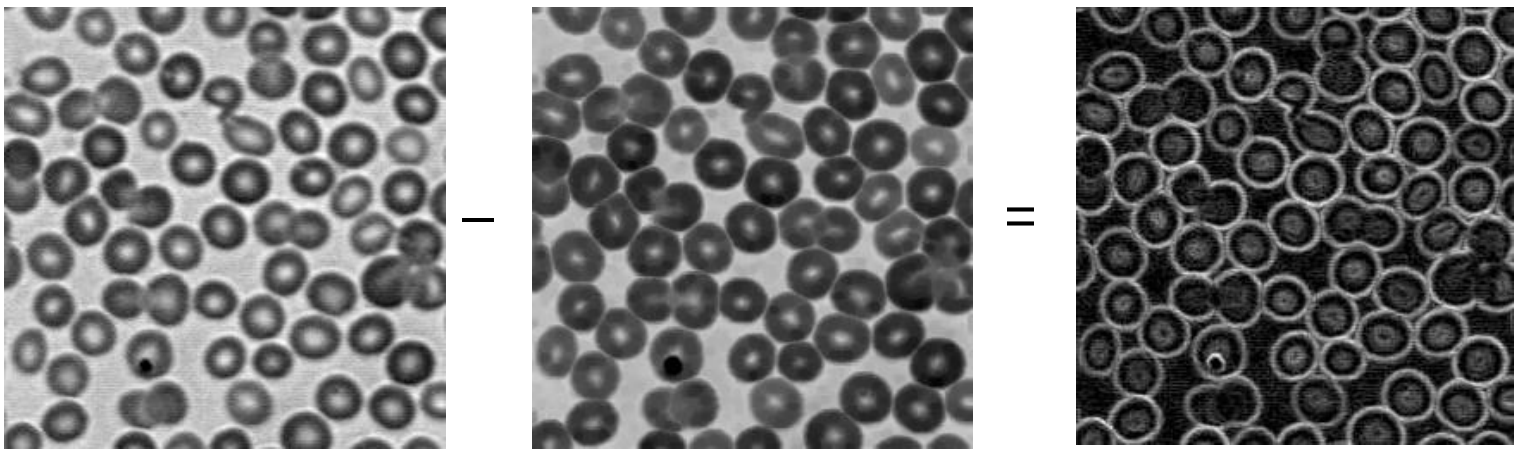
Other Definition of Gradients

- **External Boundary**

$$\partial_B^{ext}(f) = D_B(f) - f$$

- **Internal Boundary**

$$\partial_B^{int}(f) = f - \mathrm{E}_B(f)$$

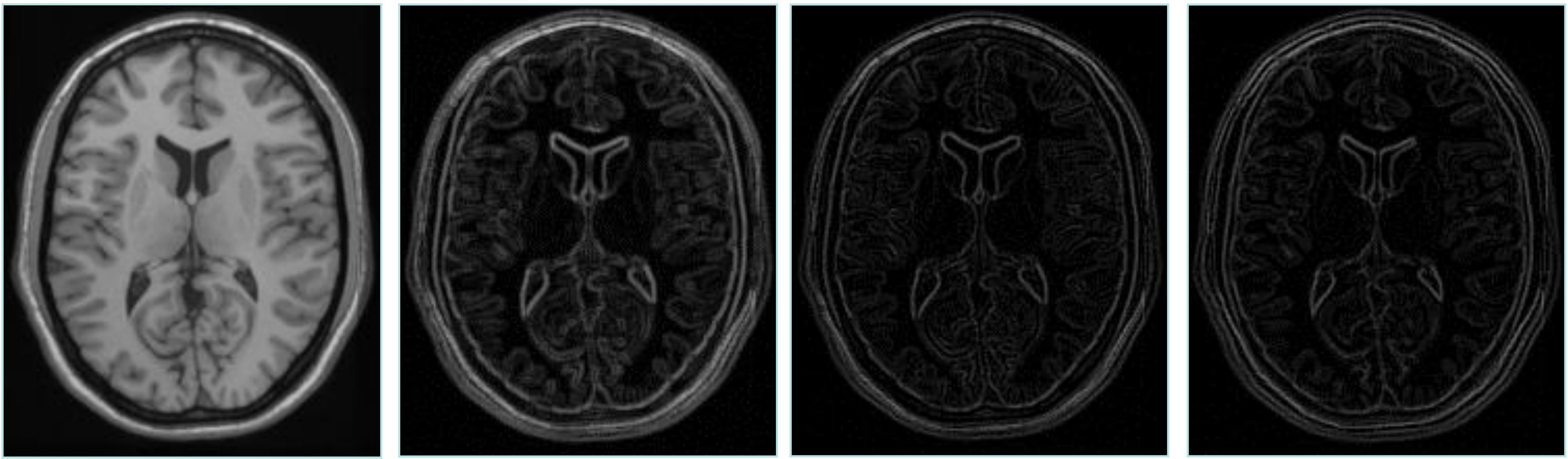


- **Boundary**

$$\partial_B(f) = \partial_B^{int}(f) + \partial_B^{ext}(f)$$

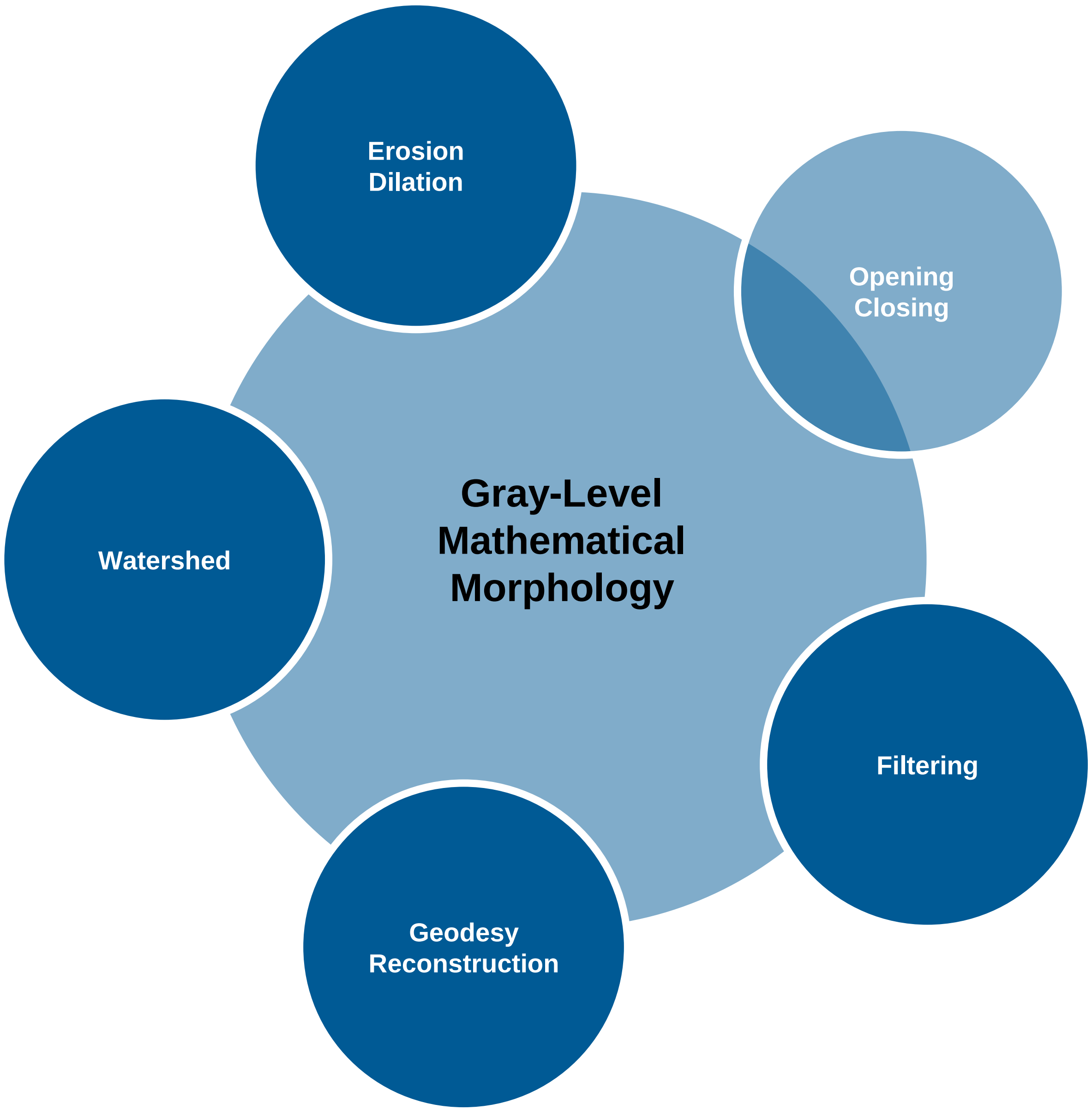
Boundary Extraction

- **Comparison**
  - Brain image



f $\quad$ $\partial_B(f)$ $\quad$ $\partial_B^{int}(f)$ $\quad$ $\partial_B^{ext}(f)$

Gray-Level Mathematical Morphology

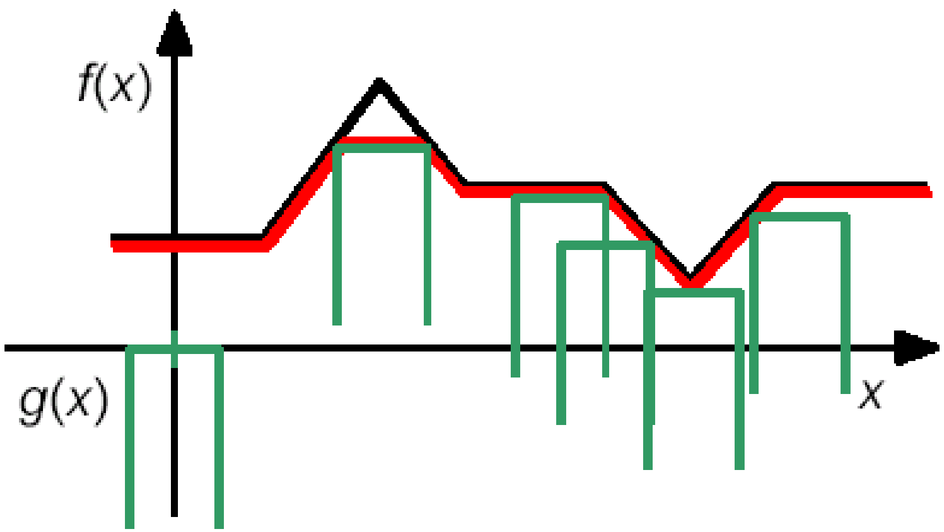


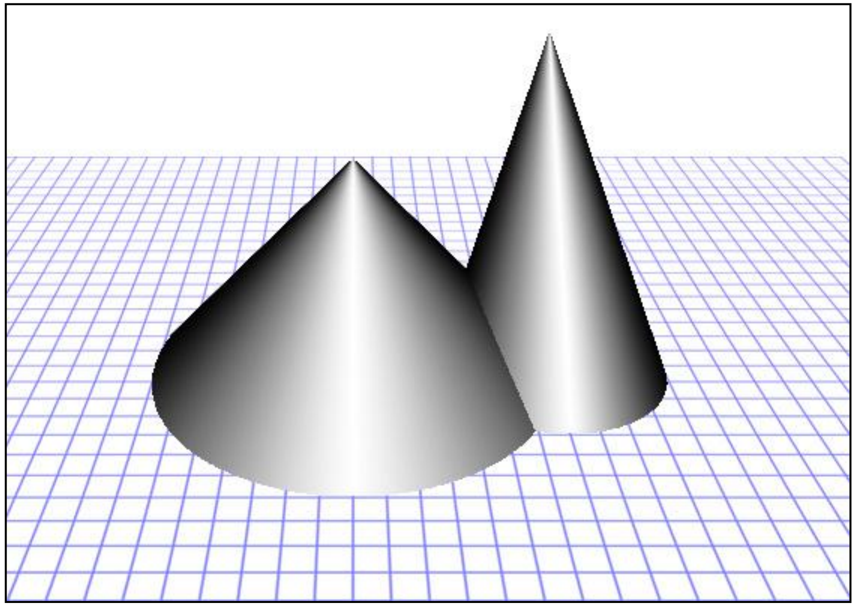
Opening

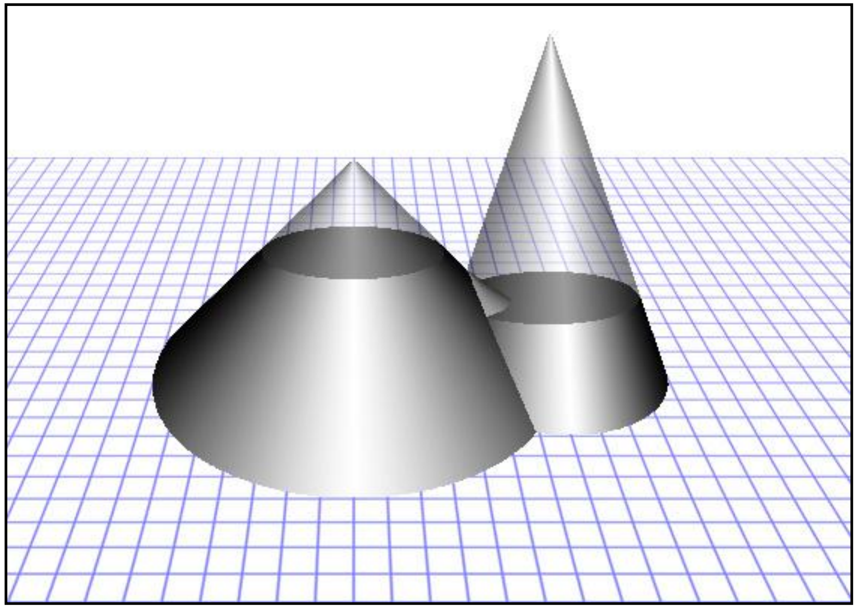
- **Definition**
  - For a flat structuring function

$$O_B(f)(x) = D_{\breve{B}}(E_B(f))$$

- **Illustration**





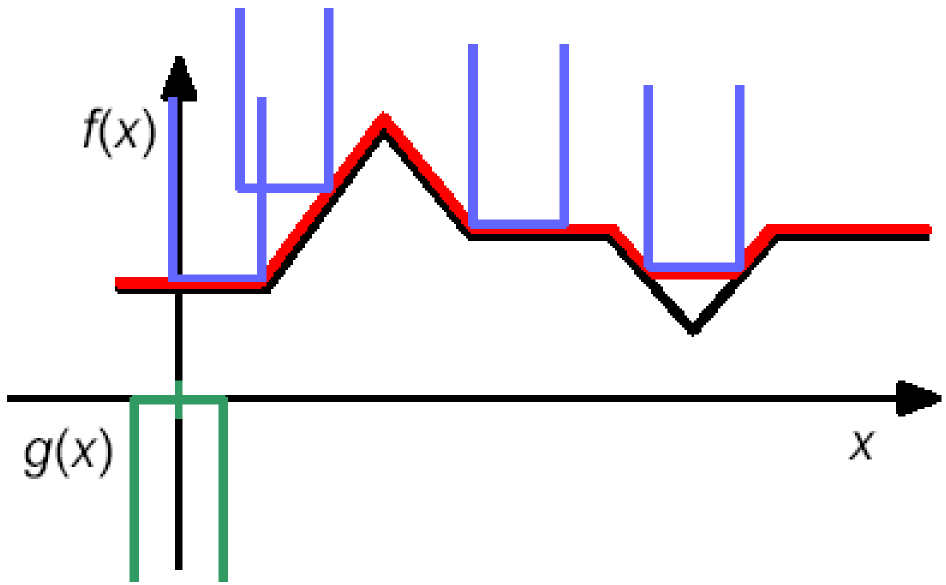


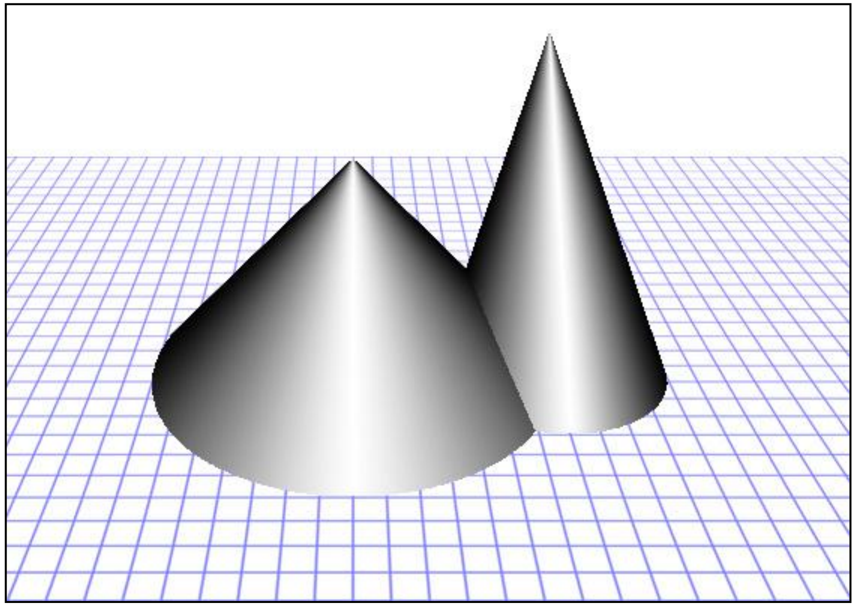
Closing

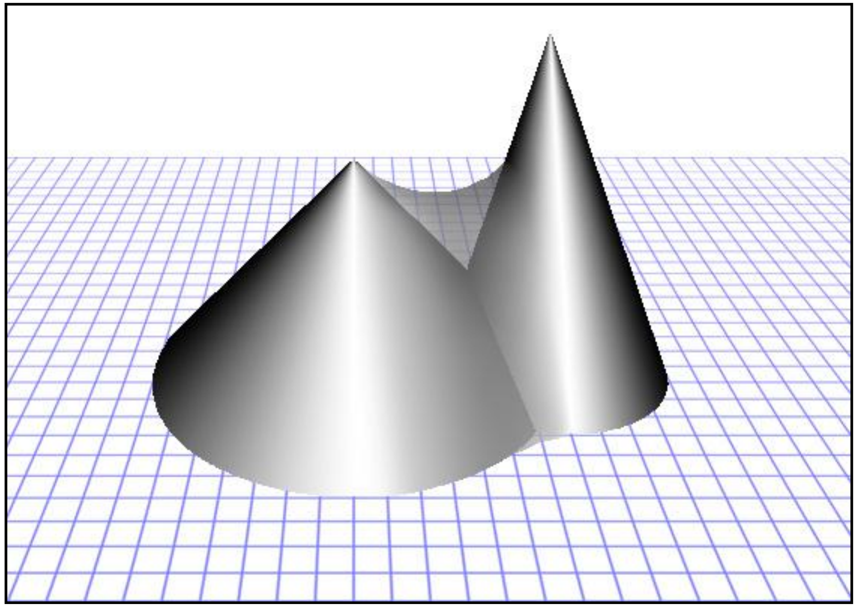
- **Definition**
  - For a flat structuring function

$$C_B(f)(x) = E_{\breve{B}}(D_B(f))$$

- **Illustration**







Main Properties

- **Extensivity, Anti-Extensivity**

$$O_B(f) \leq f \leq C_B(f)$$

- **Increasing**

$$f \leq g \ \Rightarrow O_B(f) \leq O_B(g)$$
$$f \leq g \ \Rightarrow C_B(f) \leq C_B(g)$$

- **Idempotence**

$$O_B(O_B(f)) = O_B(f)$$
$$C_B(C_B(f)) = C_B(X)$$

- **Duality with respect to the Complementation**

$$C_B(f) = M - O_B(M - f)$$

- **Duality with respect to the Adjunction**

$$C_{\check{B}}(f) \leq g \Leftrightarrow f \leq O_B(g)$$

- **Translation Invariance**
- **Compatibility with Scales**

- **Decreasing, Increasing with respect to the SE**

$$B \leq B' \ \Rightarrow O_{B'}(f) \leq O_B(g)$$
$$B \leq B' \ \Rightarrow C_B(f) \leq C_{B'}(g)$$

Illustration

- **Reducing or Removing Bright / Dark Details**


original image


opened image


closed image

Illustration

- **Reducing or Removing Bright / Dark Details**

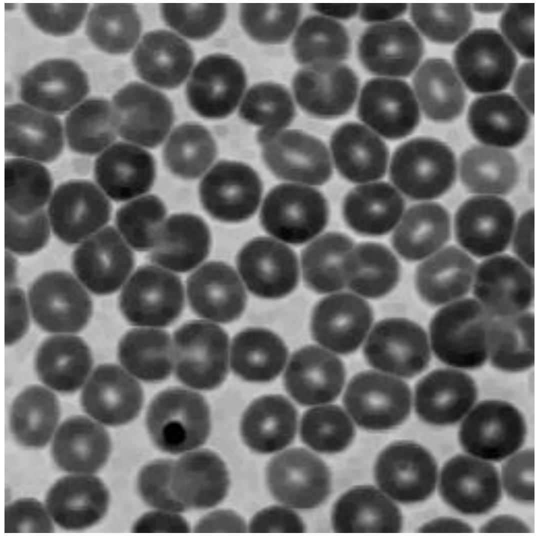
original image

opened image

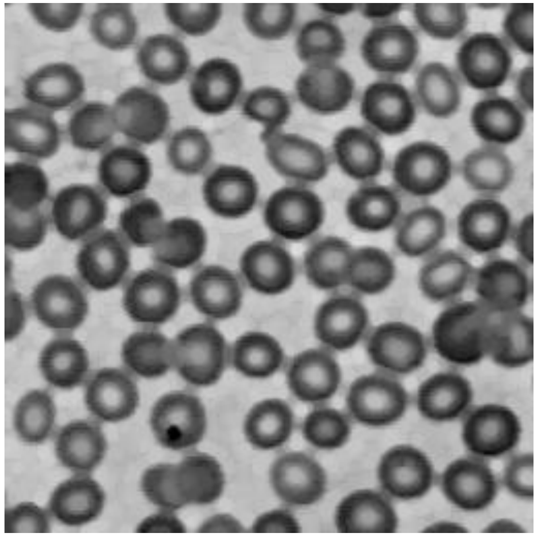
closed image

Illustration

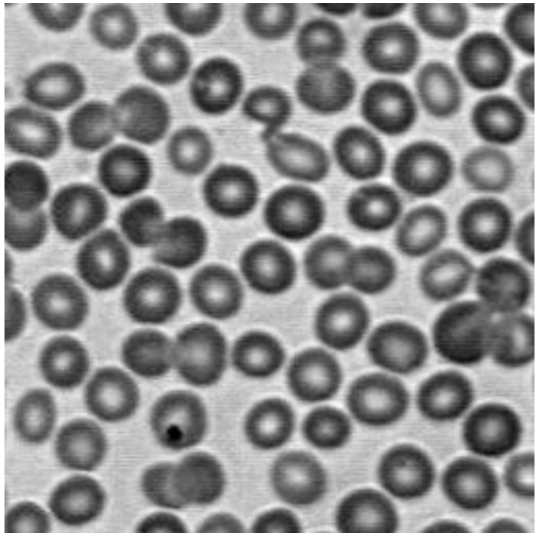
- **Comparison**



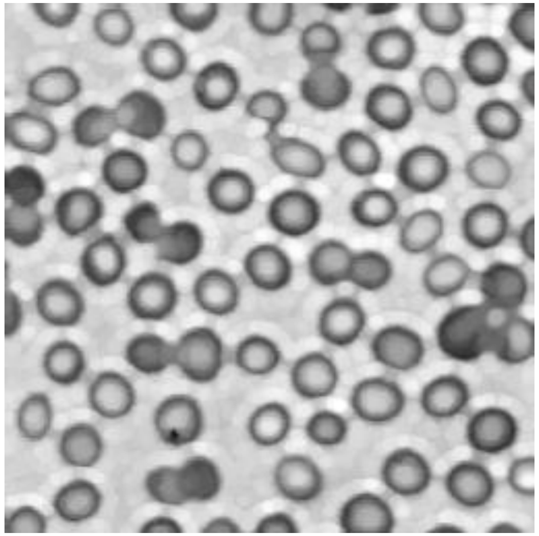
erosion



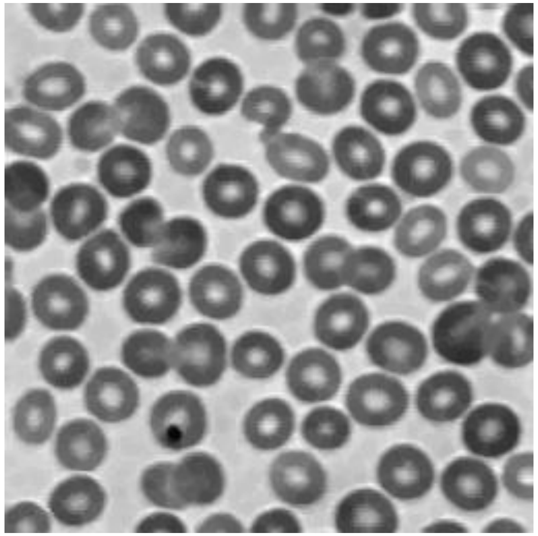
opening



original



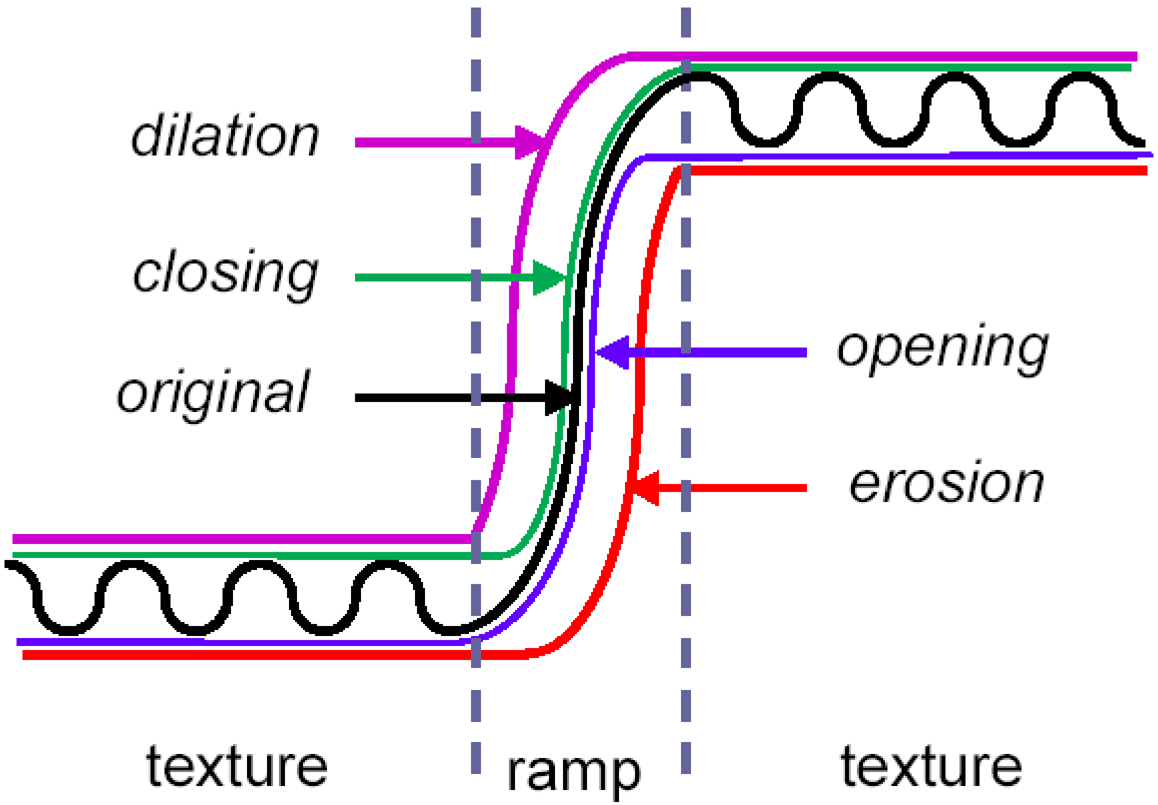
dilation



closing

Application to Smoothing

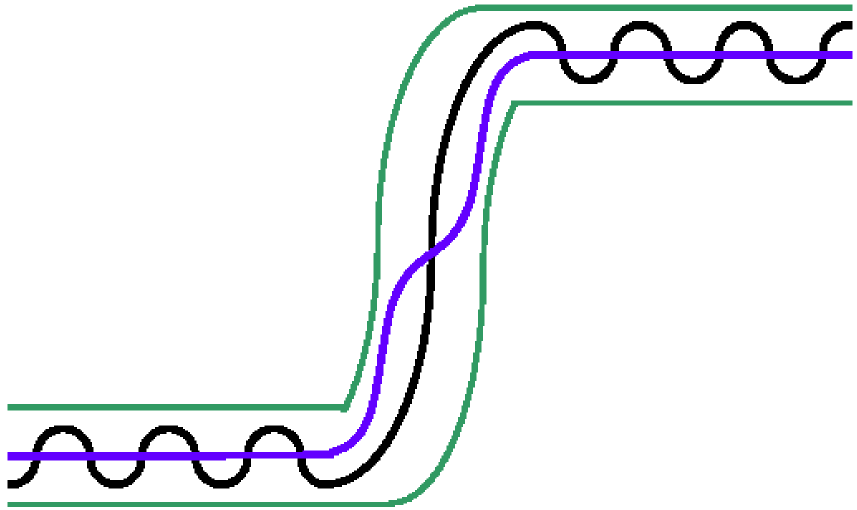
- **Ramp and Texture**
  - Morphological filters can separate ramps and textures
  - Textures are not distinct to noise



Application to Smoothing

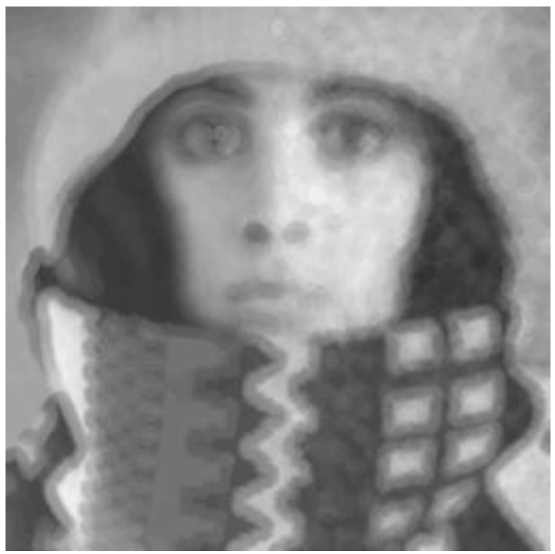
- **Dynamic Smoothing**

$$Dys_B(f) = \frac{1}{2}(D_B(f) + E_B(f))$$



- **Illustration**

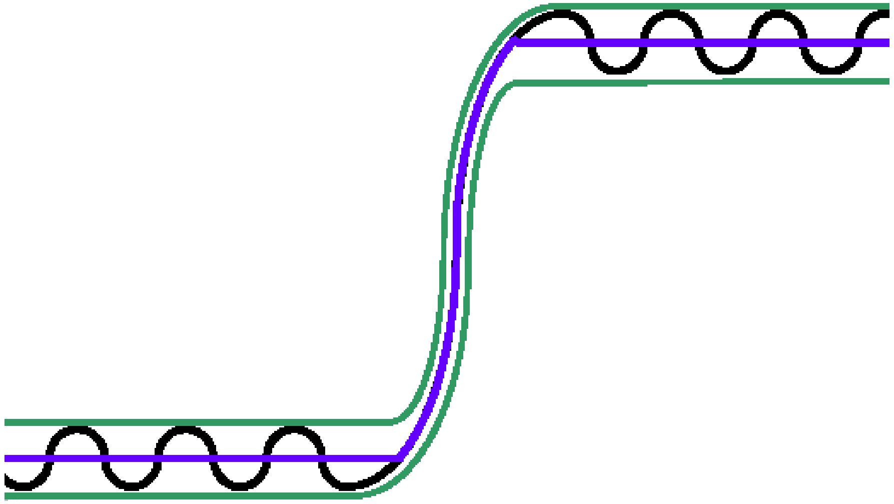

original


dynamic smoothing

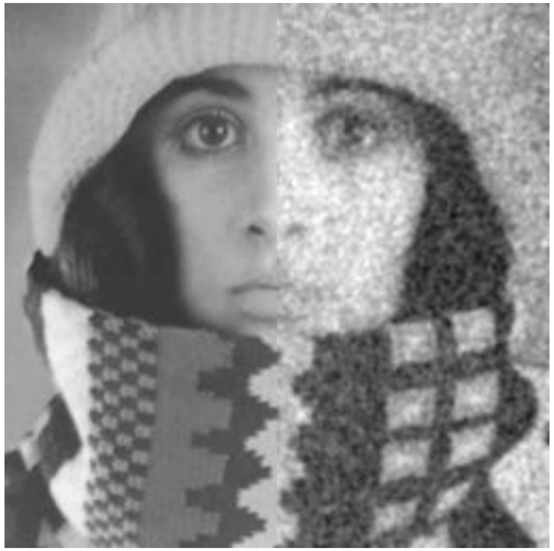
Application to Smoothing

- **Texture Smoothing**

$$Tes_B(f) = \frac{1}{2}(C_B(f) + O_B(f))$$



- **Illustration**

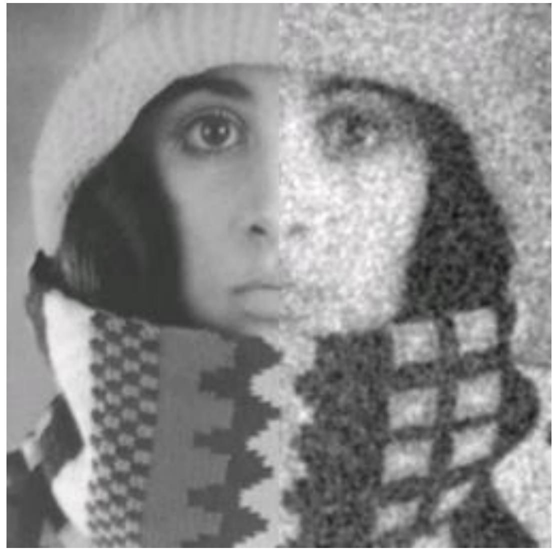


original

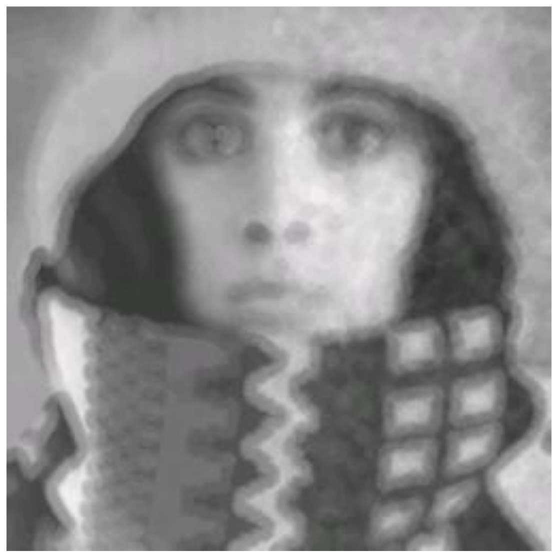
texture smoothing

Application to Smoothing

- Comparison



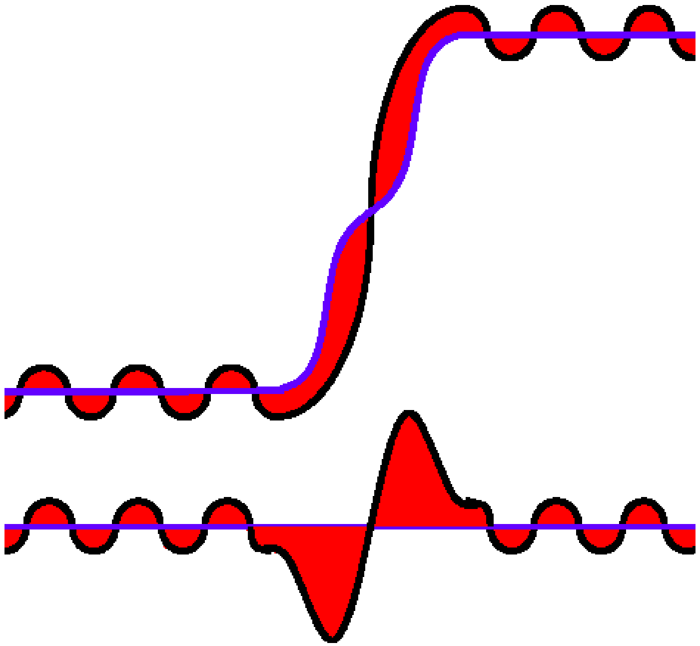
Gaussian smoothing



dynamic smoothing

texture smoothing

Second-Order Gradients



- **Dynamic Gradient**

$$Dyg_B(f) = f - Dys_B(f)$$

  - Isolates all edges

- **Illustration**

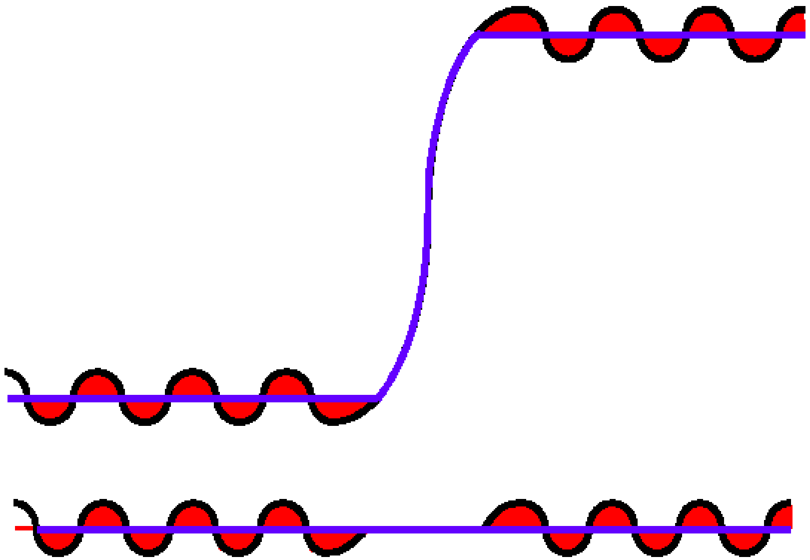
original   dynamic gradient

Second-Order Gradients

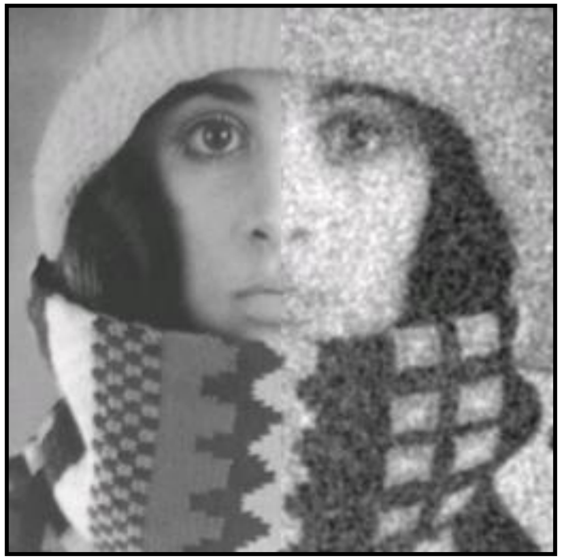
- **Texture Gradient**

$$Teg_B(f) = f - Tes_B(f)$$

  - Isolates the non-ramp edges, instead of all edges



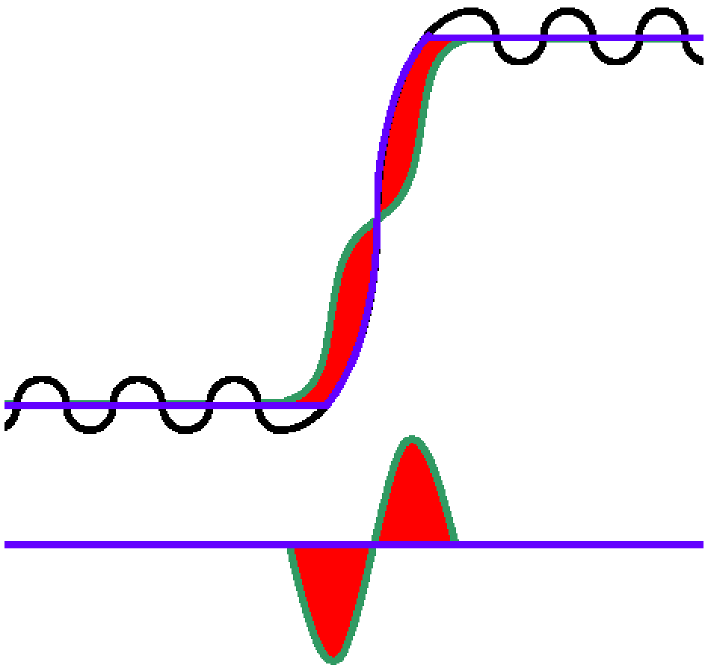
- **Illustration**



original

texture gradient

Second-Order Gradients

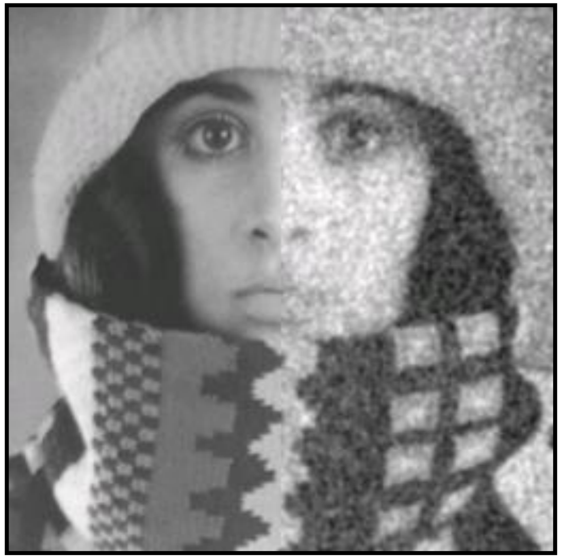


- **Ramp Gradient**

$$Rag_B(f) = Dyg_B(f) - Teg_B(f)$$

  - Isolates the ramp edges, excluding texture or noise

- **Illustration**



original

ramp gradient

Second-Order Gradients

- **Comparison**

original

Laplacian

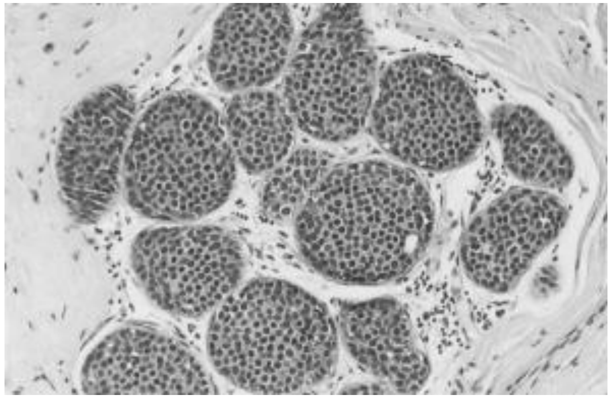
dynamic gradient

texture gradient

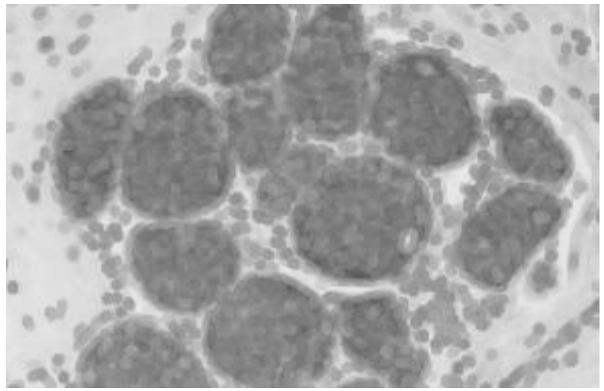
ramp gradient

Second-Order Gradients

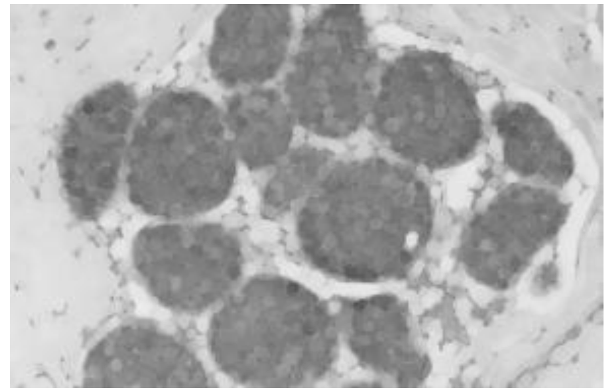
- **Comparison**



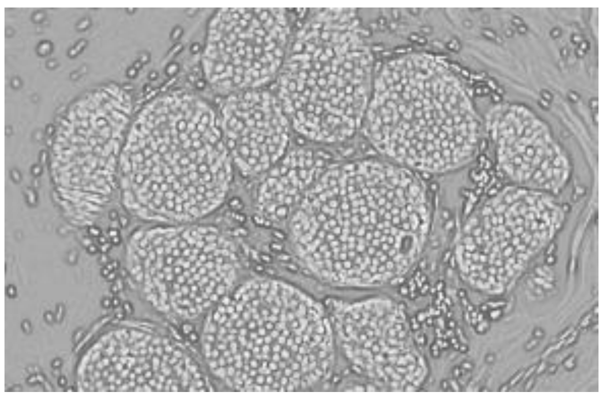
original image



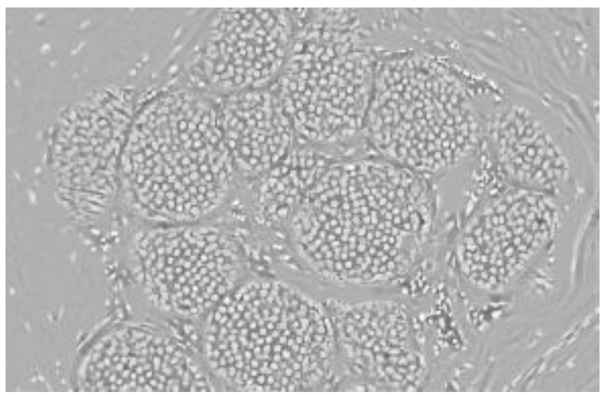
dynamic smoothing



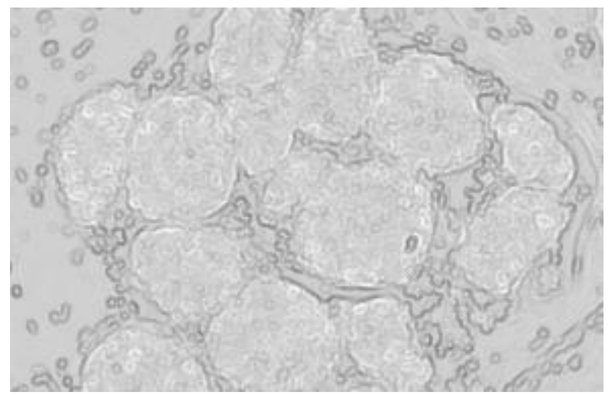
texture smoothing



dynamic gradient



texture gradient



ramp gradient

Top-Hat and Bottom-Hat

- **Extraction of Peaks and Valleys**
  - The choice of a given morphological filter is driven by the available knowledge about the shape, size and orientation of the spatial structures to be filtered.
  - Morphological top/bottom-hats proceed a contrario.
  - The approach undertaken with top/bottom-hats consists of using knowledge about shape characteristics that are not shared by their relevant image structures.
  - It is sometimes easier to remove relevant structures than trying to directly suppress irrelevant objects.

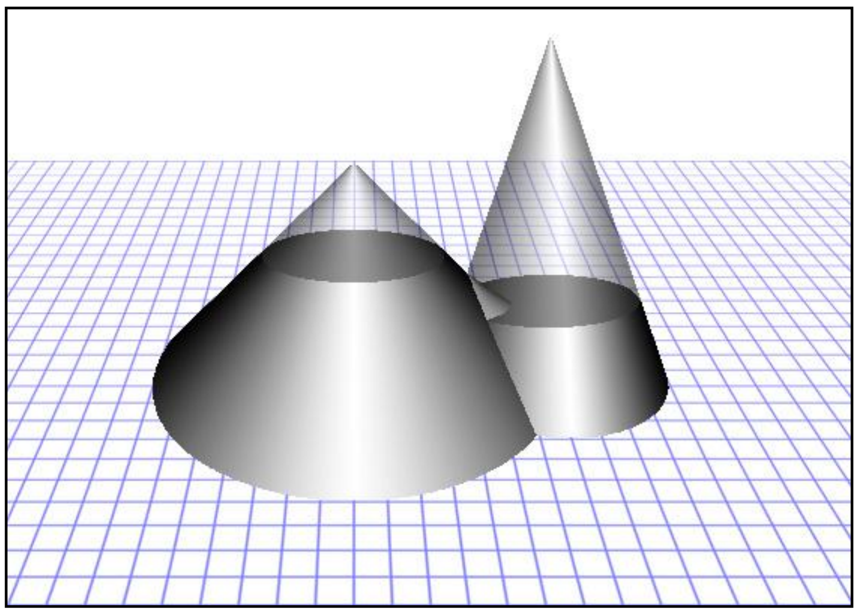
Top-Hat

- **Definition**

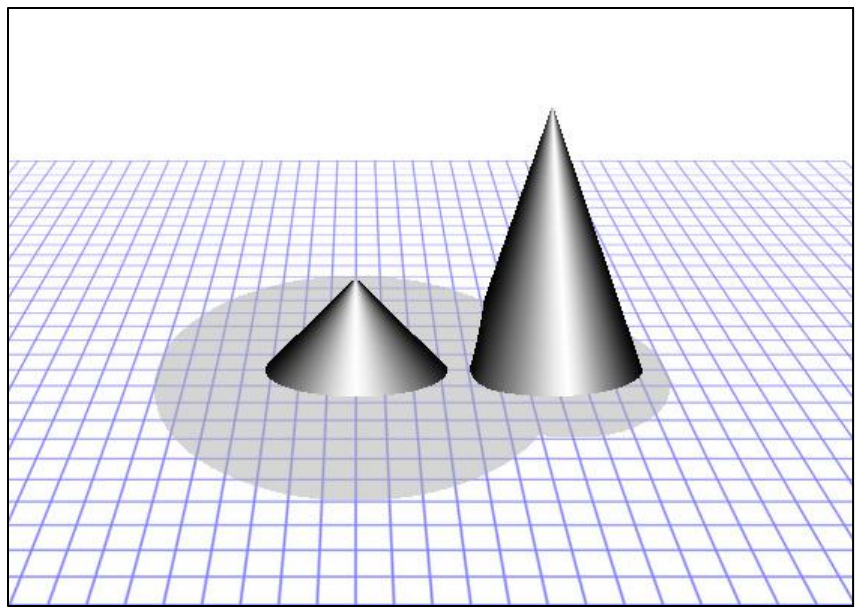
$$TH_B(f) = f - O_B(f)$$

- **Extracts the peaks**

- **Illustration**



opening



top-hat

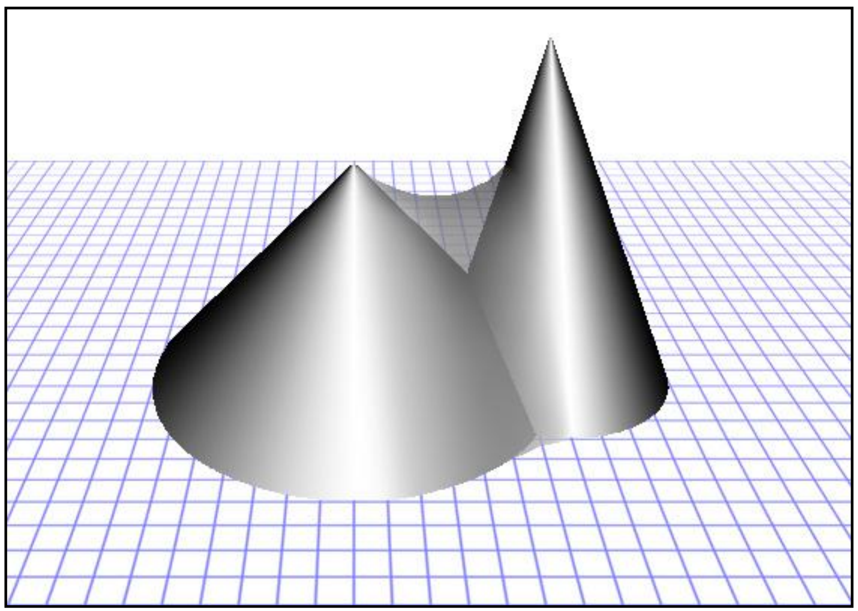
Bottom-Hat

- **Definition**

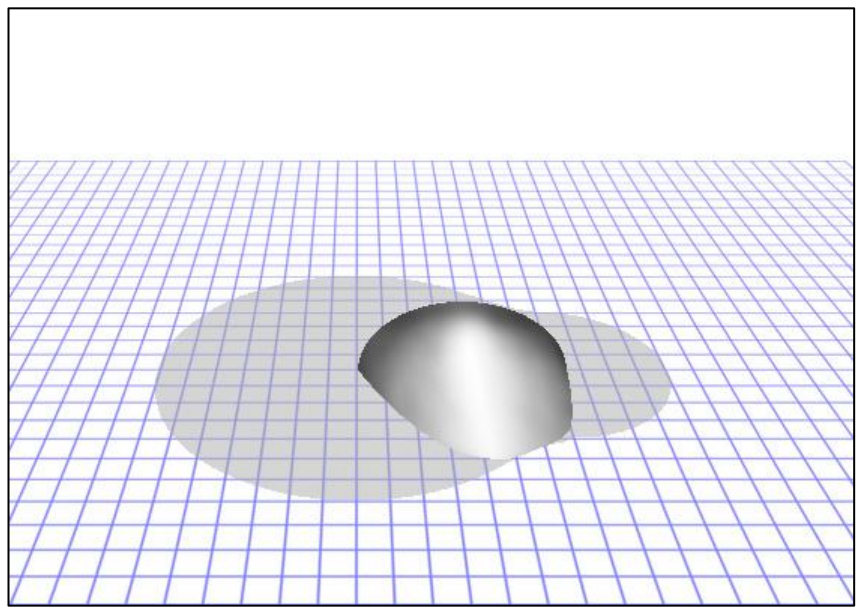
$$BH_B(f) = C_B(f) - f$$

- **Extracts the valleys**

- **Illustration**



closing

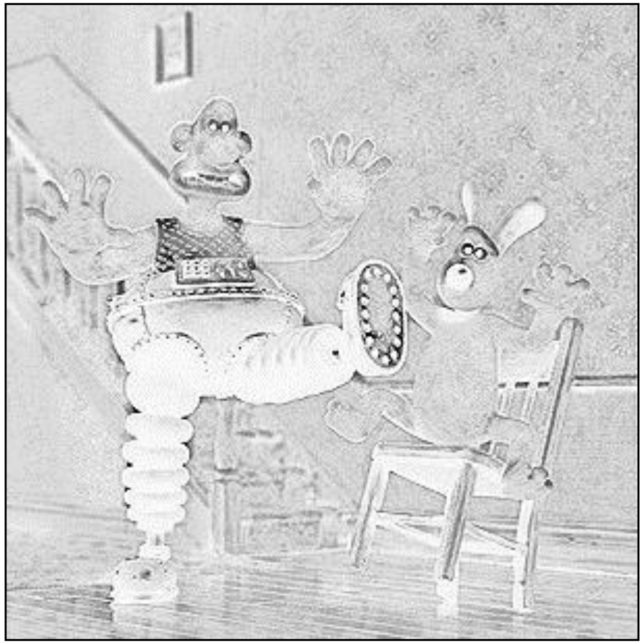


bottom-hat

Top-Hat and Bottom-Hat

- **Illustration**
  - Shown as negatives for more visibility

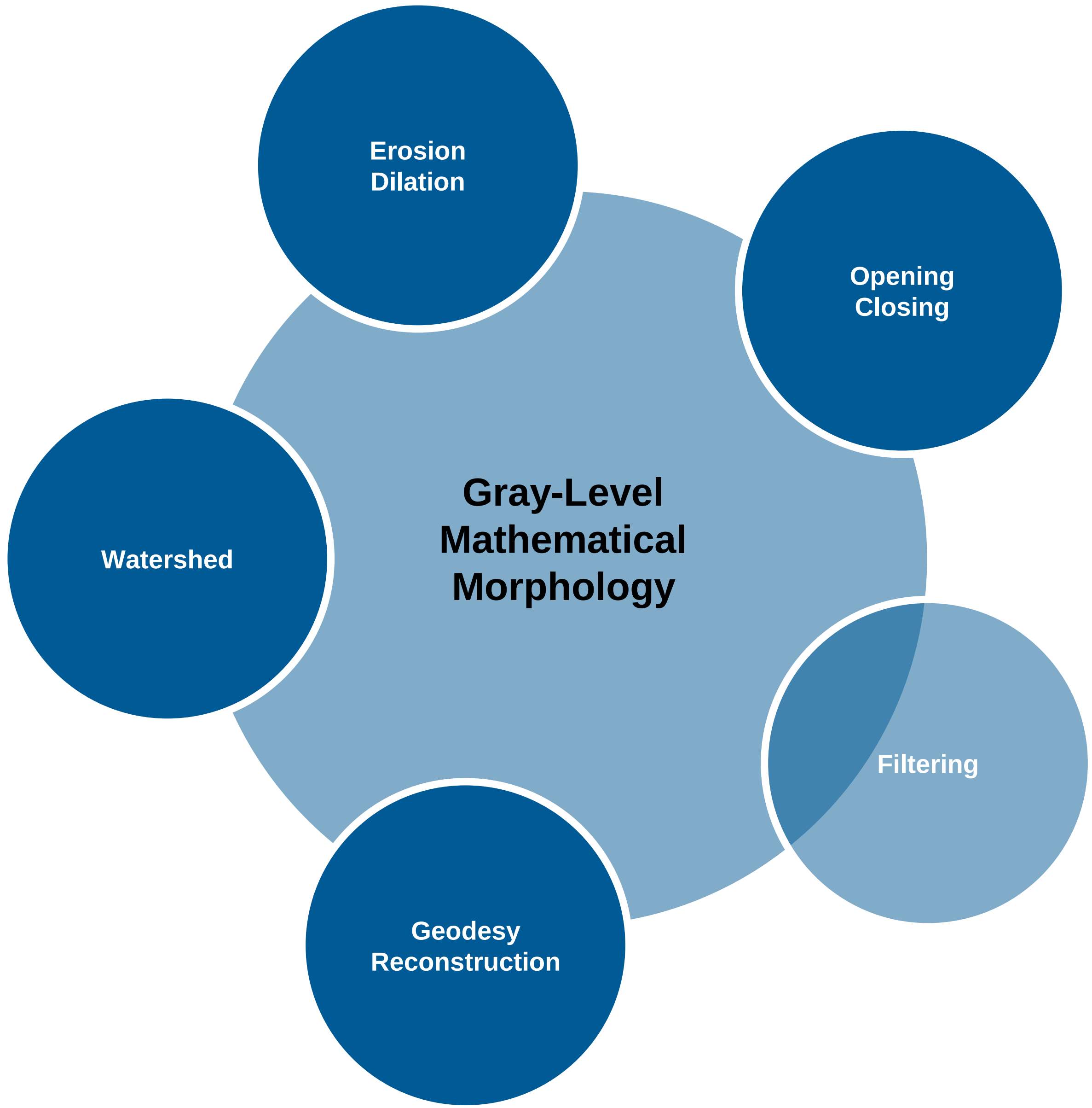
original



top-hat

bottom-hat

Gray-Level Mathematical Morphology



Morphological Filtering

- ○ **Linear Image Processing**
  - Removing some frequential components
  - Convolution
- ○ **Morphological Image Processing**
  - Removing some geometrical components
  - Morphological filter

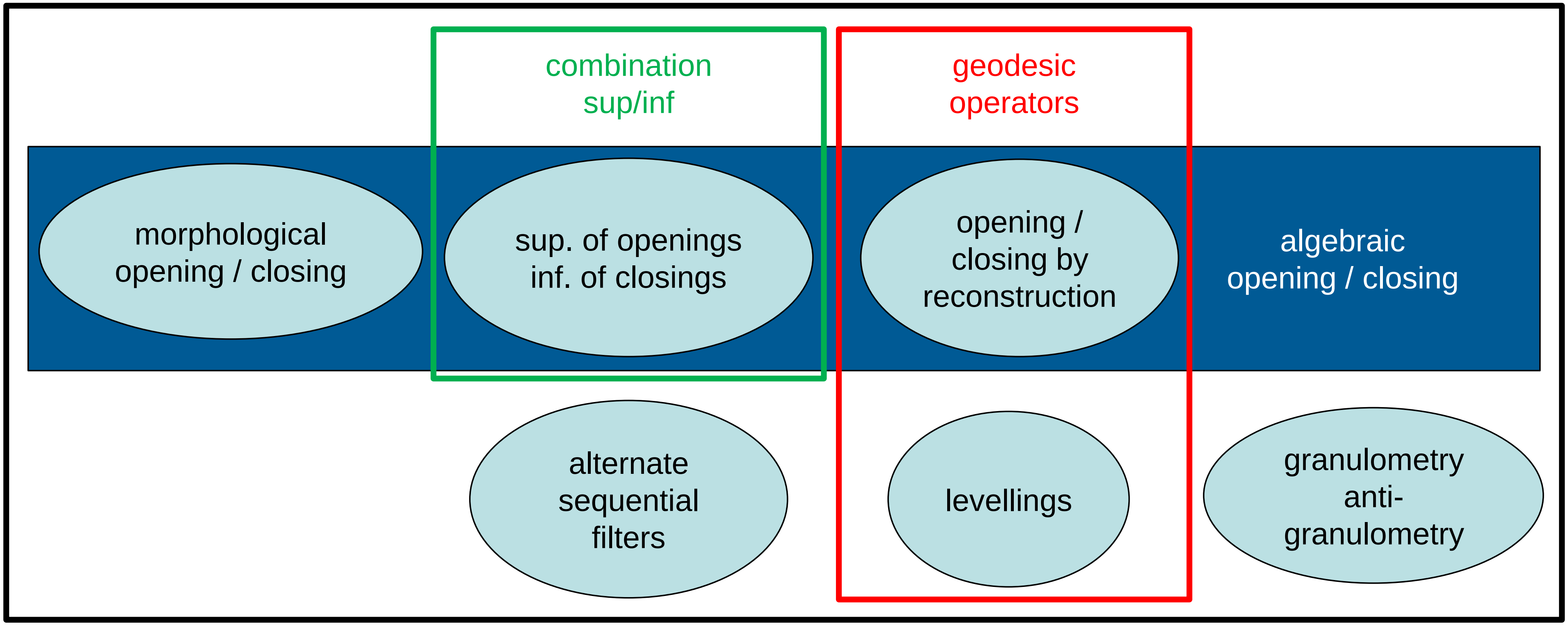
linear filtering

original

morphological filtering

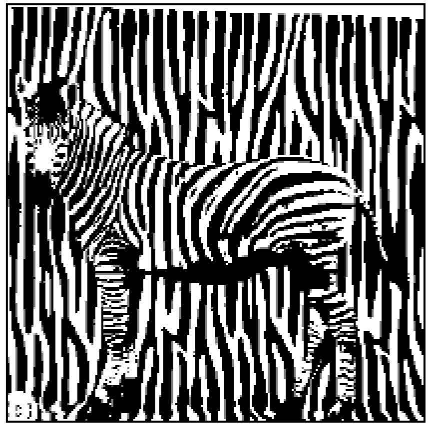
Morphological Filters

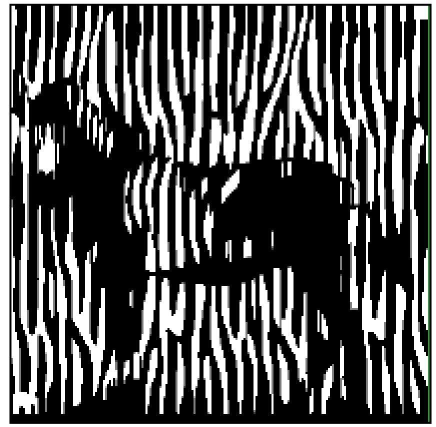
- **Properties of a (Morphological) Filter**
  - Increasing (simplification, loss of information)
  - Idempotence (stability, invariance)

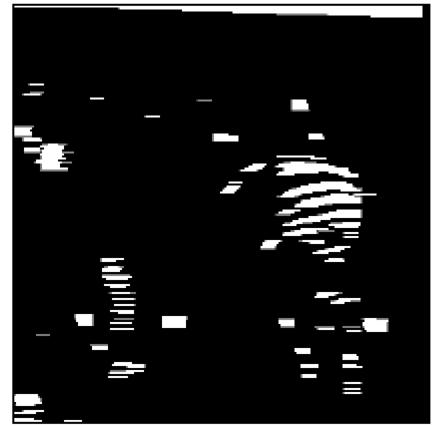


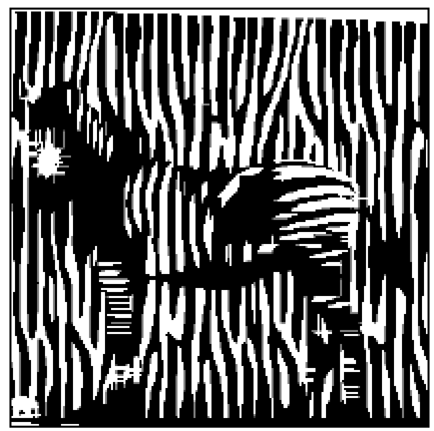
Algebraic Openings and Closings

- **Algebraic Opening**
  - Increasing
  - Idempotence
  - Anti-extensivity
- **Algebraic Closing**
  - Increasing
  - Idempotence
  - Extensivity
- **Examples**
  - Morphological opening / closing
  - $\gamma = \delta\varepsilon, \varphi = \varepsilon\delta$ where (δ, ε) = adjunction
  - $\bigvee_i \gamma_i$ where (γ_i) = algebraic openings
  - $\bigwedge_i \varphi_i$ where (φ_i) = algebraic closings


original


morph. opening / vertical segment


morph. opening / horizontal segment


alg. opening / union

Alternate Filters

- **Theorem: Suppose that $\boldsymbol{\varphi}$ and $\boldsymbol{\gamma}$ are filters such that $\boldsymbol{\varphi \geq \gamma}$**
  - $\varphi \geq \varphi\gamma\varphi \geq \varphi\gamma \vee \gamma\varphi \geq \varphi\gamma \wedge \gamma\varphi \geq \gamma\varphi\gamma \geq \gamma$
  - $\varphi\gamma$, $\gamma\varphi$, $\varphi\gamma\varphi$, $\gamma\varphi\gamma$ are (alternate) filters
  - $\varphi\gamma\varphi = \mathrm{Inf}\{g \text{ filter}; g \geq \varphi\gamma \vee \gamma\varphi\}$
  - $\gamma\varphi\gamma = \mathrm{Sup}\{g \text{ filter}; g \leq \varphi\gamma \wedge \gamma\varphi\}$
  - $(\varphi\gamma\varphi = \gamma\varphi) \Leftrightarrow (\gamma\varphi\gamma = \varphi\gamma)$

- **Examples**
  - C_B = morphological closing
  - O_B= morphological opening
  - $C_B O_B C_B$, $O_B C_B O_B$, $O_B C_B$, $C_B O_B$ are alternate filters

Alternate Filters

- **Illustration**
  - Dark and bright details are filtered

original

open-close

close-open

Alternate Sequential Filters

Filtering of an Image containing Dark and Bright Noise

- When the level of noise is high: it contains noisy structures over a wide range of scales
- A unique close-open or open-close filter with a large SE does not lead to acceptable results.
- A solution to this problem is to alternate closings and openings, beginning with a small SE and then proceeding with ever-increasing SE until a given size is reached.

ASF

- This sequential application of open-close (or close-open) filters is called an Alternating Sequential Filter.

Alternate Sequential Filters

- **Defined from Granulometry / Anti-Granulometry**
  - $\lambda \geq \mu > 0 \Rightarrow O_{\lambda B} \leq O_{\mu B} \leq C_{\mu B} \leq C_{\lambda B}$

- **Definition**

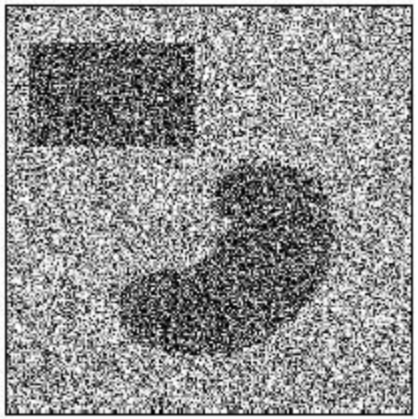
$$\Xi_\lambda = C_{\lambda B} O_{\lambda B} \ \dots \ C_{2B} O_{2B} C_B O_B$$

$$\Theta_\lambda = O_{\lambda B} C_{\lambda B} \ \dots \ O_{2B} C_{2B} O_B C_B$$

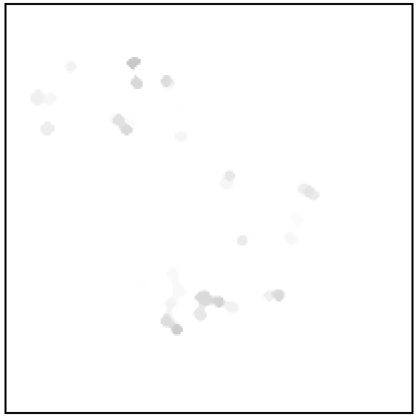
- **Absorption Property**

$$\lambda \geq \mu \Rightarrow \begin{cases} \Theta_\lambda \Theta_\mu = \Theta_\lambda \\ \Theta_\mu \Theta_\lambda \leq \Theta_\lambda \\ \Xi_\lambda \Xi_\mu = \Xi_\lambda \\ \Xi_\mu \Xi_\lambda \leq \Xi_\lambda \end{cases}$$

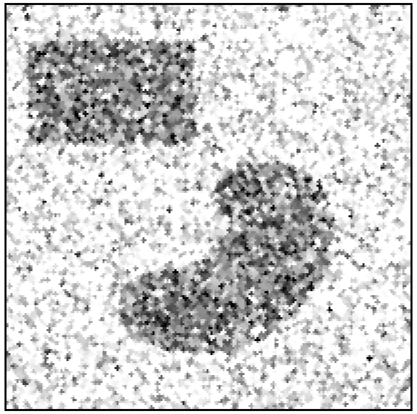
Application to Noise Reduction



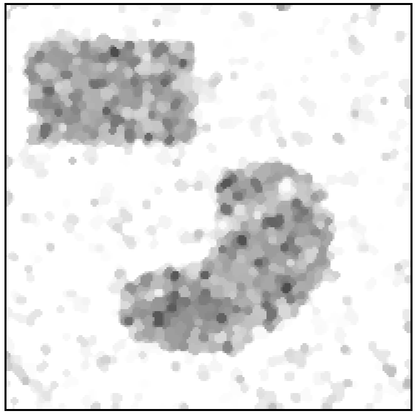
original f



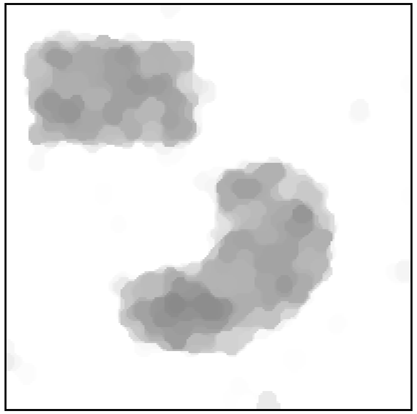
$O_{4B}(C_{4B}(f))$



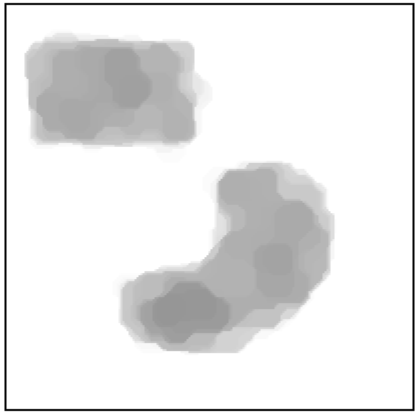
$\Theta_1(f)$



$\Theta_2(f)$



$\Theta_5(f)$



$\Theta_8(f)$

Morphological Center

- **Kind of Morphological Median filter**

- **Auto-Dual Filter**
  - Operator which is independant of the local contrast, acting similarly on bright and dark areas

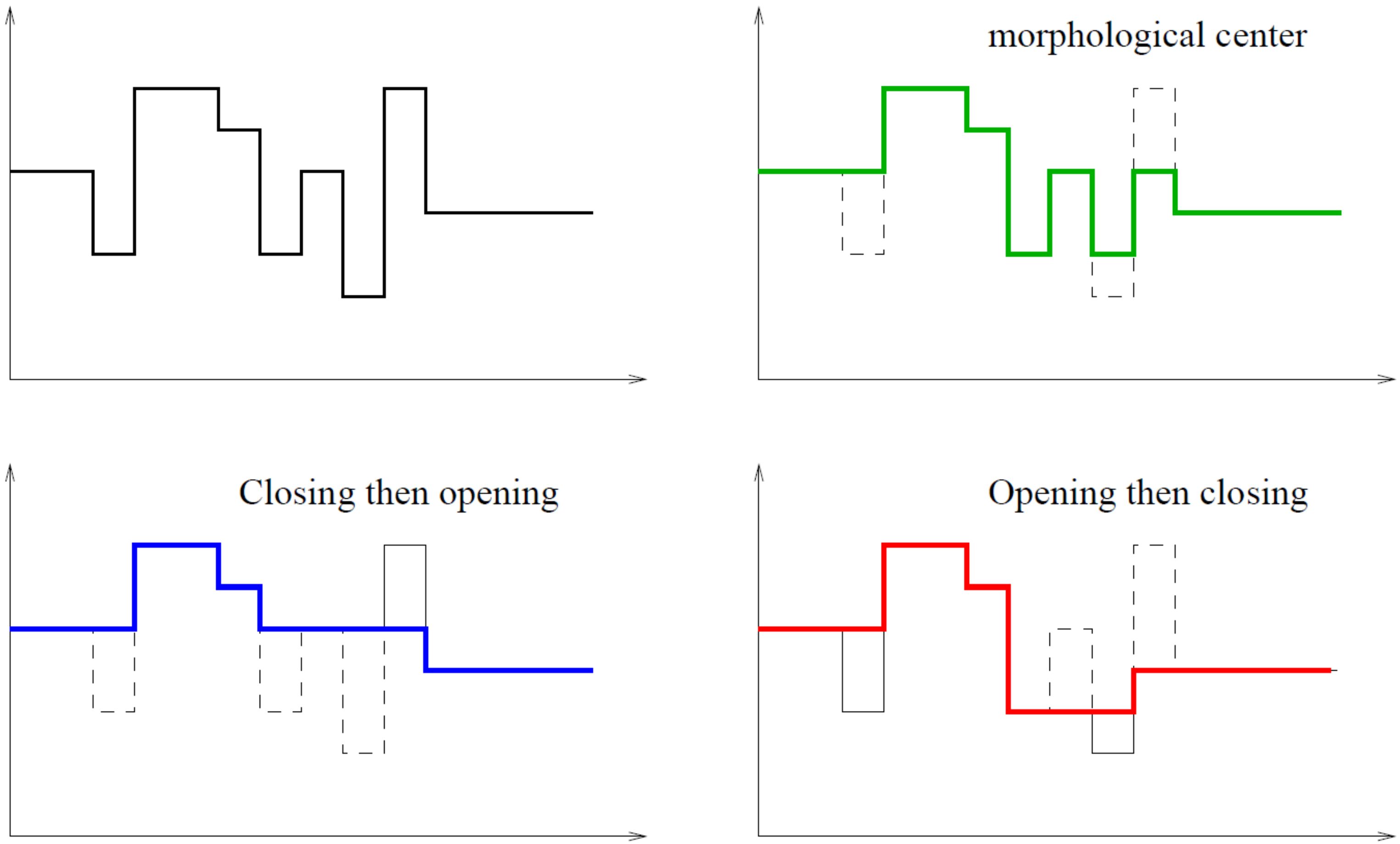
- **Definition**
  - For operators $\{\psi_1, \psi_2, \ldots \psi_n\}$

$$\Psi(f) = \left(f \vee \bigwedge_i \psi_i\right) \wedge \bigvee_i \psi_i$$

- **Properties**
  - Increasing
  - Idempotent for $\psi_1 = O_B C_B O_B$ and $\psi_2 = C_B O_B C_B$

Morphological Center

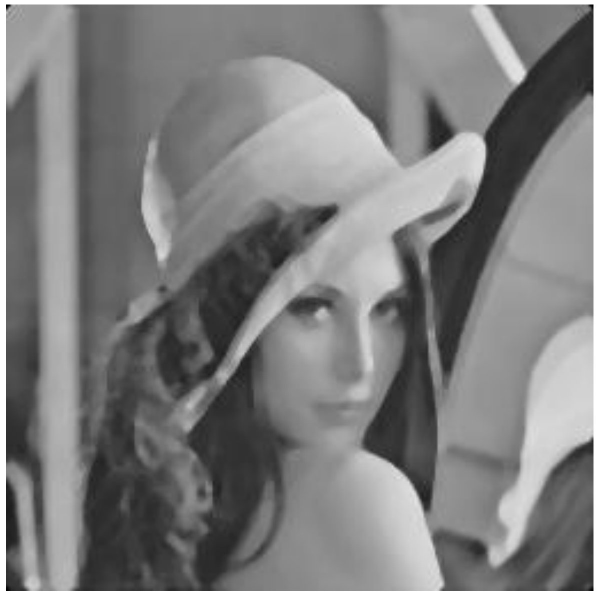
- **Illustration**
  - $\Psi_1 = O_B C_B$
  - $\Psi_2 = C_B O_B$



Application to Image Denoising

- **Salt and Pepper Noise**

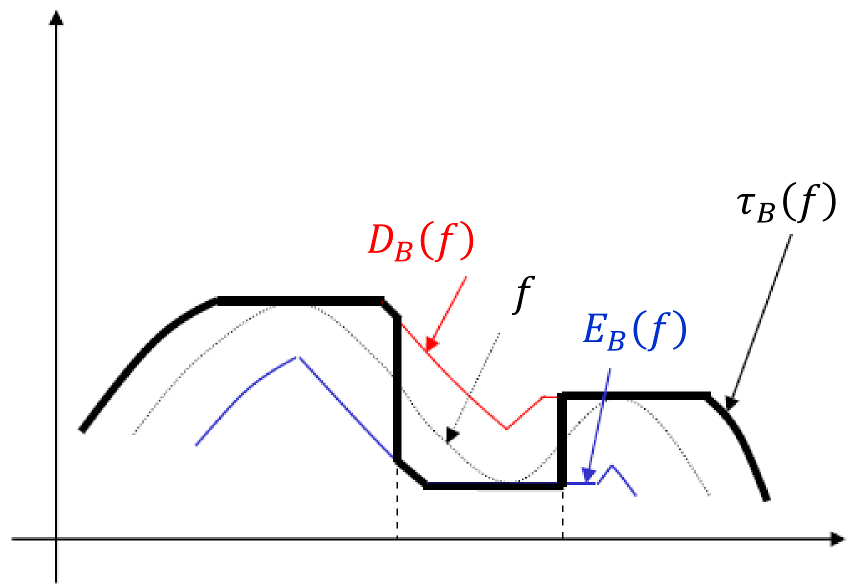
original



median

center

Toggle Contrast



Enhancement Filter

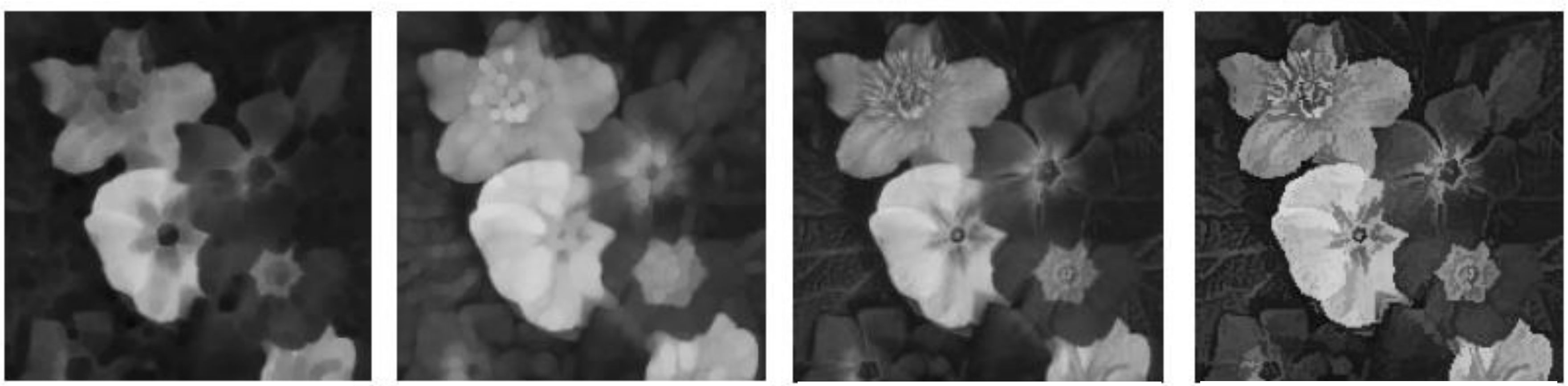
- Kind of anti-morphological centre

$$\tau_B(f) = \begin{cases} \Psi_1(f) & \text{if } |\Psi_1(f) - f| \le |\Psi_2(f) - f| \\ \Psi_2(f) & \text{if } |\Psi_1(f) - f| > |\Psi_2(f) - f| \end{cases}$$

Properties

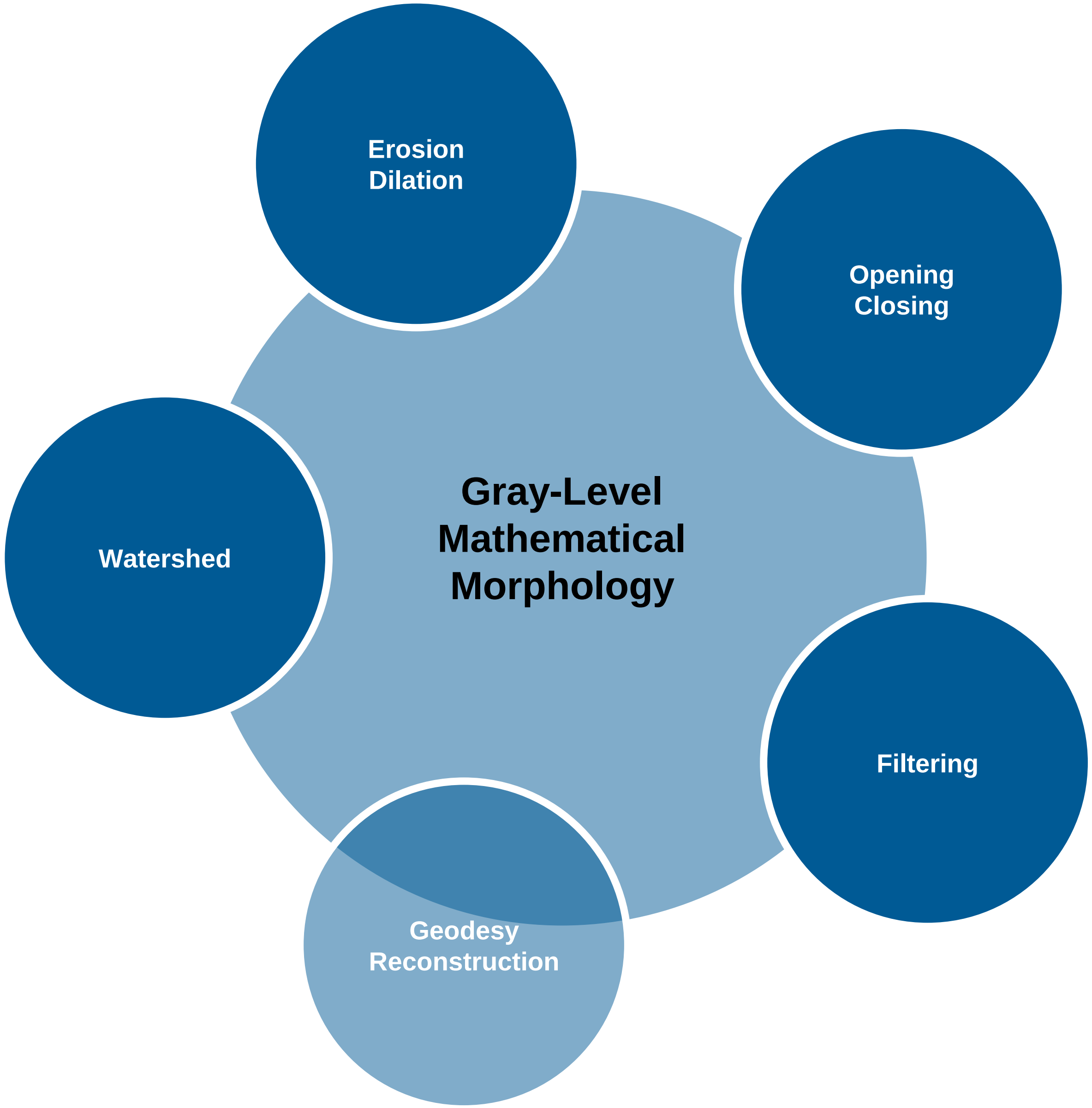
- If $\Psi_1 = O_B, \Psi_2 = C_B$: idempotent but not increasing

Illustration



$E_B(f)$ $D_B(f)$ f $\tau_B(f)$

Gray-Level Mathematical Morphology

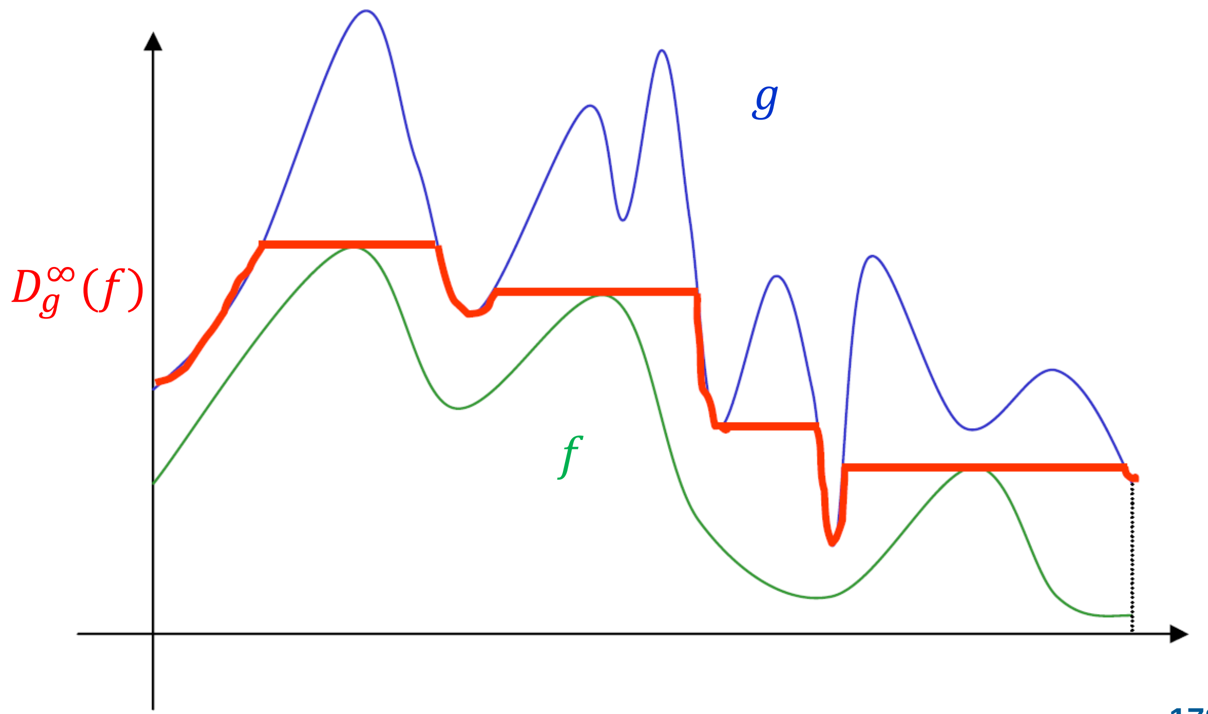


Geodesic Operators

- **Geodesic Dilation and Erosion**
  - Using level sets $f_\lambda = \{x; f(x) = \lambda\}$
  - $\left[D_{B,g}(f)\right]_\lambda = D_{B,g_\lambda}(f_\lambda)$ for $f \leq g$
  - $\left[E_{B,g}(f)\right]_\lambda = E_{B,g_\lambda}(f_\lambda)$ for $f \leq g$

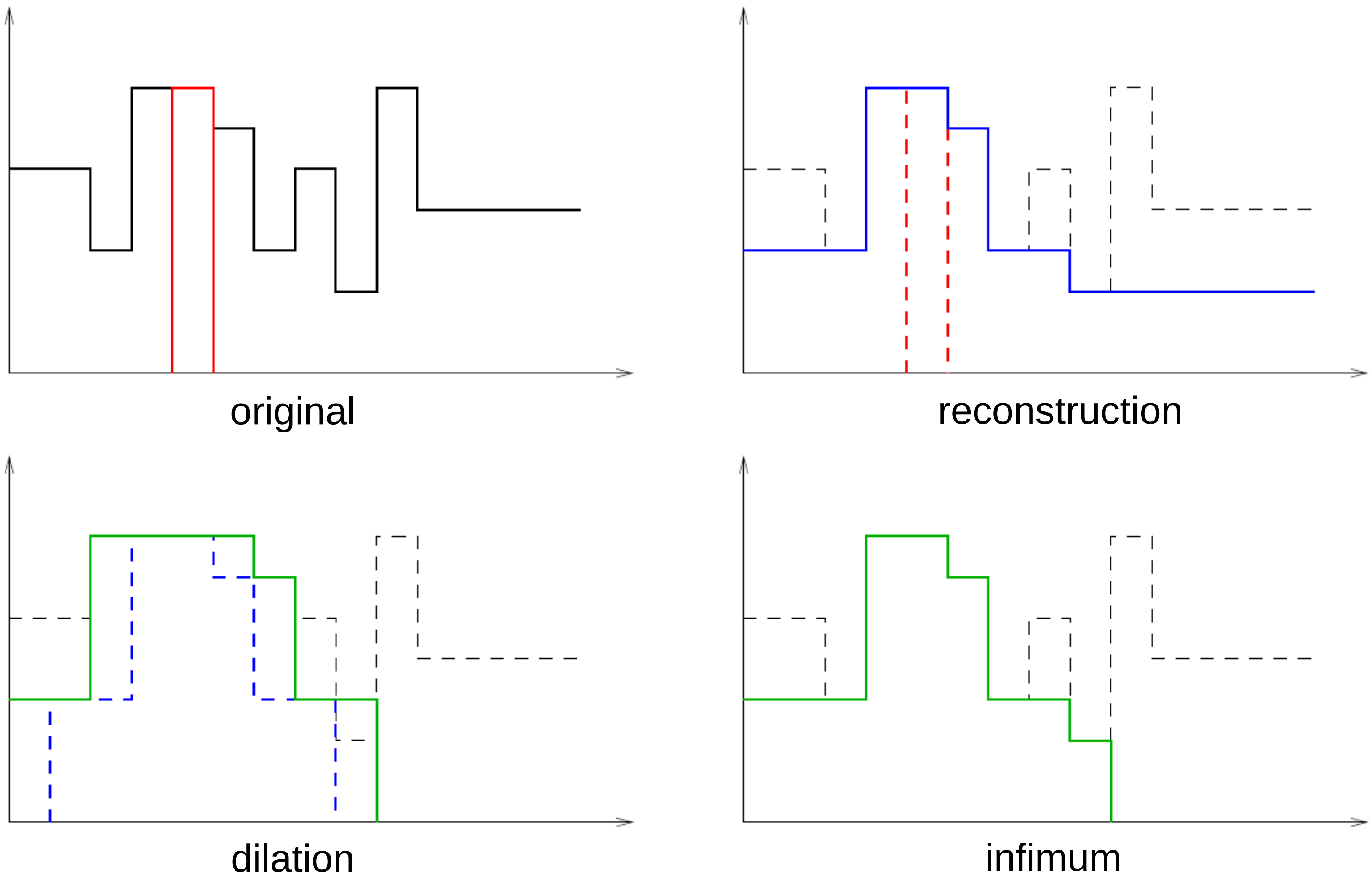
- **Digital case**
  - $D_{rB(1),g}(f) = \left[D_{B(1)}(f) \wedge g\right]^r$
  - $E_{rB(1),g}(f) = \left[E_{B(1)}(f) \vee g\right]^r$

- **Reconstruction**
  - $D_g^\infty(f) = \left[D_{B(1)}(f) \wedge g\right]^\infty$
  - $E_g^\infty(f) = \left[E_{B(1)}(f) \vee g\right]^\infty$



Reconstruction

- **Illustration**

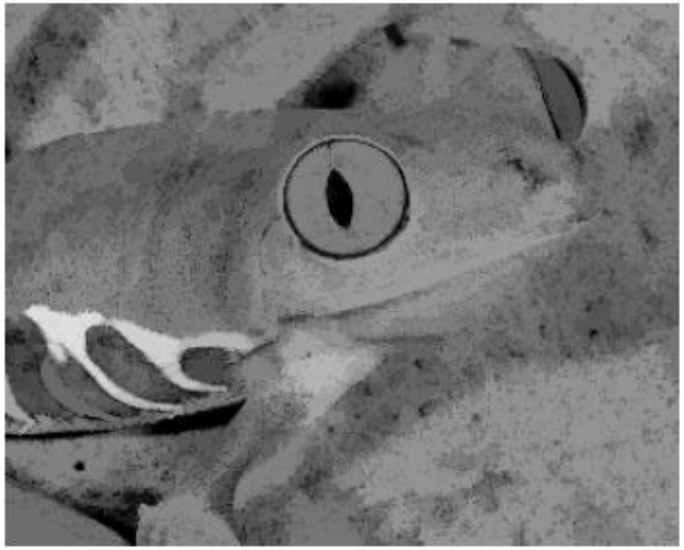


Opening and Closing by Reconstruction

- **Definition**
  - $O_{B(r)}^{\infty}(f) = \left[D_{B(1)}\left(E_{B(r)}(f)\right) \wedge f\right]^{\infty}$
  - $C_{B(r)}^{\infty}(f) = \left[E_{B(1)}\left(D_{B(r)}(f)\right) \vee f\right]^{\infty}$

- **Illustration**
  - Details are removed while preserving the image contours

original



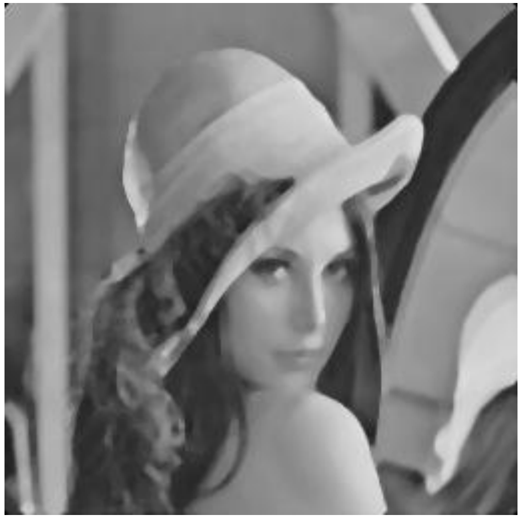
opening by reconstruction

closing by reconstruction

Application to Image Denoising

- **Salt and Pepper Noise**

noisy



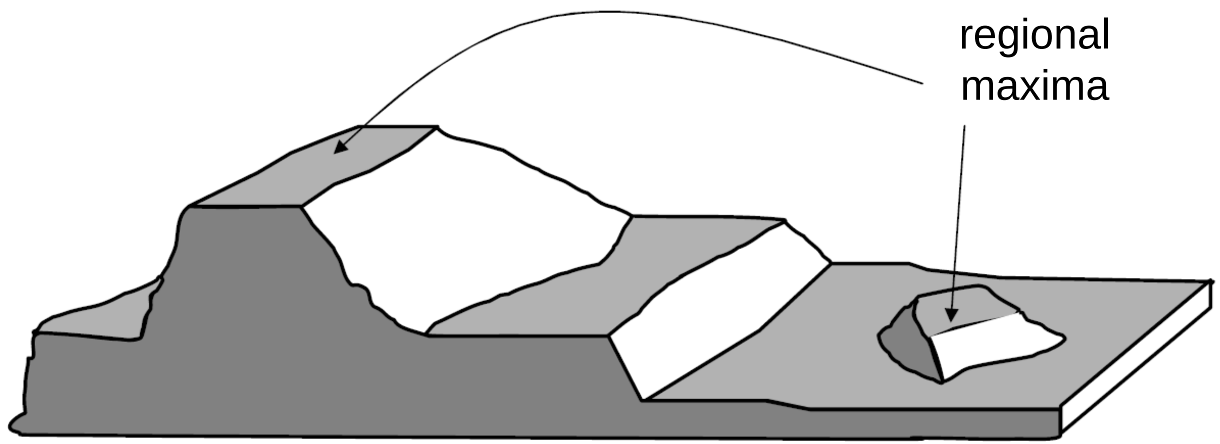
median

closing by reconstruction

opening by reconstruction

Application to Regional Maxima Extraction

- **Regional Maximum *X***
  - $\forall x \in X, f(x) = \lambda$ and $X = CC(f_\lambda)$

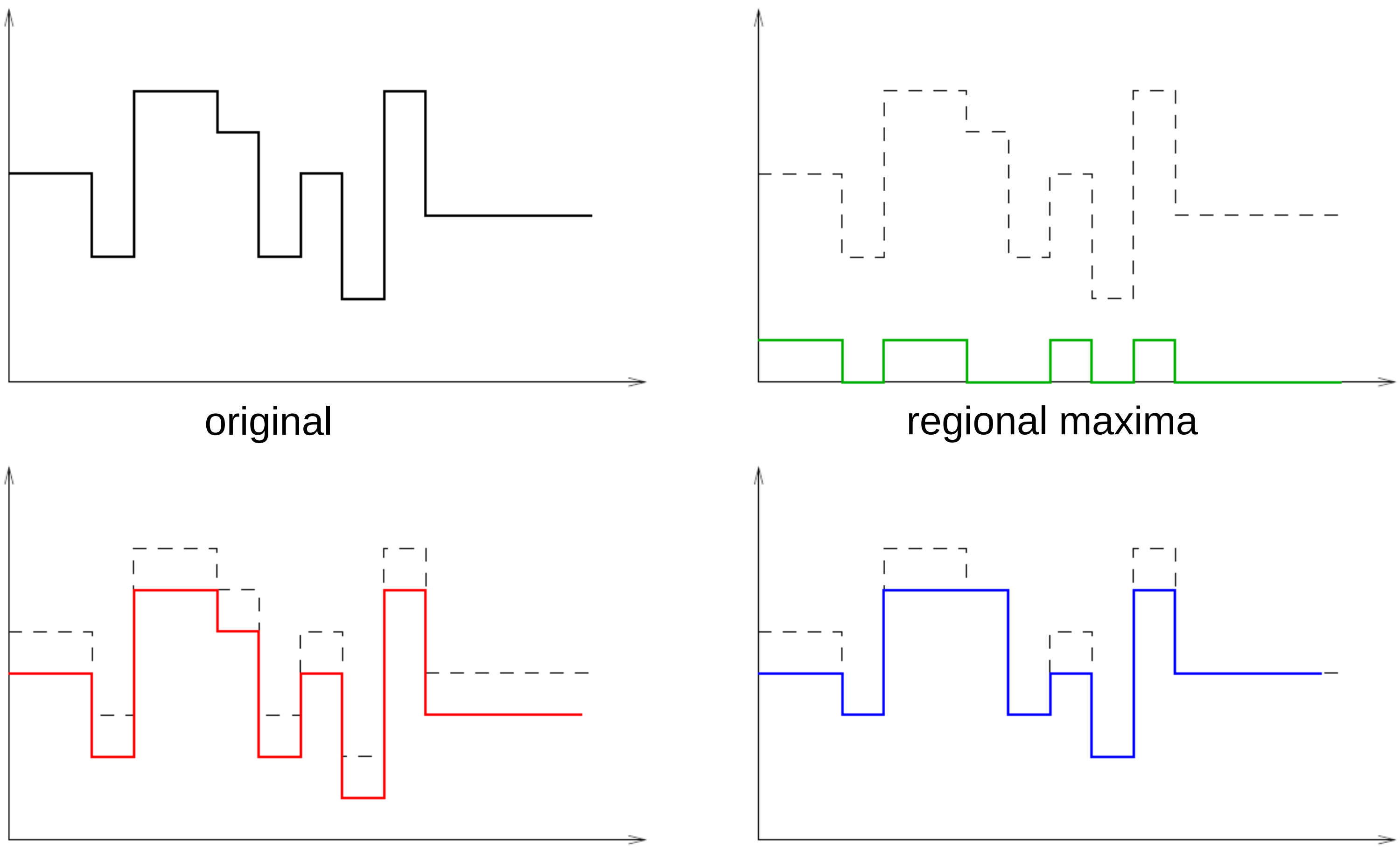


- **Computation of Regional Maxima**
  - $f - D_f^\infty(f-1)$

- **H-Maxima are more Robust**
  - Regional maxima of $D_f^\infty(f-h)$

Application to Regional Maxima Extraction

- **Illustration**
  - Regional maxima



original

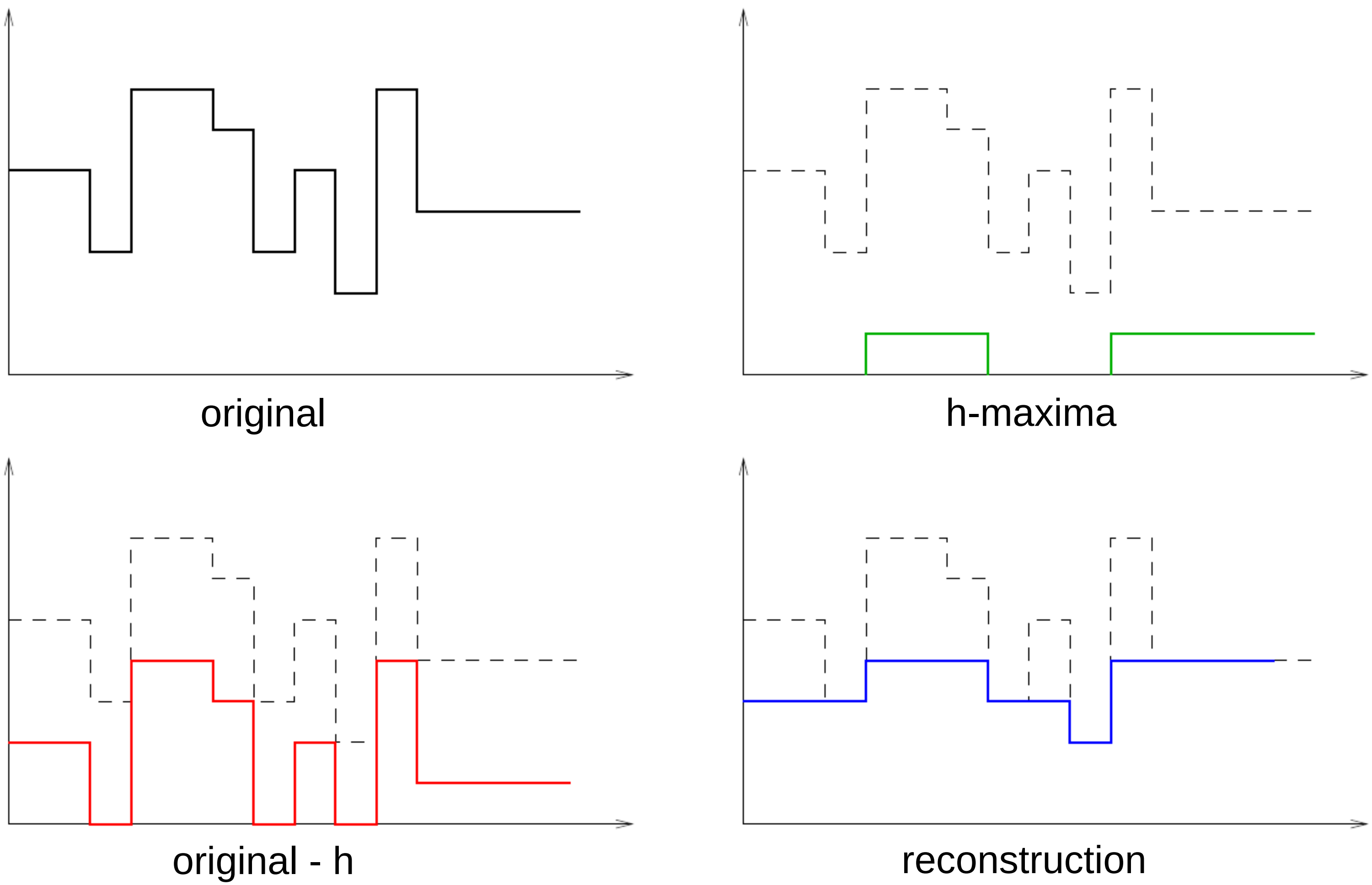
regional maxima

original - 1

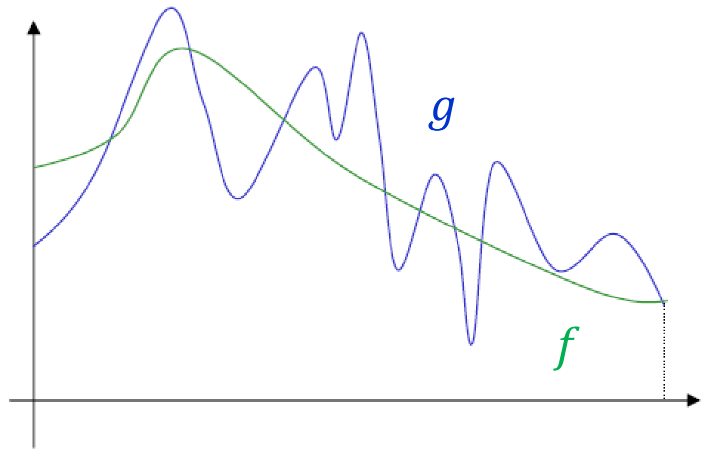
reconstruction

Application to Regional Maxima Extraction

- **Illustration**
  - H-maxima

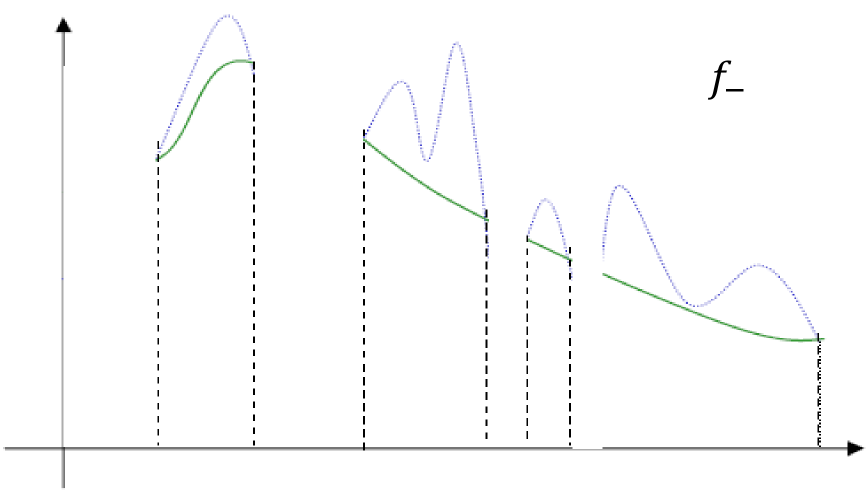


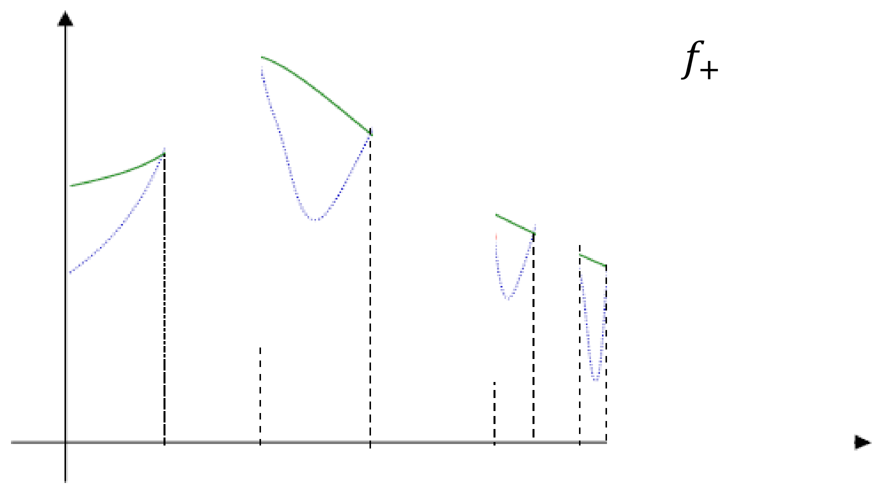
Levellings



- **Reconstruction when *f* and *g* are not Ordered?**
- **Acting on Bright and Dark Areas**

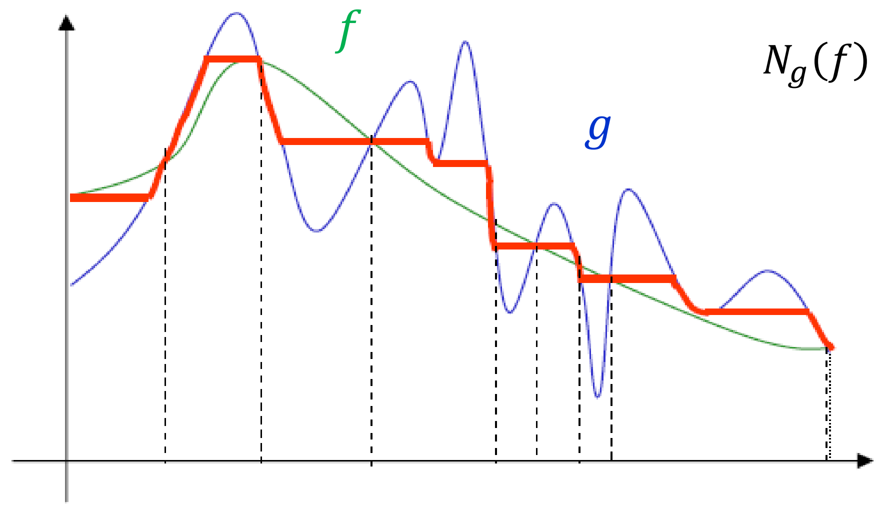
- **Decomposition of *f* with respect to *g***
  - $f = f_- + f_+$
  - $f_-(\mathrm{x}) = \begin{cases} f(x) \text{ if } f(x) \leq g(x) \\ 0 \text{ otherwise} \end{cases}$
  - $f_+(\mathrm{x}) = \begin{cases} f(x) \text{ if } f(x) > g(x) \\ 0 \text{ otherwise} \end{cases}$

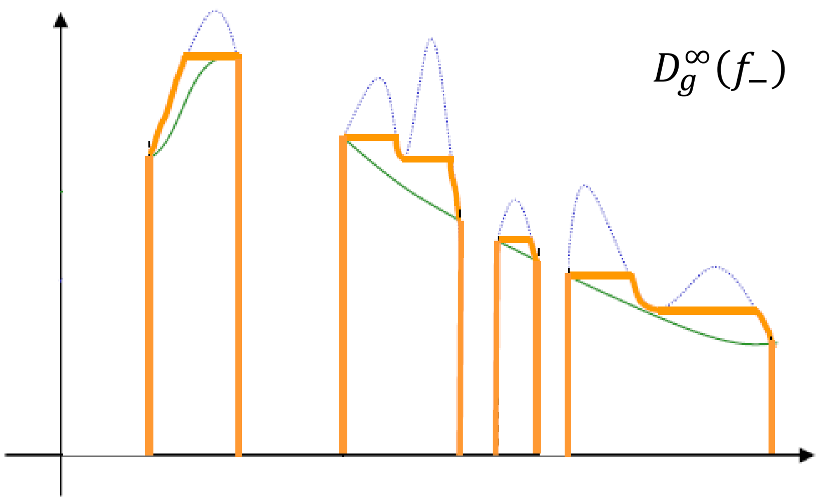


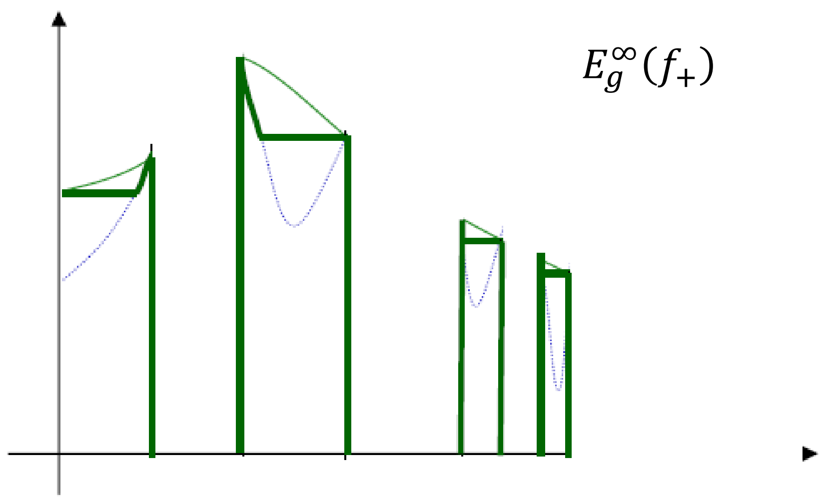


Levellings

- **Definition**
  - $N_g(f) = D_g^{\infty}(f_-) + E_g^{\infty}(f_+)$







Application to Image Denoising

- **Salt and Pepper Noise Reduction**

original g

Gaussian filtering f

Levelling $N_g(f)$

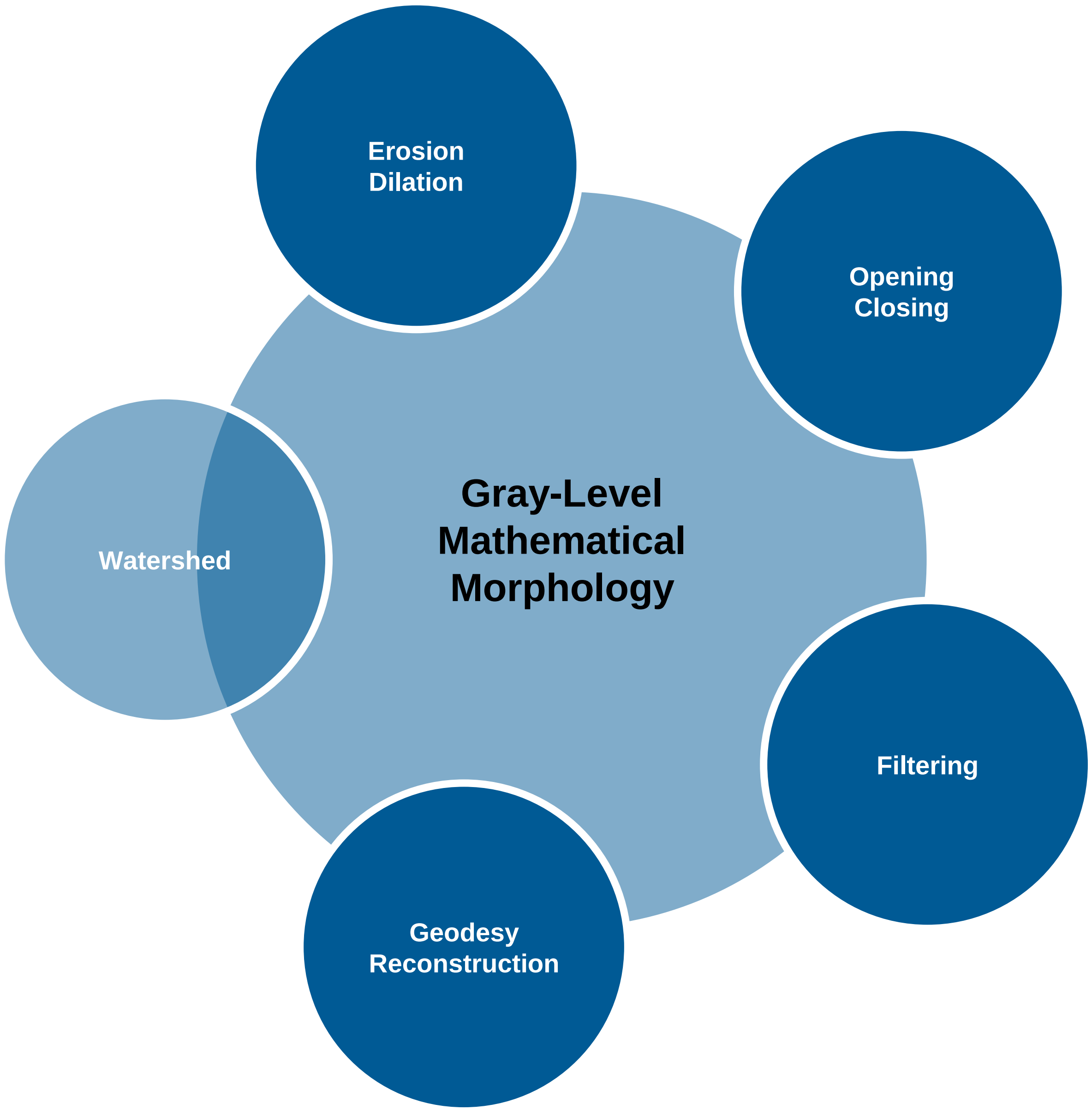
Application to Multiscale Image Representation

- Levellings of ASF

Application to Multiscale Image Representation

- Comparison with Classical ASF

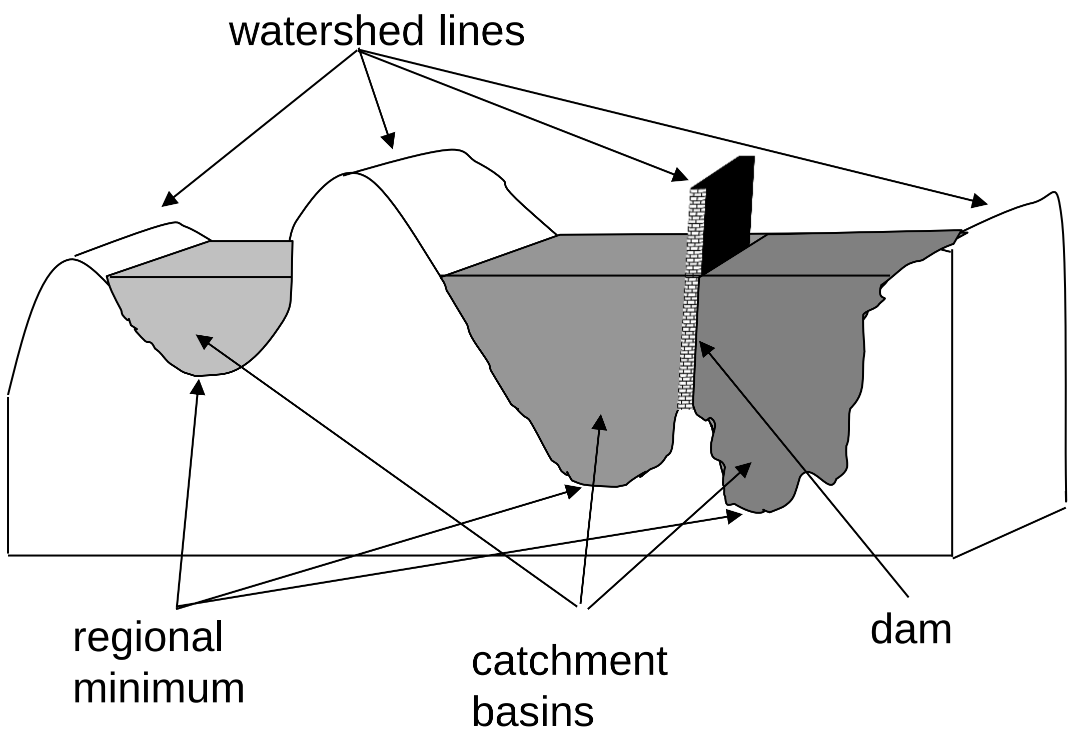
Gray-Level Mathematical Morphology



Morphological Approach for Image Segmentation

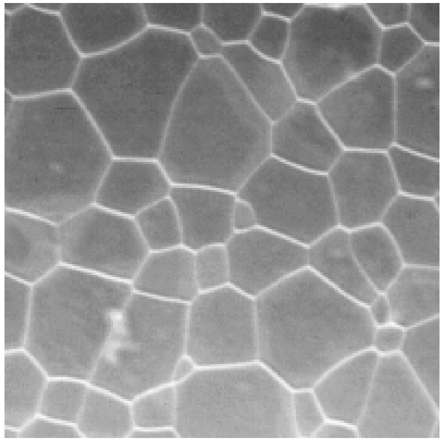
- **Basic Idea**
  - Imagine holes in regional minima
  - Progressively, the topography is flooded from below
  - Dams are built to prevent merging
  - Final dams correspond to watershed lines

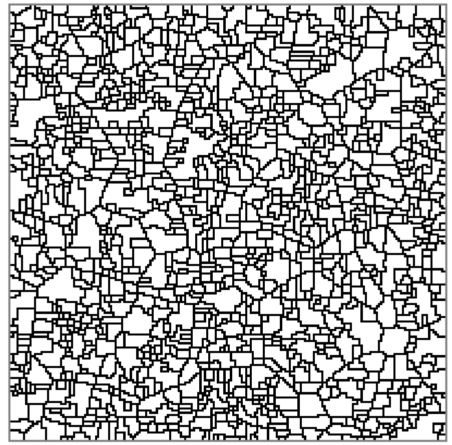
Often applied to image gradient rather than image!



Illustration

- **Oversegmentation Problem**
  - Two many local minima result in oversegmentation

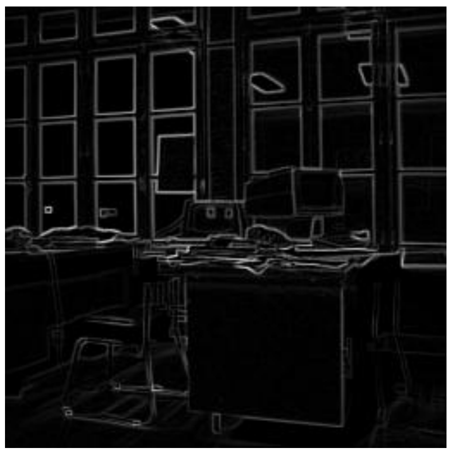


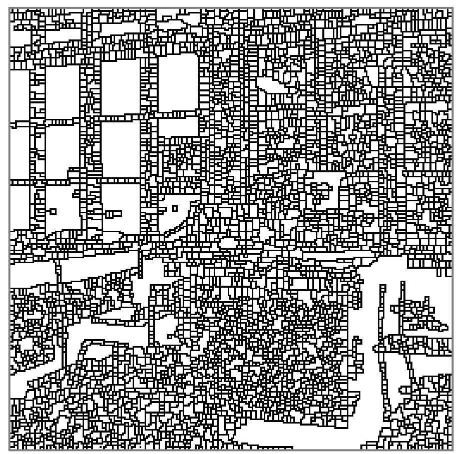


original

gradient

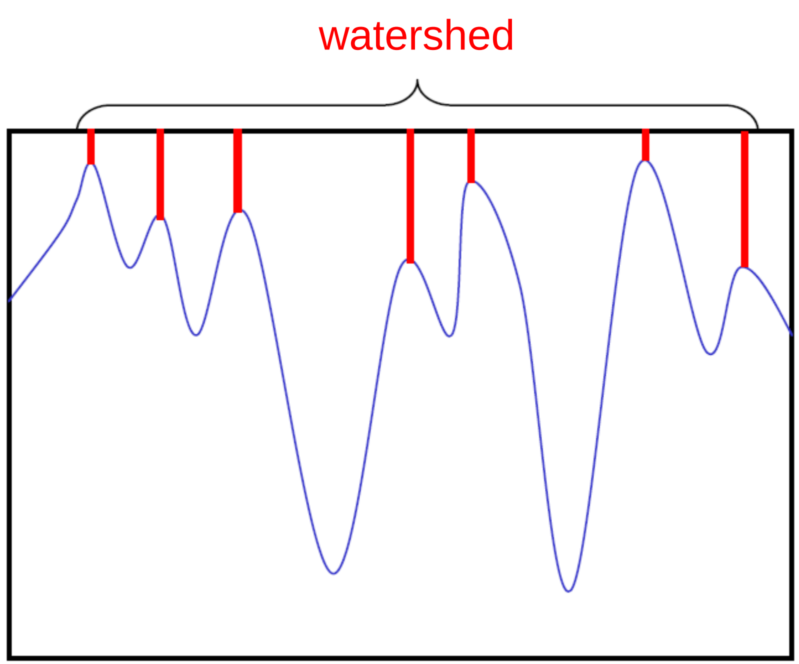
watershed

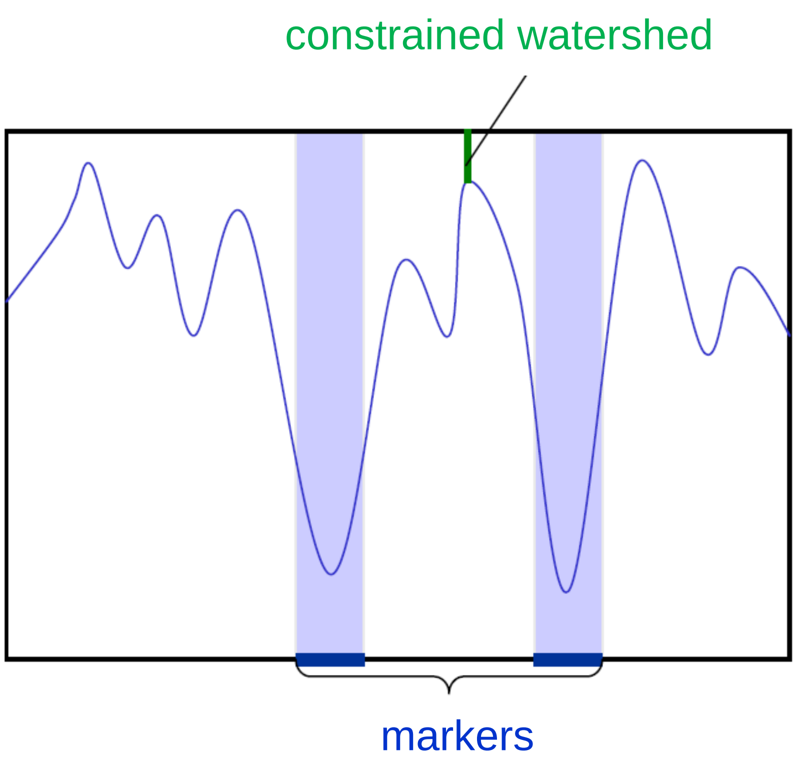




Watershed using Markers

- **Constrained Watershed**
  - The minima are imposed: markers





Watershed using Markers

- **Marker Imposition using Closing by Reconstruction**
  - The markers are the minima of the resulting filtered image

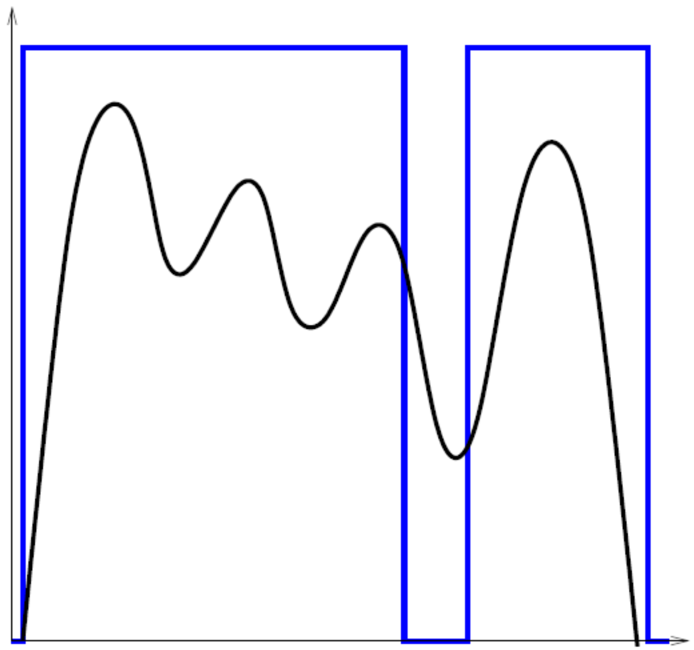
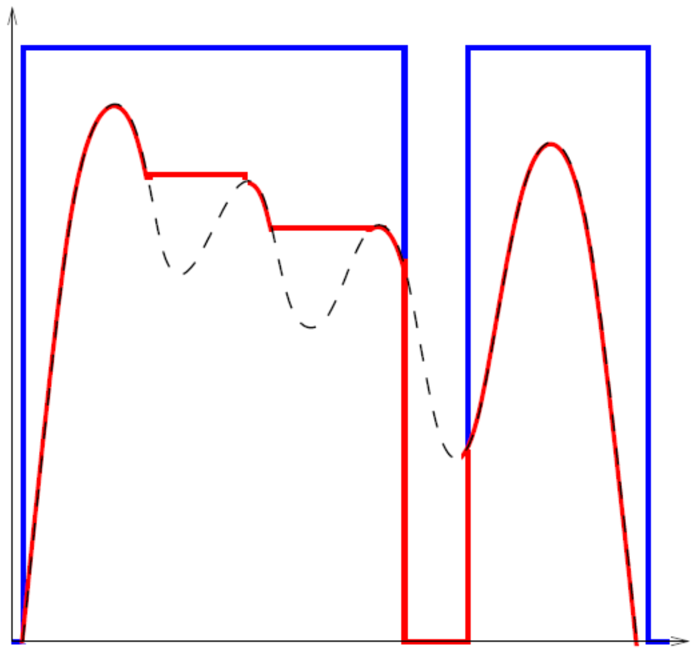


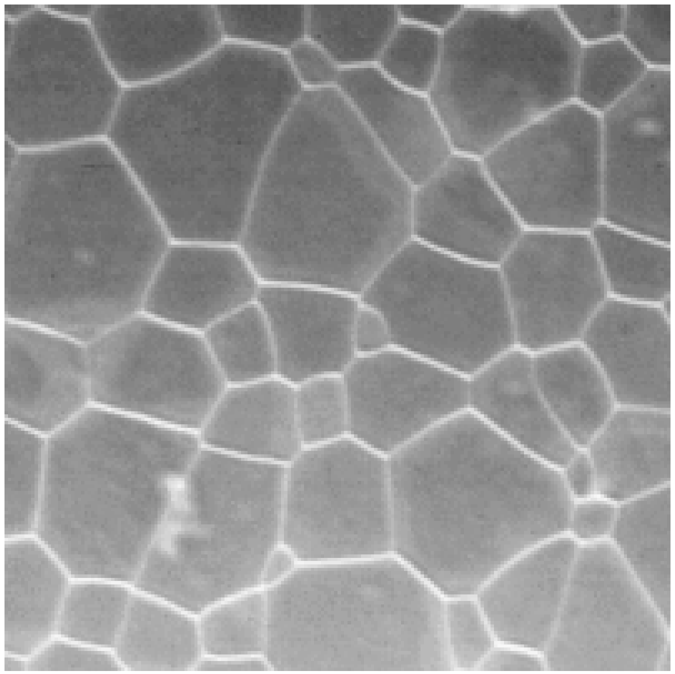
image + markers

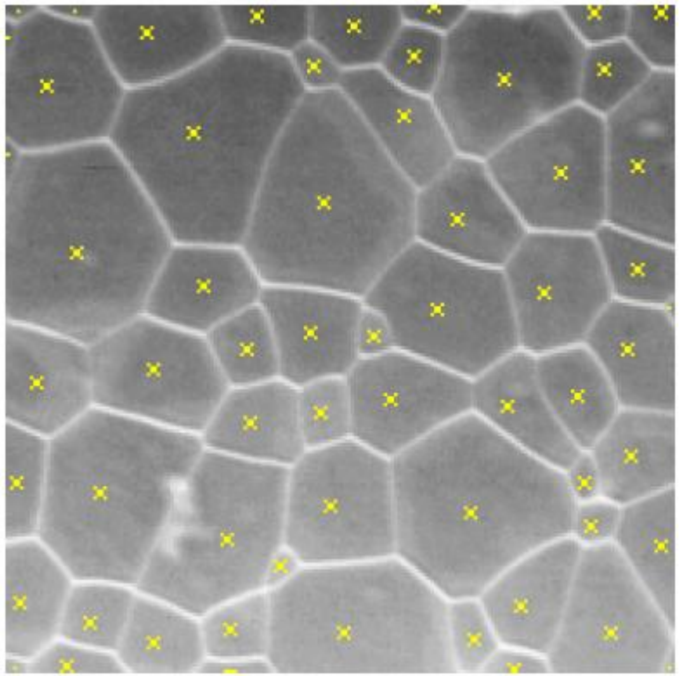


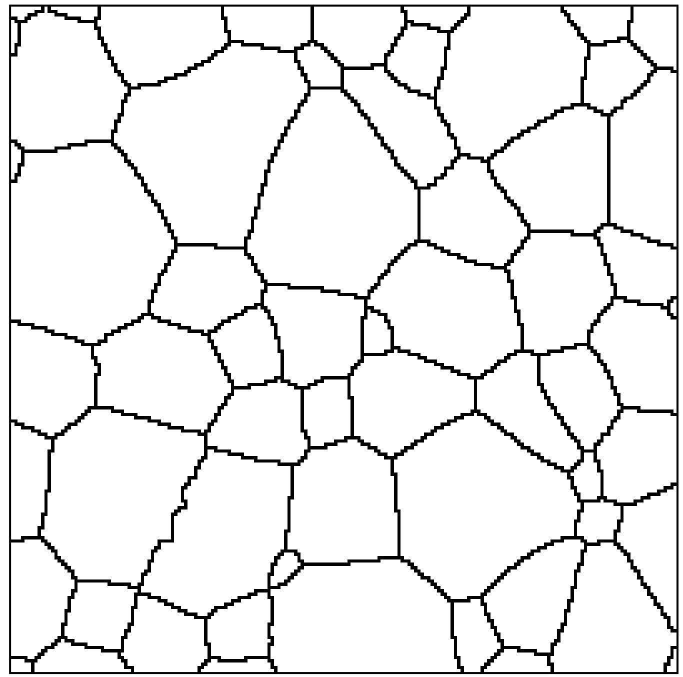
reconstructed image

Watershed using Markers

- **Illustration**

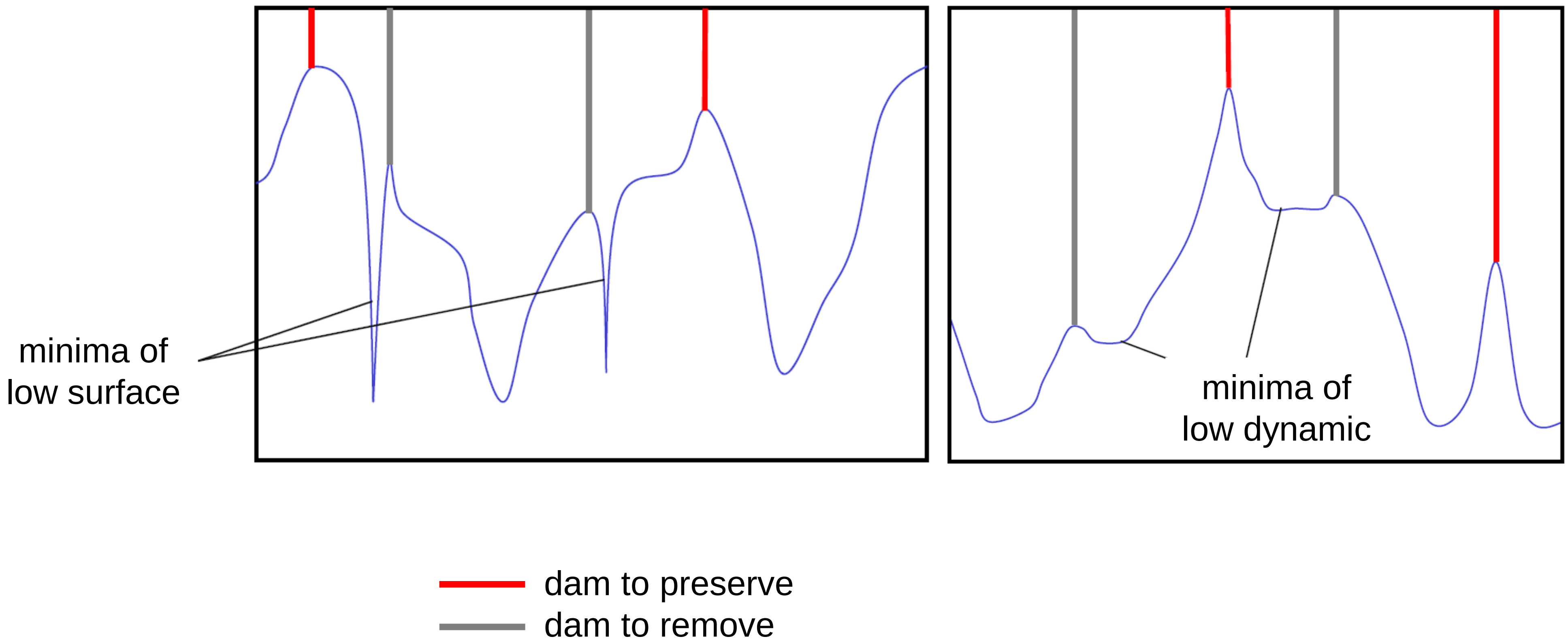

original


markers


constrained watershed

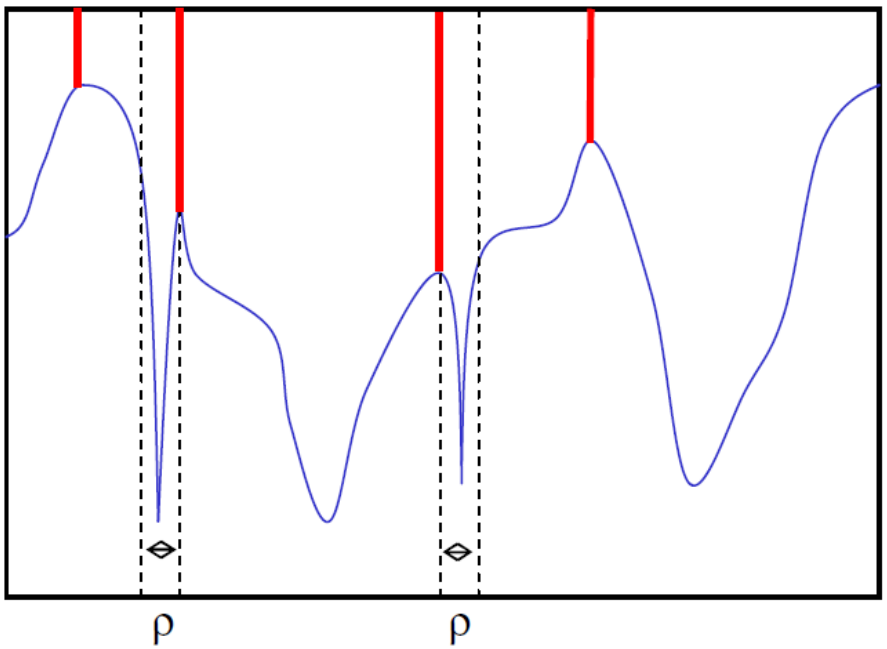
Watershed using Filtering

- **Filtering for Reducing the Numbers of Minima**
  - Surface and dynamic criteria

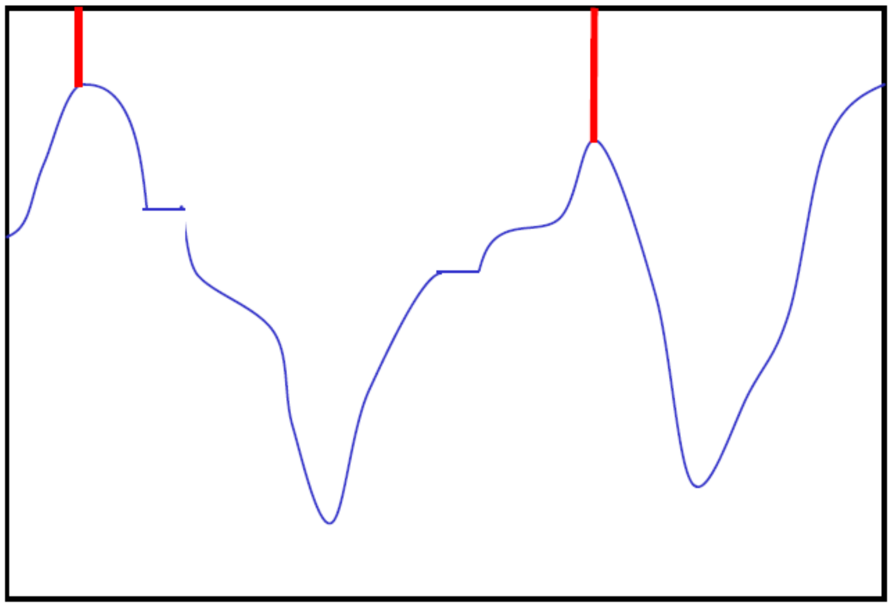


Watershed using Filtering

- **Removing of Minima with Low Surface**
  - Spatial filtering / Closing by reconstruction



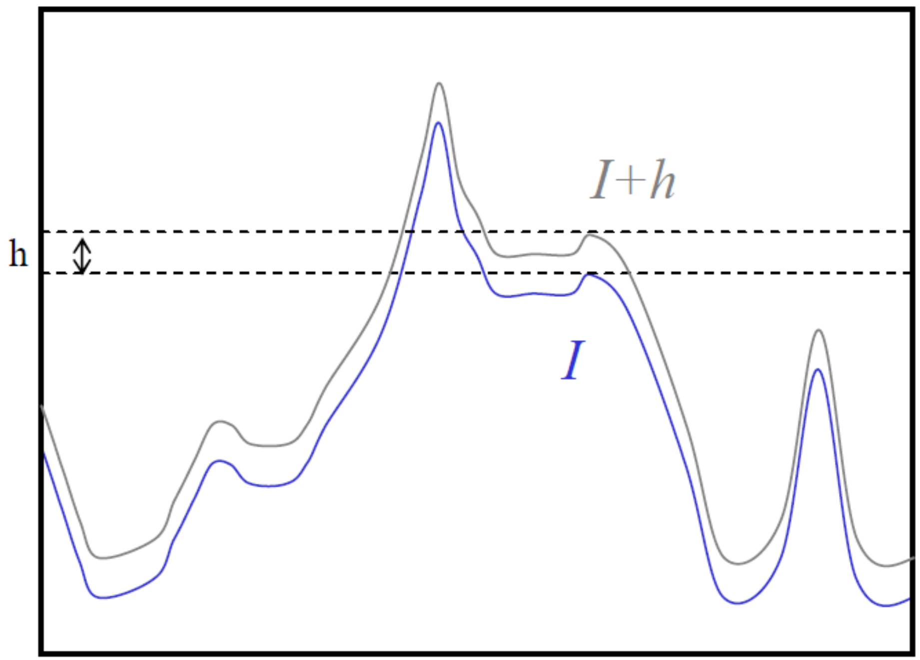
original



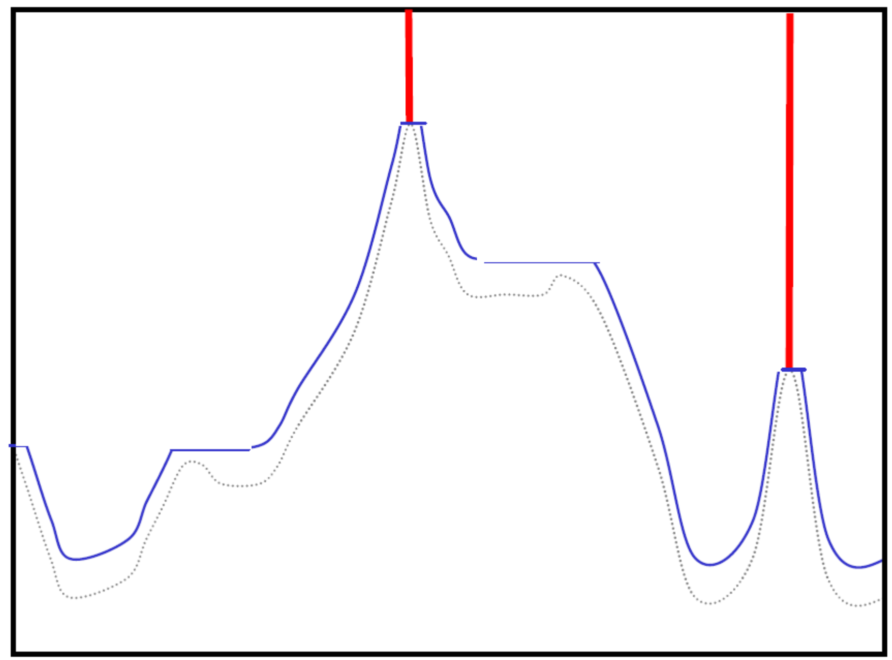
filtered image

Watershed using Filtering

- **Removing of Minima with Low Dynamic**
  - Dynamic filtering / Reconstruction of *f*+*h* over *f*
  - Removes the minima of height < *h*



original

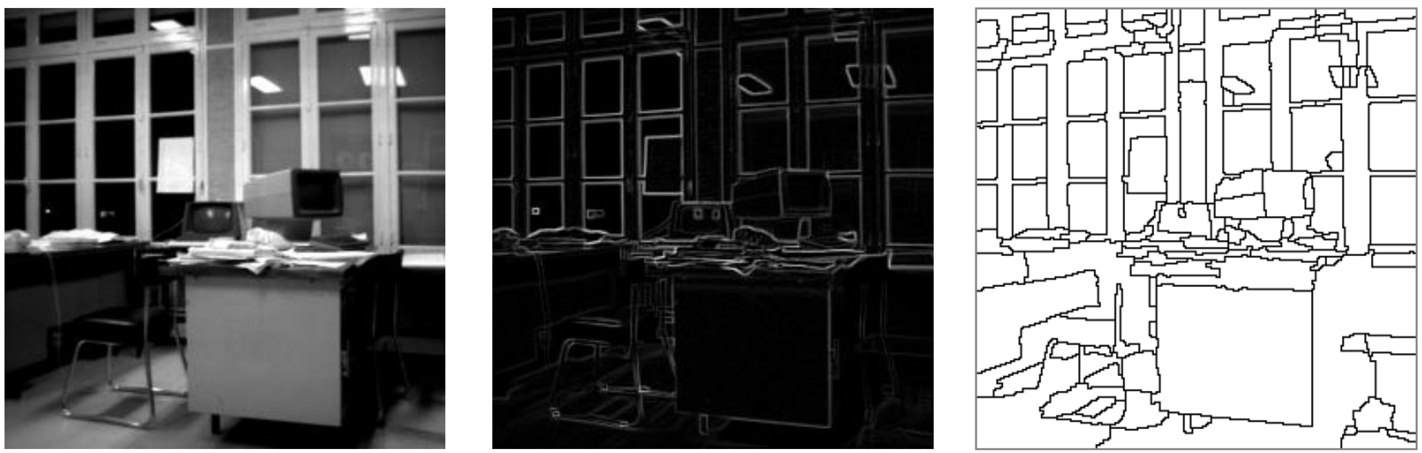


filtered image

Watershed using Filtering

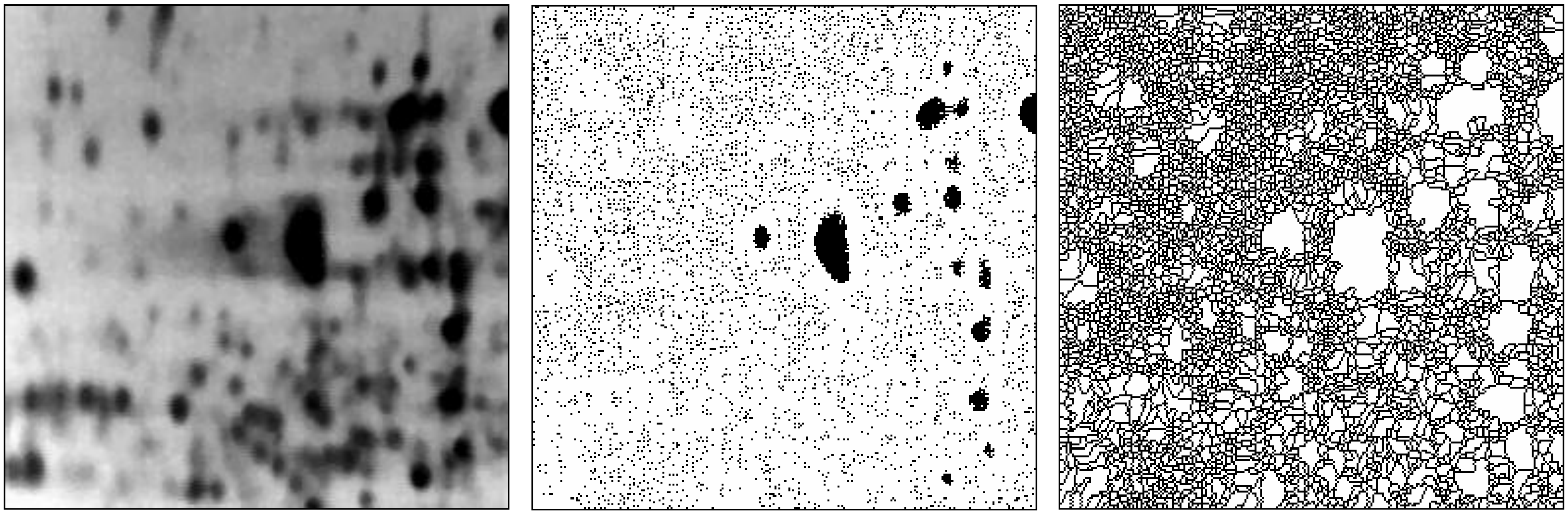
- **Illustration**

original gradient watershed



Application Example

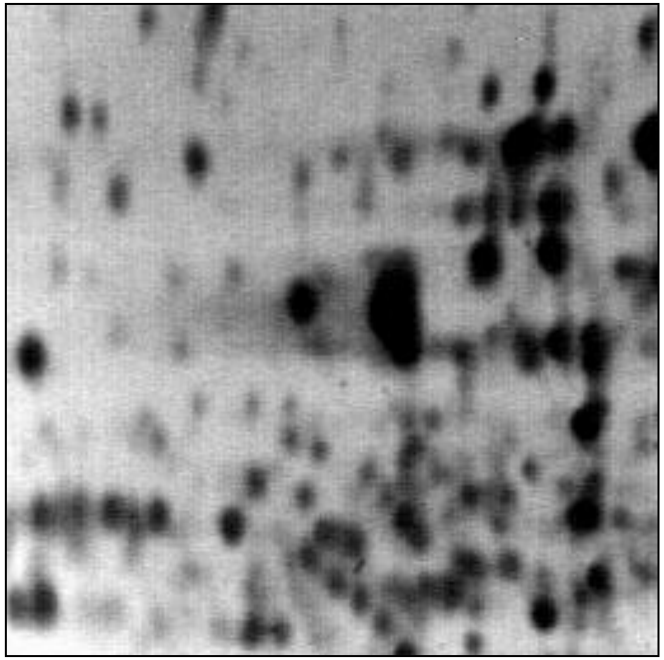
- **Segmentation of Gel Electrophoresis Image (S. Beucher and F. Meyer)**
  - Using directly the watershed

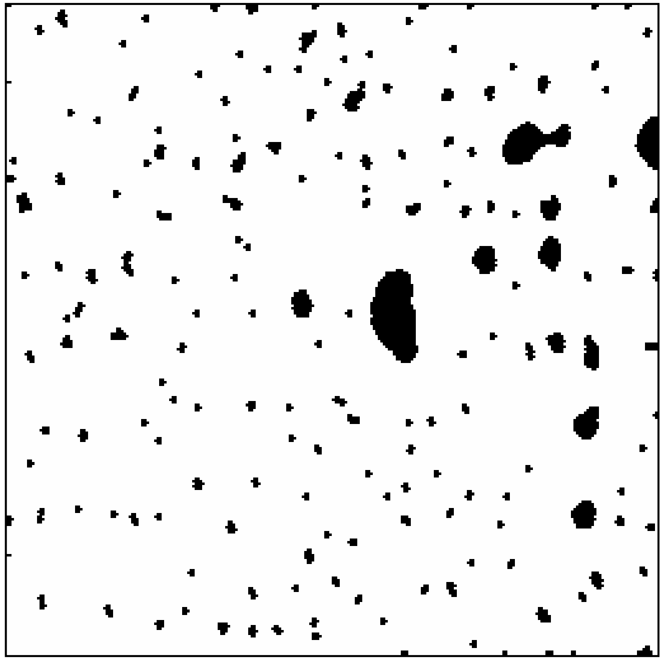


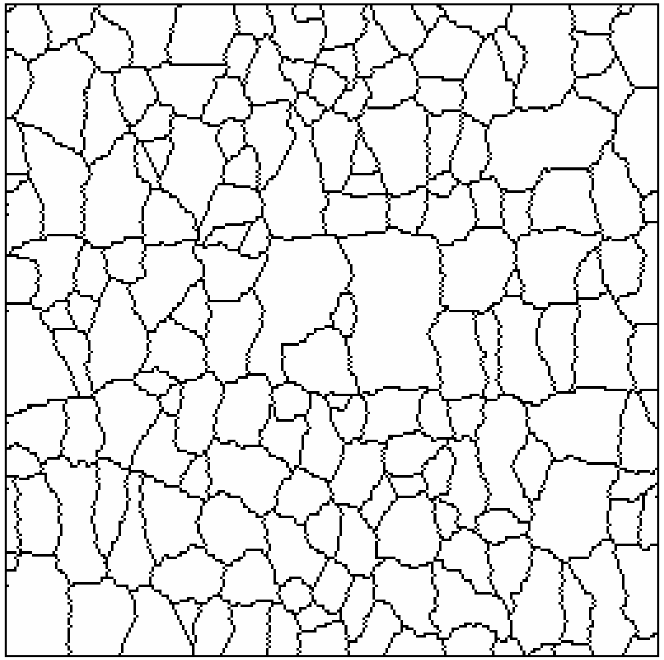
(a) original image (b) minima of the original image (c) resulting watershed

Application Example

- **Segmentation of Gel Electrophoresis Image (S. Beucher and F. Meyer)**
  - Using watershed of the filtered image

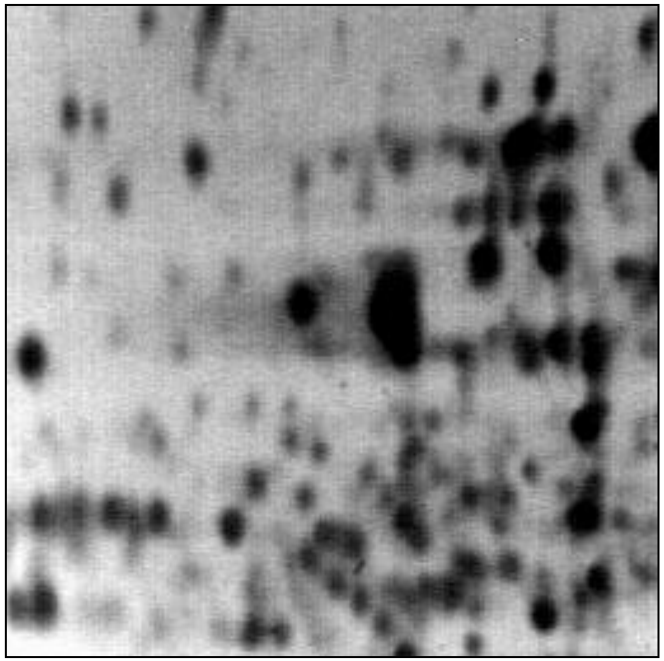

(d) alternate filtered image


(e) minima of the filtered image

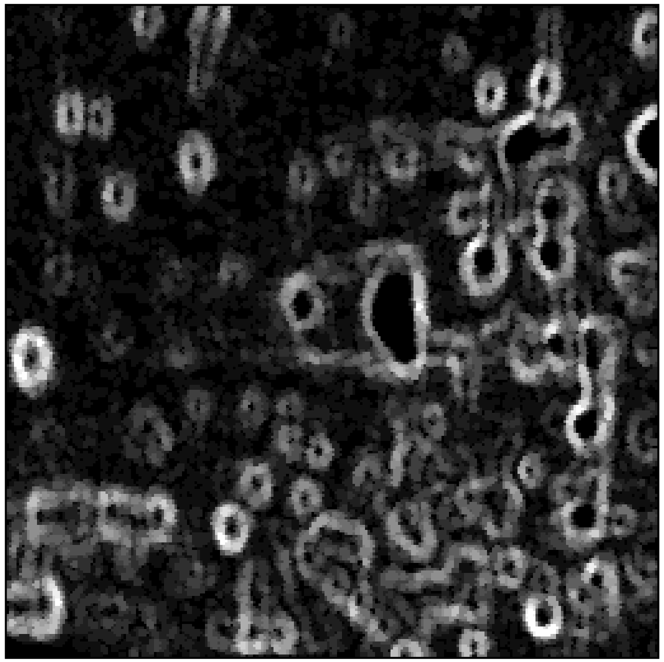

(f) resulting watershed

Application Example

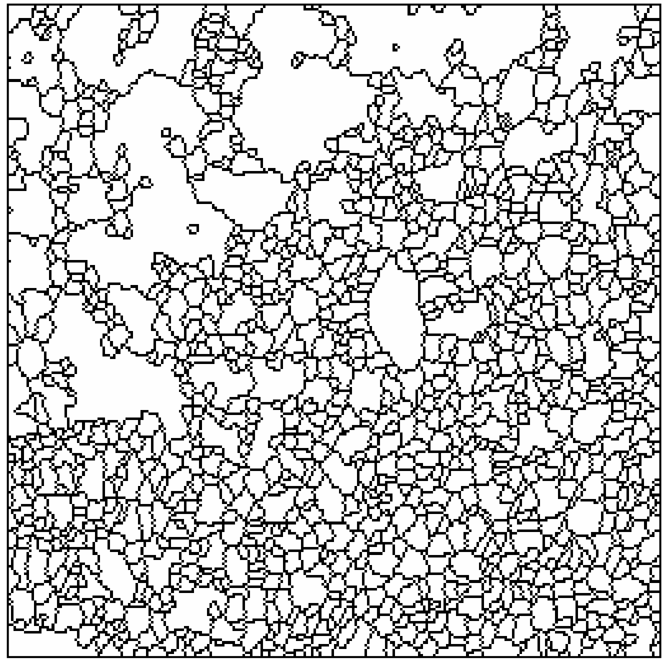
- **Segmentation of Gel Electrophoresis Image (S. Beucher and F. Meyer)**
  - Using watershed of the gradient of the filtered image



(d) alternate filtered image



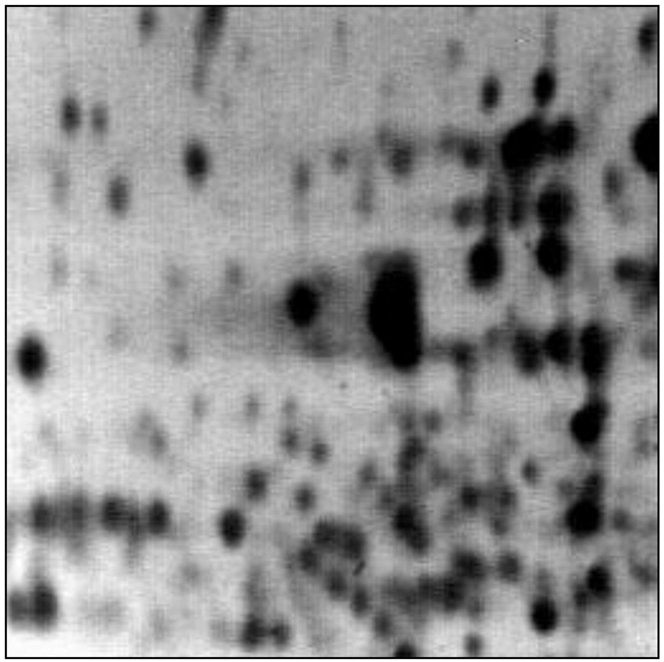
(g) gradient of the filtered image



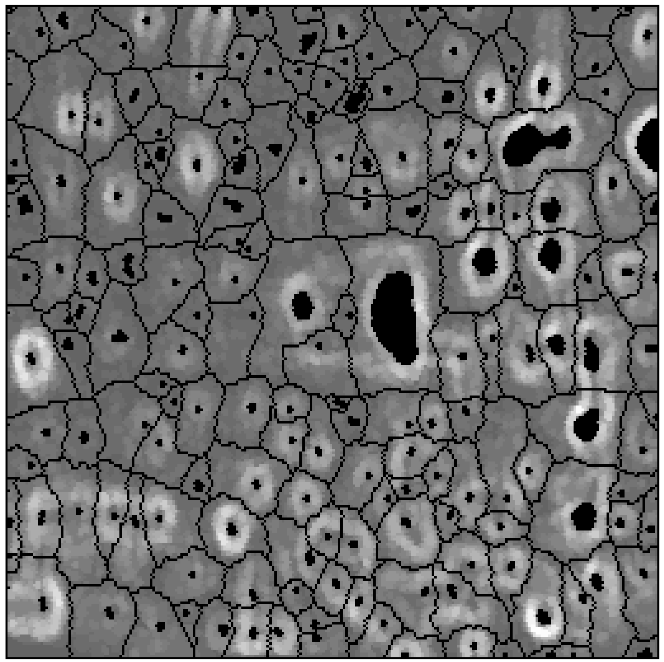
(h) resulting watershed

Application Example

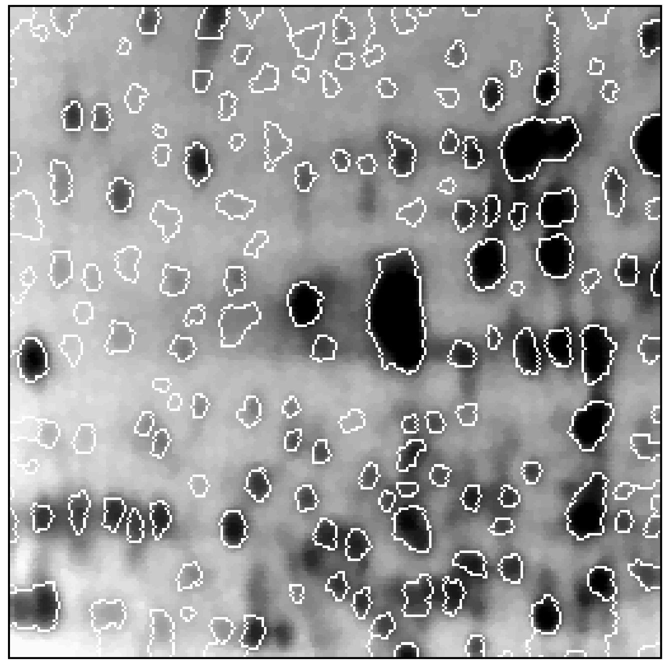
- **Segmentation of Gel Electrophoresis Image (S. Beucher and F. Meyer)**
  - Using constrained watershed with markers (e) and (f)



(a) original image



(g) markers (e,f) imposed on the gradient of the filtered image



(h) resulting watershed superimposed on the original image

Sources Used

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- **Isabelle BLOCH (Telecom ParisTech)**
- **Antoine MANZANERA (ENSTA ParisTech)**
- **Jean SERRA (ESIEE / Mines ParisTech)**
- **Marek A. PERKOWSKI (Portland State University, USA)**