Lights Out

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12-17-10

The Game

Figure: A picture of the original Tiger Toys game.

The Game

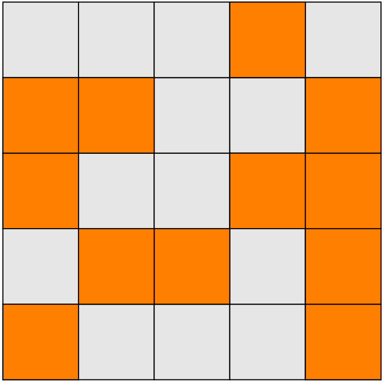

Figure: One of the 2^{25} possible configuations.

Corner Button

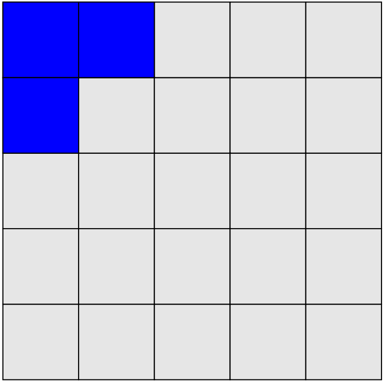

Figure: Pressing a corner button toggles adjacent cells.

Edge Button

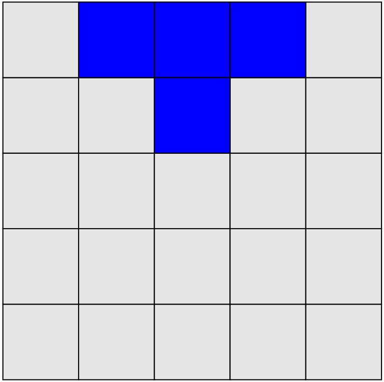

Figure: Pressing an edge button toggles adjacent cells.

Interior Button

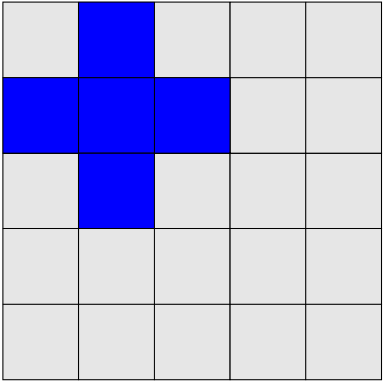

Figure: Pressing an interior button toggles adjacent cells.

Describing the Game with Vectors and Matrices

b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
b_{21}	b_{22}	b_{23}	b_{24}	b_{25}
b_{31}	b_{32}	b_{33}	b_{34}	b_{35}
b_{41}	b_{42}	b_{43}	b_{44}	b_{45}
b_{51}	b_{52}	b_{53}	b_{54}	b_{55}

Figure: This shows our scheme for labeling each button on the game board.

Describing the Game with Vectors and Matrices

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}$$

To describe the game we will use the notation b_{ij} to describe the state of the light in the ith row and jth column. Arraging these entries into a vector, we have a configuration for the game:

$$\mathbf{b} = (b_{11}, b_{12}, b_{13} \cdots b_{21}, b_{22} \cdots b_{55})$$

We will describe a initial configuration by **b** and a solution by **x**.

$A\mathbf{x} = \mathbf{b}$

We will use a 25x25 matrix A to model the relationship between between the solution $\mathbf{x}$ and the initial configuration $\mathbf{b}$. A square on the board is affected only by the adjacent squares and thus, the sum of the presses to the adjacent buttons gives the state of a button.

$$
\begin{aligned}
b_{11} &= x_{11} + x_{12} + x_{21} \\
b_{12} &= x_{11} + x_{12} + x_{13} + x_{22} \\
b_{13} &= x_{12} + x_{13} + x_{14} + x_{23} \\
b_{14} &= x_{13} + x_{14} + x_{15} + x_{24} \\
b_{15} &= x_{14} + x_{15} + x_{25}
\end{aligned}
$$

$A\mathbf{x} = \mathbf{b}$

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ \vdots \end{bmatrix}
=
\begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{14} \\ b_{15} \\ \vdots \end{bmatrix}
$$

This matrix describes the relationship between $\mathbf{x}$ and the first five entries of $\mathbf{b}$. The separation highlights the two very important blocks of the matrix.

What is A?

If we expand out every **b** and continue A all way down we get this matrix:

$$A = \begin{bmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{bmatrix}$$

Where I is a 5x5 identity matrix, O is a 5x5 matrix of zeros and B is:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Is it Solvable?

Note that $B = B^T, I = I^T, O = O^T$ so,

$$A^T = \begin{bmatrix} B^T & I^T & O^T & O^T & O^T \\ I^T & B^T & I^T & O^T & O^T \\ O^T & I^T & B^T & I^T & O^T \\ O^T & O^T & I^T & B^T & I^T \\ O^T & O^T & O^T & I^T & B^T \end{bmatrix} = \begin{bmatrix} B & I & O & O & O \\ I & B & I & O & O \\ O & I & B & I & O \\ O & O & I & B & I \\ O & O & O & I & B \end{bmatrix} = A$$

Thus, A is symmetric. This allows us to get some important information out of it.

Is it Solvable?

Because A is symmetric:

1. The Column Space of A is equal to the Row Space of A.
2. **b** is in the Column Space of A only if it is orthogonal to the Null Space of A.

Nullspace of A

- We deduce that $\mathbf{b}$ can be solved only if its dot product with the elements of N(A) is zero.
- A is a rank 23 matrix so the Nullspace only contains 2 vectors. We will call them $\mathbf{n_1}$ and $\mathbf{n_2}$.
- This gives us a way to check a configuration to see if it is solvable. If one of the dot products is not zero, there is no solution.

The Solutions

Assume $\mathbf{x}$ is a solution.

$$\begin{aligned} A(\mathbf{x} + \alpha\mathbf{n_1} + \beta\mathbf{n_2}) =& \mathbf{b} \\ A\mathbf{x} + A\alpha\mathbf{n_1} + A\beta\mathbf{n_2} =& \mathbf{b} \\ A\mathbf{x} + \mathbf{0} + \mathbf{0} =& \mathbf{b} \\ A\mathbf{x} =& \mathbf{b} \end{aligned}$$

$$\alpha, \beta = 1, 0$$

$$\mathbf{n_1}, \mathbf{n_2} \in N(A)$$

We conclude that adding either of the nullspace vectors to a solution still gives us a solution.

The Best Solution

If our winning strategy is $\mathbf{x}$. We can see that there are actually 4 winning strategies:

$$\mathbf{x}$$
$$\mathbf{x} + \mathbf{n_1}$$
$$\mathbf{x} + \mathbf{n_2}$$
$$\mathbf{x} + \mathbf{n_1} + \mathbf{n_2}$$

From these we will chose the shortest solution. This is whatever vector has the least number of nonzero entries.

Finding A Solution

$$A\mathbf{x} = \mathbf{b}$$

- We use a modified binary RREF program which does not reduce the 26th column, only preforms the same row operations on it.
- Finally to get a solution we simply put A and $\mathbf{b}$ into an augmented matrix and perform elimination on it.
- After elimination, the 26th column of the augmented matrix is the strategy vector.

Winning the Game

We then take the shortest solution and think of it as a 5x5 matrix again.

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} \end{bmatrix}$$

In each entry of the matrix:

1=push

0= do not push

Once all these are carried out, the game will be solved.

Bibliography

- D. Arnold. *2001 Writing Scientific Papers in Latex.*
- Marlow Anderson, Todd Feil *Turning Lights Out with Linear Algebra, Mathematics Magazine, Vol. 71, No. 4, October 1998*
- Oscar Martin-Sanchez, Cristobal Pareja-Flores *Two Reflected Analyses of Lights Out, Mathematics Magazine, Vol. 74, No. 4, October 2001*