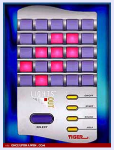
Lights Out

Reed Williston
Noah Flemens

12/17/2010

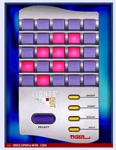


Introduction
Works Cited

Home Page
Title Page
Page 1 of 11
Go Back
Full Screen
Close
Quit

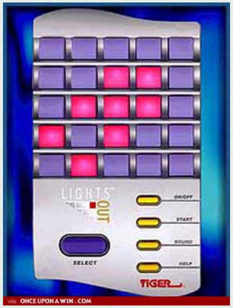
Abstract

The game Lights Out is played on a 5×5 grid. This set up makes it easy to model the game using vectors and matricies. In this paper we will discuss how to model and manipulate the game using linear algebra. We will discuss how to determine whether or not a given configuration can be solved and how to find the solution to that configuration. This paper contains all the tools necessary to analyze and solve the game of Lights Out.



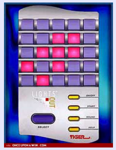
Introduction
Works Cited

Introduction



Lights out is an electronic game consisting of a 5×5 array of nodes. A node is represented by a button that toggles a light on and off with each press. Each node can be in one of two states at a given time. In the first state the node is lit, signifying it is on. When the node is unlit, it is considered to be in the second state: off. A typical configuration of a Lights Out game is represented by Figure 1. When a node is pressed, its state and the state of all nodes adjacent to the node that has been pressed, are changed to their opposite state. This can be seen in Figures 2 , 3 and 4. This effect is strictly limited to the touching squares. That is to say, there isn't a wrap around effect when a node is pressed. There are versions of the game that are played on a toroid, so activating a node in the top right corner of the board will also activate the nodes in the bottom right and top left corners of the board. The game that we will be analyzing is one that uses the standard set of rules that does not allow the wrap around effect.

The game begins with some configuration of the 25 nodes in each state. In order for a game to be solved, all of the nodes must be switched to the off state. Not all configurations are solvable. However, the game only uses solvable configurations for



Introduction
Works Cited

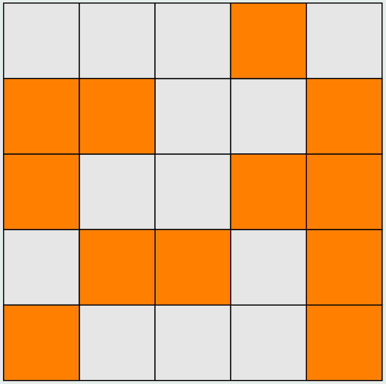


Figure 1: Orange squares are in the on state whereas grey squares are in the off state.

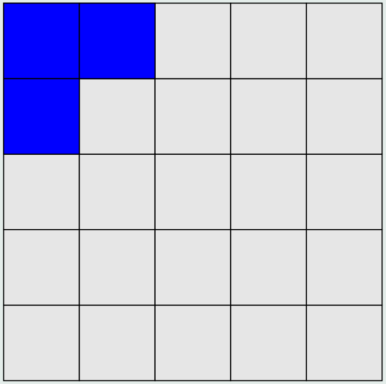


Figure 2: Pressing a corner button toggles adjacent cells.

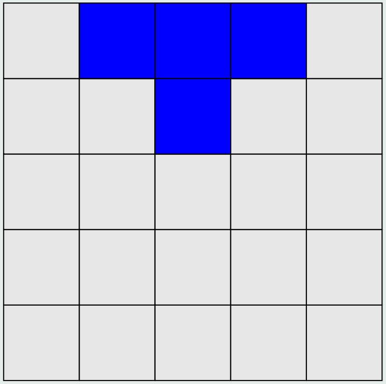


Figure 3: Pressing an edge button toggles adjacent cells.

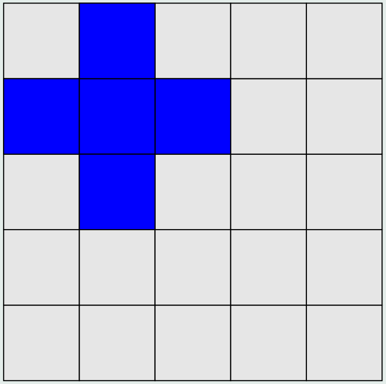
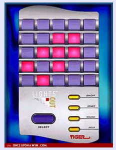


Figure 4: Pressing an interior button toggles adjacent cells.



the sake of practicality. This paper will explore what constitutes a solvable array of nodes and what makes particular combinations of nodes unsolvable.

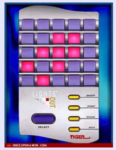
Decoding the Game

There are two observations that can be made about finding a solution to a game. The first observation is that a node need only be toggled once during the course of a game. Toggling a node once changes its state, pressing it again places the node in its initial state. Toggling a node a third time is the same as toggling it once. Thus, toggling a node an even number of times is equivalent to no change in state. While activating a state change an odd number of times is the same as if the node has been changed once. This fact, and the fact that there are only two possible states a node can be in means that all of the calculations we make must be in modulo two. The second observation is that the order in which the nodes are activated does not result in a unique outcome. This is because the adjacent nodes are going to be toggled an equal number of times regardless of the order in which each node experiences a change in state. These two observations are key to the way in which linear algebra is used to find a solution to a given configuration of node states.

To describe the game, we will use the notation b_{ij} to indicate the position of a node in the ith row and jth column. This can be seen in Figure 5 and the corresponding matrix is below.

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}$$

An element of this matrix is 1 if the corresponding node is on and 0 if the node is



b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
b_{21}	b_{22}	b_{23}	b_{24}	b_{25}
b_{31}	b_{32}	b_{33}	b_{34}	b_{35}
b_{41}	b_{42}	b_{43}	b_{44}	b_{45}
b_{51}	b_{52}	b_{53}	b_{54}	b_{55}

Figure 5: Each cell contains its name as it is represented in the vector **b**.

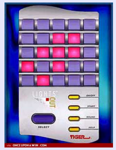
off. By arranging these entries into a 25×1 column vector by stripping off each of the row of this matrix, the configuration can be denoted by:

$$\mathbf{b} = (b_{11}, b_{12}, b_{13} \cdots b_{21}, b_{22} \cdots b_{55})^T$$

Now that we have our configuration vector **b** we will begin developing the notion of a vector that represents a strategy for solving this given configuration. We'll call this vector **x**, where **x** is a 25×1 column vector. We use the same notation as in Figure 5. It is an important point to note that a solution **x** will solve a given configuration, and will also create that configuration from a solved state.

$$\mathbf{x} = (x_{11}, x_{12}, x_{13} \cdots x_{21}, x_{22} \cdots x_{55})^T$$

An element x_{ij} is 1 if the corresponding node b_{ij} needs to be activated to effectively solve the configuration. An element is 0 if the corresponding node need not be activated



to solve the configuration. To see how a node is effected during game play, the adjacent nodes have to be taken into consideration. So the node b_{11} is effected by the strategy elements x_{11}, x_{12}, and x_{21}. Take note that these calculations are done in modulo two. For each time an adjacent node is pressed, b_{11} changes state. So if $x_{11} = 1$, signifying that it has been pressed, then b_{11} is toggled on. However, if x_{12} is pressed as well, then the sum of x_{11} and x_{12} is two, which in modulo two is equal to zero. So these two moves do not change the state of b_{11}. It follows that the state of b_{11} is the sum of the presses on the adjacent nodes: x_{11}, x_{12}, and x_{21}.

$$b_{11} = x_{11} + x_{12} + x_{21}$$

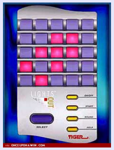
Similarly, b_{12} is effected by the strategy elements x_{11}, x_{12}, x_{13}, and x_{22}. Its state is the sum of the presses of these adjacent nodes. For example, if we toggle the nodes x_{12}, x_{13}, and x_{22}, their sum will be three which is one in modulo two. Thus, b_{12} will be toggled. So the equation pertaining to the state of b_{12} will be:

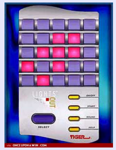
$$b_{12} = x_{11} + x_{12} + x_{13} + x_{22}$$

The equations for the rest of the nodes follow from the previous examples. These are the equations for the first five nodes and the rest are similar:

$$\begin{aligned}
b_{11} &= x_{11} + x_{12} + x_{21} \\
b_{12} &= x_{11} + x_{12} + x_{13} + x_{22} \\
b_{13} &= x_{12} + x_{13} + x_{14} + x_{23} \\
b_{14} &= x_{13} + x_{14} + x_{15} + x_{24} \\
b_{15} &= x_{14} + x_{15} + x_{25}
\end{aligned}$$

Thus, the relationship between the strategy $\mathbf{x}$ and the initial configuration $\mathbf{b}$ can be modeled by a 25×25 matrix we will call A. From these equations we can see that





elements of $\mathbf{b}$ are all linear combinations of the elements of the strategy vector $\mathbf{x}$. The matrix equation $A\mathbf{x} = \mathbf{b}$ correctly describes this relationship.

To get an idea of what A looks like, we will look at the section that shows the relationship between $\mathbf{x}$ and the first couple rows of $\mathbf{b}$

$$\begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ b_{15} \\ \\ b_{21} \\ b_{22} \\ b_{23} \\ b_{24} \\ b_{25} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0\cdots \\ 1 & 1 & 1 & 0 & 0 & & 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & 1 & 1 & 1 & 0 & & 0 & 0 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & 0 & 1 & 1 & 1 & & 0 & 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0\cdots \\ 0 & 0 & 0 & 1 & 1 & & 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & 0\cdots \\ \\ 1 & 0 & 0 & 0 & 0 & & 1 & 1 & 0 & 0 & 0 & & 1 & 0 & 0 & 0 & 0\cdots \\ 0 & 1 & 0 & 0 & 0 & & 1 & 1 & 1 & 0 & 0 & & 0 & 1 & 0 & 0 & 0\cdots \\ 0 & 0 & 1 & 0 & 0 & & 0 & 1 & 1 & 1 & 0 & & 0 & 0 & 1 & 0 & 0\cdots \\ 0 & 0 & 0 & 1 & 0 & & 0 & 0 & 1 & 1 & 1 & & 0 & 0 & 0 & 1 & 0\cdots \\ 0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 1 & 1 & & 0 & 0 & 0 & 0 & 1\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{24} \\ x_{25} \\ \vdots \end{bmatrix}$$

The separation between some of the elements in A highlight the very important blocks of the matrix. A can be more easily represented by the 5×5 block matrix:

$$A = \begin{bmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{bmatrix}$$

I is a 5×5 identity matrix, O is a 5×5 zero matrix, and B is the symmetric matrix:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

To figure out the strategy vector that will solve a given configuration, we must find a solution to $A\mathbf{x} = \mathbf{b}$. This means that $\mathbf{b}$ has to be in the column space $C(A)$ in order for a configuration to have a solution. To determine if $\mathbf{b}$ is in $C(A)$ we might want to know a bit about A. Note that $B = B^T, I = I^T, O = O^T$ so:

$$A^T = \begin{bmatrix} B^T & I^T & O^T & O^T & O^T \\ I^T & B^T & I^T & O^T & O^T \\ O^T & I^T & B^T & I^T & O^T \\ O^T & O^T & I^T & B^T & I^T \\ O^T & O^T & O^T & I^T & B^T \end{bmatrix} = \begin{bmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{bmatrix} = A$$

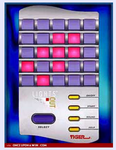
Using this information, we can now say that the row space $R(A)$ is equal to $C(A)$ because:

$$R(A) = C(A^T) = C(A)$$

Considering the fact that R(A) is the orthogonal complement to the nullspace, we can say that $\mathbf{b}$ is in $C(A)$ if and only if $\mathbf{b}$ is orthogonal to the basis of $N(A)$.

As we mentioned before, not every configuration is solvable. In order for a configuration to be solvable we have to know whether or not $\mathbf{b}$ is in C(A). One way to do this is to show that the dot product of $\mathbf{b}$ with every element of the nullspace is 0.

We use a modified RREF routine to perform row reduction in modulo 2. When this routine in run on A we can see that there are 2 free columns left in A. Thus A is a rank



Introduction
Works Cited
Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 8 of 11
Go Back
Full Screen
Close
Quit

23 matrix and the 2 free columns are:

$$[0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,0,0]^T$$
$$[1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,0]^T$$

By taking these two vectors and putting a one in the spot corresponding to the 24^{th} and 25^{th} variables respectively, we are able to obtain two vectors that form the $\mathscr{B}_{N(A)}$.

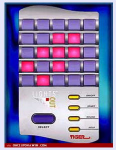
$$\mathbf{n_1} = \left[0,1,1,1,0,1,0,1,0,1,1,1,0,1,1,1,0,1,0,1,0,1,1,\overset{\downarrow}{1},0\right]^T$$

$$\mathbf{n_2} = \left[1,0,1,0,1,1,0,1,0,1,0,0,0,0,0,1,0,1,0,1,1,0,1,0,\overset{\downarrow}{1}\right]^T$$

Finding a Solution

We now have a method for determining whether or not a given configuration has a solution. Our next task is to actually solve the game. Before we look for a solution there are a few things we would like to discuss. Lets assume that we have a vector $\mathbf{x}$ that is a solution to our configuration. We can see that adding a multiple of one of the vectors in the nullspace, $\mathbf{n_1}$ or $\mathbf{n_1}$, still gives us a solution. Assume $\alpha, \beta \in \mathscr{R}$.

$$\begin{aligned} A(\mathbf{x} + \alpha\mathbf{n_1} + \beta\mathbf{n_2}) =& \mathbf{b} \\ A\mathbf{x} + A\alpha\mathbf{n_1} + A\beta\mathbf{n_2} =& \mathbf{b} \\ A\mathbf{x} + \mathbf{0} + \mathbf{0} =& \mathbf{b} \\ A\mathbf{x} =& \mathbf{b} \end{aligned}$$



Introduction
Works Cited
Home Page
Title Page
Page 9 of 11
Go Back
Full Screen
Close
Quit

Thus, if we have a strategy vector, we can see that there are actually 4 possible strategy vectors for the same configuation:

$$\begin{gathered}\mathbf{x}\\ \mathbf{x}+\mathbf{n_1}\\ \mathbf{x}+\mathbf{n_2}\\ \mathbf{x}+\mathbf{n_1}+\mathbf{n_2}\end{gathered}$$

From these we will choose the shortest solution. This would be whichever vector has the least number of nonzero entries.

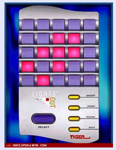
Finally, we need to get a solution. To do this we put A and $\mathbf{b}$ into an augmented matrix.

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$$

Then put it in reduced row echelon form with our modified RREF routine. This routine performs standard elimination on the first 25 columns, and simply does the same row operations on the 26th column. The 26th column is our strategy vector.

As we discussed before, this vector is a solution, but not necessarily the best one. It is easy to check each of the 4 possible solutions with a computer and choose the shortest one. We will then take our strategy vector and think of it as a 5×5 matrix again by filling out the first row of the matrix with the first five elements of $\mathbf{x}$, the second row of the matrix with the second five elements of $\mathbf{x}$, and so on.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15}\\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25}\\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35}\\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45}\\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$



Each element of $\mathbf{x}$ corresponds to a node on the board. If $\mathbf{x_{ij}}$ is 1 the corresponding node in the i^{th} row and j^{th} column must be activated. If $\mathbf{x_{ij}}$ is 0 then the corresponding node must remain unactivated in order for the winning conditions to be met. Once all these moves are carried out, the game will be solved.

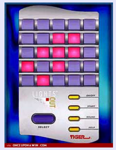
Works Cited

References

[1] D. Arnold. *2001 Writing Scientific Papers in Latex.*

[2] Marlow Anderson, Todd Feil *Turning Lights Out with Linear Algebra, Mathematics Magazine, Vol. 71, No. 4, October 1998*

[3] Oscar Martin-Sanchez, Cristobal Pareja-Flores *Two Reflected Analyses of Lights Out, Mathmatics Magazine, Vol. 74, No. 4, October 2001*



Introduction
Works Cited

Home Page

Title Page

Page 11 of 11

Go Back

Full Screen

Close

Quit