UNIVERSITY OF BURGUNDY, MSCV5

Wavelets Homework Report

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1 FILTER BANK

1.1 DEFINITION

"Filter bank is an array of band-pass filters". Each band-pass filter is the selection of the different frequency components. The idea filter bank perform a more compact representation of signal, which allowing a perfect reconstruction.

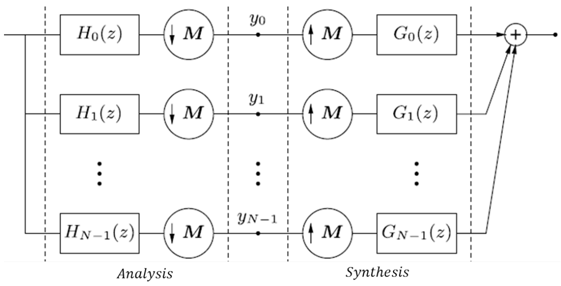


Figure 1.1: A general multidimentional filter bank. Image source: wiki

Above figure shows a general multidimentional filter bank system with N channels and a common sampling matrix M. Where the $analysis$ is signal decomposition and $synthesis$ the signal reconstruction. The down-arrow means downsampling and up-arrow means upsampling.

1.2 DOWNSAMPLING AND UPSAMPLING

The definition of downsampling in time domain and frequency domain is given by:

$$y[n] = x[Mn] \tag{1.1}$$

$$Y[\omega] = \sum_{m=Mn} x[m]e^{-i\omega m/M} = \frac{1}{M}\sum_{k=0}^{M-1} X(\frac{\omega + 2\pi k}{M}) \tag{1.2}$$

This process is called decimation. Let $M = 3$, the behavior of this system is shown in following figure:

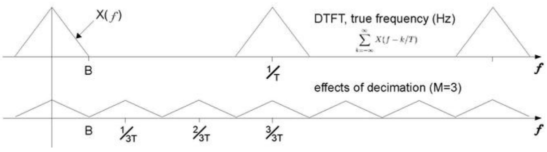


Figure 1.2: M=3 $X[\omega]$:upper $Y[\omega]$:bottom Image Source: wiki

Similary, the upsampling by L is defined as:

$$y[n] = \begin{cases} x[n/L] & ; \quad \text{n = mL} \\ 0 & ; \quad \text{otherwise} \end{cases} \tag{1.3}$$

$$Y[\omega] = \sum_{n=mL} x[n/L]e^{-i\omega n} = X[L\omega] \tag{1.4}$$

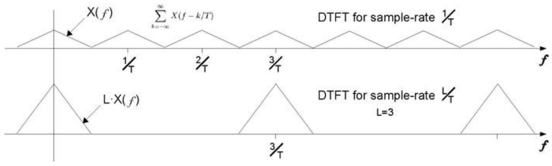


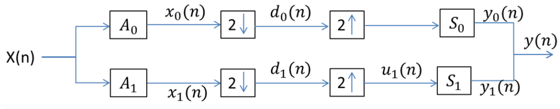
Figure 1.3: L=3 $X[\omega]$:upper $Y[\omega]$:bottom Image Source: wiki

From Figure 2.3 and Figure 2.4, it is clear that if downsampling M equal to upsampling L, the signal can be restored. And if $G(z)$ is the inverse of $H(z)$, the signal can be reconstructed.

1.3 PERFECT RECONSTRUCTION

To achieve perfect reconstruction, the system should be no distortion and satisfy alias cancellation. By using Z transformation we can figure out the condition of perfect reconstruction. The Z transformation is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{1.5}$$



A Multirate two-channel system

Figure 1.4: 2-channel filter bank low-pass:upper high-pass:bottom Image Source: wiki

As Figure 2.5 shows, we simplify the filter bank to 2 channels, which means it only has low-pass filter A_0 and high-pass filter A_1. The S_0 and S_1 is the inverse of A_0 and A_1, respectively. In the above system, the signal in Z domain at each moment is given as follow:

After band-pass filter A_i, we have:

$$X_i(z) = A_i(z)X(z) \quad i = 0,1 \tag{1.6}$$

After downsampling:

$$D_i(z) = \frac{1}{2}(X_i(z^{\frac{1}{2}}) + X_i(-z^{\frac{1}{2}})) \quad i = 0,1 \tag{1.7}$$

After upsampling:

$$U_i(z) = D_i(z^2) \quad i = 0,1 \tag{1.8}$$

After band-pass filter S_i:

$$Y_i(z) = S_i(z)U_i(z) \quad i = 0,1 \tag{1.9}$$

Signal reconstruction:

$$\begin{aligned} Y(z) &= \sum Y_i(z) \\ &= \frac{1}{2}(X(z)(S_0(z)A_0(z) + S_1(z)A_1(z)) + X(-z)(S_0(z)A_0(-z) + S_1(z)A_1(-z))) \end{aligned} \tag{1.10}$$

For perfect reconstruction, we expect $y(n) = x(n-m)$ where the m is the delay of output. Hence, $Y(z) = z^{-m}X(z)$. Consequently, the condition of perfect reconstruction is:

No distortion (The factor of $X(z)$ is the delay z^{-n}):

$$S_0(z)A_0(z) + S_1(z)A_1(z) = 2z^{-n} \tag{1.11}$$

Alias cancellation (No $X(-z)$):

$$S_0(z)A_0(-z) + S_1(z)A_1(-z) = 0 \tag{1.12}$$

To achieve alias cancellation, let:

$$\begin{aligned} S_0(z) &= A_1(-z) \\ S_1(z) &= A_0(-z) \end{aligned} \tag{1.13}$$

In time domain, it can be presented by:

$$\begin{aligned} S_0(n) &= (-1)^n A_1(n) \\ S_1(n) &= (-1)^{n+1} A_0(n) \end{aligned} \tag{1.14}$$

Therefor, if we design $A_0 = (x_0, y_0, z_0)$ and $A_1 = (x_1, y_1, z_1)$, perfect reconstruction demands $S_0 = (x_1, -y_1, z_1)$ and $S_1 = (-x_0, y_0, -z_0)$.

2 Wavelets

2.1 Wavelets Transform

In fourier transformation theory, a signal is approximation of the sum of different sin and cos waves, which is periodically and time infinitely. However, in wavelet theory, they are be replaced by scalable wavelet which is a finite and non periodic function. A variable window also be introduced.

The child wavelets is define as:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \tag{2.1}$$

where a is positive and defines the scale and b is any real number and defines the shift.

In discrete time signal processing, the formula of child wavelets is given by:

$$\Psi(k,n,m) = \frac{1}{\sqrt{c_0^n}} \cdot \Psi\left[\frac{k - mc_0^n}{c_0^n} T\right] = \frac{1}{\sqrt{c_0^n}} \cdot \Psi\left[\left(\frac{k}{c_0^n} - m\right) T\right] \tag{2.2}$$

Then Wavelet transformation is given by:

$$X(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \overline{\Psi\left(\frac{t-b}{a}\right)} x(t)\, dt \tag{2.3}$$

The formula in discrete time:

$$X_{DW}(n,m) = \frac{1}{\sqrt{c_0^n}} \cdot \sum_{k=0}^{K-1} x(k) \cdot \Psi\left[\left(\frac{k}{c_0^n} - m\right) T\right] \tag{2.4}$$

3 Filter Bank Implementation of Wavelets

Filter bank has many application in wavelets, for instence, the sub-band coding, multiresolution analysis and signal/image compression, etc. The main idea is using quadrature mirror filter to decompose signal and Mallat's pyramidal algorithm to do wavelet transform. This process obtain the approximation and detail of signal by a series of low-pass (LP) and high-pass (HP) filters.

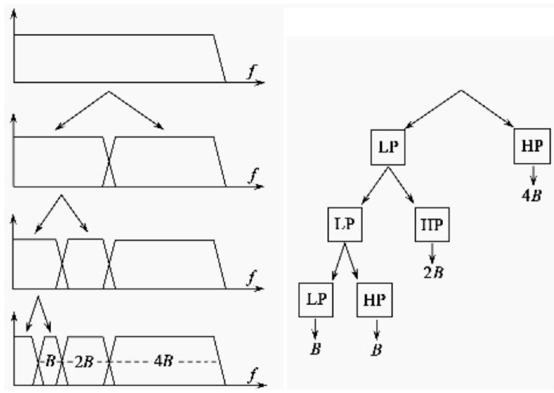


Figure 3.1: Spectrum segmentation(left) and Filter bank architecture(right)   B: band wide

Above figure actually present the discrete wavelet transform, with 32 samples, 3 levels of decomposition, 4 output scales. The same with FFT, discrete wavelet transform (DWT) was invented to speed up the transformation. Hungarian mathematician AlfrÃl’d Haar represent the input signal by a list of 2^n number by using filter bank, known as Haar wavelet transform. "The filterbank implementation of the Discrete Wavelet Transform takes only $O(N)$ in certain cases, as compared to $O(NlogN)$ for the fast Fourier transform" . It is one of the successful application. With it, the signal processing has reached the new era.

The 1D wavelet transform base on filter bank (**FWT_1D**) can be easily used for image analysis as following figure shows.

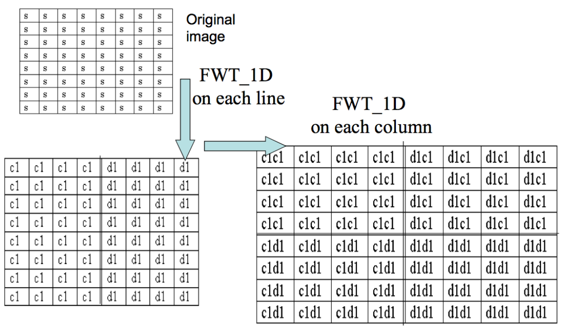


Figure 3.2: image wavelet transform

4 HOMEWORK RESULT

File **run_me.m** is the demonstration of image wavelet transform and denoising.

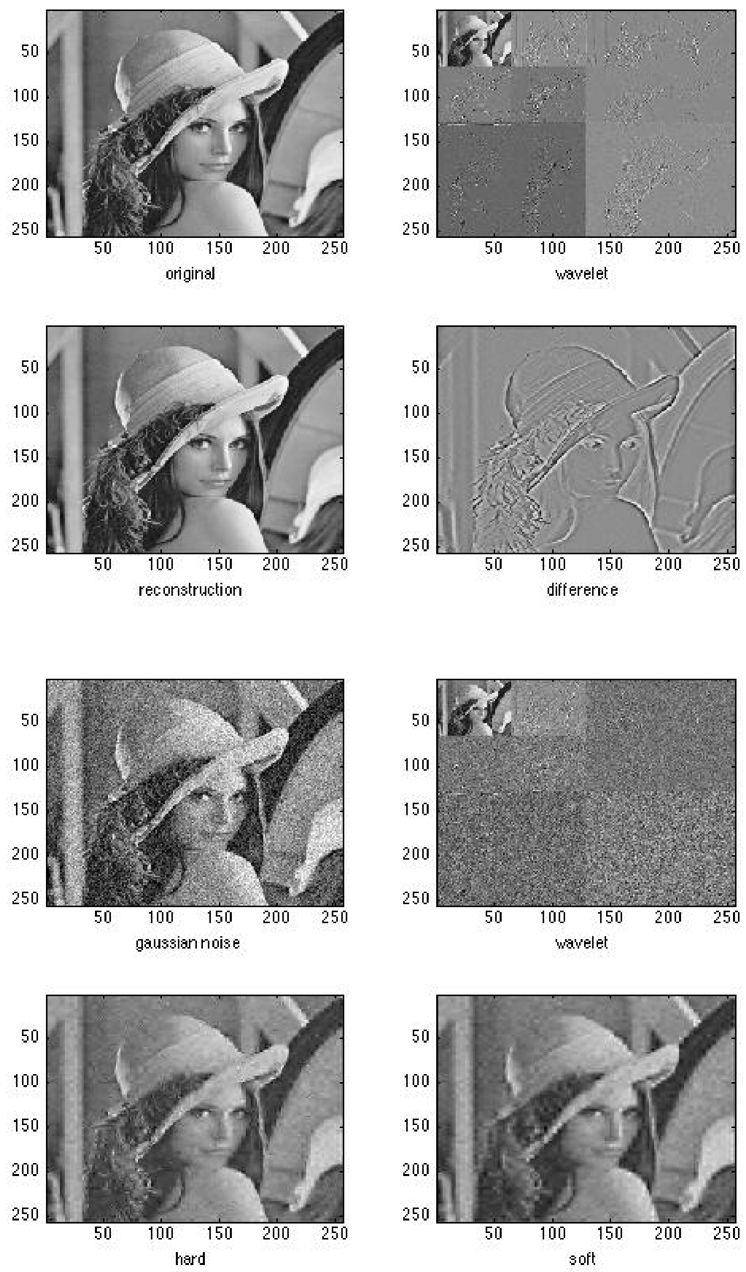


Figure 4.1: image wavelet transform, reconstraction and denoising